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FULL GROUPS OF MINIMAL HOMEOMORPHISMS

TOMÁS IBARLUCIA AND JULIEN MELLERAY

ABSTRACT. We study full groups of minimal actions of countable groups by homeomorphisms on a Cantor space X , showing that these groups do not admit a compatible Polish group topology and, in the case of \mathbf{Z} -actions, are coanalytic non-Borel inside $\text{Homeo}(X)$. We then focus on the closure of the full group of a uniquely ergodic homeomorphism, elucidating under which conditions this group has a comeager (or, equivalently in that case, dense) conjugacy class, and point out that when that happens the closure of the full group is simple and satisfies the automatic continuity property. We also prove that the full group of a uniquely ergodic homeomorphism is topologically simple.

1. INTRODUCTION

When studying a mathematical structure, one is often led to consider the properties of its automorphism group, and then it is tempting to ask to what extent the group characterizes the structure. A particularly striking example is provided by a theorem of Dye ([D], [D2]) in ergodic theory: assume that two countable groups Γ, Δ act on the unit interval $[0, 1]$ by measure-preserving automorphisms, without any non-trivial invariant sets (i.e. the actions are *ergodic*), and consider the groups $[\Gamma]$ (resp. $[\Delta]$) made up of all measurable bijections that map each Γ -orbit (resp. Δ -orbit) onto itself. Then the groups $[\Gamma]$ and $[\Delta]$ are isomorphic if, and only if, there exists a measure-preserving bijection of $[0, 1]$ which maps Γ -orbits onto Δ -orbits. One then says that the relations are orbit equivalent; $[\Gamma]$ is called the *full group* of the action. Using this language, Dye's theorem says that the full group of an ergodic action of a countable group on a standard probability space completely remembers the associated equivalence relation up to orbit equivalence.

This result was the motivation for an intensive study of full groups in ergodic theory, for which we point to [K2] as a general reference. More recently, it came to light, initially via the work of Giordano–Putnam–Skau, that a similar phenomenon takes place in *topological dynamics*. In that context, one still considers actions of countable groups, replacing probability-measure-preserving actions with actions by homeomorphisms of a Cantor space. The two settings are related: for instance, when Γ is a countable group, one could consider the Bernoulli shift action of Γ on $\{0, 1\}^\Gamma$ as a measure-preserving action (say, for the $(1/2, 1/2)$ -Bernoulli measure) or as an action by homeomorphisms. As in the measure-theoretic setting, one can define the full group of an action of a countable group Γ on a Cantor space X : this time, it is made up of all homeomorphisms of X which map Γ -orbits onto themselves. The counterpart of ergodicity here is *minimality*, i.e. the assumption that all the orbits of the action are dense; the analog of Dye's theorem for minimal group actions was proved by Giordano–Putnam–Skau [GPS2].

The measure-theoretic and topological settings may appear, at first glance, to be very similar; however, there are deep differences, for instance all ergodic group actions of countable amenable groups are orbit equivalent (Connes–Feldman–Weiss [CFW]) while there exists a continuum of pairwise non-orbit equivalent actions of \mathbf{Z} by homeomorphisms of a Cantor space. Still, it is interesting to investigate properties of full groups in topological dynamics, which has been done by several authors over the last twenty years or so.

In both contexts discussed above, it is natural to consider the full group of an action as a topological group, the topology being induced by the topology of the ambient Polish group (measure-preserving bijections of $[0, 1]$ in one case, homeomorphisms of the Cantor space in the other). The usefulness of this approach is however limited by the fact that the full group of an ergodic group action, or a minimal group action, is not closed in the ambient group; in the first case the full group is dense, in the second it seems that the closure is currently only understood for actions of \mathbf{Z}^d .

It then comes as a blessing that, in the measure-theoretic context, one can endow the full group with a stronger topology which turns it into a Polish group: the *uniform topology*, induced by the distance given by $d(g, h) = \mu(\{x : g(x) \neq h(x)\})$. This paper grew out of the following question: can one do the same thing in the topological context? It is interesting to note that, shortly after the publication of [GPS2], Bezyglyi and Kwiatkowski [BK] introduced an analog of the uniform topology in the context of topological dynamics, which is however far from being as nice as the uniform topology of ergodic theory. This provides further motivation for trying to understand whether a nice group topology exists at all.

Theorem. *Let Γ be a countable group acting minimally by homeomorphisms on a Cantor space X . Then any Hausdorff, Baire group topology on $[\Gamma]$ must extend the topology of pointwise convergence for the discrete topology on X . Consequently, there is no second countable, Hausdorff, Baire group topology on $[\Gamma]$.*

This is bad news, but certainly not surprising —if a Polish group topology existed for that group, it would have been considered a long time ago. In the same spirit, one can then wonder about the complexity of full groups inside the ambient automorphism group; in ergodic theory, full groups are always fairly tame, in the sense that they can be written as countable intersections of countable unions of closed sets [W]. Yet again, the situation turns out to be more dire in topological dynamics.

Theorem. *The full group of a minimal homeomorphism of a Cantor space X is a coanalytic non-Borel subset of $\text{Homeo}(X)$.*

This led us to study the *closure* of a full group inside the homeomorphism group; this is a Polish group, and is also a complete invariant for orbit equivalence if one is willing to restrict one’s attention to actions of \mathbf{Z} , which we do in the last section of this article. It follows from a theorem of Glasner–Weiss [GW] that the closure of the full group of a minimal homeomorphism φ coincides with the group of homeomorphisms which preserve all φ -invariant measures. Restricting our scope again, we focus on the case where the set of invariant measures is a singleton (in other words, we study uniquely ergodic homeomorphisms); one can summarize the main results of the last section as follows.

Theorem. *Let φ be a uniquely ergodic (hence, minimal) homeomorphism of a Cantor space X , and denote by H the closure of the full group of φ inside $\text{Homeo}(X)$. Then the set of elements of finite order is dense in H , and H has a comeager conjugacy class exactly when it has a dense conjugacy class.*

One can give a combinatorial characterization¹ (in terms of the values taken on clopen sets by the unique φ -invariant measure) of uniquely ergodic homeomorphisms such that H has a comeager conjugacy class; when that happens, H is simple and has the automatic continuity property.

Theorem. *The full group of a uniquely ergodic homeomorphism of the Cantor space is topologically simple.*

There is some overlap between the second-to-last theorem above and results of Akin [A]; the techniques are however different. It is an open problem whether the full group of a minimal homeomorphism is simple or even, as far as we are aware, whether it is always topologically simple.

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2. BACKGROUND AND TERMINOLOGY

We now go over some background material and discuss in more detail some facts that were mentioned briefly in the introduction.

Recall that a *Cantor space* is a nonempty, zero-dimensional, perfect compact metrizable space; any two Cantor spaces are homeomorphic. Given a Cantor space X , we denote by $\text{Clo}(X)$ the Boolean algebra of all clopen subsets of X , and by $\text{Homeo}(X)$ the group of homeomorphisms of X . This group can be endowed with the topology whose basic open sets are of the form

$$\{g \in \text{Homeo}(X) : \forall i \in \{1, \dots, n\} g(U_i) = V_i\},$$

where n is an integer and U_i, V_i are clopen subsets of X . This turns $\text{Homeo}(X)$ into a *topological group*, namely the group operations $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are continuous with respect to this topology.

Definition 2.1. A *Polish group* is a topological group whose topology is induced by a complete, separable metric.

Picking a compatible distance d on X , one can check that the topology defined above on $\text{Homeo}(X)$ is a Polish group topology, a compatible complete distance being given by

$$d(g, h) = \max_{x \in X} d(g(x), h(x)) + \max_{y \in X} d(g^{-1}(y), h^{-1}(y)).$$

¹this characterization is a little technical to state here; we refer the reader to Theorem 5.12.

It might be a bit surprising at first that the two topologies we defined coincide; actually, this is a hint of a more general phenomenon: the unique second-countable group topologies on $\text{Homeo}(X)$ are the coarse topology and the Polish group topology we defined above (this follows from results of [A2], [G] and [RS]).

Polish groups form a fairly general class of groups, yet the combination of separability and the use of Baire category methods make them relatively tame. The fact that the Baire category theorem holds in Polish groups is particularly important; we recall that, whenever G is a topological group for which the Baire category theorem holds, H is a separable topological group and $\varphi: G \rightarrow H$ is a Borel homomorphism, then φ must actually be continuous (see e.g. [K, Theorem 9.10]).

Definition 2.2. Let Γ be a countable group acting by homeomorphisms on a Cantor space X . We denote by R_Γ the associated equivalence relation and define its *full group* as the group of all homeomorphisms of X which preserve each Γ -orbit; in symbols,

$$[R_\Gamma] = \{g \in \text{Homeo}(X) : \forall x \in X \exists \gamma \in \Gamma g(x) = \gamma \cdot x\}.$$

As is the case in ergodic theory, the full group of an action of a countable group by homeomorphisms of a Cantor space X completely remembers the associated equivalence relation, a fact made precise by the following definition and theorem.

Definition 2.3. Let Γ_1, Γ_2 be two countable groups acting by homeomorphisms on a Cantor space X , and let $R_{\Gamma_1}, R_{\Gamma_2}$ be the associated equivalence relations. We say that R_{Γ_1} and R_{Γ_2} are *orbit-equivalent* if there is a homeomorphism g of X such that

$$\forall x, y \in X (xR_{\Gamma_1}y) \Leftrightarrow (g(x)R_{\Gamma_2}g(y)).$$

Theorem 2.4 ([GPS2], [M]). *Let Γ_1, Γ_2 be countable groups acting by homeomorphisms on a Cantor space X ; assume that all orbits for both actions have cardinality at least 3, and for any nonempty $U \in \text{Clop}(X)$ and $i = 1, 2$ there exists $x \in U$ such that $\Gamma_i \cdot x$ intersects U in at least two points.*

Denote by R_{Γ_1} and R_{Γ_2} the associated equivalence relations, and suppose that there exists an isomorphism Φ from $[R_{\Gamma_1}]$ to $[R_{\Gamma_2}]$. Then there must exist $g \in \text{Homeo}(X)$ such that $\Phi(h) = ghg^{-1}$ for all $h \in [R_{\Gamma_1}]$.

Consequently, $[R_{\Gamma_1}]$ and $[R_{\Gamma_2}]$ are isomorphic if, and only if, R_{Γ_1} and R_{Γ_2} are orbit equivalent.

The above result was first proved by Giordano–Putnam–Skau [GPS2] for *minimal* actions, which we define now, and then extended by Medynets [M].

Definition 2.5. Let Γ be a countable group acting by homeomorphisms on a Cantor space. We say that the action is *minimal* if every point has a dense orbit.

Minimal actions are particularly well-studied when $\Gamma = \mathbf{Z}$. In that case the action is simply induced by one homeomorphism φ ; accordingly, we will use the notation $[\varphi]$ to denote the full group of the associated equivalence relation. Similarly, when the \mathbf{Z} -action associated to a homeomorphism φ is minimal we simply say that φ is minimal. In the case of minimal actions of \mathbf{Z} , a particular subgroup plays an important role and is well understood, mainly thanks to work of Matui.

Definition 2.6. Let φ be a homeomorphism of a Cantor space X . Its *topological full group* $[[\varphi]]$ is the set of elements $g \in \text{Homeo}(X)$ for which there is a finite clopen partition U_1, \dots, U_n of X such that on each U_i g coincides with some power of φ .

The topological full group $[[\varphi]]$ is a countable subgroup of $[\varphi]$; note that, if all orbits of φ are infinite, then for any element g in $[\varphi]$ there exists a unique $n_x \in \mathbf{Z}$ such that $g(x) = \varphi^{n_x}(x)$; the group $[[\varphi]]$ is simply made up of all g for which the associated cocycle $x \mapsto n_x$ is continuous. Another equivalent (though apparently weaker) definition is that $[[\varphi]]$ is exactly the set of elements of $[\varphi]$ for which the map $x \mapsto n_x$ has a finite range. Indeed, each set of the form $\{x \in X: g(x) = \varphi^n(x)\}$ is closed, and these sets cover X if g belongs to $[\varphi]$, so if there are only finitely many nonempty such sets then they are clopen and g belongs to $[[\varphi]]$.

The next result elucidates the action of the full group of a minimal homeomorphism on the algebra of clopen sets. Before stating it we set some notation for the sequel.

Notation. Let X be a Cantor space and φ a homeomorphism of X . A Borel probability measure μ on X is φ -invariant if $\mu(A) = \mu(\varphi^{-1}(A))$ for any Borel subset $A \subseteq X$ (if this equality holds for clopen sets then it must hold for all Borel sets). Given a homeomorphism φ , we denote by \mathcal{M}_φ its set of invariant probability measures, which is a nonempty compact, convex subset of the space of all probability measures on X .

Theorem 2.7 ([GW, Lemma 2.5 and Proposition 2.6]). *Let φ be a minimal homeomorphism of a Cantor space X , and A, B be clopen subsets of X . Then the following facts hold:*

- If $\mu(A) < \mu(B)$ for all $\mu \in \mathcal{M}_\varphi$ then there exists $g \in [[\varphi]]$ such that $g(A) \subset B$.
- $(\forall \mu \in \mathcal{M}_\varphi \mu(A) = \mu(B)) \Leftrightarrow (\exists g \in [\varphi] g(A) = B)$.

Remark 1. In both cases, one can add the assumption that $g^2 = 1$ to the right-hand statement. It is useful that in the first statement above, one can find g in the topological full group and not merely in the full group; this is not always possible when it comes to the second statement.

3. TOPOLOGIES ON FULL GROUPS.

Throughout this section we let Γ denote a countable group acting on a Cantor space X , and R denote the associated equivalence relation. We make the following assumption on the action of Γ : given any nonempty open $U \subseteq X$, there exist $x \neq y \in U$ and $\gamma \in \Gamma$ such that $\gamma \cdot x = y$. Equivalently, there exists no nonempty open subset $U \subseteq X$ such that the restriction of R to U is trivial.

Note that this assumption implies that, given any nonempty open U , there exists $g \in [R]$ and a clopen V such that $g^2 = 1$, $g(V)$ and V are nonempty disjoint clopen subsets of U and g coincides with the identity outside of $V \cup g(V)$. We now point out a further consequence.

Lemma 3.1. *The set $\Omega = \{x \in X: x \text{ is an accumulation point of } \Gamma \cdot x\}$ is dense G_δ in X .*

Proof. Fix a compatible metric d on X . Recall that x is an accumulation point of $\Gamma \cdot x$ if, and only if, the following condition holds:

$$\forall \varepsilon > 0 \exists \gamma \in \Gamma (\gamma \cdot x \neq x \text{ and } d(\gamma \cdot x, x) < \varepsilon)$$

The condition between brackets is open, showing that Ω is indeed G_δ . By the Baire category theorem, to check that Ω is dense we only need to prove that for any $\varepsilon > 0$ the set $\{x: \exists \gamma \in \Gamma \gamma \cdot x \neq x \text{ and } d(\gamma \cdot x, x) < \varepsilon\}$ is dense, and this follows immediately from our assumption on the action. \square

For the next two lemmas and proposition, we let τ denote a group topology on $[R]$ which is Hausdorff and such that $([R], \tau)$ is a Baire space.

Lemma 3.2. *For any nonempty clopen subset U of X , the set $\Omega_U = \{g \in [R] : g|_U = id|_U\}$ is τ -closed.*

Proof. We claim that $g \in [R]$ coincides with the identity on U if, and only if, $gh = hg$ for any $h \in [R]$ whose support is contained in U . Each set $\{g : gh = hg\}$ is closed since τ is a Hausdorff group topology, hence if we prove this claim we can conclude that Ω_U is an intersection of closed subsets of $[R]$ so Ω_U is closed.

Now to the proof of the claim: one inclusion is obvious; to see the converse, assume that there exists $x \in U$ such that $x \neq g(x)$. This gives us a clopen subset W of U such that W and $g(W)$ are disjoint. By assumption, there exists a clopen subset V of W and an involution $h \in [R]$ with support contained in W (hence in U) and such that V and $h(V)$ are disjoint subsets of W . Then $hg(V) = g(V)$ is disjoint from $gh(V)$, showing that g and h do not commute. \square

Lemma 3.3. *For any clopen subset U of X , the set $\Sigma_U = \{g \in [R] : g(U) = U\}$ is τ -closed.*

Proof. We may assume that U is nonempty; also, since τ is a group topology and $(g(U) = U) \Leftrightarrow (g(U) \subseteq U \text{ and } g^{-1}(U) \subseteq U)$, we only need to show that $\{g \in [R] : g(U) \subseteq U\} = \Sigma'_U$ is closed in $[R]$. To that end, one can use the same strategy as above: this time, we claim that $g \in \Sigma'_U$ if and only if, for any h which coincides with the identity on U , $g^{-1}hg$ coincides with the identity on U . Proving this will show that Σ'_U is an intersection of closed sets (by Lemma 3.2), which gives the result.

Again, one inclusion is obvious; to see the converse, we assume that $g(U)$ is not contained in U . Then there exists a nonempty clopen subset W of U such that $g(W) \cap U = \emptyset$. One can find a nontrivial involution h with support in $g(W)$. This gives us a nonempty clopen $V \subseteq U$ such that $hg(V)$ and $g(V)$ are disjoint, hence $g^{-1}hg$ does not coincide with the identity on U . \square

Proposition 3.4. *The set $\{g \in [R] : g(x) = x\}$ is τ -clopen for all $x \in X$.*

Proof. The result of Lemma 3.3 shows that the natural inclusion map from $([R], \tau)$ to $\text{Homeo}(X)$ is Borel. Since τ is assumed to be Baire (this is the first time we are using that assumption) and $\text{Homeo}(X)$ is separable, the inclusion map must be continuous, showing that each set $\{g \in [R] : g(x) = x\}$ is τ -closed. Now, fix $x \in X$ and let H denote the permutation group of the countable set $\Gamma \cdot x$, endowed with its permutation group topology. Since $\{g \in [R] : g(\gamma_1 \cdot x) = \gamma_2 \cdot x\}$ is closed for all $\gamma_1, \gamma_2 \in \Gamma$, we see that the natural homomorphism from $([R], \tau)$ to H (given by $g \mapsto (\gamma \cdot x \mapsto g(\gamma \cdot x))$) is Borel. Thus this homomorphism must be continuous, and $\{g \in [R] : g(x) = x\}$ is τ -clopen. \square

Let us sum up what we just proved.

Theorem 3.5. *Let Γ be a countable group acting by homeomorphisms on a Cantor space X . Assume that the restriction of the associated equivalence relation to a nonempty open subset of X is never trivial.*

Then any Hausdorff, Baire group topology on $[R]$ must extend the topology of pointwise convergence for the discrete topology on X . Consequently, there is no second countable, Hausdorff, Baire group topology on $[R]$.

Proof. The statement in the first sentence corresponds exactly to the result of Proposition 3.4. To see why the second statement holds, let us proceed by contradiction and assume that there exists a second countable, Hausdorff, Baire group topology on $[R]$.

We recall the result of Lemma 3.1: the set Ω made up of all x such that x is an accumulation point of $\Gamma \cdot x$ is dense G_δ in X , thus in particular uncountable. Assuming τ is second countable, the Lindelöf property implies that there exists a sequence $(x_i)_{i < \omega}$ of elements of Ω such that

$$\{g \in [R] : \exists x \in \Omega g(x) = x\} = \bigcup_{i < \omega} \{g \in [R] : g(x_i) = x_i\}.$$

However, we claim that, for any countable subset $\{x_i\}_{i < \omega}$ of Ω and any $x \in \Omega \setminus \{x_i\}_{i < \omega}$, there exists $g \in [R]$ such that $g(x) = x$ and $g(x_i) \neq x_i$ for all i ; granting this, we obtain the desired contradiction.

To conclude the proof, we briefly explain why the claim holds. Using the fact that each x_i is an accumulation point of $\Gamma \cdot x_i$, one can construct inductively a sequence of clopen sets U_j and elements γ_j of Γ with the following properties:

- For all i $x_i \in \bigcup_{j \leq i} (U_j \cup \gamma_j U_j)$.
- For all j , the diameter of $U_j \cup \gamma_j U_j$ is less than 2^{-j} , and $\gamma_j U_j \cap U_j = \emptyset$.
- For all $j \neq k$, $(U_j \cup \gamma_j U_j) \cap (U_k \cup \gamma_k U_k) = \emptyset$.
- For all j $x \notin \gamma_j U_j \cup U_j$.

One can then define a bijection g of X by setting

$$g(y) = \begin{cases} \gamma_j(y) & \text{if } y \in U_j \text{ for some } j \\ \gamma_j^{-1}(y) & \text{if } y \in \gamma_j(U_j) \text{ for some } j \\ y & \text{otherwise} \end{cases}.$$

The fact that the diameter of $U_j \cup \gamma_j U_j$ vanishes ensures that g is continuous, so g belongs to $[R]$ and satisfies $g(x) = x$, $g(x_i) \neq x_i$ for all i . □

Remark 2. It is not clear whether the Hausdorffness assumption is really needed: if $[R]$ is a simple group, then a non-Hausdorff group topology is necessarily the coarse topology; indeed, the elements that cannot be separated from 1 by an open subset form a normal subgroup of $[R]$. So, if $[R]$ is simple, then the above result says that the unique Baire, second countable group topology on $[R]$ is the coarse topology. However, it is an open question whether $[R]$ is simple, even in the case when R is induced by a minimal action of \mathbf{Z} .

The techniques of this section are close to those employed by Rosendal in [R], but it seems that his results do not cover the case studied here.

4. BOREL COMPLEXITY OF THE FULL GROUP OF A MINIMAL HOMEOMORPHISM.

The following question was suggested to us by T. Tsankov: what is the complexity (in the sense of descriptive set theory) of the full group of an equivalence relation induced by a minimal action of a countable group on the Cantor space? We answer that question for $\Gamma = \mathbf{Z}$. Below we use standard results and notations of descriptive set-theory, borrowed from KeCHRIS's book [K].

In particular, we recall that if A is a countable set then a *tree* on A is a subset T of the set $A^{<\omega}$ of finite sequences of elements of A , closed under taking initial segments (see [K, Section 2] for information on descriptive-set-theoretic trees and a detailed exposition of related notions). The set \mathcal{T} of trees on A can be endowed with a topology that turns it into a Cantor space, by setting as basic open sets all sets of the form

$$\{T \in \mathcal{T} : \forall s \in S \ s \in T \text{ and } \forall s' \in S' \ s' \notin T\},$$

where S and S' are finite subsets of $A^{<\omega}$.

A tree T is said to be *well-founded* if it has no infinite branches; in this case one can define inductively the *rank* of an element s of $A^{<\omega}$ by setting

$$\rho_T(s) = \sup\{\rho(s \frown a) + 1 : s \frown a \in T\}.$$

In particular, elements not in T and terminal nodes in T all have rank 0; then one defines the rank $\rho(T)$ of T as being equal to the supremum of all $\rho_T(s) + 1$ for $s \in A^{<\omega}$; when T is nonempty, this supremum is equal to $\rho_T(\emptyset) + 1$.

All this being said, we can begin to work, which we do by pointing out the obvious: whenever Γ is a countable group acting by homeomorphisms on a Cantor space X , the full group of the associated equivalence relation R is a co-analytic subset of the Polish group $\text{Homeo}(X)$. This is simply due to the fact that each set $\{(g, x) \in \text{Homeo}(X) \times X : g(x) = \gamma \cdot x\}$ is closed, and for all $g \in \text{Homeo}(X)$ one has

$$g \in [R] \Leftrightarrow \forall x \in X \exists \gamma \in \Gamma \ g(x) = \gamma \cdot x.$$

The above line shows that $[R]$ is the co-projection of an F_σ subset of $\text{Homeo}(X) \times X$, hence is co-analytic.

One might expect that the descriptive complexity of $[R]$ is not that high in the Borel hierarchy. For instance, in the measure-preserving context, full groups are always Borel of very low complexity: it is shown in [W] that the full group of an aperiodic, probability-measure-preserving equivalence relation is a Π_3^0 -complete subset of the group of measure-preserving automorphisms (that is, in that case the full group is a countable intersection of countable unions of closed sets). Perhaps surprisingly, it turns out that full groups of minimal homeomorphisms are not Borel.

Below, we denote by φ a minimal homeomorphism of a Cantor space X and recall that $[\varphi]$ denotes the full group of the associated equivalence relation. We also denote by \mathcal{T} the space of all trees on $\text{Clop}(X)$, endowed with the topology discussed above.

Definition 4.1. To each $g \in \text{Homeo}(X)$ we associate a tree T_g on $\text{Clop}(X)$ as follows: for any sequence (U_0, \dots, U_n) of clopen sets, (U_0, \dots, U_n) belongs to T_g iff each U_j is nonempty, $U_{j+1} \subseteq U_j$ for all $j \in \{0, \dots, n-1\}$ and $g(x) \neq \varphi^{\pm j}(x)$ for all $x \in U_j$.

Lemma 4.2. *The map $g \mapsto T_g$ is a Borel mapping from $\text{Homeo}(X)$ to \mathcal{T} . For any $g \in \text{Homeo}(X)$, g belongs to $[\varphi]$ if, and only if, T_g is well-founded.*

Proof. We need to prove that for any finite sequence of nonempty clopen subsets (U_0, \dots, U_n) the set $\{g \in \text{Homeo}(X) : (U_0, \dots, U_n) \in T_g\}$ is Borel. For this, it is enough to show that for any nonempty clopen subset U of X , the set $\{g : \forall x \in U \ g(x) \neq x\}$ is Borel. The complement of this set is $\{g : \exists x \in U \ g(x) = x\}$,

which is closed because U is clopen in X and thus compact: if g_n is a sequence of homeomorphisms of X such that for all n there exists $x_n \in U$ such that $g_n(x_n) = x_n$, and g_n converges to g in $\text{Homeo}(X)$, then up to some extraction we can assume that x_n converges to $x \in U$; the distance from $g(x_n)$ to $g_n(x_n)$ must converge to 0, so $g_n(x_n)$ converges to $g(x)$, showing that $g(x) = x$. This concludes the proof that $g \mapsto T_g$ is Borel.

Next we fix $g \in \text{Homeo}(X)$. We first assume that g does not belong to $[\varphi]$, i.e. there exists $x \in X$ such that $g(x) \neq \varphi^n(x)$ for all $n \in \mathbf{Z}$. Then, using the continuity of g and φ , one can build by induction a decreasing sequence of clopen neighborhoods U_i of x such that for all i and all $y \in U_i$ one has $g(y) \neq \varphi^{\pm i}(y)$, which yields an infinite branch of T_g . Conversely, assume that T_g is not well-founded and let $(U_i)_{i < \omega}$ be an infinite branch of T_g . Then $F = \bigcap_{i < \omega} U_i$ is nonempty, and for all $x \in F$ $g(x)$ is different from $\varphi^n(x)$ for all $n \in \mathbf{Z}$, showing that g does not belong to $[\varphi]$. \square

If $[\varphi]$ were Borel, the set $\mathcal{T}_\varphi = \{T_g : g \in [\varphi]\}$ would be an analytic subset of \mathcal{T} , hence the boundedness principle for coanalytic ranks (see [K, Theorem 35.23]) would imply the existence of a countable ordinal α such that the rank of any element of \mathcal{T}_φ is less than α . We want to prove that it is not the case, so we need to produce elements of $[\varphi]$ such that the associated tree has arbitrarily large rank. Let us introduce some notation in order to make the work ahead simpler.

Definition 4.3. For any $g \in [\varphi]$ we let $\rho(g)$ denote the rank of T_g . For any finite sequence of clopen sets (U_0, \dots, U_n) , we let $\rho_g(U_0, \dots, U_n)$ denote the rank of (U_0, \dots, U_n) with regard to the tree T_g . If α and β are ordinals, we write $\alpha \sim \beta$ to express that there are only finitely many ordinals between them; we write $\alpha \gtrsim \beta$ when $\alpha \geq \beta$ or $\alpha \sim \beta$.

Note, as an encouraging sign that our introduction of ρ is a good way to capture information about elements of $[\varphi]$, that the topological full group $[[\varphi]]$ is exactly made up of all $g \in \text{Homeo}(X)$ such that $\rho(g) < \omega$.

In order to build elements of $[\varphi]$ such that the associated tree has arbitrarily large rank, the following observation will be crucial.

Lemma 4.4. *Let g be an element of $[\varphi]$, and assume that $\rho(g) \geq \omega$. Then for any $h \in [[\varphi]]$ one has $\rho(g) \sim \rho(hg)$.*

Since $\rho(g) = \rho(g^{-1})$, the above lemma also holds true when multiplying on the right by an element of the topological full group.

Proof. For $k < \omega$, let

$$T_g^k = \{(U_k, \dots, U_n) : (U_0, \dots, U_n) \in T_g \text{ for some } U_0, \dots, U_{k-1}\}.$$

If $h \in [[\varphi]]$, then for some $k < \omega$ and for all $x \in X$ there is j such that $|j| \leq k$ and $h(x) = \varphi^j(x)$. So if $g(x) \neq \varphi^j(x)$ for all $|j| \leq n$ but $hg(x) = \varphi^m(x)$ for some m , then $|m| > n - k$. This implies that $T_g^k \subseteq T_{hg}$. Since $\rho(T_g^k) \geq \rho(T_g) - k$, we get $\rho(T_{hg}) \geq \rho(T_g) - k$, and similarly $\rho(T_g) = \rho(T_{h^{-1}hg}) \geq \rho(T_{hg}) - k$, proving the claim. \square

When U is a subset of X and g belongs to $[\varphi]$, we set

$$n(g, U) = \min(\{|k| : \exists x \in U \ g(x) = \varphi^k(x)\})$$

Lemma 4.5. *Let α be an infinite ordinal belonging to $\{\rho(g) : g \in [\varphi]\}$, and N be an integer. For any nonempty clopen $U \subseteq X$, there exists $h \in [\varphi]$ with support S contained in U (in particular, $h(U) = U$), such that $\rho(h) \gtrsim \alpha$ and $n(h, S) > N$.*

Proof. Pick $g \in [\varphi]$ such that $\rho(g) = \alpha$ is infinite and fix a nonempty clopen $U \subseteq X$ and an integer N . Using compactness of the space of φ -invariant probability measures and the fact that they are all atomless, one can find a nonempty clopen $\tilde{U} \subseteq U$ such that $(2N + 2)\mu(\tilde{U}) < \mu(U)$ for any φ -invariant measure μ .

Since φ is minimal, there exist i_1, \dots, i_n such that $X = \bigcup_{j=1}^n \varphi^{i_j}(\tilde{U})$. For all $j \in \{1, \dots, n\}$, denote $U_j = \varphi^{i_j}(\tilde{U})$, and consider the tree T_j defined by

$$(V_0, \dots, V_n) \in T_j \Leftrightarrow (V_0, \dots, V_n) \in T_g \text{ and } V_0 \subseteq U_j.$$

Denote by ρ_j the rank function associated to the well-founded tree T_j , and by $\rho(T_j)$ the rank of T_j . For any finite sequence (V_0, \dots, V_k) of clopen subsets of X , we have

$$(\forall j \in \{1, \dots, n\} \rho_j(V_0 \cap U_j, \dots, V_k \cap U_j) = 0) \Rightarrow \rho_g(V_0, \dots, V_k) = 0.$$

From this, we see by transfinite induction that $\rho(g) = \max\{\rho(T_j) : j \in \{1, \dots, n\}\}$, so there exists j such that $\rho(T_j) = \alpha$. Fix such a j ; any element of $[\varphi]$ coinciding with g on U_j must have rank larger than α .

Applying the Glasner–Weiss result recalled as Theorem 2.7, we can find $f \in [[\varphi]]$ such that $f(U_j) = W \subseteq U$. We also have

$$\mu(g(U_j)) < \mu\left(U \setminus \bigcup_{i=-N}^N \varphi^i(W)\right)$$

for any φ -invariant μ , so applying Theorem 2.7 again we can find $k \in [[\varphi]]$ such that $k(g(U_j))$ is contained in U and disjoint from $\bigcup_{i=-N}^N \varphi^i(W)$. Now, let h be equal to kgf^{-1} on W , to $fg^{-1}k^{-1}$ on $kgf^{-1}(W)$, and to the identity elsewhere. We set $S = W \cup h(W)$. Using the fact that f, k belong to $[[\varphi]]$ and Lemma 4.4, we see that $\rho(h) \gtrsim \rho(g)$. The construction ensures that $h(W)$ is disjoint from $\bigcup_{i=-N}^N \varphi^i(W)$, so $n(h, W) > N$; since h is an involution, $n(h, h(W)) = n(h, W)$ is also strictly larger than N . This ensures that $n(h, S) > N$ and all the desired conditions are satisfied. \square

Theorem 4.6. *The full group of a minimal homeomorphism of a Cantor space X is a coanalytic non-Borel subset of $\text{Homeo}(X)$.*

Proof. Let φ be a minimal homeomorphism of a Cantor space X . We explain how to produce elements of $[\varphi]$ with arbitrarily large rank. To that end, we fix for the remainder of the proof a compatible distance on X , an element g of $[\varphi]$ and a countable family (V_i) of nonempty disjoint clopen subsets of X with the following property: the tree generated by terminal nodes (U_0, \dots, U_n) of T_g such that $U_n = V_i$ for some i has rank at least ω . Note that the value n associated to such a terminal node is determined by i : we must have $g = \varphi^{\pm(n+1)}$ on V_i . We note $n = N_i$, and our hypothesis is that (N_i) is unbounded.

We then pick an infinite sequence $(W_i)_{i < \omega}$ of nonempty clopen subsets of X such that the diameter of each W_i is less than 2^{-i} and $W_i \subseteq V_i$ for all i . Now, let g_i be any sequence of elements of $[\varphi]$ of infinite rank; using Lemma 4.5, we can find

elements h_i of $[\varphi]$ with support S_i contained in W_i and such that $\rho(h_i) \gtrsim \rho(g_i)$. We shall also ask that $n(h_i, S_i) > 2N_i + 1$. We then define $h: X \rightarrow X$ by setting

$$h(x) = \begin{cases} gh_i(x) & \text{if } x \text{ belongs to some } W_i \\ g(x) & \text{otherwise} \end{cases}.$$

Note that, since the sets W_i are pairwise disjoint, h is well-defined. We next show that h is continuous. Let (x_i) be a sequence of elements of X converging to some $x \in X$. If x belongs to W_j for some j then $x_i \in W_j$ for i large enough and continuity of g and h_j ensure that $h(x_i)$ converges to $h(x)$. So we may assume that x does not belong to $\cup W_j$; in that case $h(x) = g(x)$ and since g is continuous we may also assume that x_i belongs to some W_{j_i} for all i . Each W_i is clopen, so we must have $j_i \rightarrow +\infty$, hence the diameter of W_{j_i} converges to 0. Since g is uniformly continuous, the diameter of $g(W_{j_i})$ also converges to 0; $h(x_i)$ and $g(x_i)$ both belong to this set, showing that $d(g(x_i), h(x_i))$ converges to 0. Hence $h(x_i)$ converges to $g(x) = h(x)$, proving that h is continuous. The construction also ensures that h is bijective, so h is a homeomorphism of X , and h belongs to $[\varphi]$. The definition of h and the argument of Lemma 4.4 (using the fact that $g = \varphi^{\pm(N_i+1)}$ on W_i) ensure that $\rho(h) \gtrsim \rho(h_i)$ for all i . If we have $\rho(g_i) \geq \alpha_i$ for limit ordinals α_i , we then obtain $\rho(h) \geq \sup \alpha_i$.

Now let α be a countable limit ordinal, and f an element of $[\varphi]$ such that $\rho(f) \geq \alpha$; we now explain how to produce an element of $[\varphi]$ with rank greater than $\alpha + \omega$, which will conclude the proof. The element in question, again denoted by h , is obtained by applying the construction above with $g_i = f$ for all i . Let (U_0, \dots, U_n) , $U_0 = U_n = V_i$, be a terminal node of T_g , so in particular $g = \varphi^{\pm(n+1)}$ on U_0 , $n = N_i$. Since $n(h_i, S_i) > 2n + 1$, the only way to have $gh_i(x) = \varphi^m(x)$ for $x \in U_0$ is with $|m| > n$, and this says that (U_0, \dots, U_n) belongs to T_{gh_i} (and to T_h) as well. Once we know this, the argument of Lemma 4.4 implies that $\rho_h(U_0, \dots, U_n) \gtrsim \rho(h_i) \geq \alpha$: if $(U'_0, \dots, U'_k) \in T_{h_i}$, $U'_0 \subset U_n$, $2n + 1 < k$, then $(U_0, \dots, U_n, U'_{2n+2}, \dots, U'_k) \in T_h$. We conclude that $\rho(h) \geq \alpha + n = \alpha + N_i$. Since this is true for every i , we get $\rho(h) \geq \alpha + \omega$, as expected. \square

Remark 3. Given the result we just proved, it seems likely that the full group of an equivalence relation induced by a minimal action of a countable group Γ on a Cantor space is never Borel. The above argument may be adapted in large part, but it is not clear to the authors how one can modify Lemma 4.5 in a context where Theorem 2.7 does not hold.

One can nevertheless note that the above result extends to relations induced by actions of \mathbf{Z}^d for all integers d , though this extension of the result is not really meaningful, indeed it is trivial once one knows that full groups associated to minimal \mathbf{Z}^d -actions are the same as full groups associated to minimal \mathbf{Z} -actions, a powerful result proved in [GMPS].

5. CLOSURES OF FULL GROUPS.

We saw above that there does not exist a Hausdorff, Baire group topology on the full group of a minimal homeomorphism φ of a Cantor space X . This precludes the usage of Baire category methods; however, the same cannot be said of the closure of $[\varphi]$, which is of course a Polish group since it is a closed subgroup of $\text{Homeo}(X)$ (and the arguments of Section 3 show that the topology induced by

that of $\text{Homeo}(X)$ is the unique Polish topology on the closure of $[\varphi]$ which is compatible with the group operations). As pointed out in [GPS2], the closure of $[\varphi]$ is easy to describe thanks to Theorem 2.7: letting \mathcal{M}_φ denote the (compact, convex) set of all φ -invariant probability measures, we have

$$\overline{[\varphi]} = \{g \in \text{Homeo}(X) : \forall \mu \in \mathcal{M}_\varphi \ g_*\mu = \mu\} .$$

Notation. Below we denote the closure of the full group of φ in $\text{Homeo}(X)$ by G_φ .

This group is relevant when studying topological orbit equivalence of minimal homeomorphisms, because of a theorem of Giordano–Putnam–Skau which implies the following result.

Proposition 5.1. *Let φ_1, φ_2 be two minimal homeomorphisms of a Cantor space X ; assume that G_{φ_1} and G_{φ_2} are isomorphic (as abstract groups). Then φ_1 and φ_2 are orbit equivalent.*

Proof. Assume that $\Phi: G_{\varphi_1} \rightarrow G_{\varphi_2}$ is a group isomorphism. First, the usual reconstruction techniques (see e.g. [M]) mean that there must exist a homeomorphism $h \in \text{Homeo}(X)$ such that $\Phi(g) = hgh^{-1}$ for all $g \in G_{\varphi_1}$.

Then h must be such that

$$\forall g \in \text{Homeo}(X) \quad g \in G_{\varphi_1} \Leftrightarrow hgh^{-1} \in G_{\varphi_2} .$$

Since \mathcal{M}_{φ_i} is equal to the set of measures which are invariant under translation by elements of G_{φ_i} (for $i = 1, 2$), this means that

$$\forall \mu \quad \mu \in \mathcal{M}_{\varphi_1} \Leftrightarrow h_*\mu \in \mathcal{M}_{\varphi_2} .$$

Then ([GPS, Theorem 2.2(iii)]) implies that φ_1 and φ_2 are orbit equivalent . □

Of course, the converse of the above statement is true: if φ_1 and φ_2 are orbit equivalent, then their full groups are conjugated inside $\text{Homeo}(X)$, so the closures of the full groups are also conjugated. However, the statement above is only valid a priori for actions of \mathbf{Z} : while it is true that for any minimal actions of countable groups an isomorphism between the closures of the respective full groups must be implemented by an homeomorphism of X , there is no reason why this homeomorphism would be sufficient to prove that the full groups themselves are isomorphic. Indeed, using ideas from ergodic theory, one can see that there are plenty of examples of actions of countable groups Γ_1, Γ_2 on a Cantor space X such that $\overline{[R_{\Gamma_1}]} = \overline{[R_{\Gamma_2}]}$, yet the two associated relations are not orbit equivalent. The example below was explained to us by Damien Gaboriau.

Proposition 5.2. *There exists an action of \mathbf{Z} and an action of the free group F_3 on three generators on a Cantor space X , such that the closures of the full groups of the two actions coincide, yet the relations are not orbit equivalent.*

Proof. Let \mathbf{Z} act on the Cantor space $\{0, 1\}^\omega$ via the usual odometer map. Consider the free group F_2 on two generators acting by the Bernoulli shift on $\{0, 1\}^{F_2}$; using a bijection between ω and F_2 , one can see this as an action of F_2 on $\{0, 1\}^\omega = X$. Let $F_3 = F_2 * \mathbf{Z}$ act on X , where the action of F_2 is the Bernoulli shift and the action of \mathbf{Z} is via the odometer map. Then the actions of \mathbf{Z} and F_3 on $\{0, 1\}^\omega$ both preserve the $(1/2, 1/2)$ -Bernoulli measure μ on 2^ω ; since the odometer is uniquely ergodic,

we see that for both actions the closure of the full group is equal to the set of all homeomorphisms which preserve μ . Yet, there cannot even exist a μ -preserving bijection h of X such that, for μ -almost all $x, x' \in X$, one has

$$(xR_{\mathbf{Z}}x') \Leftrightarrow (h(x)R_{F_3}h(x')).$$

Indeed, the relation induced by the action of \mathbf{Z} is hyperfinite, while the relation induced by the action of F_3 contains a subrelation which is induced by a free action of F_2 , so it cannot be hyperfinite (see for instance [K2] for information on probability-measure-preserving group actions and the properties we use here without details). Since a homeomorphism realizing an orbit equivalence between $R_{\mathbf{Z}}$ and R_{F_3} would have to preserve μ , we see that while the closures of both full groups coincide, the associated relations cannot be orbit equivalent. \square

In view of this, the following question might be interesting.

Question 5.3. Let Γ_1, Γ_2 be two countable *amenable* groups acting minimally on a Cantor space X . Assume that the closures of the corresponding full groups are isomorphic as abstract groups. Must the two actions be orbit equivalent?

If one knew that any minimal action of a countable amenable group is orbit equivalent to a \mathbf{Z} -action then the answer to the question above would be positive; the result of [GMPS] mentioned at the end of the previous section implies that the above question has a positive answer when Γ_1, Γ_2 are finitely generated free abelian groups.

We already pointed out that it is unknown whether the full group of a minimal homeomorphism φ is simple. This reduces to deciding whether the full group coincides with its derived subgroup. Indeed, it is proved in [BM, Theorem 3.4] that any normal subgroup of $[\varphi]$ contains its derived subgroup; the same is true for G_φ , as can be seen by following the proof of [BM].

Unfortunately, it seems to be hard in general to decide which elements of $[\varphi]$ are products of commutators (though one might conjecture that every element has this property; partial results in this direction can be found in [BM]). The use of Baire category methods might make things simpler in the case of G_φ , especially in view of the following folklore result.

Proposition 5.4. *Let G be a Polish group; assume that G has a comeager conjugacy class. Then every element of G is a commutator.*

Proof. Assume that Ω is a comeager conjugacy class in G , and let $g \in G$. The intersection $g\Omega \cap \Omega$ is nonempty; picking an element g_0 in this intersection, we see that there exists $k \in G$ such that $gkg_0k^{-1} = g_0$, in other words $g = g_0kg_0^{-1}k^{-1}$. \square

It is thus interesting to understand when G_φ has a comeager conjugacy class, even more so because of the following observation.

Proposition 5.5. *Let φ be a minimal homeomorphism of a Cantor space X , and assume that G_φ has a comeager conjugacy class. Then G_φ has the automatic continuity property, i.e. any homomorphism from G_φ to a separable topological group is continuous.*

Proof. The argument in [RS, Theorem 12] adapts straightforwardly. \square

From now on, we focus on the case when φ is uniquely ergodic, i.e. there is a unique φ -invariant probability measure.

Definition 5.6. A Borel probability measure μ on a Cantor space X is said to be a *good measure* if μ is atomless, has full support, and satisfies the following property: whenever A, B are clopen subsets of X such that $\mu(A) \leq \mu(B)$, there exists a clopen subset C of B such that $\mu(C) = \mu(A)$.

Note that in the definition above the fact that A, B, C are clopen is essential. Good measures are relevant in our context because of the following fact.

Theorem 5.7 (Akin [A]; Glasner–Weiss [GW]). *Let μ be a probability measure on a Cantor space X . There exists a minimal homeomorphism φ of X such that $\{\mu\} = \mathcal{M}_\varphi$ if, and only if, μ is a good measure.*

The fact that the goodness of μ is a necessary condition in the result above is due to Glasner–Weiss (it follows directly from the result we recalled as Theorem 2.7); the fact that is sufficient is due to Akin.

It is natural to ask the following question, which we only mention in passing.

Question 5.8. Can one give a similar characterization of compact, convex subsets K of the set of probability measures on a Cantor space X for which there exists a minimal homeomorphism φ of X such that K is the set of all φ -invariant measures?

The following invariant of good measures is very useful.

Definition 5.9 (Akin [A]). Let μ be a good measure on a Cantor space X . Its *clopen value set* is the set

$$V(\mu) = \{r \in [0, 1] : r = \mu(A) \text{ for some clopen } A \subseteq X\}.$$

A good measure μ on a Cantor space X is completely characterized by its clopen value set, in the sense that for any two good measures μ, ν on X with the same clopen value set there must exist a homeomorphism g of X such that $g_*\mu = \nu$ (see [A, Theorem 2.9]). If μ is a good measure, then $V(\mu)$ is the intersection of a countable subgroup of $(\mathbf{R}, +)$ and $[0, 1]$, contains 1, and is dense in the interval; conversely it is not hard to see that any such set is the clopen value set of some good measure μ . The density condition corresponds to the fact that μ is atomless, and is equivalent (since $1 \in V$) to saying that V is not contained in $\frac{1}{p}\mathbf{Z}$ for any integer p .

Definition 5.10. Given a good measure μ on a Cantor space X , we follow [A] and denote by H_μ the set of all homeomorphisms of X which preserve μ . For a countable $V \subset [0, 1]$, we denote by $\langle V \rangle$ the intersection of the subgroup of $(\mathbf{R}, +)$ generated by $V \cup \{1\}$ with $[0, 1]$; we say that V is *group-like* when V is not contained in $\frac{1}{p}\mathbf{Z}$ for any integer p and $V = \langle V \rangle$. In that case, we denote by μ_V the good measure whose clopen value set is equal to V .

Of course, μ_V above is only defined up to isomorphism; since we focus on isomorphism-invariant properties we allow ourselves this small abuse of terminology.

We would like to understand when there exists a comeager conjugacy class in H_μ ; Akin [A, Theorem 4.17] proved that this holds true whenever $V(\mu) + \mathbf{Z}$ is a \mathbf{Q} -vector subspace of \mathbf{R} , or equivalently whenever any clopen subset can be partitioned into m clopen subsets of equal measure for any integer m . One can check that this also holds true, for instance, when μ is a Bernoulli measure (this is

explicitly pointed out in [KR]), and it was our hope that this property would be satisfied by all good measures. We start with the following observation, which is the main result of this section. While the result is about the full group of a minimal homeomorphism, the proof is actually about good measures.

Theorem 5.11. *Let φ be a uniquely ergodic homeomorphism of a Cantor space X . Then the set $\{g \in [\varphi] : g \text{ has finite order}\}$ is dense in $[\varphi]$.*

Proof. We denote by μ the unique φ -invariant measure. Pick $g \in [\varphi]$ and let $U_0, \dots, U_{n-1}, V_0, \dots, V_{n-1}$ be clopen partitions of X such that $g(U_i) = V_i$, (thus $\mu(U_i) = \mu(V_i)$), for $i = 0, \dots, n-1$. We have to find a finite order homeomorphism $h \in [\varphi]$ also satisfying $h(U_i) = V_i$ for each i . Let B_0, \dots, B_{m-1} be an enumeration of the atoms of the finite boolean algebra generated by the sets U_i, V_i . We will find a further finer clopen partition \mathcal{B}' and a measure-preserving permutation $h' : \mathcal{B}' \rightarrow \mathcal{B}'$ such that $h'(B') \subseteq V_i$ for $B' \subseteq U_i$, whence the desired h can be obtained by applying Theorem 2.7 to find an element of $[\varphi]$ mapping B' to $h'(B')$ for all $B' \in \mathcal{B}'$.

It will be useful to introduce the following terminology: a clopen subset $S \subseteq V_i$ is an *initial fragment* of V_i if there is $j_0 < m$ such that $B_j \cap V_i \subseteq S$ for every $j < j_0$ and $B_j \cap S = \emptyset$ for $j_0 < j$.

To construct the desired \mathcal{B}' and h' we start by building a family of clopen sets $T_0 = \{C_\sigma \in \text{Clop}(X) : \sigma \in m^{<\omega}\}$, each of its elements contained in some atom B_j , with the following properties:

- (1) $C_\emptyset = \emptyset, C_j = V_0 \cap B_j$,
- (2) if $C_\sigma \subseteq U_i, \sigma \neq \emptyset$, then $C_{\sigma j} \subseteq V_i \cap B_j$,
- (3) if $C_\sigma \subseteq U_0, \sigma \neq \emptyset$, then $C_{\sigma j} = \emptyset$ for each j ,
- (4) if $C_\sigma \not\subseteq U_0$, then $\mu(C_\sigma) = \mu(\bigcup_{j < m} C_{\sigma j})$,
- (5) if $\sigma \neq \tau$, then $C_\sigma \cap C_\tau = \emptyset$,
- (6) for $k < \omega, \bigcup\{C_\sigma : |\sigma| \leq k, C_\sigma \subseteq V_i\}$ is an initial fragment of V_i ,
- (7) for $0 < i < n$ and $k < \omega$,

$$\mu\left(\bigcup\{C_\sigma : |\sigma| < k, C_\sigma \subseteq U_i\}\right) = \mu\left(\bigcup\{C_\sigma : |\sigma| \leq k, C_\sigma \subseteq V_i\}\right),$$

- (8) for $0 < k < \omega$,

$$\mu(U_0) = \mu\left(\bigcup\{C_\sigma : |\sigma| = k, \text{ or else } |\sigma| < k \text{ and } C_\sigma \subseteq U_0\}\right),$$

- (9) C_σ is empty for all but finitely many σ .

Note that the above conditions imply that the set $\mathcal{S} = \{\emptyset\} \cup \{\sigma : C_\sigma \neq \emptyset\}$ must then be a subtree of $m^{<\omega}$, which is finite from condition (9).

The starting point is given by the first item. Suppose we have defined C_σ for all σ of length $|\sigma| \leq k, k \geq 1$, and assume inductively items (2)-(7) to hold so far. Let

$$E_i^k = \{C_\sigma : |\sigma| = k, C_\sigma \subseteq U_i\}, F_i^k = \{C_\sigma : |\sigma| \leq k, C_\sigma \subseteq V_i\}.$$

For $C_\sigma \in E_0^k$ we define $C_{\sigma j} = \emptyset$ for each j . For $i > 0$, items (5), (7) and the fact that $\mu(U_i) = \mu(V_i)$ and that μ is a good measure allow us to find a clopen subset S of $V_i \setminus \bigcup F_i^k$ of measure $\mu(\bigcup E_i^k)$. Again because μ is a good measure, we can partition S into clopen sets S_σ such that $\mu(S_\sigma) = \mu(C_\sigma)$ for each $C_\sigma \in E_i^k$; we then define $C_{\sigma j} = S_\sigma \cap B_j$. One should only take care to choose S so that (6) be true for $k+1$;

the rest of the items (2)-(8) are preserved automatically. So, as for the choice of S , let $j_0 < m$ be least such that

$$\mu\left(\bigcup F_i^k\right) + \mu\left(\bigcup E_i^k\right) \leq \mu\left(\bigcup_{j \leq j_0} B_j \cap V_i\right).$$

Then choose —once more, because μ is good— a clopen subset $S_0 \subset B_{j_0} \setminus \bigcup F_i^k$ of measure $\mu\left(\bigcup_{j \leq j_0} B_j \cap V_i\right) - \mu\left(\bigcup F_i^k\right) - \mu\left(\bigcup E_i^k\right)$, and define $S = ((B_{j_0} \setminus S_0) \cup \bigcup_{j < j_0} (B_j \cap V_i)) \setminus \bigcup F_i^k$. We should remark that it is here that we need to restrict to a single good measure (thus to a uniquely ergodic homeomorphism); we do not know whether the argument can be adapted to a *good family* of measures.

The tree already constructed, we should argue that item (9) holds. Let b_k be the number of atoms B_j intersecting $\bigcup\{C_\sigma : |\sigma| \leq k\}$; let c_k be the number of non-empty sets C_σ with $|\sigma| = k$, plus the number of non-empty sets $C_\sigma \subseteq U_0$ with $|\sigma| < k$. By construction, especially item (6), $c_k < c_{k+1}$ only if $b_k < b_{k+1}$: a set C_σ has more than one non-empty successor only if some new atom B_j is attained. As $\{b_k\}$ is bounded by m , $\{c_k\}$ is bounded. This means that, after a certain point, each node has at most one non-empty successor, which must be of equal measure. Since distinct nodes are disjoint, this is possible only if (9) holds.

The family $\{C_\sigma : \sigma \text{ is a leaf of } \mathcal{S}\}$ forms a finite partition of U_0 . Indeed: they are disjoint, by (5). If a non-empty node N is not inside U_0 , then it has a non-empty successor by (4); thus the leaves lie inside U_0 . Finally, they add up the measure of U_0 , by (8), and a clopen set of μ -measure zero must be empty.

Starting from the leaves and following the tree backwards, we now introduce clopen partitions of the elements of T_0 , and thus a partition of $\bigcup T_0$ itself; this will be an h' -invariant part of our intended refinement \mathcal{B}' , so we simultaneously define how h' acts on this part. Let $\rho : m^{<\omega} \rightarrow \omega$ be the rank function of \mathcal{S} . Zero-rank nodes (the leaves) get the trivial partition; we postpone the definition of h' on them, which will be given at the end of the construction. If we have already defined a partition $\mathcal{P}(\tau)$ of C_τ for each τ such that $\rho(\tau) < r$, and $\rho(\sigma) = r$, we use that μ is a good measure to get a partition

$$\mathcal{P}(\sigma) = \{P_Q \subset C_\sigma : Q \in \mathcal{P}(\sigma j), j = 0, \dots, m-1\}$$

such that $\mu(P_Q) = \mu(Q)$ for each atom P_Q . At the same time we define $h'(P_Q) = Q$. Moreover, we can associate an element $L(P)$ to each $P \in \mathcal{P}(\sigma)$, so that $L(C_\sigma) = C_\sigma$ if σ is a leaf of \mathcal{S} , and $L(P_Q) = L(Q)$ in the inductive step. We stop the induction just before the root \emptyset , and finally define, for each leaf σ , $h'(C_\sigma) = P$ if P is the (only) atom in the families $\mathcal{P}(0), \dots, \mathcal{P}(m-1)$ such that $L(P) = C_\sigma$.

Collecting the atoms in each $\mathcal{P}(\sigma)$ we now have a partition of $\bigcup T_0$, which we take to be the restriction of \mathcal{B}' to this set; we know how to define h' on this part. The whole process must now be repeated on $X \setminus \bigcup T_0$, starting with the sets $U'_i = U_i \setminus \bigcup T_0$, $V'_i = V_i \setminus \bigcup T_0$ and $B'_j = B_j \setminus \bigcup T_0$, the set U'_1 playing the role of U_0 . Remark simply that $\mu(U'_i) = \mu(V'_i)$, by (7). And so on; after less than n iterations of this process we reach the definition of the intended \mathcal{B}' and h' . \square

Remark 4. The above result might be rephrased as follows: for a good measure μ , a *partial* μ -automorphism of a finite algebra $\mathcal{A} \subset \text{Clop}(X)$ can always be extended to a μ -automorphism of a finite algebra $\mathcal{B} \subset \text{Clop}(X)$ extending \mathcal{A} .

Thinking of the existence of a dense conjugacy class in terms of the *joint embedding property* introduced in [KR], we see that H_μ has a dense conjugacy class if and only if the following condition is satisfied: whenever \mathcal{A}, \mathcal{B} are finite subalgebras of $\text{Clop}(X)$, and a, b are automorphisms of $(\mathcal{A}, \mu), (\mathcal{B}, \mu)$ respectively, there exists a finite subalgebra \mathcal{C} of $\text{Clop}(X)$ and an automorphism c of (\mathcal{C}, μ) such that there exist μ -preserving embeddings $\alpha: \mathcal{A} \rightarrow \mathcal{C}$ and $\beta: \mathcal{B} \rightarrow \mathcal{C}$ satisfying $c(\alpha(A)) = \alpha(a(A))$ for all $A \in \mathcal{A}$, and $c(\beta(B)) = \beta(b(B))$ for all $B \in \mathcal{B}$.

Following the terminology of [KR], we say that μ satisfies the *joint embedding property for partial automorphisms* if the above condition is verified (strictly speaking, this condition should involve partial automorphisms of finite algebras, but because of Theorem 5.11 one can simply consider automorphisms of finite algebras).

The *weak amalgamation property* for partial automorphisms, which along with the joint embedding property characterizes the existence of comeager conjugacy classes [KR, Theorem 3.4], is always satisfied in our case, again because of Theorem 5.11. Thus we see from the results of [KR] that H_μ has a comeager conjugacy class if, and only if, it has a dense conjugacy class, which is equivalent to saying that μ satisfies the joint embedding property for partial automorphisms.

Unfortunately, this property is not always satisfied. Indeed, assume that μ satisfies the joint embedding property for partial automorphisms, and that there exists $A \in \text{Clop}(X)$ such that $\mu(A) = \frac{1}{n}$ for some integer n . Then, there exists an element $a \in H_\mu$ such that X is the disjoint union of $A, \dots, a^{n-1}(A)$. Let now r be any element of $V(\mu)$, B a clopen subset of X such that $\mu(B) = r$, and consider:

- the algebra \mathcal{A} generated by $A, \dots, a^{n-1}(A)$, with the automorphism a ;
- the algebra \mathcal{B} made up of B and its complement, with the identity automorphism b .

Assume one can jointly embed (\mathcal{A}, a) and (\mathcal{B}, b) in (\mathcal{C}, c) ; identify \mathcal{A}, \mathcal{B} with the subalgebras of \mathcal{C} associated with these embeddings. Then $B = B \cap \sqcup_{i=0}^{n-1} c^i(A) = \sqcup_{i=0}^{n-1} c^i(B \cap A)$, so B is cut into n clopen subsets of equal measure. This means that $\frac{r}{n}$ must belong to $V(\mu)$. Hence, the joint embedding property fails for instance when $V = \langle \frac{1}{2}, \frac{1}{\pi} \rangle$.

Analysing the above example, one can extract a combinatorial condition on V which is equivalent to the existence of a comeager conjugacy class in H_{μ_V} .

Proposition 5.12. *Let V be a group-like subset of $[0, 1]$. Then there is a comeager conjugacy class in H_{μ_V} if, and only if, V satisfies the following condition: whenever $a_i, b_j \in V$ and $n_i, m_j \in \mathbf{N}$ are such that $\sum_{i=1}^p n_i a_i = 1 = \sum_{j=1}^q m_j b_j$, there exist $c_{i,j} \in V$ such that*

$$\forall j \ m_j b_j = \sum_{i=1}^p \text{lcm}(n_i, m_j) c_{i,j} \quad \text{and} \quad \forall i \ n_i a_i = \sum_{j=1}^q \text{lcm}(n_i, m_j) c_{i,j}.$$

This holds true in particular when $V + \mathbf{Z}$ is a \mathbf{Q} -vector subspace of \mathbf{R} , and when $V + \mathbf{Z}$ is a subring of \mathbf{R} .

As we already mentioned above, the fact that H_μ has a comeager conjugacy class when $V(\mu) + \mathbf{Z}$ is a \mathbf{Q} -vector subspace of \mathbf{R} was first proved by Akin [A]

Before giving the proof, we set a definition.

Definition 5.13. For a clopen set U , a *cycle* on U will be an automorphism a of a finite algebra all of whose atoms are contained in U except for the atom $X \setminus U$,

such that a acts transitively on the atoms that are contained in U and fixes $X \setminus U$. We say that the number p of atoms of the partition contained in U is the *order* of the cycle, and that a is a p -cycle on U . An homeomorphism g of X might be called a p -cycle on U as well, provided that its support is contained in U and that, for some clopen $A \subset U$, $U = \sqcup_{i=0}^{p-1} g^i(A)$ and $g^p = id$.

Proof of Proposition 5.12. To simplify the notation below we sometimes do not mention the measure; in particular, all automorphisms are to be understood as preserving μ .

Assume that the joint embedding property for partial automorphisms holds, and consider $(a_i, n_i)_{1 \leq i \leq p}$, $(b_j, m_j)_{1 \leq j \leq q}$ as above. Then one can consider a finite algebra \mathcal{A} with clopen atoms $A_{i,k}$ for $k \in \{0, \dots, n_i - 1\}$ such that each $A_{i,k}$ has measure a_i , and an automorphism a of \mathcal{A} such that $a(A_{i,k}) = A_{i,k+1}$ for all i, k (where addition is to be understood modulo n_i); similarly one can consider a finite algebra \mathcal{B} with clopen atoms $B_{j,k}$ ($k \in \{0, \dots, m_j - 1\}$) and the corresponding automorphism b of \mathcal{B} . For all i we let $A_i = \cup A_{i,k}$ and $B_j = \cup B_{j,k}$.

Then we pick (\mathcal{C}, c) such that (\mathcal{A}, a) and (\mathcal{B}, b) can be embedded in (\mathcal{C}, c) , where c is an automorphism of the finite algebra \mathcal{C} , and we identify them with the corresponding subalgebras of \mathcal{C} . If for some i, j $A_i \cap B_j$ is nonempty, then it is a c -invariant clopen set. Any atom of \mathcal{C} contained in some $A_{i,k} \cap B_{j,l}$ must have an orbit whose cardinality is a multiple of $\text{lcm}(n_i, m_j)$, so $c_{i,j} = \frac{1}{\text{lcm}(n_i, m_j)} \mu(A_i \cap B_j)$ belongs to V . Then we have for all i :

$$n_i a_i = \mu(A_i) = \sum_{j=1}^q \mu(A_i \cap B_j) = \sum_{j=1}^q c_{i,j} \text{lcm}(n_i, m_j).$$

The same reasoning holds for $m_j b_j$.

This proves one implication; to prove the converse, let us first note that, given a clopen U and two cycles a, b on U of orders n, m respectively and such that $\frac{1}{\text{lcm}(n, m)} \mu(U)$ belongs to V , there exists a cycle on U of order $N = \text{lcm}(n, m)$ in which both a and b embed. Such a cycle is obtained by cutting U in N disjoint pieces C_i ($0 \leq i \leq N - 1$) of equal measure, and setting $c(C_i) = C_{i+1}$ (modulo N). Then, let $N = nr = ms$; letting A_0, \dots, A_{n-1} denote the atoms contained in U of the algebra on which a is defined, one obtains the desired embedding by identifying each A_i with $\sqcup_{k=0}^{r-1} C_{nk+i}$, and each B_j with $\sqcup_{k=0}^{s-1} C_{mk+j}$.

Now, let α, β in H_{μ_V} be such that $X = \sqcup_{i=1}^p A_i$, where each A_i is clopen and α is a product of cycles α_i of order n_i on A_i , and $X = \sqcup_{j=1}^q B_j$, where each B_j is clopen and β is a product of cycles β_j of order m_j on B_j . From the proof of Theorem 5.11, we see that it is enough to prove that α, β embed in a common element of H_{μ_V} . Let $n_i a_i = \mu(A_i)$ and $m_j b_j = \mu(B_j)$, and apply our assumption on V to get $c_{i,j}$ as in the lemma's statement. Let I denote the set of all (i, j) such that $c_{i,j} \neq 0$; we may find a finite subalgebra of $\text{Clop}(X)$ whose atoms $C_{i,j}^k$ ($(i, j) \in I, 1 \leq k \leq \text{lcm}(n_i, m_j)$) are of measure $c_{i,j}$. For each $(i, j) \in I$, set

$$D_{i,j} = \bigsqcup_{k=1}^{\text{lcm}(n_i, m_j)} C_{i,j}^k$$

We saw that there exists a cycle $\delta_{i,j}$ on $D_{i,j}$, of order $\text{lcm}(n_i, m_j)$, in which a cycle $\alpha_{i,j}$ on $D_{i,j}$ of order n_i and a cycle $\beta_{i,j}$ on $D_{i,j}$ of order m_j both embed. Let δ be the product of all $\delta_{i,j}$; α embeds in δ as the product of all $\alpha_{i,j}$, and β embeds in δ as the product of all $\beta_{i,j}$.

To see that the property under discussion holds true when $V + \mathbf{Z}$ is a subring of \mathbf{R} , simply note that in that case $a_i b_j$ belongs to V ; thus $c_{i,j} = a_i b_j \frac{n_i m_j}{\text{lcm}(n_i, m_j)} = a_i b_j \text{gcd}(n_i, m_j)$ works.

When $V + \mathbf{Z}$ is a \mathbf{Q} -vector subspace of \mathbf{R} , which is equivalent to saying that $\frac{a}{n} \in V$ for any positive integer n and any $a \in V$, it is simpler to prove directly the joint embedding property: notice first that in that case any partial automorphism embeds in a cycle (combined with Theorem 5.11, we thus see that cyclic automorphisms of X are dense in H_μ , a fact first proved in [A] using a different method). Hence it is enough to check that two cycles can be embedded in a common cycle, which we already proved above. □

Even though we do not know whether G_φ or $[\varphi]$ are simple in general, we can use Theorem 5.11 to prove that these groups do not have any non-trivial closed normal subgroups.

Theorem 5.14. *Let φ be a uniquely ergodic homeomorphism of a Cantor space X . Then G_φ and $[\varphi]$ are topologically simple.*

Proof. We show that the derived subgroup of $[\varphi]$ is dense, which implies the simplicity of both groups by the result of Bezuglyi–Medynets recalled in the paragraph before Proposition 5.4. The proof of Theorem 5.11 shows that products of p -cycles are dense in $[\varphi]$, so it is enough for our purposes to show that p -cycles are products of commutators.

Let g be a p -cycle on a clopen U , with $U = \sqcup_{i=0}^{p-1} g^i(A)$. Given a permutation σ belonging to the permutation group S_p on p elements, we denote by g_σ the element of $[\varphi]$ defined by setting $g_\sigma(x) = x$ for all x outside U and

$$\forall i \in \{0, \dots, p-1\} \forall x \in g^i(A) \quad g_\sigma(x) = g^{\sigma(i)-i}(x).$$

The map $\sigma \mapsto g_\sigma$ is a homomorphism from S_p to $[\varphi]$. Since the commutator subgroup of S_p is the alternating subgroup A_p , we thus see that whenever σ belongs to A_p g_σ is a product of commutators. In particular, g has this property if p is odd. If p is even, let τ be the transposition of S_p which exchanges 0 and 1. Then [BM, Corollary 4.8] tells us that g_τ is a product of 10 commutators in $[\varphi]$; since $g g_\tau = g_\sigma$ for some $\sigma \in A_p$, g is also a product of commutators. □

We conclude these notes with another reason why we think it might be interesting to further study the properties of closures of full groups.

Proposition 5.15. *Let φ be a minimal homeomorphism of a Cantor space X . Then the closure of its full group is an amenable subgroup of $\text{Homeo}(X)$.*

Proof. One may assume, replacing φ by a minimal homeomorphism which is orbit equivalent to it (which does not affect the full group) that the topological full group $[[\varphi]]$ is dense in $[\varphi]$, hence also in G_φ (see [BK, Theorem 1.6]). Juschenko and

Monod recently proved in [JM] that $[[\varphi]]$ is amenable as a discrete group, hence G_φ is an amenable topological group. \square

There is probably a direct proof of the amenability of G_φ not using [JM]. This result is particularly interesting in view of a question of Angel–Kechris–Lyons [AKL, Question 15.1] asking whether, whenever an amenable Polish group has a metrizable universal minimal flow, the universal minimal flow is uniquely ergodic. A positive answer to the following problem would then show that the answer to Angel–Kechris–Lyons’ question is negative.

Question 5.16. Let φ be a minimal homeomorphism of a Cantor space X . Is the universal minimal flow of G_φ metrizable?

Remark 5. Proving that there exists *one* minimal homeomorphism φ which is not uniquely ergodic, yet has a metrizable universal minimal flow would be enough to answer negatively the question of Angel–Kechris–Lyons mentioned above. In the opposite direction, proving that the universal minimal flows of these groups are not metrizable as soon as the homeomorphism is not uniquely ergodic, and are metrizable otherwise, would point towards a positive answer to their question.

At the moment, this seems out of reach: for instance, when φ is equal to the usual binary odometer, G_φ is just the set of all homeomorphisms of the Cantor space $\{0, 1\}^\omega$ which preserve the usual $(1/2, 1/2)$ -Bernoulli measure on $\{0, 1\}^\omega$. Identifying the universal minimal flow of this group is already a very complicated problem, studied in [KST] where a candidate (which is metrizable) is proposed. Thus it seems that the current state of the art does not, for the moment, allow us to hope for an easy answer to our question.

REFERENCES

- [A1] E. Akin, *Good measures on Cantor space*, Trans. Amer. Math. Soc. **357** (2005), no. 7, 2681–2722 (electronic). MR2139523 (2006e:37003)
- [A2] R. D. Anderson, *The algebraic simplicity of certain groups of homeomorphisms*, Amer. J. Math. **80** (1958), 955–963. MR0098145 (20 #4607)
- [AKL] O. Angel, A. S. Kechris, and R. Lyons, *Random orderings and unique ergodicity of automorphism groups*, 2012. Preprint.
- [BK] S. Bezuglyi and J. Kwiatkowski, *Topologies on full groups and normalizers of Cantor minimal systems*, Mat. Fiz. Anal. Geom. **9** (2002), no. 3, 455–464. MR1949801 (2003j:37014)
- [BM] S. Bezuglyi and K. Medynets, *Full groups, flip conjugacy, and orbit equivalence of Cantor minimal systems*, Colloq. Math. **110** (2008), no. 2, 409–429. MR2353913 (2009b:37011)
- [CFW] A. Connes, J. Feldman, and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Ergodic Theory Dynamical Systems **1** (1981), no. 4, 431–450 (1982). MR662736 (84h:46090)
- [D1] H. A. Dye, *On groups of measure preserving transformations. I*, Amer. J. Math. **81** (1959), 119–159. MR0131516 (24 #A1366)
- [D2] ———, *On groups of measure preserving transformations. II*, Amer. J. Math. **85** (1963), 551–576. MR0158048 (28 #1275)
- [G] D. Gamarnik, *Minimality of the group $\text{Autohomeom}(\mathbb{C})$* , Serdica **17** (1991), no. 4, 197–201. MR1201115 (94a:54095)
- [GMPS] T. Giordano, H. Matui, I. F. Putnam, and C. F. Skau, *Orbit equivalence for Cantor minimal \mathbb{Z}^d -systems*, Invent. Math. **179** (2010), no. 1, 119–158. MR2563761 (2011d:37013)
- [GPS1] T. Giordano, I. F. Putnam, and C. F. Skau, *Topological orbit equivalence and C^* -crossed products*, J. Reine Angew. Math. **469** (1995), 51–111. MR1363826 (97g:46085)
- [GPS2] ———, *Full groups of Cantor minimal systems*, Israel J. Math. **111** (1999), 285–320. MR1710743 (2000g:46096)

- [GW] E. Glasner and B. Weiss, *Weak orbit equivalence of Cantor minimal systems*, *Internat. J. Math.* **6** (1995), no. 4, 559–579. MR1339645 (96g:46054)
- [JM] K. Juschenko and N. Monod, *Cantor systems, piecewise translations and simple amenable groups*, 2012. To appear in *Annals of Mathematics*.
- [K1] A. S. Kechris, *Classical descriptive set theory*, *Graduate Texts in Mathematics*, vol. 156, Springer-Verlag, New York, 1995. MR1321597 (96e:03057)
- [K2] ———, *Global aspects of ergodic group actions*, *Mathematical Surveys and Monographs*, vol. 160, American Mathematical Society, Providence, RI, 2010. MR2583950
- [KR] A. S. Kechris and C. Rosendal, *Turbulence, amalgamation and generic automorphisms of homogeneous structures*, *Proc. Lond. Math. Soc.* **94** (2007), no. 2, 302–350.
- [KST] A. S. Kechris, M. Sokić, and S. Todorćević, *Ramsey properties of finite measure algebras and topological dynamics of the group of measure preserving automorphisms: some results and an open problem*, 2012. preprint.
- [M] K. Medynets, *Reconstruction of orbits of Cantor systems from full groups*, *Bull. Lond. Math. Soc.* **43** (2011), no. 6, 1104–1110. MR2861532
- [R] C. Rosendal, *On the non-existence of certain group topologies*, *Fund. Math.* **187** (2005), no. 3, 213–228. MR2213935 (2006k:03091)
- [RS] C. Rosendal and S. Solecki, *Automatic continuity of homomorphisms and fixed points on metric compacta*, *Israel J. Math.* **162** (2007), 349–371. MR2365867
- [W] T.-J. Wei, *Descriptive properties of measure preserving actions and the associated unitary representations*, Ph.D. Thesis, 2005.

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