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► **To cite this version:**

François Dubois. On hyperbolic systems with entropy velocity covariant under the action of a group. 2013. hal-00860736v1

HAL Id: hal-00860736

<https://hal.science/hal-00860736v1>

Preprint submitted on 11 Sep 2013 (v1), last revised 28 Mar 2022 (v2)

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On hyperbolic systems with entropy velocity covariant under the action of a group

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10 september 2013 *

Abstract. – For hyperbolic systems of conservation laws in one space dimension with a mathematical entropy, we define the notion of entropy velocity. Then we give sufficient conditions for such a system to be covariant under the action of a group of space-time transformations. These conditions naturally introduce a representation of the group in the space of states. We construct such hyperbolic system from the knowledge of data on the manifold of null velocity. We apply these ideas for Galileo, Lorentz and circular groups. We focus on particular non trivial examples for two of two systems of conservation laws.

Keywords: Galileo group, Lorentz group, circular group, mathematical entropy.

AMS classification: 35L40, 58J45.

* This contribution in homage to Jean-Marie Souriau (1922 - 2012) is issued from a lecture entitled “Sur les systèmes hyperboliques à vitesse entropique invariants sous l’action d’un groupe” given on tuesday 28 august 2012 at the 56th “Colloque International de Théories Variationnelles”, La Baume, Aix en Provence, France.

1) Introduction

- The first link between covariance under the action of a group and conservation laws is the classic result of Emmy Noether [16]: when the action is invariant under the action of a group, an associated moment is conserved along the time evolution. This result is well understood for systems that possess some action. This result is also fundamental for all the modern physics. It has been intensively studied and generalized by Jean-Marie Souriau in his fundamental book [19] (see also the work of Kostant [11]). The fact to consider Galilean relativity is natural in this kind of work. In order to explicit the mass as an invariant, the Bargmann group [1] is necessary in this framework and we refer to the work of de Saxcé and Vallée [4] for the introduction of the second principle of thermodynamics. Observe also that very early Souriau has considered not only Galilean relativity but also Einstein's relativity [18, 20, 21]. An other contribution is due to Vallée [22] relative to continuum media in the framework of special relativity.

- Our scientific background is first concerned with the approximation of gas dynamics problems with numerical methods [5]. The notion of first order hyperbolic system of conservation laws is fundamental for this study and the contribution of Peter lax (see *e.g.* [13]) is essential for all that follows. The introduction of the second principle of thermodynamics in this framework is due to Godunov [10] and to Friedrichs and Lax [8] with the notion of “mathematical entropy”. A fundamental result of existence and uniqueness of an entropy solution in the scalar case is due to Kružkov [12]. Galilean conservation for hyperbolic systems has been proposed by Ruggeri [17]. We have considered the same question with enforcing the structure of the mathematical entropy in [6]. We consider an extension of this approach in this contribution.

- The summary of our study is the following. We introduce the entropy velocity at Section 2 for hyperbolic systems with a mathematical entropy. Then we define in Section 3 the notion of covariance for an hyperbolic system with entropy velocity under the action of a group and propose sufficient conditions that make in evidence an algebraic structure. The manifold of null velocity defined at Section 4 is a natural notion in ths context. Then precise constraints for the entropy variables are derived at Section 5. In Section 6, we study the particular case of systems of order two. Some words of conclusion are proposed at Section 7.

2) Entropy velocity

- Let N be a positive integer and Ω a non void convex open part of \mathbb{R}^N . The physical flux $f(W) \in \mathbb{R}^N$ of a state $W \in \Omega$ is a regular function $f : \Omega \rightarrow \mathbb{R}^N$. We study in the following the system of conservation laws in one space dimension with unknown $\mathbb{R} \times [0, +\infty[\ni (x, t) \mapsto W(x, t) \in \Omega$ associated with this nonlinear flux function:

$$(1) \quad \frac{\partial W}{\partial t} + \frac{\partial}{\partial x} f(W) = 0.$$

- As proposed by Godunov [10], Friedrichs and Lax [8] and Boillat [2], a mathematical entropy for the system (1) is a regular strictly convex function $\eta : \Omega \rightarrow \mathbb{R}$ such that for

any regular solution $W(x, t)$ of the conservation law (1), there exists a complementary conservation equation

$$(2) \quad \frac{\partial \eta}{\partial t} + \frac{\partial \zeta}{\partial x} = 0.$$

The space derivative of the second term in (2) is the entropy flux, a regular function $\zeta : \Omega \rightarrow \mathbb{R}$ that satisfies the identity

$$(3) \quad d\zeta \equiv d\eta \bullet df.$$

It is well known (see *e.g.* Godlewski - Raviart [9] or [5]) that a system of conservation law which admits a mathematical entropy is hyperbolic: the jacobian $df(W)$ is diagonalisable in \mathbb{R}^N . Observe also that the restriction to only one space dimension is not restrictive in principle if we adopt a point of view founded on mathematical entropy. The generalization to several space dimensions does not add in principle new fundamental difficulties. As well known, the equality (2) has to be replaced by an inequality (taken in an appropriate weak sense) when discontinuous solutions of the conservation law (1) are considered. The co-vector φ of entropy variables is the jacobian of the entropy relatively to the variation of the state W :

$$(4) \quad d\eta \equiv \varphi \bullet dW, \quad \forall W \in \Omega.$$

The dual entropy η^* is the Legendre [14] - Fenchel [7] - Moreau [15] transform of the convex function η . It is a function of the entropy variables:

$$(5) \quad \eta^*(\varphi) \equiv \sup_{W \in \Omega} (\varphi \bullet W - \eta(W))$$

and we have

$$(6) \quad d\eta^*(\varphi) = d\varphi \bullet W.$$

- In the following, we define the “entropy velocity” u as the quotient of the entropy flux divided by the entropy. In other terms, we suppose that the system (1) (2) admits an entropy velocity u if there exists a regular function $u : \Omega \rightarrow \mathbb{R}$ such that the entropy flux ζ can be written under the form ηu :

$$(7) \quad \zeta(W) \equiv \eta(W) u(W), \quad \forall W \in \Omega.$$

This definition has been previously proposed in [6]. For gas dynamics this entropy velocity is the usual velocity because the specific entropy is advected by the flow (see *e.g.* [5] or [9]). Once the velocity is defined, it is natural, as suggested by Ruggeri [17], to decompose the physical flux f into a convective part uW and a complementary contribution g . We define the “thermodynamic flux” $g(W)$ by the relation

$$(8) \quad f(W) \equiv u(W)W + g(W), \quad \forall W \in \Omega.$$

- **Proposition 1. Compatibility relation**

With the above framework, the compatibility relation (3) is satisfied if and only if

$$(9) \quad \varphi \bullet dg + \eta^* du \equiv 0.$$

Proof of Proposition 1.

The proof has been explicited in [6]; we detail it for completeness. We have, due to the relations (3) to (8), the following calculus:

$$\begin{aligned} d\zeta - d\eta \bullet df &= d(\eta u) - d\eta \bullet d(uW + g) = \eta du + u d\eta - \varphi \bullet [(du)W + u dW + dg] \\ &= (\eta - \varphi \bullet W) du + u (d\eta - \varphi \bullet dW) - \varphi \bullet dg = -\eta^* du - \varphi \bullet dg \end{aligned}$$

and the conclusion is clear. \square

3) Covariance under the action of a group

- We introduce a group \mathcal{G} with one real parameter θ composed by transformations G_θ of space-time defined by a two by two matrix:

$$(10) \quad G_\theta = \begin{pmatrix} \alpha_\theta & \delta_\theta \\ \beta_\theta & \gamma_\theta \end{pmatrix}, \quad G_\theta \in \mathcal{G}, \quad \theta \in \mathbb{R}.$$

At a fixed θ , the transformation $(x, t) \mapsto (x', t')$ is defined by the relations:

$$(11) \quad \begin{pmatrix} x' \\ t' \end{pmatrix} = G_\theta \begin{pmatrix} x \\ t \end{pmatrix}; \quad \begin{cases} x' = \alpha_\theta x + \delta_\theta t \\ t' = \beta_\theta x + \gamma_\theta t. \end{cases}$$

In the following, we consider three particular groups, defined by the conditions

$$\alpha_\theta = \gamma_\theta = C_\theta, \quad \beta_\theta = \frac{\varepsilon}{c} S_\theta, \quad \delta_\theta = c S_\theta$$

id est by the following matrices

$$(12) \quad G_\theta = \begin{pmatrix} C_\theta & c S_\theta \\ \frac{\varepsilon}{c} S_\theta & C_\theta \end{pmatrix}.$$

The Galileo group corresponds to

$$(13) \quad \varepsilon = 0, \quad C_\theta \equiv 1, \quad S_\theta \equiv \theta,$$

the Lorentz group to

$$(14) \quad \varepsilon = 1, \quad C_\theta \equiv \cosh \theta, \quad S_\theta \equiv \sinh \theta,$$

and a third one here called the ‘‘circular group’’ to

$$(15) \quad \varepsilon = -1, \quad C_\theta \equiv \cos \theta, \quad S_\theta \equiv \sin \theta.$$

We remark that we have the elementary relations,

$$(16) \quad C_\theta^2 - \varepsilon S_\theta^2 \equiv 1, \quad \frac{dS_\theta}{d\theta} \equiv C_\theta, \quad \frac{dC_\theta}{d\theta} \equiv \varepsilon S_\theta, \quad \varepsilon \in \{-1, 0, 1\}.$$

- We say here that the system (1) associated with the conservation of entropy

$$(17) \quad \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(\eta u) = 0$$

is covariant under the action of the group \mathcal{G} if for each $\theta \in \mathbb{R}$ and for each state $W \in \Omega$, we can define a new state W' such that after the change of variables (11), the system of equations (1) (17) take the form

$$(18) \quad \begin{cases} \frac{\partial W'}{\partial t'} + \frac{\partial}{\partial x'} f(W') = 0 \\ \frac{\partial}{\partial t'} \eta(W') + \frac{\partial}{\partial x'} (\eta(W') u(W')) = 0 \end{cases}$$

with the *same* flux function $f(\bullet)$, the *same* mathematical entropy $\eta(\bullet)$ and the *same* entropy velocity $u(\bullet)$.

- We present now an elementary calculus. We have the chain rule

$$\begin{cases} \frac{\partial}{\partial x} = \frac{\partial x'}{\partial x} \frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x} \frac{\partial}{\partial t'} = \alpha_\theta \frac{\partial}{\partial x'} + \beta_\theta \frac{\partial}{\partial t'} \\ \frac{\partial}{\partial t} = \frac{\partial x'}{\partial t} \frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t} \frac{\partial}{\partial t'} = \delta_\theta \frac{\partial}{\partial x'} + \gamma_\theta \frac{\partial}{\partial t'}. \end{cases}$$

We inject these operators inside the relation (1):

$$\left(\delta_\theta \frac{\partial}{\partial x'} + \gamma_\theta \frac{\partial}{\partial t'} \right) W + \left(\alpha_\theta \frac{\partial}{\partial x'} + \beta_\theta \frac{\partial}{\partial t'} \right) f(W) = 0$$

and we put in evidence the new partial derivatives:

$$\frac{\partial}{\partial t'} (\gamma_\theta W + \beta_\theta f(W)) + \frac{\partial}{\partial x'} (\delta_\theta W + \alpha_\theta f(W)) = 0.$$

We introduce a bijective linear operator Y_θ depending on $\theta \in \mathbb{R}$, acting from \mathbb{R}^N into \mathbb{R}^N and independent on x' and t' :

$$(19) \quad Y_\theta \in \mathcal{GL}(\mathbb{R}^N), \quad \theta \in \mathbb{R}.$$

After applying Y_θ to the previous equation, we obtain a new conservation law equivalent to (1):

$$(20) \quad \frac{\partial}{\partial t'} (\gamma_\theta Y_\theta \bullet W + \beta_\theta Y_\theta \bullet f(W)) + \frac{\partial}{\partial x'} (\delta_\theta Y_\theta \bullet W + \alpha_\theta Y_\theta \bullet f(W)) = 0.$$

A *sufficient* condition to establish the first equation of (18) is to identify the arguments of the two partial derivatives $\partial_{t'}$ and $\partial_{x'}$ in the previous equation:

$$(21) \quad \begin{cases} W' = \gamma_\theta Y_\theta \bullet W + \beta_\theta Y_\theta \bullet f(W) \\ f(W') = \delta_\theta Y_\theta \bullet W + \alpha_\theta Y_\theta \bullet f(W). \end{cases}$$

The relation (21) can be written in a more compact form:

$$(22) \quad \begin{pmatrix} f(W') \\ W' \end{pmatrix} = G_\theta Y_\theta \bullet \begin{pmatrix} f(W) \\ W \end{pmatrix} \equiv G_\theta \begin{pmatrix} Y_\theta \bullet f(W) \\ Y_\theta \bullet W \end{pmatrix}.$$

Under the ‘‘linearity hypothesis’’ (19), the relation (22) gives geometrical constraints for the transformation $\Omega \ni W \mapsto W' \in \Omega$ associated with the action of the group \mathcal{G} .

- We make an analogous calculus for the second equation of (18) relative to entropy. After the transformation of partial derivatives, the relation (17) takes the form

$$\left(\delta_\theta \frac{\partial}{\partial x'} + \gamma_\theta \frac{\partial}{\partial t'} \right) \eta(W) + \left(\alpha_\theta \frac{\partial}{\partial x'} + \beta_\theta \frac{\partial}{\partial t'} \right) (\eta(W) u(W)) = 0$$

and can be written as

$$(23) \quad \frac{\partial}{\partial t'} \left(\gamma_\theta \eta(W) + \beta_\theta \eta(W) u(W) \right) + \frac{\partial}{\partial x'} \left(\delta_\theta \eta(W) + \alpha_\theta \eta(W) u(W) \right) = 0.$$

This is a scalar conservation law. This equation is identical to the second equation of (18) if the following *sufficient* conditions are satisfied:

$$(24) \quad \begin{cases} \eta(W') &= \gamma_\theta \eta(W) + \beta_\theta \eta(W) u(W) \\ \eta(W') u(W') &= \delta_\theta \eta(W) + \alpha_\theta \eta(W) u(W). \end{cases}$$

With a notation that makes the action of the group \mathcal{G} explicit:

$$(25) \quad \begin{pmatrix} \eta(W') u(W') \\ \eta(W') \end{pmatrix} = G_\theta \begin{pmatrix} \eta(W) u(W) \\ \eta(W) \end{pmatrix}.$$

• **Proposition 2. Linear representation**

If the relation (22) is satisfied for every $\theta \in \mathbb{R}$, then the application $\mathcal{G} \ni G_\theta \mapsto Y_\theta \in \mathcal{GL}(\mathbb{R}^N)$ is a linear representation of the group \mathcal{G} :

$$(26) \quad Y_{\theta+\theta'} = Y_\theta \bullet Y_{\theta'}, \quad \theta, \theta' \in \mathbb{R}.$$

Proof of Proposition 2.

We first consider the relation (22) and an analogous relation obtained by replacing the variable θ by the variable θ' :

$$\begin{pmatrix} f(W'') \\ W'' \end{pmatrix} = G_{\theta'} Y_{\theta'} \bullet \begin{pmatrix} f(W') \\ W' \end{pmatrix} \equiv G_{\theta'} \begin{pmatrix} Y_{\theta'} \bullet f(W') \\ Y_{\theta'} \bullet W' \end{pmatrix}.$$

We compose this relation with (22) and we obtain

$$\begin{pmatrix} f(W'') \\ W'' \end{pmatrix} = G_{\theta'} G_\theta \begin{pmatrix} Y_{\theta'} \bullet Y_\theta \bullet f(W) \\ Y_{\theta'} \bullet Y_\theta \bullet W \end{pmatrix}.$$

When we apply the relation (22) with the argument $\theta + \theta'$, we have directly

$$\begin{pmatrix} f(W'') \\ W'' \end{pmatrix} = G_{\theta+\theta'} \begin{pmatrix} Y_{\theta+\theta'} \bullet f(W) \\ Y_{\theta+\theta'} \bullet W \end{pmatrix}.$$

We look precisely to the two previous relations. The scalars α_θ , β_θ , γ_θ and δ_θ commute with the linear operator Y_θ and we obtain the relation (26) because the two previous equalities are satisfied for an arbitrary state W . \square

4) Manifold of null velocity

• In the following we denominate by Ω_0 the set of states with a velocity equal to zero. Because the mapping $\Omega \ni W \mapsto u(W) \in \mathbb{R}$ is scalar, it is natural to suppose that Ω_0 is a manifold in \mathbb{R}^N of codimension one. We observe also that the flux function $f(\bullet)$ is reduced to its thermodynamic contribution $g(\bullet)$ on the manifold of null velocity. Our program is to construct the states W , the flux function $f(\bullet)$ and the mathematical entropy $\eta(\bullet)$ when we suppose that the flux function $\Omega_0 \ni W_0 \mapsto g_0(W_0) \in \mathbb{R}^N$ and the mathematical entropy $\Omega_0 \ni W_0 \mapsto \sigma(W_0) \in \mathbb{R}$ are given on the manifold of null velocity.

The idea is to use the group invariance to link a given state W with an appropriate state W_0 on the manifold Ω_0 as presented in Figure 1. The conditions (22) and (25) are geometric constraints that will allow essentially to solve the problem.

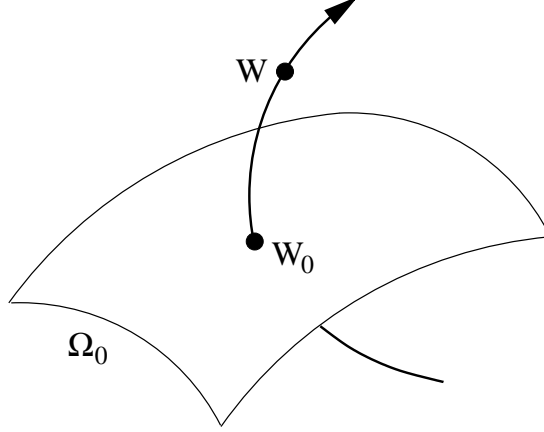


Figure 1. Fiber above the manifold of null velocity Ω_0 . Thanks to the action of the group, a state W_0 of null velocity Ω_0 generates a current state W .

- **Proposition 3. Velocity field**

With the hypothesis (21) and if the mathematical entropy σ on the manifold of null velocity is not the null function, the velocity field $u(W)$ is necessarily given by the relation

$$(27) \quad u(W) = \frac{\delta_\theta}{\gamma_\theta}.$$

Proof of Proposition 3.

We consider a given state W_0 on the manifold Ω_0 and a running state $W \in \Omega$ obtained by the relations (21). With the actual notations, we have

$$(28) \quad W = \gamma_\theta Y_\theta \bullet W_0 + \beta_\theta Y_\theta \bullet g_0(W_0), \quad f(W) = \delta_\theta Y_\theta \bullet W_0 + \alpha_\theta Y_\theta \bullet g_0(W_0).$$

The analogous relations (24) relative to the entropy take a simple form

$$(29) \quad \eta(W) = \gamma_\theta \sigma(W_0), \quad \eta(W) u(W) = \delta_\theta \sigma(W_0).$$

because the velocity is also null on Ω_0 . We consider a state W_0 such that $\sigma(W_0) \neq 0$. If γ_θ is null, then $\eta(W) = 0$ is null and $\delta_\theta = 0$ because $\sigma(W_0) \neq 0$. This contradicts the fact that the matrix defining G_θ by the relation (10) is invertible. Then $\gamma_\theta \neq 0$ and the relation (27) is obtained by taking the ratio of the two relations of (29). \square

- With the example proposed in (12), we have

$$(30) \quad u = c \frac{S_\theta}{C_\theta}.$$

We observe that we have also established the following property:

- **Proposition 4. Entropy field**

With the above hypotheses, if the state W is given from $W_0 \in \Omega_0$ with the help of the relations (28), we have necessarily

$$(31) \quad \eta(W) = \gamma_\theta \sigma(W_0).$$

- **Hypothesis 1. The null-velocity manifold Ω_0 is included in an hyperplane**

We remark that if a convex restriction of the mathematical entropy $\Omega_0 \ni W_0 \mapsto \sigma(W_0) \in \mathbb{R}$ is given on the manifold Ω_0 , a natural hypothesis is to suppose Ω_0 convex. We will do this hypothesis in the following. Moreover, we suppose that Ω_0 is flat; this is expressed by our hypothesis 1.

- **Proposition 5. Thermodynamic flux**

With the above framework (28), the thermodynamic flux $g(W)$ is given from the datum $g_0(W_0)$ according to

$$(32) \quad g(W) = (\alpha_\theta - u(W) \beta_\theta) Y_\theta \bullet g_0(W_0).$$

Proof of Proposition 5.

The second relation of (28) and the definition of the thermodynamic flux (8) allows us to evaluate the function $g(\bullet)$. We have

$$\begin{aligned} g(W) &= f(W) - u(W) W \\ &= \delta_\theta Y_\theta \bullet W_0 + \alpha_\theta Y_\theta \bullet g_0(W_0) - u(W) (\gamma_\theta Y_\theta \bullet W_0 + \beta_\theta Y_\theta \bullet g_0(W_0)) \\ &= (\alpha_\theta - u(W) \beta_\theta) Y_\theta \bullet g_0(W_0) \quad \text{due to (27)} \end{aligned}$$

and the relation (32) is established. \square

- For the groups proposed in (12), we have $\alpha_\theta - u(W) \beta_\theta = C_\theta - c \frac{S_\theta}{C_\theta} \frac{\varepsilon}{c} S_\theta = \frac{1}{C_\theta}$ due to (16). Then the relation (32) takes the form

$$(33) \quad g(W) = \frac{1}{C_\theta} Y_\theta \bullet g_0(W_0).$$

5) Entropy variables

- Even if the state and the entropy are only partially known with the help of relations (28) and (31), its jacobian φ introduced in (4) relative to the conserved variables W can be essentially determined. We first consider the dynamical equations present in this study. First, the relation

$$(34) \quad G_{\theta+\theta'} = G_\theta \bullet G_{\theta'}$$

and the relation (26) for the matrices Y_θ can be differentiated relatively to θ . We have

$$(35) \quad \frac{dG_\theta}{d\theta} = G'_0 G_\theta = G_\theta G'_0, \quad \frac{dY_\theta}{d\theta} = Y'_0 Y_\theta = Y_\theta Y'_0.$$

With the choice (12) of one classical group, we have

$$(36) \quad G'_0 = \begin{pmatrix} 0 & c \\ \frac{\varepsilon}{c} & 0 \end{pmatrix}$$

and we can precise how to extend the differential relations (34) and (35).

• **Proposition 6. Differential relations for the entropy variables**

With the above context, we have

$$(37) \quad G'_0 \begin{pmatrix} u \eta^* + \varphi g \\ \eta^* \end{pmatrix} d\theta + \begin{pmatrix} \varphi Y'_0 f(W) \\ \varphi Y'_0 W \end{pmatrix} d\theta + G_\theta \begin{pmatrix} \varphi Y_\theta dg_0 \\ \varphi Y_\theta dW_0 - d\sigma \end{pmatrix} = 0$$

Proof of Proposition 6.

We write the relation (28) under a matricial form

$$(38) \quad \begin{pmatrix} f(W) \\ W \end{pmatrix} = G_\theta \begin{pmatrix} Y_\theta g_0(W_0) \\ Y_\theta W_0 \end{pmatrix}.$$

We differentiate the relation (38), taking into account the two contributions relative to the group \mathcal{G} on one hand and along the manifold Ω_0 on the other hand, as illustrated on the figure 1. Then

$$\begin{pmatrix} df \\ dW \end{pmatrix} = G'_0 G_\theta \begin{pmatrix} Y_\theta g_0 \\ Y_\theta W_0 \end{pmatrix} d\theta + G_\theta \begin{pmatrix} Y'_0 Y_\theta g_0 \\ Y'_0 Y_\theta W_0 \end{pmatrix} d\theta + G_\theta \begin{pmatrix} Y_\theta dg_0 \\ Y_\theta dW_0 \end{pmatrix}$$

and because

$$\begin{aligned} G_\theta \begin{pmatrix} Y'_0 Y_\theta g_0 \\ Y'_0 Y_\theta W_0 \end{pmatrix} &= \begin{pmatrix} \alpha_\theta Y'_0 Y_\theta g_0 + u \gamma_\theta Y'_0 Y_\theta W_0 \\ \beta_\theta Y'_0 Y_\theta g_0 + \gamma_\theta Y'_0 Y_\theta W_0 \end{pmatrix} = \begin{pmatrix} Y'_0 (\alpha_\theta Y_\theta g_0 + u \gamma_\theta Y_\theta W_0) \\ Y'_0 (\beta_\theta Y_\theta g_0 + \gamma_\theta Y_\theta W_0) \end{pmatrix} \\ &= \begin{pmatrix} Y'_0 f(W) \\ Y'_0 W \end{pmatrix}, \end{aligned}$$

we have

$$(39) \quad \begin{pmatrix} df \\ dW \end{pmatrix} = G'_0 \begin{pmatrix} f \\ W \end{pmatrix} d\theta + \begin{pmatrix} Y'_0 f \\ Y'_0 W \end{pmatrix} d\theta + G_\theta \begin{pmatrix} Y_\theta dg_0 \\ Y_\theta dW_0 \end{pmatrix}.$$

We multiply each line of the relation (39) by the line vector φ . We remark that

$$\begin{aligned} \varphi G'_0 \begin{pmatrix} df \\ dW \end{pmatrix} &= \varphi \begin{pmatrix} \alpha'_0 & (u \gamma)'_0 \\ \beta'_0 & \gamma'_0 \end{pmatrix} \begin{pmatrix} df \\ dW \end{pmatrix} = \varphi \begin{pmatrix} \alpha'_0 f + (u \gamma)'_0 W \\ \beta'_0 f + \gamma'_0 W \end{pmatrix} \\ &= \begin{pmatrix} \alpha'_0 \varphi f + (u \gamma)'_0 \varphi W \\ \beta'_0 \varphi f + \gamma'_0 \varphi W \end{pmatrix} = G'_0 \begin{pmatrix} \varphi f \\ \varphi W \end{pmatrix} \end{aligned}$$

and in a similar way,

$$\begin{aligned} \varphi G_\theta Y_\theta \begin{pmatrix} dg_0 \\ dW_0 \end{pmatrix} &= \varphi \begin{pmatrix} \alpha_\theta Y_\theta dg_0 + (u \gamma)_\theta Y_\theta dW_0 \\ \beta_\theta Y_\theta dg_0 + \gamma_\theta Y_\theta dW_0 \end{pmatrix} \\ &= \begin{pmatrix} \alpha_\theta \varphi Y_\theta dg_0 + (u \gamma)_\theta \varphi Y_\theta dW_0 \\ \beta_\theta \varphi Y_\theta dg_0 + \gamma_\theta \varphi Y_\theta dW_0 \end{pmatrix} = G_\theta \begin{pmatrix} \varphi Y_\theta dg_0 \\ \varphi Y_\theta dW_0 \end{pmatrix}. \end{aligned}$$

Then

$$(40) \quad \begin{pmatrix} \varphi df \\ \varphi dW \end{pmatrix} = G'_0 \begin{pmatrix} \varphi f \\ \varphi W \end{pmatrix} d\theta + \begin{pmatrix} \varphi Y'_0 f \\ \varphi Y'_0 W \end{pmatrix} d\theta + G_\theta \begin{pmatrix} \varphi Y_\theta dg_0 \\ \varphi Y_\theta dW_0 \end{pmatrix}.$$

• We develop the same calculus for the entropy. We start with a matricial form of the relation (29):

$$(41) \quad \begin{pmatrix} \eta u \\ \eta \end{pmatrix} = G_\theta \begin{pmatrix} 0 \\ \sigma \end{pmatrix}$$

Then

$$(42) \quad \begin{pmatrix} d(\eta u) \\ d\eta \end{pmatrix} = G'_0 \begin{pmatrix} \eta u \\ \eta \end{pmatrix} d\theta + G_\theta \begin{pmatrix} 0 \\ d\sigma \end{pmatrix}.$$

We make the difference between the relations (42) and (40). Due to (3) and (4) the left hand side is equal to zero! Then the right hand side can be written as

$$G'_0 \begin{pmatrix} \varphi f - \eta u \\ \varphi W - \eta \end{pmatrix} d\theta + \begin{pmatrix} \varphi Y'_0 f \\ \varphi Y'_0 W \end{pmatrix} d\theta + G_\theta \begin{pmatrix} \varphi Y_\theta dg_0 \\ \varphi Y_\theta dW_0 - d\sigma \end{pmatrix}.$$

The relation (37) is now easily obtained by the explicitation of the dual entropy introduced in (5) and the thermodynamic flux defined in (8). \square

- **Proposition 7. Constraints for the entropy variables**

With the above framework, if we suppose moreover that the derivative of the group \mathcal{G} for $\theta = 0$ is given by the relation (36), we have the following relations for the entropy variables

$$(43) \quad c\eta^* + \varphi Y'_0 f(W) = 0$$

$$(44) \quad \frac{\varepsilon}{c} (u\eta^* + \varphi g(W)) + \varphi Y'_0 W = 0$$

$$(45) \quad \varphi Y_\theta dg_0 = 0$$

$$(46) \quad \varphi Y_\theta dW_0 - d\sigma = 0.$$

Proof of Proposition 7.

With the matrix G'_0 chosen according to (36), the relation (37) becomes

$$\begin{pmatrix} c\eta^* \\ \frac{\varepsilon}{c} (u\eta^* + \varphi g(W)) \end{pmatrix} d\theta + \begin{pmatrix} \varphi Y'_0 f(W) \\ \varphi Y'_0 W \end{pmatrix} d\theta + G_\theta \begin{pmatrix} \varphi Y_\theta dg_0 \\ \varphi Y_\theta dW_0 - d\sigma \end{pmatrix} = 0$$

and the relations (43) to (46) are clear. \square

- The relation (46) can be expressed with the help of the gradient φ_0 of the entropy restricted on the manifold Ω_0 :

$$(47) \quad d\sigma \equiv \varphi_0 \bullet dW_0.$$

Then the relation (46) express that the difference $\varphi Y_\theta - \varphi_0$ belongs to the polar set Ω_0^0 of Ω_0 : we can write $\varphi Y_\theta - \varphi_0 = \mu r_0$ for some scalar μ and some non-null linear form r_0 identically equal to zero on Ω_0 . We make also a new hypothesis.

- **Hypothesis 2. The entropy σ and the thermodynamic flux g_0 are weakly decoupled on the hyperplane Ω_0**

In other terms,

$$(48) \quad (\varphi_0, dg_0 \bullet \rho_0) = 0, \quad \forall \rho_0 \in \Omega_0.$$

- **Proposition 8. Explicitation of the entropy variables**

We make the hypotheses 1 and 2 and suppose moreover

$$(49) \quad \exists \rho_0 \in \Omega_0, \quad dg_0 \bullet \rho_0 \neq 0.$$

Then the relations (43) to (46) imply a simple form for the entropy variables:

$$(50) \quad \varphi Y_\theta = \varphi_0.$$

Proof of Proposition 8.

Consider an arbitrary vector $\rho_0 \in \Omega_0$. When we apply the relation $\varphi Y_\theta - \varphi_0 = \mu r_0$ established previously to the vector $dg_0 \bullet \rho_0$, we find zero due to the constraint (45) and the hypothesis 2 with (48). Then $\mu < r_0, dg_0 \bullet \rho_0 > = 0$. With the choice of ρ_0 suggested in (49), the duality product $\langle r_0, dg_0 \bullet \rho_0 \rangle$ is not null. So $\mu = 0$ and the relation (50) is established. \square

- **Hypothesis 3. The entropy σ and the thermodynamic flux g_0 are strongly decoupled on the hyperplane Ω_0**

In other terms,

$$(51) \quad \varphi_0(W_0) \bullet g_0(\widetilde{W}_0) = 0, \quad \forall W_0, \widetilde{W}_0 \in \Omega_0.$$

We observe that this strong hypothesis (51) implies (48) by differentiation relatively to the variable \widetilde{W}_0 in Ω_0 . In consequence, we have a new property for the entropy variables.

- **Proposition 9. The entropy variables in the polar set of the thermo flux**

We make the hypotheses 1 and 3 and suppose also the technical condition (49) proposed previously. Then we have a complete decoupling:

$$(52) \quad \varphi(W) \bullet g(\widetilde{W}) \equiv 0 \quad \forall W, \widetilde{W} \in \Omega.$$

Proof of Proposition 9.

We just have to precise the expression (33) of the thermodynamic flux. Then we have

$$\varphi(W) \bullet g(\widetilde{W}) = \varphi(W) Y_\theta Y_{-\theta} g(\widetilde{W}) = \varphi_0(W_0) \frac{1}{C_\theta} g_0(\widetilde{W}_0) = 0$$

due to (52). \square

- **Proposition 10. Constraints for the representation Y**

Under the same hypotheses that the ones done for Proposition 9, we have

$$(53) \quad c\eta^* + C_\theta \varphi_0 Y'_0 g_0 = 0$$

$$(54) \quad \varphi_0 Y'_0 W_0 = 0 \quad \text{on } \Omega_0.$$

Proof of Proposition 10.

We first have the following calculus:

$$\varphi Y'_0 g = \varphi Y_\theta Y_{-\theta} Y'_0 g = \varphi Y_\theta Y'_0 Y_{-\theta} g = \varphi_0 Y'_0 \frac{1}{C_\theta} g_0 = \frac{1}{C_\theta} \varphi_0 Y'_0 g_0.$$

Then we multiply the equation (44) by the opposite of the velocity u and we add it to (43). With the definition (8) of the thermodynamic flux, we have, taking into account the “trigonometrical relations” (16) and the expression (27) of the velocity:

$$0 = \eta^* \left(c - \frac{\varepsilon u^2}{c} \right) + \varphi Y'_0 g = \eta^* \left(c - \frac{\varepsilon c S_\theta^2}{C_\theta^2} \right) + \frac{1}{C_\theta} \varphi_0 Y'_0 g_0 = \frac{c\eta^*}{C_\theta^2} + C_\theta \varphi_0 Y'_0 g_0$$

and the relation (53) is established. We report now the expression (28) of a current state W inside the equation (44):

$$\begin{aligned} 0 &= \frac{\varepsilon}{c} u \eta^* + \varphi Y'_0 Y_\theta \bullet \left(C_\theta W_0 + \frac{\varepsilon}{c} S_\theta g_0 \right) = C_\theta \varphi Y_\theta Y'_0 W_0 + \frac{\varepsilon}{c} \left(u \eta^* + S_\theta \varphi Y_\theta Y'_0 g_0 \right) \\ &= C_\theta \varphi Y_\theta Y'_0 W_0 + \frac{\varepsilon}{c} \left(u \eta^* - S_\theta \frac{c}{C_\theta} \eta^* \right) = C_\theta \varphi Y_\theta Y'_0 W_0 + \varepsilon \eta^* \left(\frac{u}{c} - \frac{S_\theta}{C_\theta} \right) \\ &= C_\theta \varphi Y_\theta Y'_0 W_0 \end{aligned}$$

and the relation (54) follows from (50). \square

• **Proposition 11. Dual entropy**

Under the hypotheses proposed at Proposition 9, we denote by σ^* the dual of the entropy σ restricted on the manifold of null velocity:

$$(55) \quad \sigma^* \equiv \varphi_0 \bullet W_0 - \sigma(W_0) \quad \text{on } \Omega_0, \quad \text{with } \varphi_0 = \sigma'(W_0).$$

Then we have

$$(56) \quad \eta^* = C_\theta \sigma^*$$

and the relation (53) takes the form

$$(57) \quad c \sigma^* + \varphi_0 Y'_0 g_0 = 0 \quad \text{on } \Omega_0.$$

Proof of Proposition 11.

We have, thanks to (5), (31), (38) and (50) :

$$\eta^* = \varphi \bullet W - \eta(W) = \varphi_0 \bullet Y_{-\theta} Y_\theta \left(C_\theta W_0 + \frac{\varepsilon}{c} S_\theta g_0 \right) - C_\theta \sigma = C_\theta (\varphi_0 \bullet W_0 - \sigma)$$

due to Hypothesis 3 (relation (51)). The end of the proof is clear. \square

6) Hyperbolic systems of order two

• If we suppose $N = 2$, the matrix Y_θ is a two by two matrix. We suppose here that this matrix has the same algebraic form that the form (12) suggested for G_θ . We set for this contribution

$$(58) \quad Y_\theta = \begin{pmatrix} \tilde{C}_\theta & \frac{\tilde{\varepsilon}}{a} \tilde{S}_\theta \\ a \tilde{S}_\theta & \tilde{C}_\theta \end{pmatrix}, \quad \theta \in \mathbb{R}, \quad a > 0.$$

In the relation (58), the functions \tilde{S}_θ and \tilde{C}_θ are analogous to the ones proposed in (13), (14), (15) and (16) for S_θ and C_θ . We have in particular

$$(59) \quad \tilde{C}_\theta^2 - \tilde{\varepsilon} \tilde{S}_\theta^2 \equiv 1.$$

The nilpotent representation of the group \mathcal{G} corresponds to $\tilde{\varepsilon} = 0$, the hyperbolic representation to $\tilde{\varepsilon} = 1$ and the elliptic one to $\tilde{\varepsilon} = -1$. The derivative Y'_θ of the matrix Y_θ proposed in (58) at $\theta = 0$ is given by

$$(60) \quad Y'_0 = \begin{pmatrix} 0 & \frac{\tilde{\varepsilon}}{a} \\ a & 0 \end{pmatrix}.$$

- We particularize the system by a simple choice for the null-velocity manifold. As in our previous work [6], we suppose

$$(61) \quad W_0 = \begin{pmatrix} \rho_0 \\ 0 \end{pmatrix} \in \Omega_0$$

and

$$(62) \quad g_0(W_0) = \begin{pmatrix} 0 \\ p_0 \end{pmatrix} \quad \text{for } W_0 \in \Omega_0.$$

With this choice, the hypotheses 2 and 3 are satisfied. The variable p_0 in the right hand side of the relation (62) is the thermodynamic pressure.

- **Proposition 12. Pressure as a thermodynamic variable**

With the above hypotheses, we have the relation

$$(63) \quad \sigma^* + \frac{\tilde{\varepsilon}}{a c} \sigma' p_0 = 0.$$

Proof of Proposition 12.

Just write the relation (57) with the help of (60) and (61) :

$$c \sigma^* + \varphi_0 Y'_0 g_0 = c \sigma^* + (\sigma' \ 0) \begin{pmatrix} 0 & \tilde{\varepsilon} \\ a & 0 \end{pmatrix} \begin{pmatrix} 0 \\ p_0 \end{pmatrix} = c \sigma^* + \frac{\tilde{\varepsilon}}{a} \sigma' p_0$$

and the relation (63) is established. \square

- The nilpotent case ($\tilde{\varepsilon} = 0$) has no interest when $N = 2$ because the dual of the entropy is necessary null, as we have remarked in [6] in the particular case of the Galileo group ($\varepsilon = 0$). We can explicit the first relations of (28). With the notation

$$(64) \quad W \equiv \begin{pmatrix} \rho \\ J \end{pmatrix}$$

we have

$$(65) \quad \rho = C_\theta \tilde{C}_\theta \rho_0 + \frac{\varepsilon \tilde{\varepsilon}}{a c} S_\theta \tilde{S}_\theta p_0, \quad \frac{J}{a} = C_\theta \tilde{S}_\theta \rho_0 + \frac{\varepsilon}{a c} S_\theta \tilde{C}_\theta p_0.$$

- In the case of Galileo group, we have $\varepsilon = 0$, $C_\theta = 1$, $S_\theta = \theta$ and $u = c\theta$. The explicitation of the parameter θ and the state of null velocity ρ_0 is simple by resolution of the system (65). We have

$$\rho = \tilde{C}_\theta \rho_0, \quad \frac{J}{a} = \tilde{S}_\theta \rho_0, \quad \varepsilon = 0, \quad \text{Galileo group.}$$

In the hyperbolic case, we have

$$\tanh \theta = \frac{J}{a \rho}, \quad u = c\theta, \quad \varepsilon = 0, \quad \tilde{\varepsilon} = 1$$

and in the elliptic one,

$$\text{tg } \theta = \frac{J}{a \rho}, \quad u = c\theta, \quad \varepsilon = 0, \quad \tilde{\varepsilon} = -1$$

as we have explained in [6].

- In the following, we focus on hyperbolic ($\tilde{\varepsilon} = 1$) and elliptic ($\tilde{\varepsilon} = -1$) representations of the Lorentz ($\varepsilon = 1$) and circular ($\varepsilon = -1$) groups. With other words,

$$(66) \quad \varepsilon^2 = \tilde{\varepsilon}^2 = 1.$$

We can replace the pressure p_0 with an appropriate thermodynamic quantity because

$$\frac{\sigma^*}{\sigma'} = \frac{\rho_0 \sigma' - \sigma}{\sigma'} = \rho_0 - \frac{\sigma}{\sigma'}.$$

Then the relations (65) take the form

$$(67) \quad \rho = (C_\theta \tilde{C}_\theta - \varepsilon S_\theta \tilde{S}_\theta) \rho_0 + \varepsilon S_\theta \tilde{S}_\theta \frac{\sigma}{\sigma'}, \quad \frac{J}{a} = (C_\theta \tilde{S}_\theta - \varepsilon \tilde{\varepsilon} S_\theta \tilde{C}_\theta) \rho_0 + \varepsilon \tilde{\varepsilon} S_\theta \tilde{C}_\theta \frac{\sigma}{\sigma'}.$$

For the Lorentz ($\varepsilon = 1$) or the circular ($\varepsilon = -1$) group, the resolution of the system (67) with unknowns θ and ρ_0 is absolutely non trivial. In order to be able to differentiate the solution (θ, ρ_0) relatively to (ρ, J) , we have the following result.

• **Proposition 13. Jacobian**

We denote by ζ_0 the partial derivative of the ratio σ/σ' relatively to the density ρ_0 at rest :

$$(68) \quad \zeta_0 \equiv \frac{\partial}{\partial \rho_0} \left(\frac{\sigma}{\sigma'} \right).$$

If the hypothesis (66) is satisfied and if the following jacobian determinant

$$(69) \quad \Delta \equiv \frac{\partial(\rho, J)}{\partial(\theta, \rho_0)} = a \frac{\sigma}{\sigma'} \left[(\varepsilon \tilde{\varepsilon} - 1) C_\theta^2 \frac{\sigma^*}{\sigma} + \tilde{\varepsilon} S_\theta^2 \zeta_0 - (C_\theta^2 + \tilde{\varepsilon} S_\theta^2) \right]$$

is not null, the solution (θ, ρ_0) of the system (67) is locally unique and we can differentiate the parameters θ and ρ_0 relatively to the state W introduced in (64).

Proof of Proposition 13.

We have

$$\left\{ \begin{array}{l} \frac{\partial \rho}{\partial \theta} = (\tilde{\varepsilon} - \varepsilon) C_\theta \tilde{S}_\theta \rho_0 + \varepsilon (C_\theta \tilde{S}_\theta + S_\theta \tilde{C}_\theta) \frac{\sigma}{\sigma'} \\ \frac{\partial \rho}{\partial \rho_0} = (C_\theta \tilde{C}_\theta - \varepsilon S_\theta \tilde{S}_\theta) + \varepsilon S_\theta \tilde{S}_\theta \zeta_0 \\ \frac{1}{a} \frac{\partial J}{\partial \theta} = (1 - \varepsilon \tilde{\varepsilon}) C_\theta \tilde{C}_\theta \rho_0 + (\varepsilon \tilde{\varepsilon} C_\theta \tilde{C}_\theta + \varepsilon S_\theta \tilde{S}_\theta) \frac{\sigma}{\sigma'} \\ \frac{1}{a} \frac{\partial J}{\partial \rho_0} = (C_\theta \tilde{S}_\theta - \varepsilon \tilde{\varepsilon} S_\theta \tilde{C}_\theta) + \varepsilon \tilde{\varepsilon} S_\theta \tilde{C}_\theta \zeta_0. \end{array} \right.$$

Then the determinant $\Delta \equiv \frac{\partial(\rho, J)}{\partial(\theta, \rho_0)}$ can be evaluated as follows :

$$\begin{aligned} \Delta &= \frac{\partial \rho}{\partial \theta} \frac{\partial J}{\partial \rho_0} - \frac{\partial \rho}{\partial \rho_0} \frac{\partial J}{\partial \theta} \\ &= a \left[\left((\tilde{\varepsilon} - \varepsilon) C_\theta \tilde{S}_\theta \rho_0 + \varepsilon (C_\theta \tilde{S}_\theta + S_\theta \tilde{C}_\theta) \frac{\sigma}{\sigma'} \right) \left((C_\theta \tilde{S}_\theta - \varepsilon \tilde{\varepsilon} S_\theta \tilde{C}_\theta) + \varepsilon \tilde{\varepsilon} S_\theta \tilde{C}_\theta \zeta_0 \right) \right] \\ &\quad - a \left[\left((C_\theta \tilde{C}_\theta - \varepsilon S_\theta \tilde{S}_\theta) + \varepsilon S_\theta \tilde{S}_\theta \zeta_0 \right) \left((1 - \varepsilon \tilde{\varepsilon}) C_\theta \tilde{C}_\theta \rho_0 + (\varepsilon \tilde{\varepsilon} C_\theta \tilde{C}_\theta + \varepsilon S_\theta \tilde{S}_\theta) \frac{\sigma}{\sigma'} \right) \right] \end{aligned}$$

$$= a \left[(\varepsilon \tilde{\varepsilon} - 1) C_\theta^2 (\tilde{C}_\theta^2 - \tilde{\varepsilon} \tilde{S}_\theta^2) \rho_0 + (S_\theta^2 + \varepsilon C_\theta^2) (\tilde{S}_\theta^2 - \tilde{\varepsilon} \tilde{C}_\theta^2) \frac{\sigma}{\sigma'} \right. \\ \left. + \tilde{\varepsilon} S_\theta^2 (\tilde{C}_\theta^2 - \tilde{\varepsilon} \tilde{S}_\theta^2) \frac{\sigma}{\sigma'} \zeta_0 \right]$$

and due to the identity (59) and the hypothesis (66), we have the intermediate result :

$$(70) \quad \Delta = a \left[(\varepsilon \tilde{\varepsilon} - 1) C_\theta^2 \rho_0 - \tilde{\varepsilon} (S_\theta^2 + \varepsilon C_\theta^2) \frac{\sigma}{\sigma'} + \tilde{\varepsilon} S_\theta^2 \frac{\sigma}{\sigma'} \zeta_0 \right].$$

Then

$$\Delta = a \frac{\sigma}{\sigma'} \left[(\varepsilon \tilde{\varepsilon} - 1) C_\theta^2 \left(\frac{\sigma^*}{\sigma} + 1 \right) - \tilde{\varepsilon} (S_\theta^2 + \varepsilon C_\theta^2) + \tilde{\varepsilon} S_\theta^2 \zeta_0 \right] \\ = a \frac{\sigma}{\sigma'} \left[(\varepsilon \tilde{\varepsilon} - 1) C_\theta^2 \frac{\sigma^*}{\sigma} - (C_\theta^2 + \tilde{\varepsilon} S_\theta^2) + \tilde{\varepsilon} S_\theta^2 \zeta_0 \right]$$

and the expression (69) is established. \square

- After determining the parameters θ and ρ_0 it is possible to evaluate the mathematical entropy η thanks to the relation (31). Then we can differentiate this mathematical entropy relatively the density ρ and the momentum J . We have the following remarkable coherence property.

- **Proposition 14. Entropy variables**

If the jacobian determinant (69) is not null, then the entropy variables $\varphi \equiv (\alpha, \beta)$ can be determined according to the relations (50). In other terms,

$$(71) \quad \alpha \equiv \frac{\partial \eta}{\partial \rho} = \tilde{C}_\theta \sigma'(\rho_0), \quad \beta \equiv \frac{\partial \eta}{\partial J} = -\frac{\tilde{\varepsilon}}{a} \tilde{S}_\theta \sigma'(\rho_0).$$

Proof of Proposition 14.

We first invert the jacobian matrix :

$$\begin{pmatrix} \frac{\partial \rho}{\partial \theta} & \frac{\partial \rho}{\partial \rho_0} \\ \frac{\partial J}{\partial \theta} & \frac{\partial J}{\partial \rho_0} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\partial \theta}{\partial \rho} & \frac{\partial \theta}{\partial J} \\ \frac{\partial \rho_0}{\partial \rho} & \frac{\partial \rho_0}{\partial J} \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} \frac{\partial J}{\partial \rho_0} & -\frac{\partial \rho}{\partial \rho_0} \\ -\frac{\partial J}{\partial \theta} & \frac{\partial \rho}{\partial \theta} \end{pmatrix}.$$

Then

$$\left\{ \begin{array}{l} \frac{\partial \theta}{\partial \rho} = \frac{1}{\Delta} \frac{\partial J}{\partial \rho_0} = \frac{a}{\Delta} \left[(C_\theta \tilde{S}_\theta - \varepsilon \tilde{\varepsilon} S_\theta \tilde{C}_\theta) + \varepsilon \tilde{\varepsilon} S_\theta \tilde{C}_\theta \zeta_0 \right] \\ \frac{\partial \rho_0}{\partial \rho} = -\frac{1}{\Delta} \frac{\partial J}{\partial \theta} = -\frac{a}{\Delta} \left[(1 - \varepsilon \tilde{\varepsilon}) C_\theta \tilde{C}_\theta \rho_0 + (\varepsilon \tilde{\varepsilon} C_\theta \tilde{C}_\theta + \varepsilon S_\theta \tilde{S}_\theta) \frac{\sigma}{\sigma'} \right] \\ \frac{\partial \theta}{\partial J} = -\frac{1}{\Delta} \frac{\partial \rho}{\partial \rho_0} = -\frac{1}{\Delta} \left[(C_\theta \tilde{C}_\theta - \varepsilon S_\theta \tilde{S}_\theta) + \varepsilon S_\theta \tilde{S}_\theta \zeta_0 \right] \\ \frac{\partial \rho_0}{\partial J} = \frac{1}{\Delta} \frac{\partial \rho}{\partial \theta} = \frac{1}{\Delta} \left[(\tilde{\varepsilon} - \varepsilon) C_\theta \tilde{S}_\theta \rho_0 + \varepsilon (C_\theta \tilde{S}_\theta + S_\theta \tilde{C}_\theta) \frac{\sigma}{\sigma'} \right]. \end{array} \right.$$

We first observe that

$$\frac{\partial \eta}{\partial \theta} = \frac{\partial}{\partial \theta} (C_\theta \sigma(\rho_0)) = \varepsilon S_\theta \sigma, \quad \frac{\partial \eta}{\partial \rho_0} = \frac{\partial}{\partial \rho_0} (C_\theta \sigma(\rho_0)) = C_\theta \sigma'.$$

Then with the chain rule, we have

$$\alpha = \frac{\partial \eta}{\partial \rho} = \frac{\partial \eta}{\partial \theta} \frac{\partial \theta}{\partial \rho} + \frac{\partial \eta}{\partial \rho_0} \frac{\partial \rho_0}{\partial \rho}$$

$$\begin{aligned}
 &= \frac{a}{\Delta} \left[\varepsilon S_\theta \sigma \left((C_\theta \tilde{S}_\theta - \varepsilon \tilde{\varepsilon} S_\theta \tilde{C}_\theta) + \varepsilon \tilde{\varepsilon} S_\theta \tilde{C}_\theta \zeta_0 \right) \right. \\
 &\quad \left. - C_\theta \sigma' \left((1 - \varepsilon \tilde{\varepsilon}) C_\theta \tilde{C}_\theta \rho_0 + (\varepsilon \tilde{\varepsilon} C_\theta \tilde{C}_\theta + \varepsilon S_\theta \tilde{S}_\theta \frac{\sigma}{\sigma'}) \right) \right] \\
 &= \frac{a}{\Delta} \left[(\varepsilon \tilde{\varepsilon} - 1) C_\theta^2 \tilde{C}_\theta \rho_0 \sigma' + \tilde{\varepsilon} S_\theta^2 \tilde{C}_\theta \zeta_0 \sigma - \varepsilon \tilde{C}_\theta (S_\theta^2 + \varepsilon C_\theta^2) \sigma \right] \\
 &= \frac{a}{\Delta} \tilde{C}_\theta \sigma' \left[(\varepsilon \tilde{\varepsilon} - 1) C_\theta^2 \rho_0 + \tilde{\varepsilon} S_\theta^2 \zeta_0 \frac{\sigma}{\sigma'} - \varepsilon (S_\theta^2 + \varepsilon C_\theta^2) \frac{\sigma}{\sigma'} \right] = \tilde{C}_\theta \sigma'
 \end{aligned}$$

due to (70). In an analogous way, we have

$$\begin{aligned}
 \beta &= \frac{\partial \eta}{\partial J} = \frac{\partial \eta}{\partial \theta} \frac{\partial \theta}{\partial J} + \frac{\partial \eta}{\partial \rho_0} \frac{\partial \rho_0}{\partial J} \\
 &= \frac{1}{\Delta} \left[-\varepsilon S_\theta \sigma \left((C_\theta \tilde{C}_\theta - \varepsilon S_\theta \tilde{S}_\theta) + \varepsilon S_\theta \tilde{S}_\theta \zeta_0 \right) \right. \\
 &\quad \left. + C_\theta \sigma' \left((\tilde{\varepsilon} - \varepsilon) C_\theta \tilde{S}_\theta \rho_0 + \varepsilon (C_\theta \tilde{S}_\theta + S_\theta \tilde{C}_\theta) \frac{\sigma}{\sigma'} \right) \right] \\
 &= \frac{\tilde{\varepsilon}}{\Delta} \left[(1 - \varepsilon \tilde{\varepsilon}) C_\theta^2 \tilde{S}_\theta \rho_0 \sigma' + \tilde{\varepsilon} (S_\theta^2 + \varepsilon C_\theta^2) \tilde{S}_\theta \sigma - \tilde{\varepsilon} S_\theta^2 \tilde{S}_\theta \zeta_0 \sigma \right] \\
 &= \frac{\tilde{\varepsilon}}{\Delta} \tilde{S}_\theta \sigma' \left[(1 - \varepsilon \tilde{\varepsilon}) C_\theta^2 \rho_0 + \tilde{\varepsilon} (S_\theta^2 + \varepsilon C_\theta^2) \frac{\sigma}{\sigma'} - \tilde{\varepsilon} S_\theta^2 \zeta_0 \frac{\sigma}{\sigma'} \right] = -\frac{\tilde{\varepsilon}}{a} \tilde{S}_\theta \sigma'
 \end{aligned}$$

due to (70). The proof is completed. \square

• We particularize our study and wish to construct nontrivial hyperbolic systems with entropy velocity covariant under the Lorentz group and / or the circular group. It seems essential to maintain a constant sign for the determinant Δ presented at the relation (69). Moreover, we have to enforce the strict convexity of the mathematical entropy $(\rho, J) \mapsto \eta(\rho, J)$. In this contribution, we only consider the particular case

$$(72) \quad \varepsilon = \tilde{\varepsilon}.$$

We focus on “circular elliptic” and “Lorentz hyperbolic” two by two systems of conservation laws. Observe first that due to the condition (72), the expression (67) of the conserved variables simplify into

$$(73) \quad \rho = \rho_0 + \varepsilon S_\theta^2 \frac{\sigma}{\sigma'}, \quad J = a S_\theta C_\theta \frac{\sigma}{\sigma'}.$$

Then the mathematical entropy η constrained to the condition (31) is strictly convex if and only if the determinant of its hessian $\det(d^2\eta)$ is strictly positive and if the second derivative $\frac{\partial^2 \eta}{\partial \rho^2}$ is also strictly positive.

• **Proposition 15. Technical lemma for convexity conditions**

Under the hypotheses (66) and (72), the expression of the jacobian determinant Δ is given according to

$$(74) \quad \Delta = a \delta \frac{\sigma}{\sigma'}$$

with

$$(75) \quad \delta = \varepsilon S_\theta^2 \zeta_0 - (C_\theta^2 + \varepsilon S_\theta^2).$$

The two signed expressions are given according to

$$(76) \quad \det(d^2\eta) = \frac{\varepsilon \sigma' \sigma''}{a \Delta} = \frac{\varepsilon}{a^2} (\sigma')^2 \frac{\sigma''}{\sigma} \frac{1}{\delta}$$

and

$$(77) \quad \frac{\partial^2 \eta}{\partial \rho^2} = \frac{\partial \alpha}{\partial \rho} = \frac{C_\theta}{\delta} \left(\varepsilon S_\theta^2 \zeta_0 \frac{\sigma'^2}{\sigma} - (C_\theta^2 + \varepsilon S_\theta^2) \sigma'' \right).$$

Proof of Proposition 15.

The expression δ in (75) is just a specific name for the big bracket in right hand side of the expression (69) and the relation (74) is satisfied. Under the condition (72), the expression δ is clearly reduced to the expression (75). We establish now the expression (76) of the determinant of the hessian matrix. From the chain rule, we have

$$\begin{aligned} \det(d^2\eta) &= \frac{\partial(\alpha, \beta)}{\partial(\rho, J)} = \frac{\partial(\alpha, \beta)}{\partial(\theta, \rho_0)} \frac{\partial(\theta, \rho_0)}{\partial(\rho, J)} = \frac{1}{\Delta} \det \begin{pmatrix} \frac{\partial \alpha}{\partial \theta} & \frac{\partial \alpha}{\partial \rho_0} \\ \frac{\partial \beta}{\partial \theta} & \frac{\partial \beta}{\partial \rho_0} \end{pmatrix} \\ &= \frac{1}{\Delta} \det \begin{pmatrix} \varepsilon S_\theta \sigma' & C_\theta \sigma'' \\ -\frac{\varepsilon}{a} C_\theta \sigma' & -\frac{\varepsilon}{a} S_\theta \sigma'' \end{pmatrix} = -\frac{1}{\Delta} \frac{\varepsilon}{a} \sigma' \sigma'' \det \begin{pmatrix} \varepsilon S_\theta & C_\theta \\ C_\theta & S_\theta \end{pmatrix} \end{aligned}$$

due to (71) and (72). Then $\det(d^2\eta) = \frac{1}{a\delta} \frac{\sigma'}{\sigma} \frac{\varepsilon}{a} \sigma' \sigma''$ is a consequence of the trigonometrical property (16) and the expression (76) follows. The evaluation of (77) is also elementary. From the chain rule, we have

$$\begin{aligned} \frac{\partial \alpha}{\partial \rho} &= \frac{\partial \alpha}{\partial \theta} \frac{\partial \theta}{\partial \rho} + \frac{\partial \alpha}{\partial \rho_0} \frac{\partial \rho_0}{\partial \rho} = \frac{a}{\Delta} \left[\varepsilon S_\theta \sigma' S_\theta C_\theta \zeta_0 - C_\theta \sigma'' (C_\theta^2 + \varepsilon S_\theta^2) \frac{\sigma}{\sigma'} \right] \\ &= \frac{1}{\delta} \frac{\sigma'}{\sigma} C_\theta \left[\varepsilon S_\theta^2 \sigma' \zeta_0 - \sigma'' (C_\theta^2 + \varepsilon S_\theta^2) \frac{\sigma}{\sigma'} \right] \end{aligned}$$

and the expression (77) is clear. \square

- We detail now two explicit examples of hyperbolic system of order two. The first one is invariant under the action of the circular group and the second under the Lorentz group.

• **Proposition 16. Exponential entropy for a “circular-elliptic” system**

We introduce two strictly positive constants ρ_* and $\bar{\sigma}$ and an exponential entropy on the manifold Ω_0 thanks to

$$(78) \quad \sigma(\rho_0) = \bar{\sigma} \exp\left(\frac{\rho_0}{\rho_*}\right).$$

We set

$$(79) \quad \begin{cases} \Psi = \frac{2J}{a\rho_*}, & \rho_0 = \rho + \frac{1}{2}(1 - \sqrt{1 - \Psi^2})\rho_*, & \theta = \frac{1}{2} \text{Arcsin } \Psi, \\ u = c \frac{J}{|J|} \sqrt{\frac{1 - \sqrt{1 - \Psi^2}}{1 + \sqrt{1 - \Psi^2}}} = c \text{sgn}(J) \tan \theta, & p_0 = a c (\rho_0 - \rho_*). \end{cases}$$

The system of conservation laws

$$(80) \quad \begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \left(\frac{c}{a} J \right) = 0 \\ \frac{\partial J}{\partial t} + \frac{\partial}{\partial x} (u J + p_0) = 0 \end{cases}$$

is hyperbolic if we suppose

$$(81) \quad |\Psi| \leq 1, \quad |\theta| \leq \frac{\pi}{4}.$$

It admits a mathematical entropy given by the expression

$$(82) \quad \eta(\rho, J) \equiv \sqrt{\frac{1}{2}(1 + \sqrt{1 - \Psi^2})} \sigma\left(\rho + \frac{\rho_*}{2}(1 - \sqrt{1 - \Psi^2})\right)$$

and the dissipation of entropy of entropy solutions is given by the inequality

$$(83) \quad \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(u \eta) \leq 0.$$

Proof of Proposition 16.

With the choice (78) of an exponential entropy, the parameter ζ_0 introduced at the relation (68) is null. Because $\bar{\sigma}$ is supposed strictly positive, σ is a convex function and the condition $\det(d^2\eta) > 0$ is reduced to $\delta < 0$ if we look to a “circular-elliptic” hyperbolic system, *id est* if we suppose $\varepsilon = -1$ (see (76)). Then with the notation Ψ introduced in (79), the relations (73) can be written

$$\sin(2\theta) = \Psi, \quad \rho = \rho_0 - \sin^2\theta \rho_*.$$

and the calculus of θ proposed in (79) is natural. We have the same remark for the expression of ρ_0 in (79) as a consequence of the identity $\sin^2\theta = \frac{1}{2}(1 - \sqrt{1 - \sin^2(2\theta)})$. The second condition in (81) is a consequence of the expression of θ proposed in (79). This last condition is also necessary when we impose $\delta < 0$ because the relation (75) can be now written $\delta = -(\cos^2\theta - \sin^2\theta)$. The velocity in (79) is simply the actual formulation of the relation (30) because

$$\tan \theta = \operatorname{sgn}(\theta) \sqrt{\frac{1 - \sqrt{1 - \sin^2(2\theta)}}{1 + \sqrt{1 + \sin^2(2\theta)}}}.$$

• The static pressure p_0 is related to the entropy σ thanks to (63). The expressions (8) and (33) of the physical and thermodynamix fluxes, the algebraic forms (58) of the matrix Y_θ and the choice (62) for g_0 show that the first component f_1 of $f(W)$ is given by

$$\begin{aligned} f_1 &= \rho u - \frac{\sin \theta}{a \cos \theta} p_0 = \rho u - \frac{1}{a c} u a c (\rho_0 - \rho_*) = \rho_* u (1 - \sin^2 \theta) \\ &= \rho_* c \sin \theta \cos \theta = \frac{1}{2} \rho_* c \sin(2\theta) = \frac{1}{2} \rho_* c \Psi = \frac{1}{2} \rho_* c \frac{2 J}{a \rho_*} = \frac{c}{a} J. \end{aligned}$$

Then the relations (80) are established. The expression (82) of the mathematical entropy is then a consequence of (31) : $\eta = \cos \theta \sigma(\rho_0)$. A particular hyperbolic system with entropy velocity invariant under the action of the circular group is constructed. \square

• **Proposition 17. Homographic entropy for a “Lorentz-hyperbolic” system**

We consider two strictly positive constants ρ_* and $\bar{\sigma}$ and an homographic entropy σ on the manifold Ω_0 defined by the relation

$$(84) \quad \sigma(\rho_0) = -\bar{\sigma} \frac{\rho_0}{\rho_0 + \rho_*}, \quad \rho_0 > 0.$$

We set

$$(85) \quad \Phi(\rho_0) \equiv \frac{2J}{a} \frac{\sigma'}{\sigma} = \frac{2J}{a} \frac{\rho_*}{\rho_0(\rho_0 + \rho_*)}$$

and we suppose

$$(86) \quad \frac{|J|}{a} \leq \rho \leq \frac{1}{2} \rho_*.$$

Then the equation

$$(87) \quad F(\rho_0) \equiv \rho_0 + \frac{\sigma}{2\sigma'} \left(\sqrt{1 + \Phi(\rho_0)^2} - 1 \right) = \rho$$

has a unique solution ρ_0 such that

$$(88) \quad 0 \leq \rho_0 \leq \frac{1}{2} \rho_*.$$

We consider the expressions

$$(89) \quad p_0 = a c \frac{\rho_0^2}{\rho_*}, \quad \theta = \frac{1}{2} \text{Argsh} \Phi(\rho_0), \quad u = a \tanh \theta = c \operatorname{sgn}(J) \sqrt{\frac{\sqrt{1 + \Phi^2} - 1}{\sqrt{1 + \Phi^2} + 1}}.$$

Then the system of conservation laws

$$(90) \quad \begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \left[\left(\rho + \frac{\rho_0}{\rho_*} \right) u \right] = 0 \\ \frac{\partial J}{\partial t} + \frac{\partial}{\partial x} (u J + p_0) = 0 \end{cases}$$

is hyperbolic and admits a mathematical entropy given by the expression

$$(91) \quad \eta(\rho, J) \equiv \sqrt{\frac{1}{2} (1 + \sqrt{1 + \Phi(\rho)^2})} \sigma(\rho_0).$$

The dissipation of entropy of entropy solutions is still given by the inequality (83).

Proof of Proposition 17.

In this case of the Lorentz group, we have $\varepsilon = 1$, $S_\theta \equiv \sinh \theta$ and $C_\theta \equiv \cosh \theta$. We have chosen with (84) a negative convex function σ . Then (*c.f.* (76)) the condition $\delta < 0$ has to be satisfied to assume $\det(d^2\eta) > 0$. We have

$$\frac{\sigma}{\sigma'} = \frac{\rho_0}{\rho_*} (\rho_0 + \rho_*), \quad \zeta_0 \equiv \frac{\partial}{\partial \rho_0} \left(\frac{\sigma}{\sigma'} \right) = 1 + 2 \frac{\rho_0}{\rho_*}.$$

The relations (73) can now be written as

$$\sinh(2\theta) = \Phi(\rho_0), \quad \rho = \rho_0 + \sinh^2 \theta \frac{\sigma}{\sigma'}.$$

The identity $\sinh^2 \theta \equiv \frac{1}{2} (\sqrt{1 + \sinh^2(2\theta)} - 1)$ shows that the density ρ_0 at null velocity associated with an arbitrary state $W \equiv (\rho, J)^t$ has to solve the equation

$$\rho - \rho_0 = \frac{1}{2} \left(\sqrt{1 + \Phi(\rho_0)^2} - 1 \right) \frac{\sigma}{\sigma'}$$

which is equivalent to (87). We remark that $F(0) = |J|/a$ and the first condition of (88) is natural.

- We prove now that the equation (87) has a unique solution if the conditions (86) are satisfied. We have

$$\frac{\partial}{\partial \rho_0} \left[\frac{\sigma}{2\sigma'} \sqrt{1 + \Phi^2} \right] = \frac{1}{2} \frac{\partial}{\partial \rho_0} \sqrt{\left(\frac{\sigma}{\sigma'}\right)^2 + \frac{4J^2}{a^2}} = \frac{2}{4 \sqrt{\left(\frac{\sigma}{\sigma'}\right)^2 + \frac{4J^2}{a^2}}} \frac{\sigma}{\sigma'} \zeta_0 > 0$$

if $\rho_0 > 0$. We have also $\frac{\partial}{\partial \rho_0} \left(\rho_0 - \frac{\sigma}{2\sigma'} \right) = 1 - \frac{\zeta_0}{2} = \frac{1}{2} - \frac{\rho_0}{\rho_*} > 0$ under the condition (88) for ρ_0 . Then we have $F'(\rho_0) > 0$ and the equation (87) has a unique solution when (88) is satisfied. The maximum of the function $F(\bullet)$ on the interval $[0, \frac{1}{2} \rho_*]$ can be estimated as follows. We have first $\frac{\sigma}{\sigma'} \left(\frac{\rho_*}{2} \right) = \frac{1}{2} (\frac{1}{2} + 1) \rho_* = \frac{3}{4} \rho_*$. Then

$$F\left(\frac{\rho_*}{2}\right) = \frac{\rho_*}{2} - \frac{3}{8} \rho_* + \frac{\rho_*}{2} \sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{2J}{a\rho_*}\right)^2} \geq \frac{\rho_*}{2} - \frac{3}{8} \rho_* + \frac{3}{8} \rho_* = \frac{\rho_*}{2}$$

and the second condition of (86) is natural. The pressure p_0 is given by the relation (63) :

$$p_0 = -ac \frac{\sigma^*}{\sigma'} = ac \left(\frac{\sigma}{\sigma'} - \rho_0 \right) = ac \frac{\rho_0^2}{\rho_*}$$

and the first relation of (89) is established. The two other relations of (89) are an elementary consequence of the previous considerations.

- We now establish the algebraic form of the hyperbolic system (1). From (8) and (33), the particular expression (58) of the matrix Y_θ and the choice $g_0 \equiv (0, p_0)^\dagger$ proposed in (62) show that the first component f_1 of $f(W)$ is given by

$$f_1 = \rho u + \frac{\sinh \theta}{a \cosh \theta} p_0 = \rho u + \frac{1}{ac} u ac \frac{\rho_0^2}{\rho_*} = \left(\rho + \frac{\rho_0^2}{\rho_*} \right) u.$$

Then the first line of the relations (90) is established. The second line is straightforward to evidence. The expression of the mathematical entropy is a direct consequence of (31) and the exploitation of $\cosh \theta$ as a function of $\Phi(\rho_0)$ as proposed in (89). This remark completes the construction of an hyperbolic system with entropy velocity (90) invariant under the action of the Lorentz group. \square

7) Conclusion

- In this contribution, we have extended our previous work [6] relative to the construction of hyperbolic systems covariant under the Galileo group and space reflection. We have precised the notion of “entropy velocity” as an intrinsic velocity associated with the conservation of entropy for regular solutions. In particular, the usual velocity of gas dynamics system is an entropy velocity. An hyperbolic system of conservation laws and the associated conservation of entropy are covariant under the action of a group of space-time transformations if the algebraic form of the equations is not changed after the action of the corresponding group. We have proposed sufficient conditions in order to satisfy a group covariance property. These conditions introduce naturally a representation of the

group in the space of states that can be chosen in multiple ways. We have also introduced a natural manifold of “null velocity”. When thermodynamic data are given on this manifold, the natural question is the construction of the whole hyperbolic system from the covariance property. We have also natural constraints for the entropy variables.

- After the study of the particular case of Galileo covariance in [6], we have particularized in this contribution our study for the two-dimensional case with the Lorentz and circular groups. We have proposed nontrivial hyperbolic systems covariant under each of these two groups. For the circular example, we find a condition (81) that shows that the associated hyperbolic system is not covariant under the action of the whole circular group. The question is set [23] to know if there exists a mathematical obstruction to find hyperbolic systems covariant under the action of the whole circular group. A better knowledge of such systems of conservation laws is a possible next step. Last but not least, the link with relativistic gas dynamics (see *e.g.* Souriau [20, 21], Chiu [3] and Vallée [22]) is also a natural question. The extension of this work to several space dimensions is, in principle, straightforward.

Acknowledgments

The author thanks Géry de Saxcé and Claude Vallée for their kind invitation to participate to the 56th and 57th CITV in august 2012 and august 2013, and for stimulating discussions during these workshops.

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