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A LEAST-SQUARES FORMULATION FOR THE APPROXIMATION OF CONTROLS FOR THE STOKES SYSTEM

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Abstract. This work deals with the approximation of distributed null controls for the Stokes system. The goal is to compute an approximation of controls that drives the solution from a prescribed initial state at $t = 0$ to zero at $t = T$. The existence of square-integrable controls have been obtained in *[Fursikov & Imanuvilov, Controllability of Evolution Equations, 1996])* via Carleman type estimates. We introduce and analyze a least-squares formulation of the controllability problem, and we show that it allows the construction of convergent sequences of functions toward null controls for the Stokes system. The approach consists first in introducing a class of functions satisfying *a priori* the boundary conditions in space and time - in particular the null controllability condition at time T -, and then finding among this class one element satisfying the Stokes system. This second step is done by minimizing a quadratic functional, among the admissible corrector functions of the Stokes system. Numerical experiments for the two dimensional case are performed in the framework of finite element approximations and demonstrate the interest of the approach. The method described here does not make use of duality arguments and therefore avoid the introduction of numerical ill-posed problem, as is typical when parabolic type equation are considered. This work extends *[Munch & Pedregal, Numerical Controllability of the heat equation through a variational approach, 2011])* where the case of the heat equation is discussed.

Contents

Key Words. Stokes equations, Null controllability, Least-squares approach.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N = 2$ or $N = 3$ be a bounded connected open set whose boundary $\partial\Omega$ is Lipschitz. Let $\omega \subset \Omega$ be a (small) nonempty open subset, and assume that $T > 0$. We use the notation $Q_T = \Omega \times (0,T), q_T = \omega \times (0,T)$, $\Sigma_T = \partial\Omega \times (0,T)$ and we denote by $\mathbf{n} = \mathbf{n}(\mathbf{x})$ the

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outward unit normal to Ω at any point $\mathbf{x} \in \partial\Omega$. Bold letters and symbols denote vector-valued functions and spaces; for instance $\mathbf{L}^2(\Omega)$ is the Hilbert space of the functions $\mathbf{v} = (v_1, \dots, v_N)$ with $v_i \in L^2(\Omega)$ for all i.

This work is concerned with the null controllability problem for the non-stationary Stokes system

$$\begin{cases} \mathbf{y}_t - \nu\Delta\mathbf{y} + \nabla\pi = \mathbf{f}\, 1_\omega, \quad \nabla\cdot\mathbf{y} = 0 \quad \text{in } Q_T \\ \mathbf{y} = \mathbf{0} \quad \text{on } \Sigma_T, \quad \mathbf{y}(\cdot,0) = \mathbf{y}_0 \quad \text{in } \Omega \end{cases} \tag{1.1}$$

which describes a viscous incompressible fluid flow in the bounded domain Ω. We use as a control function the density of external forces $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ concentrated in the arbitrary subdomain ω during the time interval $(0, T)$; $\mathbf{y}$ is the vector field of the fluid velocity, and π is the scalar pressure. The real ν denotes the constant viscosity of the fluid. The symbol 1_ω stands for the characteristic function of ω. We introduce the following spaces, usual in the context of Stokes systems:

$$\begin{aligned} \mathbf{H} &= \{\boldsymbol{\varphi} \in \mathbf{L}^2(\Omega) : \nabla\cdot\boldsymbol{\varphi} = 0 \text{ in } \Omega,\ \boldsymbol{\varphi}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}, \\ \mathbf{V} &= \{\boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega) : \nabla\cdot\boldsymbol{\varphi} = 0 \text{ in } \Omega\}, \\ U &= \left\{\psi \in L^2(\Omega) : \int_\Omega \psi(\mathbf{x})\, d\mathbf{x} = 0\right\}. \end{aligned} \tag{1.2}$$

For any $\mathbf{y}_0 \in \mathbf{H}$, $T > 0$, and $\mathbf{f} \in \mathbf{L}^2(q_T)$, there exists exactly one solution $(\mathbf{y}, \pi)$ of (1.1) with the following regularity :

$$\mathbf{y} \in C^0\left([0,T];\mathbf{H}\right) \cap L^2\left(0,T;\mathbf{V}\right), \ \pi \in L^2(0,T;U)$$

(see [29]). The null controllability problem for (1.1) at time T is the following:

For any $\mathbf{y}_0 \in \mathbf{H}$, find $\mathbf{f} \in \mathbf{L}^2(q_T)$ such that the corresponding solution to (1.1) *satisfies*

$$\mathbf{y}(\cdot, T) = \mathbf{0} \quad \textit{in} \quad \Omega. \tag{1.3}$$

The controllability properties of evolution PDEs have attracted a lot of works in the last decades: some relevant references are [3, 12, 20, 21, 30]. In particular, the Stokes - and more generally the Navier-Stokes - system has received a lot of attention: we mention the references [6, 19]. Specifically, the following result is proved in [11] (see also [5, 12, 18]) by the way of Carleman estimates.

Theorem 1.1 (Fursikov-Imanuvilov)**.** *The linear system* (1.1) *is null-controllable at any time $T > 0$.*

On the other hand, to the knowledge of the author, the (numerical) approximation of (exact) controls either distributed or located on the boundary for the Stokes system has received much less attention. This is probably due to the underlying ill-posedness of the approximation. On the other hand, the literature devoted to flow control and optimization (sometimes leading to approximate controls) is much more developed, both for the system of Stokes and Navier-Stokes. We refer to [16] and their references for a review, and also to the chapter 10 of [14].

In practice, the approximation of (exact) controls is usually addressed in the framework of an optimal control reformulation. Since there are controls $\mathbf{f}$ in $\mathbf{L}^2(q_T)$ for (1.1), it is natural to look for the one with minimal square-integrable norm, that is, one seeks to minimize the quadratic functional $J(\mathbf{f}) := \frac{1}{2}\|\mathbf{f}\|^2_{\mathbf{L}^2(q_T)}$ over the non-empty set

$$\mathcal{C}(\mathbf{y}_0, T) = \{(\mathbf{y}, \mathbf{f}) : \mathbf{f} \in \mathbf{L}^2(q_T),\ \mathbf{y} \text{ solves (1.1) and satisfies (1.3)}\}.$$

Since it is difficult to construct pairs in $\mathcal{C}(\mathbf{y}_0, T)$ (and *a fortiori* minimizing sequences !), one may use, as it is by now well-known in control theory, and following [15], duality arguments to

replace the constrained minimization of J by the unconstrained minimization of its conjugate function $J^\star$ defined as

$$J^\star(\boldsymbol{\varphi_T}) = \frac{1}{2}\iint_{q_T} |\boldsymbol{\varphi}|^2\, d\mathbf{x}\, dt + \int_\Omega \mathbf{y}_0 \cdot \boldsymbol{\varphi}(\cdot,0)\, d\mathbf{x}$$

over $\boldsymbol{\varphi_T} \in \boldsymbol{\mathcal{H}}$, where $(\boldsymbol{\varphi}, \sigma)$ solves the adjoint backward Stokes system associated with (1.1) :

$$\begin{cases} -\boldsymbol{\varphi}_t - \nu\Delta\boldsymbol{\varphi} + \nabla\sigma = \mathbf{0}, \quad \nabla\cdot\boldsymbol{\varphi} = 0 \quad \text{in } Q_T \\ \boldsymbol{\varphi} = \mathbf{0} \quad \text{on } \Sigma_T, \quad \boldsymbol{\varphi}(\cdot,T) = \boldsymbol{\varphi_T} \quad \text{in } \Omega. \end{cases} \tag{1.4}$$

$\boldsymbol{\mathcal{H}}$ is the Hilbert space defined as the completion of any smooth space functions included in $\mathbf{H}$ for the norm $\|\boldsymbol{\varphi}\|_{\mathbf{L}^2(q_T)}$. The control of minimal square-integrable norm is then given by $\mathbf{f} = \hat{\boldsymbol{\varphi}}\, 1_\omega$ where $\hat{\boldsymbol{\varphi}}$ is associated with the unique minimizer $\hat{\boldsymbol{\varphi}}_{\boldsymbol{T}}$ in $\boldsymbol{\mathcal{H}}$ of $J^\star$ (see [3, 15]). The difficulty, when one wants to approximate such control in any finite dimensional space, that is when one likes to minimize numerically $J^\star$, is that the space $\boldsymbol{\mathcal{H}}$ is huge, in particular, contains $\cup_{s\in\mathbb{N}}\mathbf{H}^{-s}(\Omega)$, and even elements that may not be distributions. Numerical experiments do suggest that the minimizer $\hat{\boldsymbol{\varphi}}_{\boldsymbol{T}}$ is very singular (we refer to [2, 22, 24] for a detailed analysis in the heat case). This phenomenon is independent of the choice of J, but is related to the use of dual variables. Actually, the equality (1.3) can be viewed as an equality in a very small space (due to the strong regularization effect of the heat kernel). Accordingly, the associated multiplier $\boldsymbol{\varphi_T}$ belong to a large dual space, much larger than $\mathbf{L}^2(\Omega)$, that is hard to represent (numerically) in any finite dimensional space.

An alternative way of looking at these problems and avoiding the introduction of dual variables has been proposed in [27]. It is based on the following simple strategy. Instead of working all the time with solutions of the underlying state equation, and looking for one that may comply with the final desired state, one considers a suitable class of functions complying with required initial, boundary, final conditions and appropriate regularity, and seeks one of those that is a solution of the state equation. This is in practice accomplished by setting up an error functional defined for all feasible functions, and measuring how far those are from being a solution of the underlying state equation.

One main practical advantage of this variational approach is that the way to get closer to a solution of the problem is by minimizing a functional that cannot get stuck on local minima because the only critical points of the error turn out to be global minimizers with zero error. Therefore a general strategy for (numerical) approximation consists in using a typical descent algorithm for this error functional. This approach which has the flavor of a least-squares type method has been employed successfully in our null controllability context for the linear heat equation in [25] and for an hyperbolic system in [23].

The objective of this paper is to apply the approach to the Stokes system, the main new ingredient with respect to the heat equation being the incompressible constraint for $\mathbf{y}$. It is organized as follows. In Section 2, we describe the ingredients of the variational approach for the system (1.1) and reduce the search of one controlled trajectory for the Stokes system to the minimization of the quadratic functional E defined by (2.2) over the affine space $\mathcal{A}$ defined by (2.1). In Section 3, by a general-purpose lemma (Lemma 3.2), using the very specific structure of the functional E, we prove that we may construct minimizing sequences for the error functional E that do converge strongly to an extremal point for E (see Proposition 3.1). We then move on Section 4 to provide the details for the numerical approach based on the Polak-Ribière version of the conjugate gradient algorithm to minimize the error functional and then to discuss the $\mathbb{P}_2 - \mathbb{P}_0 - \mathbb{P}_2$ finite element approximation of the underlying elliptic problem. Section 5 presents several experiments for the 2D case ($N = 2$), that is $Q_T \subset \mathbb{R}^3$ and emphasize the practical interest of the approach. Section 6 ends with some perspectives.

Results of this work were partially announced in the note [26].

2. A least-squares reformulation

Following [25, 27], we define the non-empty space

$$\mathcal{A} = \Big\{ (\mathbf{y},\pi,\mathbf{f});\ \mathbf{y}\in \mathbf{L}^2(0,T,\mathbf{H}_0^1(\Omega)), \mathbf{y}_t\in \mathbf{L}^2(0,T;\mathbf{H}^{-1}(\Omega)), \\ \mathbf{y}(\cdot,0)=\mathbf{y}_0,\ \mathbf{y}(\cdot,T)=\mathbf{0},\ \pi\in L^2(0,T;U),\ \mathbf{f}\in \mathbf{L}^2(q_T)\Big\}. \tag{2.1}$$

Note that these hypotheses on $\mathbf{y}$ imply that it belongs to $C([0,T],\mathbf{L}^2(\Omega))$, and so the two equalities $\mathbf{y}(\cdot,0)=\mathbf{y}_0$, $\mathbf{y}(\cdot,T)=\mathbf{0}$ in $\mathbf{L}^2(\Omega)$ are appropriate. It is also worth to note that $\mathcal{A}$ is defined in agreement with the regularity of any solution $(\mathbf{y},\pi)$ of the Stokes system with a source term $\mathbf{f}\in\mathbf{L}^2(q_T)$. Then, we define the functional $E:\mathcal{A}\to\mathbb{R}^+$ by

$$E(\mathbf{y},\pi,\mathbf{f}) = \frac{1}{2}\iint_{Q_T} (|\mathbf{v}_t|^2+|\nabla\mathbf{v}|^2+|\nabla\cdot\mathbf{y}|^2)\,d\mathbf{x}\,dt \tag{2.2}$$

where the corrector $\mathbf{v}$ is the unique solution in $\mathbf{H}^1(Q_T)$ of the (elliptic) boundary value problem

$$\begin{cases} -\mathbf{v}_{tt}-\Delta\mathbf{v}+(\mathbf{y}_t-\nu\Delta\mathbf{y}+\nabla\pi-\mathbf{f}\,1_\omega)=0, \quad \text{in} \quad Q_T,\\ \mathbf{v}=0 \quad \text{on} \quad \Sigma_T, \quad \mathbf{v}_t=0 \quad \text{on} \quad \Omega\times\{0,T\}. \end{cases} \tag{2.3}$$

For any $(\mathbf{y},\pi,\mathbf{f})\in\mathcal{A}$, the term $\mathbf{y}_t-\nu\Delta\mathbf{y}+\nabla\pi-\mathbf{f}\,1_\omega$ belongs to $\mathbf{L}^2(0,T;\mathbf{H}^{-1}(\Omega))$ so that the functional E is well-defined in $\mathcal{A}$. The approach developed in this work is based on the following result.

Proposition 2.1. *$(\mathbf{y},\pi)$ is a controlled solution of the Stokes system (1.1) by the control function $\mathbf{f}\,1_\omega\in\mathbf{L}^2(q_T)$ if and only if $(\mathbf{y},\pi,\mathbf{f})$ is a solution of the extremal problem :*

$$\inf_{(\mathbf{y},\pi,\mathbf{f})\in\mathcal{A}} E(\mathbf{y},\pi,\mathbf{f}). \tag{2.4}$$

Proof- From the controllability of the Stokes system given by Theorem 1.1, the extremal problem (2.4) is well-posed in the sense that the infimum, equal to zero, is reached by any controlled solution of the Stokes system. It is worthwhile to note that, without additional assumptions, the minimizer is not unique. Conversely, we check that any minimizer for E is a solution of the (controlled) Stokes system: let $(\mathbf{Y},\Pi,\mathbf{F})\in\mathcal{A}_0$ be arbitrary where

$$\mathcal{A}_0 = \Big\{ (\mathbf{y},\pi,\mathbf{f});\ \mathbf{y}\in \mathbf{L}^2(0,T,\mathbf{H}_0^1(\Omega)),\ \mathbf{y}_t\in \mathbf{L}^2(0,T;\mathbf{H}^{-1}(\Omega)), \\ \mathbf{y}(\cdot,0)=\mathbf{y}(\cdot,T)=\mathbf{0},\ \pi\in L^2(0,T;U),\ \mathbf{f}\in \mathbf{L}^2(q_T)\Big\}. \tag{2.5}$$

The first variation of E at the point $(\mathbf{y},\pi,\mathbf{f})$ in the admissible direction $(\mathbf{Y},\Pi,\mathbf{F})$ defined by

$$\langle E'(\mathbf{y},\pi,\mathbf{f}),(\mathbf{Y},\Pi,\mathbf{F})\rangle = \lim_{\eta\to 0}\frac{E((\mathbf{y},\pi,\mathbf{f})+\eta(\mathbf{Y},\Pi,\mathbf{F}))-E(\mathbf{y},\pi,\mathbf{f})}{\eta}, \tag{2.6}$$

exists, and is given by

$$\langle E'(\mathbf{y},\pi,\mathbf{f}),(\mathbf{Y},\Pi,\mathbf{F})\rangle = \iint_{Q_T}\Big(\mathbf{v}_t\cdot\mathbf{V}_t+\nabla\mathbf{v}\cdot\nabla\mathbf{V}+(\nabla\cdot\mathbf{y})(\nabla\cdot\mathbf{Y})\Big)\,d\mathbf{x}\,dt \tag{2.7}$$

where the corrector $\mathbf{V}\in\mathbf{H}^1(Q_T)$, associated with $(\mathbf{Y},\Pi,\mathbf{F})$, is the unique solution of

$$\begin{cases} -\mathbf{V}_{tt}-\Delta\mathbf{V}+(\mathbf{Y}_t-\nu\Delta\mathbf{Y}+\nabla\Pi-\mathbf{F}\,1_\omega)=0 \quad \text{in} \quad Q_T,\\ \mathbf{V}=0 \quad \text{on} \quad \Sigma_T, \quad \mathbf{V}_t=0 \quad \text{on} \quad \Omega\times\{0,T\}. \end{cases} \tag{2.8}$$

Multiplying the main equation of this system by $\mathbf{v}$ (recall that $\mathbf{v}$ is the corrector associated with the minimizer $(\mathbf{y},\pi,\mathbf{f})$), integrating by parts, and using the boundary conditions on $\mathbf{v}$ and $\mathbf{V}$,

we get

$$\langle E'(\mathbf{y},\pi,\mathbf{f}),(\mathbf{Y},\Pi,\mathbf{F})\rangle = -\iint_{Q_T}(-\mathbf{Y}\cdot\mathbf{v}_t+\nu\nabla\mathbf{Y}\cdot\nabla\mathbf{v}-\Pi\,\nabla\cdot\mathbf{v}-\mathbf{F}\cdot\mathbf{v}\,1_\omega)\,d\mathbf{x}\,dt + \iint_{Q_T}(\nabla\cdot\mathbf{y})(\nabla\cdot\mathbf{Y})\,d\mathbf{x}\,dt,\qquad \forall(\mathbf{Y},\Pi,\mathbf{F})\in\mathcal{A}_0, \tag{2.9}$$

where we have used that

$$-\int_0^T\langle\mathbf{Y}_t,\mathbf{v}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1(\Omega)}\,dt=\iint_{Q_T}\mathbf{Y}\cdot\mathbf{v}_t\,d\mathbf{x}\,dt-\int_\Omega[\mathbf{Y}\cdot\mathbf{v}]_0^T\,d\mathbf{x}=\iint_{Q_T}\mathbf{Y}\cdot\mathbf{v}_t\,d\mathbf{x}\,dt,$$

and that

$$\int_0^T\langle\nabla\Pi,\mathbf{v}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1(\Omega)}dt=-\iint_{Q_T}\Pi\,\nabla\cdot\mathbf{v}\,d\mathbf{x}\,dt.$$

Therefore if $(\mathbf{y},\pi,\mathbf{f})$ minimizes E, the equality $\langle E'(\mathbf{y},\pi,\mathbf{f}),(\mathbf{Y},\Pi,\mathbf{F})\rangle=0$ for all $(\mathbf{Y},\Pi,\mathbf{F})\in\mathcal{A}_0$ implies that the corrector $\mathbf{v}=\mathbf{v}(\mathbf{y},\pi,\mathbf{f})$ solution of (2.3) satisfies the conditions

$$\begin{cases}\mathbf{v}_t+\nu\Delta\mathbf{v}-\nabla(\nabla\cdot\mathbf{y})=0,\quad \nabla\cdot\mathbf{v}=0, & \text{in}\quad Q_T,\\ \mathbf{v}=0, & \text{in}\quad q_T.\end{cases} \tag{2.10}$$

But from the unique continuation property for the Stokes system (see [5]), it turns out that $\mathbf{v}=0$ in Q_T and that $\nabla\cdot\mathbf{y}$ is a constant in Q_T. Eventually, the relation (2.9) is then reduced to

$$\langle E'(\mathbf{y},\pi,\mathbf{f}),(\mathbf{Y},\Pi,\mathbf{F})\rangle=(\nabla\cdot\mathbf{y})\iint_{Q_T}\nabla\cdot\mathbf{Y}\,d\mathbf{x}\,dt=0,\quad\forall(\mathbf{Y},\Pi,\mathbf{F})\in\mathcal{A}_0$$

and then implies that this constant is zero. Consequently, if $(\mathbf{y},\pi,\mathbf{f})\in\mathcal{A}$ is a minimizer for E, then $\nabla\cdot\mathbf{y}=0$ in Q_T, and the corrector $\mathbf{v}$ is zero in Q_T, so that $E(\mathbf{y},\pi,\mathbf{f})=0$. Therefore, $(\mathbf{y},\pi,\mathbf{f})$ solves (1.1), and since $(\mathbf{y},\pi,\mathbf{f})\in\mathcal{A}$, the state $\mathbf{y}$ is controlled at the time T by the function $\mathbf{f}$ which acts as a control distributed in ω. □

Remark 2.2. *It is worthwhile to notice that the proof of Proposition 2.1 only utilizes optimality of $(\mathbf{y},\pi,\mathbf{f})$ and not its minimality. Therefore in the statement of the proposition, we could have written instead : $(\mathbf{y},\pi)$ is a controlled solution of the Stokes system (1.1) by the control function $\mathbf{f}\,1_\omega\in\mathbf{L}^2(q_T)$ if and only if $(\mathbf{y},\pi,\mathbf{f})$ is a stationary point for the functional $E(\mathbf{y},\pi,\mathbf{f})$ over $(\mathbf{y},\pi,\mathbf{f})\in\mathcal{A}$. This is relevant from the perspective of the numerical simulation for it guarantees that the numerical procedure based on a descent strategy cannot get stuck in local minima.*

Remark 2.3. *For any $(\mathbf{y},\pi,\mathbf{f})\in\mathcal{A}$, the cost E can be formulated as follows*

$$E(\mathbf{y},\pi,\mathbf{f})=\frac{1}{2}\|\mathbf{y}_t-\nu\Delta\mathbf{y}+\nabla\pi-\mathbf{f}\,1_\omega\|^2_{\mathbf{H}^{-1}(Q_T)}+\frac{1}{2}\|\nabla\cdot\mathbf{y}\|^2_{\mathbf{L}^2(Q_T)}.$$

This justifies the least-squares terminology we have employed. The use of least-squares type approaches to solve linear and nonlinear problem is not new and we refer to [13] *for many applications in the last two decades (and specifically to* [1] *for numerical analysis). It seems however that the use of least-squares type approaches in the controllability context comes from* [27].

Remark 2.4. *The quasi-incompressibility case is obtained in the same way. It suffices to add $\epsilon\pi$ (for any $\epsilon>0$) to the divergence term in the functional E. This is also in practice a classical numerical trick to fix the constant of the pressure π (see Section 4).*

Remark 2.5. *The approach allows to consider compact support control jointly in time and space. It suffices to replace the function 1_ω in (1.1) by any compact support function in time and space, say $1_{\tilde{q}_T}$, where $\tilde{q}_T$ denotes a non-empty subset of Q_T. Since Theorem 1.1 holds for any controllability time T and any subset ω of Ω, the controllability of (1.1) remains true as soon as $\tilde{q}_T$ contains any non-empty cylindrical domain of the form $\omega_1\times(t_1,t_2)\subset\Omega\times(0,T)$.*

Remark 2.6. *A fortiori, the approach is well-adapted to address the direct problem (which consists,* $\mathbf{f}$ *being fixed, in solving the boundary value problem (1.1): it suffices to remove from* $\mathcal{A}$ *and* $\mathcal{A}_0$ *the condition (1.3), and fix the forcing term* $\mathbf{f}$*. In that case, the second line of (2.10) is replaced by* $\mathbf{v}(\cdot,T)=0$*, which implies with the first line, that* $\mathbf{v}$ *and* $\nabla\cdot\mathbf{y}=0$ *both vanish in* Q_T*.*

Let us mention that the global space-time least-squares approach introduced here differs from the ones introduced earlier (usually for elliptic problems) and discussed at length in the book [1]*. For instance, Section 9.2 of* [1] *discusses a least-squares method for a semi-discretization in time of (1.1). In the closed context of the heat equation, let us mention that a different so called "global space-time least-squares principle" (but still different from the approach introduced here) is mentioned and briefly discussed in* [1]*, Section 9.1.4.*

It is worth to notice that this approach allows to treat at the same time the null controllability constraint and the incompressibility one. In this sense, the pair $(\pi,\mathbf{f})$ can be regarded as a control function for the set of constraints

$$\mathbf{y}(\cdot,T)=0 \quad \text{on} \quad \Omega, \qquad \nabla\cdot\mathbf{y}=0 \quad \text{in} \quad Q_T. \tag{2.11}$$

Obviously, these two conditions are compatibles: there is no competition between them. In the uncontrolled situation, from the uniqueness, the pressure π is unique as soon as the source term (here $\mathbf{f}1_\omega$) is fixed. On the other hand, in our controllability context, the pair $(\pi,\mathbf{f})$ is not unique: the pressure π depends on $\mathbf{f}\,1_\omega$ and *vice versa*. Therefore, the optimization with respect to both variables at the same time makes sense. From this point of view, we may reformulate the problem as a general controllability problem for the heat equation:

$$\mathbf{y}_t-\nu\Delta\mathbf{y}=\boldsymbol{V}:=\mathbf{f}\,1_\omega-\nabla\pi \quad \text{in} \quad Q_T,$$

$\boldsymbol{V}$ being a control function such that (2.11) holds. The control function $\boldsymbol{V}$ acts on the whole domain, but on the other hand, should take the specific form $\boldsymbol{V}:=\mathbf{f}\,1_\omega-\nabla\pi$.

Again, this perspective is different with the classical one, which consists in finding a control $\mathbf{v}\in\mathbf{L}^2(q_T)$, such that $\mathbf{y}(\cdot,T)=0$ in Ω where $(\mathbf{y},\pi)$ solves the Stokes system (1.1). This can done by duality, penalization technique, etc. Conversely, one may also consider iteratively first the null controllability constraint, that is, for any π fixed in $L^2(0,T;U)$, find a control $\mathbf{f}_\pi\,1_\omega$ such that (1.3) holds, and then find the pressure π such that $\nabla\cdot\mathbf{y}_\pi=0$ holds in Q_T. Using again a least-squares type approach (for the heat equation, as developed in [25]), the first step reduces to solve, for any $\pi\in L^2(0,T,U)$ fixed, the problem

$$\inf_{(\mathbf{y}_\pi,\mathbf{f}_\pi)\in\mathcal{A}_1}\tilde{E}(\mathbf{y},\mathbf{f}):=\frac{1}{2}\|\mathbf{v}\|^2_{\mathbf{H}^1(Q_T)}$$

where $\mathbf{v}=\mathbf{v}(\mathbf{y}_\pi,\pi,\mathbf{f}_\pi)$ solves (2.3) and $\mathcal{A}_1$ is given by

$$\mathcal{A}_1=\left\{(\mathbf{y},\mathbf{f});\ \ \mathbf{y}\in\mathbf{L}^2(0,T,\mathbf{H}^1_0(\Omega)),\mathbf{y}_t\in\mathbf{L}^2(0,T;\mathbf{H}^{-1}(\Omega)),\mathbf{y}(\cdot,0)=\mathbf{y}_0,\ \mathbf{y}(\cdot,T)=\mathbf{0},\ \mathbf{f}\in\mathbf{L}^2(q_T)\right\}.$$

The second step consists in updating the pressure according to a descent direction for the function $G:L^2(0,T,U)\to\mathbb{R}$ defined by $G(\pi):=1/2\|\nabla\cdot\mathbf{y}_\pi\|^2_{L^2(Q_T)}$. We get that the first variation of G at π in the direction $\overline{\pi}\in L^2(0,T;U)$ is given by $<G'(\pi),\overline{\pi}>=\iint_{Q_T}\nabla\overline{\pi}\cdot\mathbf{p}\,dx\,dt$ where $\mathbf{p}$ solves

$$-\mathbf{p}_t-\nu\Delta\mathbf{p}=\nabla(\nabla\cdot\mathbf{y}_\pi) \quad \text{in} \quad Q_T, \quad \mathbf{p}(\cdot,T)=0 \quad \text{in} \quad \Omega, \quad \mathbf{p}=0 \quad \text{on} \quad \Sigma_T.$$

Again, this direct problem may be solved within the variational approach developed in this work (see Remark 2.6).

3. Convergence of some minimizing sequences for E

Proposition 2.1 reduces the approximation of a null control for (1.1) to a minimization of the functional E over the space $\mathcal{A}$. As a preliminary step, since $\mathcal{A}$ is not an Hilbert space (precisely, $\mathcal{A}$ is not a vectorial space), we note that any element of $\mathcal{A}$ can be written as the sum of one

element of $\mathcal{A}$, say $\mathbf{s}_{\mathcal{A}}$, plus any element of $\mathcal{A}_0$, which is a vectorial space. Thus we consider for any $\mathbf{s}_{\mathcal{A}} := (\mathbf{y}_{\mathcal{A}}, \pi_{\mathcal{A}}, \mathbf{f}_{\mathcal{A}}) \in \mathcal{A}$ the following problem:

$$\min_{(\mathbf{y},\pi,\mathbf{f})\in\mathcal{A}_0} E_{\mathbf{s}_{\mathcal{A}}}(\mathbf{y},\pi,\mathbf{f}), \qquad E_{\mathbf{s}_{\mathcal{A}}}(\mathbf{y},\pi,\mathbf{f}) := E(\mathbf{s}_{\mathcal{A}} + (\mathbf{y},\pi,\mathbf{f})). \tag{3.1}$$

Problems (2.4) and (3.1) are equivalent. Any solution of Problem (3.1) is a solution of the initial problem (2.4). Conversely, any solution of Problem (2.4) can be decomposed as the sum $\mathbf{s}_{\mathcal{A}}+\mathbf{s}_{\mathcal{A}_0}$, for some $\mathbf{s}_{\mathcal{A}_0}$ in $\mathcal{A}_0$.

We endow the vectorial space $\mathcal{A}_0$ with its natural norm $\|\cdot\|_{\mathcal{A}_0}$ such that :

$$\|\mathbf{y},\pi,\mathbf{f}\|^2_{\mathcal{A}_0} := \iint_{Q_T} (|\mathbf{y}|^2 + |\nabla\mathbf{y}|^2)\,d\mathbf{x}\,dt + \int_0^T \|\mathbf{y}_t(\cdot,t)\|^2_{\mathbf{H}^{-1}(\Omega)}dt + \iint_{Q_T} (|\mathbf{f}|^2 + |\pi|^2)\,d\mathbf{x}\,dt, \tag{3.2}$$

recalling that $\|\mathbf{y}_t\|_{\mathbf{H}^{-1}(\Omega)} = \|\mathbf{g}\|_{\mathbf{H}^1_0(\Omega)}$ where $\mathbf{g} \in \mathbf{H}^1_0(\Omega)$ solves $-\boldsymbol{\Delta}\mathbf{g} = \mathbf{y}_t$ in Ω. We denote $\langle , \rangle_{\mathcal{A}_0}$ the corresponding scalar product. $(\mathcal{A}_0, \|\cdot\|_{\mathcal{A}_0})$ is an Hilbert space.

The relation (2.9) allows to define a minimizing sequence in $\mathcal{A}_0$ for $E_{\mathbf{s}_{\mathcal{A}}}$, using a typical descent method.

It turns out that minimizing sequences for $E_{\mathbf{s}_{\mathcal{A}}}$ which belong to a precise subset of $\mathcal{A}_0$ remain bounded uniformly. It is worth to note that this very valuable property is not *a priori* guaranteed from the definition of $E_{\mathbf{s}_{\mathcal{A}}}$. The boundedness of $E_{\mathbf{s}_{\mathcal{A}}}$ implies only the boundedness of the corrector $\mathbf{v}$ for the $\mathbf{H}^1(Q_T)$-norm and the boundedness of the divergence $\nabla\cdot\mathbf{y}$ of the velocity field for the $L^2(Q_T)$-norm. Actually, this property is mainly due to the fact the functional $E_{\mathbf{s}_{\mathcal{A}}}$ is invariant in the subset of $\mathcal{A}_0$ which satisfies the state equations of (1.1).

In order to construct a minimizing sequence bounded in $\mathcal{A}_0$ for $E_{\mathbf{s}_{\mathcal{A}}}$, we introduce the linear continuous operator $\mathbf{T}$ which maps a triplet $(\mathbf{y},\pi,\mathbf{f}) \subset \mathcal{A}$ into the corresponding vector $(\mathbf{v}, \nabla\cdot\mathbf{y}) \in \mathbf{H}^1(Q_T) \times L^2(Q_T)$, with the corrector $\mathbf{v}$ as defined by (2.3). Then we define the space $A = \mathrm{Ker}\,\mathbf{T} \cap \mathcal{A}_0$ composed of the elements $(\mathbf{y},\pi,\mathbf{f})$ satisfying the Stokes system and such that $\mathbf{y}$ vanishes on the boundary ∂Q_T. Note that A is not the trivial space : it suffices to consider the difference of two distinct null controlled solutions of (1.1). Finally, we note $A^\perp = (\mathrm{Ker}\,\mathbf{T} \cap \mathcal{A}_0)^\perp$ the orthogonal complement of A in $\mathcal{A}_0$ and $P_{A^\perp} : \mathcal{A}_0 \to A^\perp$ the (orthogonal) projection on $A^\perp$.

We then define the following minimizing sequence $(\mathbf{y}^k,\pi^k,\mathbf{f}^k)_{k\geq 0} \in A^\perp$ as follows:

$$\begin{cases} (\mathbf{y}^0,\pi^0,\mathbf{f}^0) \text{ given in } A^\perp, \\ (\mathbf{y}^{k+1},\pi^{k+1},\mathbf{f}^{k+1}) = (\mathbf{y}^k,\pi^k,\mathbf{f}^k) - \eta_k P_{A^\perp}(\overline{\mathbf{y}}^k,\overline{\pi}^k,\overline{\mathbf{f}}^k), \quad k \geq 0 \end{cases} \tag{3.3}$$

where $(\overline{\mathbf{y}}^k,\overline{\pi}^k,\overline{\mathbf{f}}^k) \in \mathcal{A}_0$ is defined as the unique solution of the formulation

$$\langle(\overline{\mathbf{y}}^k,\overline{\pi}^k,\overline{\mathbf{f}}^k),(\mathbf{Y},\Pi,\mathbf{F})\rangle_{\mathcal{A}_0} = \langle E'_{\mathbf{s_0}}(\mathbf{y}^k,\pi^k,\mathbf{f}^k),(\mathbf{Y},\Pi,\mathbf{F})\rangle, \quad \forall(\mathbf{Y},\Pi,\mathbf{F}) \in \mathcal{A}_0. \tag{3.4}$$

η_k denotes a positive descent step. In particular, (3.4) implies that $\overline{\pi}^k = -\nabla\cdot\mathbf{v}^k \in L^2(Q_T)$ and $\overline{\mathbf{f}}^k = -\mathbf{v}^k\,1_\omega \in \mathbf{L}^2(q_T)$ (actually in $\mathbf{H}^1(q_T)$).

One main issue of our variational approach is to establish the convergence of the minimizing sequence defined by (3.3). We have the following result.

Proposition 3.1. *For any $\mathbf{s}_{\mathcal{A}} \in \mathcal{A}$ and any $\{\mathbf{y}^0,\pi^0,\mathbf{f}^0\} \in A^\perp$, the sequence $\mathbf{s}_{\mathcal{A}}+\{(\mathbf{y^k},\pi^k,\mathbf{f^k})\}_{\mathbf{k}\geq\mathbf{0}} \in \mathcal{A}$ converges strongly to a solution of the extremal problem (2.4).*

This proposition is the consequence of the following abstract result which can be adapted to many different situations where this variational perspective can be of help.

Lemma 3.2. *Suppose $\mathbf{T} : X \mapsto Y$ is a linear, continuous operator between Hilbert spaces, and $H \subset X$, a closed subspace, $u_0 \in X$. Put*

$$E : u_0 + H \mapsto \mathbb{R}^+, \quad E(u) = \frac{1}{2}\|\mathbf{T}u\|^2, \quad A = Ker\,\mathbf{T} \cap H.$$

(1) $E : u_0 + A^\perp \to \mathbb{R}$ *is quadratic, non-negative, and strictly convex, where* $A^\perp$ *is the orthogonal complement of* A *in* H.
(2) *If we regard* E *as a functional defined on* H, $E(u_0+\cdot)$, *and identify* H *with its dual, then the derivative* $E'(u_0+\cdot)$ *always belongs to* $A^\perp$. *In particular, a typical steepest descent procedure for* $E(u_0+\cdot)$ *will always stay in the manifold* $u_0 + A^\perp$.
(3) *If, in addition,* $\min_{u\in H} E(u_0+u) = 0$, *then the steepest descent scheme will always produce sequences converging (strongly in* X*) to a unique (in* $u_0 + A^\perp$*) minimizer* $u_0 + \overline{u}$ *with zero error.*

Proof of Lemma 3.2- Suppose there are $u_i \in A^\perp$, $i = 1, 2$, such that

$$E\left(u_0 + \frac{1}{2}u_1 + \frac{1}{2}u_2\right) = \frac{1}{2}E(u_0+u_1) + \frac{1}{2}E(u_0+u_2).$$

Due to the strict convexity of the norm in a Hilbert space, we deduce that this equality can only occur if $\mathbf{T}u_1 = \mathbf{T}u_2$. So therefore $u_1 - u_2 \in A \cap A^\perp = \{0\}$, and $u_1 = u_2$. For the second part, note that for arbitrary $U \in A$, $\mathbf{T}U = 0$, and so

$$E(u_0+u+U) = \frac{1}{2}\|\mathbf{T}u_0 + \mathbf{T}u + \mathbf{T}U\|^2 = \frac{1}{2}\|\mathbf{T}u_0 + \mathbf{T}u\|^2 = E(u_0+u).$$

Therefore the derivative $E'(u_0+u)$, the steepest descent direction for E at u_0+u, has to be orthogonal to all such $U \in A$.

Finally, assume $E(u_0+\overline{u}) = 0$. It is clear that this minimizer is unique in $u_0 + A^\perp$ (recall the strict convexity in (i)). This, in particular, implies that for arbitrary $u \in A^\perp$,

$$\langle E'(u_0+u), \overline{u} - u\rangle \le 0, \tag{3.5}$$

because this inner product is the derivative of the section $t \mapsto E(u_0 + t\overline{u} + (1-t)u)$ at $t = 0$, and this section must be a positive parabola with the minimum point at $t = 1$. If we consider the gradient flow

$$u'(t) = -E'(u_0+u(t)), \quad t \in [0, +\infty),$$

then, because of (3.5),

$$\frac{d}{dt}\left(\frac{1}{2}\|u(t) - \overline{u}\|^2\right) = \langle u(t) - \overline{u}, u'(t)\rangle = \langle u(t) - \overline{u}, -E'(u_0+u(t))\rangle \le 0.$$

This implies that sequences produced through a steepest descent method will be minimizing for E, uniformly bounded in X (because $\|u(t) - \overline{u}\|$ is a non-increasing function of t), and due to the strict convexity of E restricted to $u_0 + A^\perp$, they will have to converge towards the unique minimizer $u_0 + \overline{u}$. □

Remark 3.3. *Despite the strong convergence in this statement, it may not be true that the error is coercive, even restricted to* $u_0 + A^\perp$, *so that strong convergence could be very slow. Because of this same reason, it may be impossible to establish rates of convergence for these minimizing sequences.*

The element u_0 determines the non-homogeneous data set of each problem: source term, boundary conditions, initial and/or final condition, etc. The subspace H is the subset of the ambient Hilbert space X for which the data set vanishes. $\mathbf{T}$ is the operator defining the corrector, so that $\operatorname{Ker}\mathbf{T}$ is the subspace of all solutions of the underlying equation or system. The subspace A is the subspace of all solutions of the problem with vanishing data set. In some situations A will be trivial, but in some others will not be so. The important property is (iii) in the statement guaranteeing that we indeed have strong convergence in X of iterates. The main requirement for this to hold is to know, *a priori*, that the error attains its minimum value zero somewhere, which in the situation treated here is guaranteed by Theorem 1.1.

Proof of Proposition 3.1- The result is obtained by applying Lemma 3.2 as follows. If we put $B = \{\mathbf{y} \in \mathbf{L}^2(0,T,\mathbf{H}_0^1(\Omega)) : \mathbf{y}_t \in \mathbf{L}^2(0,T;\mathbf{H}^{-1}(\Omega))\}$, X is taken to be $B \times L^2(0,T;U) \times \mathbf{L}^2(q_T)$. H is taken to be $\mathcal{A}_0$ as given in (2.5) and $u_0 = \boldsymbol{s}_{\boldsymbol{\mathcal{A}}} \in \mathcal{A} \subset X$. The operator $\mathbf{T}$ maps a triplet $(\mathbf{y},\pi,\mathbf{f}) \in \mathcal{A} \subset X$ into $(\mathbf{v}, \nabla\cdot\mathbf{y}) \in Y := \mathbf{H}^1(Q_T) \times L^2(Q_T)$ as explained earlier.

Remark 3.4. *The construction of the minimizing sequence only requires the resolution of standard well-posed elliptic problems over Q_T, well-adapted to general situations (time dependent support, mesh adaptation, etc). On the other hand, it is important to highlight that the $\mathbf{L}^2(q_T)$ control function $\mathbf{f}$ obtained from the minimizing procedure does not a priori minimize any specific norm (for instance the $\mathbf{L}^2$-norm).*

Without the projection on $(\mathrm{Ker}\,\mathbf{T} \cap \mathcal{A}_0)^\perp$ in (3.3), the sequence $(\mathbf{y}^k,\pi^k,\mathbf{f}^k)$ remains a minimizing sequence for $E_{\mathbf{s}_\mathcal{A}}$: actually, the values of the cost $E_{\mathbf{s}_\mathcal{A}}$ along the sequence $(\mathbf{y}^k,\pi^k,\mathbf{f}^k)$ are equal with or without the projection. This is due to the fact that the component of the descent direction $(\overline{\mathbf{y}}^k,\overline{\pi}^k,\overline{\mathbf{f}}^k)$ on $(\mathrm{Ker}\,\mathbf{T} \cap \mathcal{A}_0)$ does not affect the value of the cost : on the other hand, without the projection, the minimizing sequence may not be bounded uniformly in $\mathcal{A}_0$, in particular the control function $\mathbf{f}$ may not be bounded in $\mathbf{L}^2(q_T)$.

The subset $A^\perp$ is not explicit, so that in practice the projection $P_{A^\perp}(\overline{\mathbf{y}}^k,\overline{\pi}^k,\overline{\mathbf{f}}^k)$ may be defined by $P_{A^\perp}(\overline{\mathbf{y}}^k,\overline{\pi}^k,\overline{\mathbf{f}}^k) = (\overline{\mathbf{y}}^k,\overline{\pi}^k,\overline{\mathbf{f}}^k) - \boldsymbol{p}$, where $\boldsymbol{p}$ solves the extremal problem :

$$\min_{\boldsymbol{p}\in A} \|\boldsymbol{p} - (\overline{\mathbf{y}}^k,\overline{\pi}^k,\overline{\mathbf{f}}^k)\|_{\mathcal{A}_0}. \tag{3.6}$$

Recalling that A is by definition the set of triplets $(\mathbf{y},\pi,\mathbf{f})$ satisfying $\mathbf{y}_t - \nu\Delta\mathbf{y} + \nabla\pi - \mathbf{f}\,1_\omega = 0$, $\nabla\cdot\mathbf{y} = 0$ in Q_T such that $\mathbf{y}$ vanishes on ∂Q_T, this extremal problem is nothing else than a controllability problem for the Stokes system, similar to the one considered in this work. Therefore, we shall bypass this projection and shall introduce instead a stopping criteria for the descent method measuring how far from $A^\perp$ the descent direction is.

4. Numerical resolution - Conjugate gradient algorithm

4.1. **Conjugate gradient algorithm.** Let us describe briefly the procedure to approximate a minimizer of E over the space $\mathcal{A}$. We refer to [25] where full details are given for the case of heat equation (see also [23]). In spite of the possible lack of coercivity of the functional E, we use the conjugate gradient algorithm which provides faster convergence.

We recall that since $\mathcal{A}$ is not an Hilbert space, we actually minimize the functional $E_{\mathbf{s}_\mathcal{A}} := E(\mathbf{s}_\mathcal{A} + \cdot)$ over $\mathcal{A}_0$, for any element $\mathbf{s}_\mathcal{A} \in \mathcal{A}$:

$$\min_{(\mathbf{y},\pi,\mathbf{f})\in\mathcal{A}_0} E_{\mathbf{s}_\mathcal{A}}(\mathbf{y},\pi,\mathbf{f}).$$

The Polak-Ribière version of the conjugate gradient (CG in the sequel) algorithm to minimize $E_{\mathbf{s}_\mathcal{A}}$ over $\mathcal{A}_0$ is as follows (see [13]): for any $\mathbf{s}_\mathcal{A} \in \mathcal{A}$

- *Step 0: Initialization* - Given any $\varepsilon > 0$ and any $\boldsymbol{z^0} = (\boldsymbol{y^0},\pi^0,\boldsymbol{f^0}) \in \mathcal{A}_0$, compute the residual $\boldsymbol{g^0} = (\overline{\boldsymbol{y}}^{\boldsymbol{0}},\overline{\pi}^0,\overline{\boldsymbol{f}}^{\boldsymbol{0}}) \in \mathcal{A}_0$ solution of

$$\langle \boldsymbol{g^0}, (\mathbf{Y},\Pi,\mathbf{F})\rangle_{\mathcal{A}_0} = \langle E'_{\mathbf{s}_\mathcal{A}}(\boldsymbol{z^0}), (\mathbf{Y},\Pi,\mathbf{F})\rangle, \quad \forall (\mathbf{Y},\Pi,\mathbf{F}) \in \mathcal{A}_0. \tag{4.1}$$

  If $\|\boldsymbol{g^0}\|_{\mathcal{A}_0}/\|\boldsymbol{z^0}\|_{\mathcal{A}_0} \le \varepsilon$ take $\boldsymbol{z} = \boldsymbol{z^0}$ as an approximation of a minimum of $E_{\mathbf{s}_\mathcal{A}}$. Otherwise, set $\boldsymbol{w^0} = \boldsymbol{g^0}$.

  For $k \ge 0$, assuming $\boldsymbol{z^k}$, $\boldsymbol{g^k}$, $\boldsymbol{w^k}$ being known with $\boldsymbol{g^k}$ and $\boldsymbol{w^k}$ both different from zero, compute $\boldsymbol{z^{k+1}}, \boldsymbol{g^{k+1}}$, and if necessary $\boldsymbol{w^{k+1}}$ as follows:

- *Step 1: Steepest descent* - Set $\boldsymbol{z^{k+1}} = \boldsymbol{z^k} - \lambda_k \boldsymbol{w^k}$ where $\lambda_k \in \mathbb{R}$ is the solution of the one-dimensional minimization problem

$$\text{minimize} \quad E_{\mathbf{s}_\mathcal{A}}(\boldsymbol{z^k} - \lambda \boldsymbol{w^k}) \quad \text{over}\, \lambda \in \mathbb{R}. \tag{4.2}$$

Then, compute the residual $\boldsymbol{g^{k+1}} \in \mathcal{A}_0$ from the relation

$$\langle \boldsymbol{g^{k+1}}, (\mathbf{Y}, \Pi, \mathbf{F})\rangle_{\mathcal{A}_0} = \langle E'_{\mathbf{s}_\mathcal{A}}(\boldsymbol{z^{k+1}}), (\mathbf{Y}, \Pi, \mathbf{F})\rangle, \quad \forall (\mathbf{Y}, \Pi, \mathbf{F}) \in \mathcal{A}_0.$$

- *Step 2: Convergence testing and construction of the new descent direction -* If $\|\boldsymbol{g^{k+1}}\|_{\mathcal{A}_0}/\|\boldsymbol{g^0}\|_{\mathcal{A}_0} \leq \varepsilon$ take $\boldsymbol{z} = \boldsymbol{z^{k+1}}$; otherwise compute

$$\gamma_k = \frac{(\boldsymbol{g^{k+1}}, \boldsymbol{g^{k+1}} - \boldsymbol{g^k})_{\mathcal{A}_0}}{(\boldsymbol{g^k}, \boldsymbol{g^k})_{\mathcal{A}_0}}, \qquad \boldsymbol{w^{k+1}} = \boldsymbol{g^{k+1}} + \gamma_k \boldsymbol{w^k}. \tag{4.3}$$

Then do $k = k + 1$, and return to step 1.

Since $E_{\mathbf{s}_\mathcal{A}}$ is a quadratic functional, one may write that (we denote by $\hat{E}$ the same functional than E but from $\mathcal{A}_0$ into $\mathbb{R}^+$)

$$E_{\mathbf{s}_\mathcal{A}}(\boldsymbol{z^k} - \lambda \boldsymbol{w^k}) = E_{\mathbf{s}_\mathcal{A}}(\boldsymbol{z^k}) - \lambda \langle E'_{\mathbf{s}_\mathcal{A}}(\boldsymbol{z^k}), \boldsymbol{w^k}\rangle + \lambda^2 \hat{E}(\boldsymbol{w^k})$$

and solve explicitly the problem (4.2): $\lambda_k = \langle E'_{\mathbf{s}_\mathcal{A}}(\boldsymbol{z^k}), \boldsymbol{w^k}\rangle/(2\hat{E}(\boldsymbol{w^k}))$. The evaluation of $\hat{E}(\boldsymbol{w^k})$ requires the computation of the corrector corresponding to $\boldsymbol{w^k}$.

Remark 4.1. *The parameter γ_n given by (4.3) corresponds to the Polak-Ribière version of the conjugate gradient algorithm. In the present quadratic-linear situation, this one should coincide with the Fletcher-Reeves conjugate algorithm for which $\gamma_k = \|\boldsymbol{g^{k+1}}\|^2_{\mathcal{A}_0}/\|\boldsymbol{g^k}\|^2_{\mathcal{A}_0}$ since gradients are conjugate to each other ($(\boldsymbol{g^k}, \boldsymbol{g^p})_{\mathcal{A}_0} = 0$ for all $k \neq p$). However, we observed that in the parabolic situation (see* [25] *and also* [9]*) the Polak-Ribière version (mainly used in nonlinear situations) allows to reduce the numerical loss of the orthogonality.*

With respect to the projection $P_{A^\perp}$ appearing in (3.3), let us comment that since $\hat{E}$ vanishes on $A = \operatorname{Ker} \mathbf{T} \cap \mathcal{A}_0$, the term $\hat{E}(\boldsymbol{w^k})$ is a measure of the distance of the descent direction $\boldsymbol{w^k} \in \mathcal{A}_0$ to the set $A^\perp \subset \mathcal{A}_0$.

Therefore, as soon as the residual $\|\boldsymbol{g^k}\|_{\mathcal{A}_0}$ (which very closed to the norm $\|\boldsymbol{w^k}\|_{\mathcal{A}_0}$) is small enough, the smallness of $\hat{E}(\boldsymbol{w^k})$ is a sure indication that the descent direction has almost no more components on $A^\perp$ and that the algorithm may be stopped. This trick avoids in practice the projection of the descent direction on $A^\perp$, that is the resolution at each iteration of the extremal problem (3.6). We shall replace the usual stopping criterion from the step 2 of the algorithm by the following one : if

$$\|\boldsymbol{g^{k+1}}\|_{\mathcal{A}_0}/\|\boldsymbol{g^0}\|_{\mathcal{A}_0} \leq \varepsilon_1 \quad \text{and} \quad \hat{E}(\boldsymbol{w^k})/E_{\boldsymbol{s}_\mathcal{A}}(\boldsymbol{z^k}) \leq \varepsilon_2$$

take $\boldsymbol{z} = \boldsymbol{z^{k+1}}$, for some $\varepsilon_1, \varepsilon_2 > 0$.

4.2. **Numerical approximation.** At the practical level, the approach requires only the resolution of elliptic problem, implying the Laplacian operator in dimension $N+1$, over the space-time bounded domain $Q_T = \Omega \times (0, T) \subset \mathbb{R}^{N+1}$, with Dirichlet and Neumann homogeneous boundary conditions, see (2.3) and (3.4). A finite element approximation is therefore very appropriate to solve our problem.

Assuming that Ω is a polygonal of $\mathbb{R}^N$, we introduce a regular triangulation $\mathcal{T}_{\Delta x,\Omega}$ of Ω so that $\overline{\Omega} = \cup_{\mathcal{T} \in \mathcal{T}_{\Delta x,\Omega}} T$, where Δx denotes as usual a parameter related to the size of the largest element of $\mathcal{T}_{\Delta x,\Omega}$. We then introduce the following conformal approximation of $\mathbf{H}^1(\Omega)$ (and also of $\mathbf{L}^2(\Omega)$)

$$\mathbf{V}_{\Delta x} = \{\mathbf{v} = (v_i)_{1 \leq i \leq N}; v_i \in C^0(\overline{\Omega}), v_{i|\mathcal{T}} \in \mathbb{P}_2^{(N)}, \forall T \in \mathcal{T}_{\Delta x,\Omega}, \forall 1 \leq i \leq N\}$$

where $\mathbb{P}_k^{(l)}$ denotes the space of the polynomials in l-variables of degree $\leq k$, $k \in \mathbb{N}$. Hence $Q_T = \Omega \times (0, T)$ is a polygonal domain of $\mathbb{R}^{N+1}$ and we can introduce a regular "triangulation" $\mathcal{T}_h$ of Q_T so that $\overline{Q_T} = \cup_{\mathcal{T} \in \mathcal{T}_h} \mathcal{T}$ (with h the approximation parameter related to the size of the largest element of $\mathcal{T}_h$). In the sequel, we note

$$h := \operatorname{diam}(\mathcal{T}_h) := \max\{\operatorname{diam}(\mathcal{T}); \mathcal{T} \in \mathcal{T}_h\}. \tag{4.4}$$

A conformal approximation of $\mathbf{H}^1(Q_T)$ is then :

$$\mathbf{V}_h = \{\mathbf{v} = (v_i)_{1\leq i\leq N}; v_i \in C^0(\overline{Q_T}), v_{i|\mathcal{T}} \in \mathbb{P}_2^{(N+1)}, \forall \mathcal{T} \in \mathcal{T}_h, \forall 1 \leq i \leq N\}.$$

We also define a conformal approximation of $L^2(Q_T)$, natural space for the pressure π as follows:

$$M_h = \{\pi; \pi \in L^2(Q_T), \pi_{|\mathcal{T}} \in \mathbb{P}_0^{(N+1)}, \forall \mathcal{T} \in \mathcal{T}_h\}.$$

We note $\mathbf{M}_h$ the corresponding approximation of $\mathbf{L}^2(Q_T)$. We then introduce the following approximation of $\mathcal{A}$:

$$\mathcal{A}_h = \left\{(\mathbf{y}, \pi, \mathbf{f});\ \mathbf{y} \in \mathbf{V}_h,\quad \mathbf{y}(\cdot, 0) = \Pi_{\Delta x}(\mathbf{y}_0),\ \mathbf{y}(\cdot, T) = \mathbf{0},\ \pi \in M_h,\ \mathbf{f} \in \mathbf{M}_h\right\} \tag{4.5}$$

where $\Pi_{\Delta x}$ denotes a projection operator from $\mathbf{L}^2(\Omega)$ to $\mathbf{V}_{\Delta x}$. A finite element approximation of the extremal problem (2.4) then reads as follows :

$$\min_{(\mathbf{y}_h, \pi_h, \mathbf{f}_h) \in \mathcal{A}_h} E(\mathbf{y}_h, \pi_h, \mathbf{f}_h). \tag{4.6}$$

The approximation of $\mathcal{A}_h$ we use is based on a $\mathbb{P}_2/\mathbb{P}_0/\mathbb{P}_2$ Lagrange finite element discretization. We have observed that this choice leads to a better decrease of the discrete minimizing sequence than, for instance, a $\mathbb{P}_2/\mathbb{P}_2/\mathbb{P}_2$ discretization. Note that within the classical Lagrangian approach (used to address the direct problem), similar strategies are generally used so as to satisfy discrete inf-sup conditions.

5. Numerical experiments

We present some experiments for the 2D case in space, for which $N = 2$; the domain Q_T is then a subset of $\mathbb{R}^3$. The computations have been performed with the FreeFem++ package developed at University Paris 6 (see [17]). Remark that the N components of the corrector $\mathbf{v}$ solution of (2.3) are independent : this reduces (by N) the size of the underlying linear systems, solved with a Cholesky method. The same remark holds for the corrector $\mathbf{V}$ solution of (2.8).

Since the support ω of the control function is time independent, a regular triangulation $\mathcal{T}_h$ of Q_T (composed of hexahedra) is easily obtained from a regular triangulation $\mathcal{T}_{\Delta x}$ of Ω (composed of planar triangles). From now, we assume that $\Omega = (0, 1)^2$ and $T = 1/2$.

According to Remark 2.4, we fixe the constant of the pressure, at any time, by replacing the divergence term $\nabla \cdot \mathbf{y}$ by $\nabla \cdot \mathbf{y} + \epsilon\pi$ with $\epsilon := 10^{-8}$.

5.1. **Direct problem.** Before to apply the variational method to the controllability problem, we first consider the direct problem for which an explicit solution is easily available. To do so, we simply fix the external forces $\mathbf{f}$ and remove from the spaces $\mathcal{A}$ and $\mathcal{A}_0$ the final condition $\mathbf{y}(\cdot, T) = 0$ (see Remark 2.6). Consequently, the descent direction in $\mathcal{A}_0$ is arbitrary at the final time. The minimizing procedure is unchanged. In particular, Lemma 3.2 applies and Proposition 3.1 holds true (from the uniqueness of the solution of (1.1), for any fixed source term $\mathbf{f}$, the space $A = \operatorname{Ker} \mathbf{T} \cap \mathcal{A}_0$ contains exactly one element and we get the strict convexity of $E_{s_{\mathcal{A}}}$ everywhere)

Let us define arbitrarily such explicit (uncontrolled) solution. For any $m, n \in \mathbb{N}^\star$, we consider the initial data

$$\mathbf{y}_0 = K\ \nabla \times \psi_0, \qquad \psi_0(x_1, x_2) = (\sin(n\pi x_1) \sin(m\pi x_2))^2, \quad (x_1, x_2) \in \Omega \tag{5.1}$$

with $K = 4/(\pi\sqrt{3(m^2 + n^2)})$. We check that $\mathbf{y}_0 \in \mathbf{H}$ and $\|\mathbf{y}_0\|_{\mathbf{L}^2(\Omega)} = 1$. Then, we take $g_1(t) = 1 + \sin(2\pi t)$, $g_2(t) = \cos(3\pi t)$ and define

$$\mathbf{y}_{ex}(x_1, x_2, t) = \mathbf{y}_0(x_1, x_2) g_1(t), \quad \pi_{ex}(x_1, x_2, t) = \sin(\pi x_1) \sin(2\pi x_2) g_2(t), \quad (x_1, x_2, t) \in Q_T. \tag{5.2}$$

Finally, we define the function $\mathbf{f} = \mathbf{y}_t - \nu\Delta\mathbf{y} + \nabla\pi$ so that the couple (5.2) is the unique solution of the Stokes system with initial condition $\mathbf{y}_0$, source term $\mathbf{f}$ and such that $\pi(0, 0, t) = 0$ for all time t.

We take $\nu = 1/40$, $m = 1$, $n = 2$, $\mathbf{s}_{\mathcal{A}} = (\mathbf{y}^0, 0)$ and initialize the conjugate gradient (CG) algorithm with $\boldsymbol{z}^{\mathbf{0}} = (\mathbf{0}, 0)$. $\varepsilon = 10^{-5}$ is the value for the stopping criterion of the CG-algorithm.

Figure 1 depicts the evolution of the cost $E(\mathbf{y}_h^k, \pi_h^k, \mathbf{f})$ and the residue $\|E'(\mathbf{y}_h^k, \pi_h^k, \mathbf{f})\|_{\mathcal{A}_0}$ corresponding to a fine regular mesh $\mathcal{T}_h$ for which $\mathrm{diam}(\mathcal{T}_h) := \max\{\mathrm{diam}(\mathcal{T}); \mathcal{T} \in \mathcal{T}_h\}$ is approximatively equal to 5.12×10^{-2} and the number of elements $\mathrm{card}(\mathcal{T}_h)$ is equal to 454 800. The convergence, that is $\|E'(\mathbf{y}_h^k, \pi_h^k, \mathbf{f}^k)\|_{\mathcal{A}_0} \leq \varepsilon$, is reached after $k = 334$ iterates and the corresponding cost is $E(\mathbf{y}_h, \pi_h, \mathbf{f}) \approx 4 \times 10^{-5}$. Actually, we observe from Figure 1 that the cost stagnates after approximatively 100 iterates. This is due to the divergence term $\nabla \cdot \mathbf{y}_h$ in the cost, which can not be, in general, arbitrarily small (for the $L^2(Q_T)$-norm) for a given finite dimensional piecewise polynomial approximation of $\mathbf{y}_h$. We obtain $\|\nabla \cdot \mathbf{y}_h\|_{L^2(Q_T)} \approx 2.91 \times 10^{-3}$, which is a comparable value with respect to the usual approximation where the null divergence constraint is weakly introduced against a Lagrangian multiplier (see again [14]).

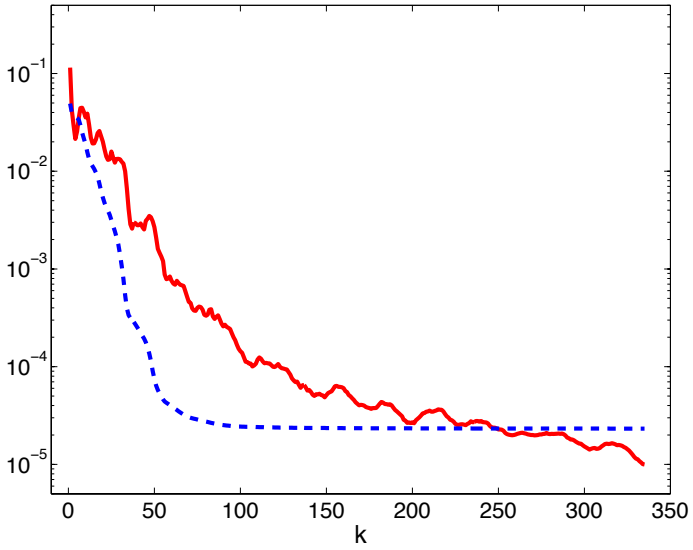

FIGURE 1. **Direct problem** ; $\nu = 1/40$; $E(\mathbf{y}_h^k, \pi_h^k, \mathbf{f})$ (**dashed line**) and $\|E'(\mathbf{y}_h^k, \pi_h^k, \mathbf{f})\|_{\mathcal{A}_0}$ (**full line**) vs. the iterates k of the CG algorithm.

From Table 1, we also observe that the number of iterates to reach the criterion increases very slowly with respect to the dimension of the discretized problem: it behaves sub-linearly with respect to the dimension of the discrete problem. This suggests that the choice we have made for the corrector system (2.3) to address the least-squares problem (2.4) is appropriate.

Table 1 also collects various norms of the solution with respect to $h := \mathrm{diam}(\mathcal{T}_h)$. In particular, we observe the convergence of the error $\|\mathbf{y}_h - \mathbf{y}_{h,ex}\|_{\mathbf{L}^2(Q_T)}$ with respect to h: we get

$$\begin{aligned} &\|\mathbf{y}_h - \mathbf{y}_{h,ex}\|_{\mathbf{L}^2(Q_T)} = O(h^{2.83}), \quad \|\mathbf{y}_h - \mathbf{y}_{ex}\|_{\mathbf{L}^2(0,T;\mathbf{H}_0^1(\Omega))} = O(h^{1.75}), \\ &\|\nabla \cdot \mathbf{y}_h\|_{L^2(Q_T)} = O(h^{1.51}), \quad \|\mathbf{v}_h\|_{\mathbf{H}^1(Q_T)} = O(h^{1.36}). \end{aligned} \tag{5.3}$$

Figure 2 depicts the various norm with respect to h and highlights the polynomial convergence.

5.2. **Controllability problem.** We now describe some experiments in the controlled situation for which $\mathbf{f}$ is an unknown. Once again, we take $\Omega = (0,1)^2$. The controllability time is $T = 1/2$ and we assume that the control acts on $\omega = (0.2, 0.6)^2$. We consider the initial data $\mathbf{y}_0$ of the previous section.

According to the observation made on the behavior of the cost E in the previous section (see Figure 1), we modify the stopping criterion: the algorithm is stopped when the cost stagnates :

$$|E(\mathbf{y}_h^{k+1}, \pi_h^{k+1}, \mathbf{f}_h^{k+1}) - E(\mathbf{y}_h^k, \pi_h^k, \mathbf{f}_h^k)| \leq \varepsilon E(\mathbf{y}_h^0, \pi_h^0, \mathbf{f}_h^0). \tag{5.4}$$

$\mathrm{diam}(\mathcal{T}_h)$	3.06×10^{-1}	1.69×10^{-1}	7.75×10^{-2}	5.12×10^{-2}	4.06×10^{-2}
$\mathrm{card}(\mathcal{T}_h)$		7 200	57 120	188 640	454 800
♯ CG iterates	163	233	303	324	334
$\|\mathbf{y}_h - \mathbf{y}_{h,ex}\|_{\mathbf{L}^2(Q_T)}$	2.56×10^{-1}	3.32×10^{-2}	5.72×10^{-3}	2.38×10^{-3}	8.68×10^{-4}
$\|\mathbf{y}_h - \mathbf{y}_{ex}\|_{\mathbf{L}^2(0,T;\mathbf{H}^1_0(\Omega))}$	5.62	1.74	5.89×10^{-1}	3.63×10^{-1}	1.70×10^{-1}
$\|\nabla \cdot \mathbf{y}_h\|_{L^2(Q_T)}$	2.61×10^{-1}	4.37×10^{-2}	1.09×10^{-2}	1.00×10^{-2}	2.91×10^{-3}
$\|\mathbf{v}_h\|_{\mathbf{H}^1(Q_T)}$	1.29×10^{-1}	4.19×10^{-2}	1.61×10^{-2}	5.45×10^{-3}	6.16×10^{-3}

TABLE 1. **Direct problem** ; $\nu = 1/40$; Numerical results with respect to $h := \mathrm{diam}(\mathcal{T}_h)$.

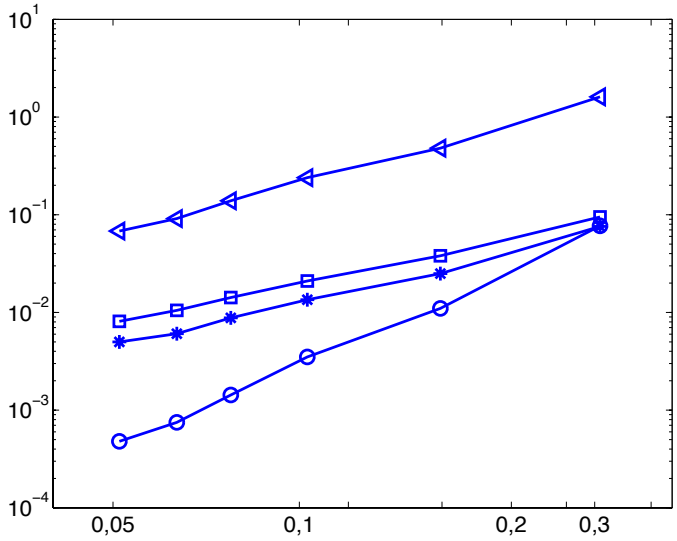

FIGURE 2. $\nu = 1/40$ - $\|\mathbf{y}_h - \mathbf{y}_{ex}\|_{\mathbf{L}^2(Q_T)}$ ($\circ$), $\|\mathbf{y}_h - \mathbf{y}_{ex}\|_{\mathbf{L}^2(0,T;\mathbf{H}^1_0(\Omega))}$ ($<$) $\|\nabla \cdot \mathbf{y}_h\|_{L^2(Q_T)}$ ($\star$) and $\|\mathbf{v}_h\|_{\mathbf{H}^1(Q_T)}$ ($\square$) w.r.t. $\mathrm{diam}(\mathcal{T}_h)$

We take $\varepsilon = 10^{-5}$ as the value for the stopping criterion of the CG-algorithm. From Section 3, any minimizing sequence takes the form $\mathbf{s}_{\mathcal{A}} + (\mathbf{y}^n, \pi^n, \mathbf{f}^n)$ with $\mathbf{s}_{\mathcal{A}} \in \mathcal{A}$ and $(\mathbf{y}^n, \pi^n, \mathbf{f}^n) \in \mathcal{A}_0$, $n \geq 0$. We take $\mathbf{s}_{\mathcal{A}} := (\mathbf{y}_0(x_1, x_2)(1 - t/T)^2, 0, \mathbf{0}) \in \mathcal{A}$. Moreover, the CG algorithm is initialized with $(\mathbf{y}^0, \pi^0, \mathbf{f}^0) = (\mathbf{0}, 0, \mathbf{0}) \in \mathcal{A}_0$.

The meshes $\mathcal{T}_h$ considered are different than the ones used in Section 5.1: precisely, as a polygonal subset of Ω, the domain ω is explicitly meshed. For each h, the mesh $\mathcal{T}_h$ is obtained from the mesh $\mathcal{T}_{\Delta x}$ by a dilatation principle in the time direction. Figures 3 and 4 provide an example of (coarse) triangulation $\mathcal{T}_{\Delta x}$ and $\mathcal{T}_h$ of Ω and Q_T respectively. The corresponding value for h is $h \approx 1.6 \times 10^{-1}$.

We first take a viscosity $\nu = 1/40$. In the uncontrolled situation ($\mathbf{f} \equiv 0$), this leads to $\|\mathbf{y}_h(\cdot, T)\|_{\mathbf{L}^2(\Omega)} \approx 1.83 \times 10^{-1}$.

Table 2 collects the main norms of the solution with respect to h. For the finer mesh, Figure 5 depicts the corresponding evolution of the cost $E(\mathbf{y}_h^k, \pi_h^k, \mathbf{f}_h^k)$, the first derivative $\|E'(\mathbf{y}_h^k, \pi_h^k, \mathbf{f}_h^k)\|_{\mathcal{A}_0}$ and $\hat{E}(\boldsymbol{w_k})$ with respect to the iterates of the algorithm. We observe that the values of $\hat{E}(\boldsymbol{w^k})$ - measure of the distance $\boldsymbol{w^k}$ from $\mathcal{A}^\perp$ - is of the order $\mathcal{O}(10^{-9})$; this is a small enough value to affirm that the descent direct has almost no components in $A^\perp$. Again, we observe that the number of iterations behaves sub-linearly with respect to h. We observe a slower decrease of the cost for the control problem than for the direct problem: the fact that the function $\boldsymbol{f}$ is an

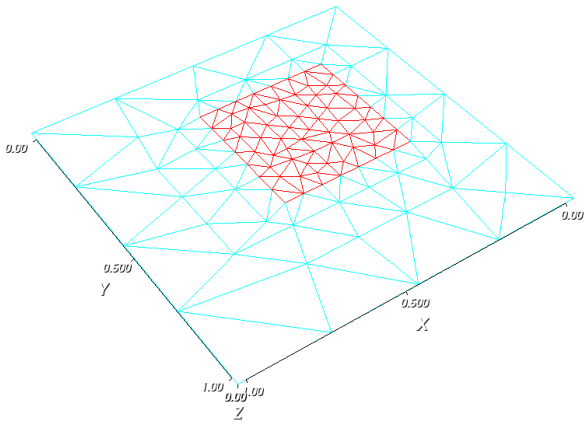

FIGURE 3. A regular triangulation $\mathcal{T}_{\Delta x}$ of $\Omega = (0,1)^2$: 214 triangles; 116 vertices.

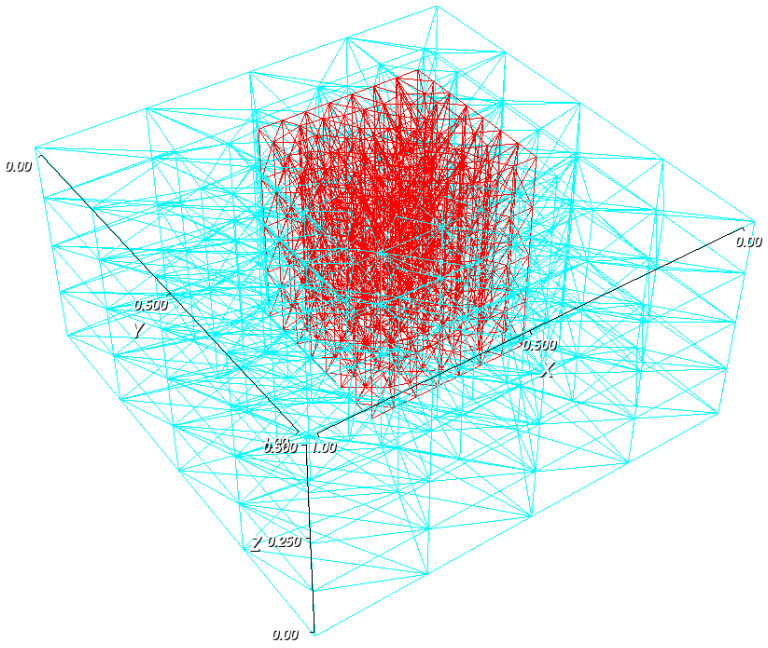

FIGURE 4. A regular triangulation $\mathcal{T}_h$ of $Q_T := \Omega \times (0,T)$: 2 568 elements; 580 vertices.

additional variable does not help to reach easier a solution of Stokes system. Recall that for the control problem, the coercivity of E is lost.

For $h \approx 4.3 \times 10^{-2}$, the stopping criterion is reached after 640 iterates: we get $\|\mathbf{v}_h\|_{\mathbf{H}_0^1(Q_T)} \approx 7.96 \times 10^{-3}$ and $\|\nabla \cdot \mathbf{y}_h\|_{\mathbf{L}^2(Q_T)} \approx 4.77 \times 10^{-3}$, which suggests that $(\mathbf{y}_h, \pi_h)$ is a very satisfactory approximation of a controlled solution for (1.1). After $k = 640$ iterates, the value of $\hat{E}(\boldsymbol{w}^{\boldsymbol{k}})$ is of the order $\mathcal{O}(10^{-9})$, so that the descent $\boldsymbol{w}^{\boldsymbol{k}}$ has almost no more components in $A^{\perp}$ and the algorithm may be stopped.

$\mathrm{diam}(\mathcal{T}_h)$	1.6×10^{-1}	1.06×10^{-1}	8.2×10^{-2}	4.3×10^{-2}
$\mathrm{card}(\mathcal{T}_h)$	3210	12 288	27 300	212 160
♯ vertices	768	2 601	5 456	38 829
♯ CG iterates	247	457	543	640
CPU time (seconds)	407	2 895	7 330	72 459
$\|\mathbf{y}_h\|_{\mathbf{L}^2(Q_T)}$	3.4×10^{-1}	3.41×10^{-1}	3.38×10^{-1}	3.29×10^{-1}
$\|\mathbf{y}_h\|_{\mathbf{L}^2(0,T;\mathbf{H}_0^1(\Omega))}$	4.48	4.49	4.47	4.59
$\|\mathbf{f}_h\|_{\mathbf{L}^2(q_T)}$	2.51	3.23	3.46	3.48
$\|\pi_h\|_{L^2(Q_T)}$	1.69×10^{-1}	1.98×10^{-1}	2.28×10^{-1}	2.77×10^{-1}
$\|\nabla \cdot \mathbf{y}_h\|_{L^2(Q_T)}$	8.7×10^{-2}	2.62×10^{-2}	1.69×10^{-2}	4.77×10^{-3}
$\|\mathbf{v}_h\|_{\mathbf{H}^1(Q_T)}$	2.5×10^{-2}	1.66×10^{-2}	1.28×10^{-2}	7.96×10^{-3}
$E(\mathbf{y}_h, \pi_h, \mathbf{f}_h)$	4.10×10^{-3}	4.82×10^{-4}	2.24×10^{-4}	4.29×10^{-5}
$\|E'(\mathbf{y}_h, \pi_h, \mathbf{f}_h)\|_{\mathcal{A}_0}$	2.98×10^{-4}	1.45×10^{-4}	1.26×10^{-4}	1.13×10^{-4}
$\hat{E}(\boldsymbol{w}^{\boldsymbol{k}})$	2.37×10^{-8}	5.5×10^{-9}	4.25×10^{-9}	3.61×10^{-9}

TABLE 2. **Control problem**; $\nu = 1/40$; Numerical results with respect to $h := \mathrm{diam}(\mathcal{T}_h)$.

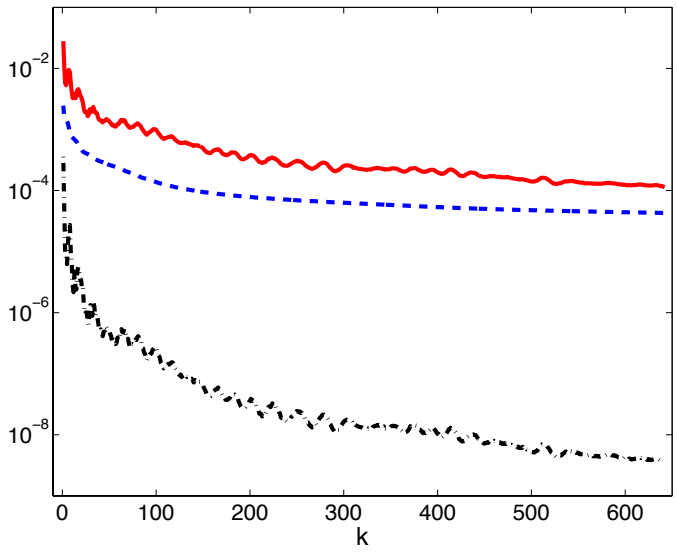

FIGURE 5. **Control problem**; $\nu = 1/40$ - $E(\mathbf{y}_h^k, \pi_h^k, \mathbf{f}_h^k)$ (**dashed line**), $\|E'(\mathbf{y}_h^k, \pi_h^k, \mathbf{f}_h^k)\|_{\mathcal{A}_0}$ (**full line**) and $\hat{E}(\boldsymbol{w}_{\boldsymbol{k}})$ (**dashed-dotted line**) vs. the iterates k of the CG algorithm; $h \approx 4.3 \times 10^{-2}$.

Figure 6 depicts the evolution of the norm $\|\mathbf{y}_h(\cdot, t)\|_{\mathbf{L}^2(\Omega)}$ of the solution $\mathbf{y}_h$ with respect to t in the controlled and uncontrolled cases. Figure 7 depicts the evolution of the $\mathbf{L}^2$-norm $\|\mathbf{f}_h(\cdot, t)\|_{\mathbf{L}^2(\omega)}$ of the control $\mathbf{f}_h$ as well as its components f_{1h}, f_{2h}. Observe that the approximation

$\mathbf{f}_h$ in $L^2(q_T)$ of a control does not enjoy any specific property of regularity. Figure 8 depicts the L^2-norm of the corresponding pressure.

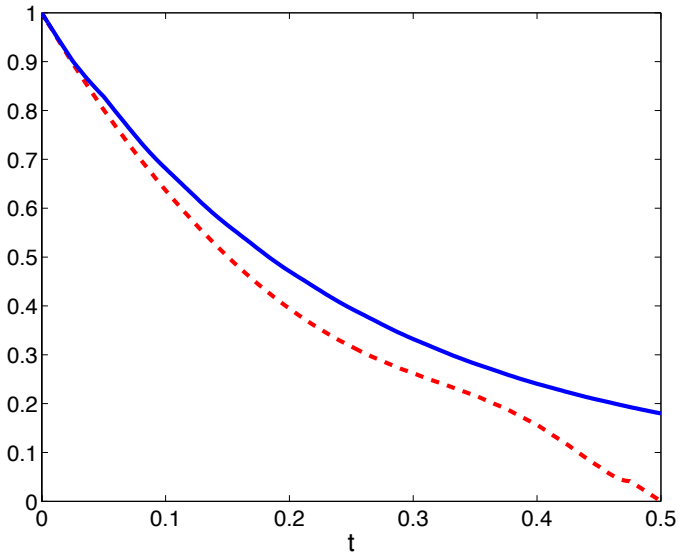

FIGURE 6. **Control problem**; $\nu = 1/40$ - Evolution of $\|\mathbf{y}_h(\cdot,t)\|_{L^2(\Omega)}$ w.r.t $t \in (0,T)$ in the uncontrolled (**full line**) and controlled (**dashed line**) situations.

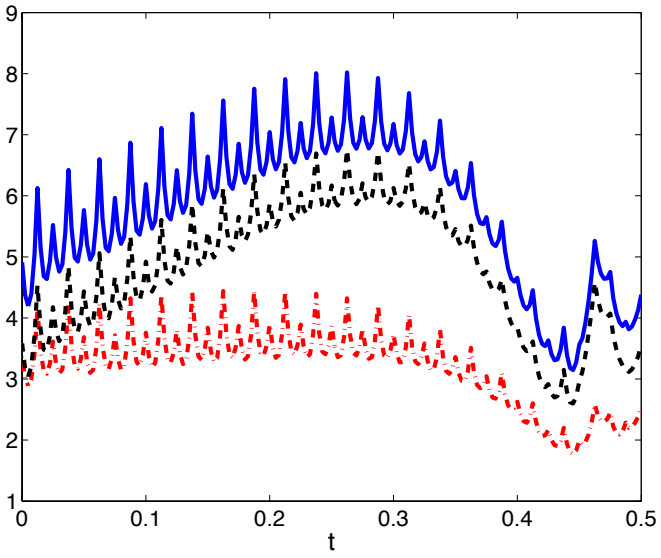

FIGURE 7. **Control problem**; $\nu = 1/40$ - Evolution of $\|\mathbf{f}_h(\cdot,t)\|_{L^2(\omega)}$ (**full line**), $\|f_1(\cdot,t)\|_{L^2(\omega)}$ (**dashed-dotted line**), $\|f_2(\cdot,t)\|_{L^2(\omega)}$ (dashed line) w.r.t $t \in (0,T)$.

Table 3 and Figure 9 give the results obtained for the viscosity $\nu = 1/40\,000$. We recall that small values of the viscosity (equivalently large values of the Reynolds number $Re := 1/\nu$) lead to numerical difficulties for the direct problem when classical Lagrangian approaches are used and need of preconditioning technics so as to speed up the convergence of Uzawa type algorithms (see [14]). The least-squares approach developed here *a priori* avoids such difficulties, since

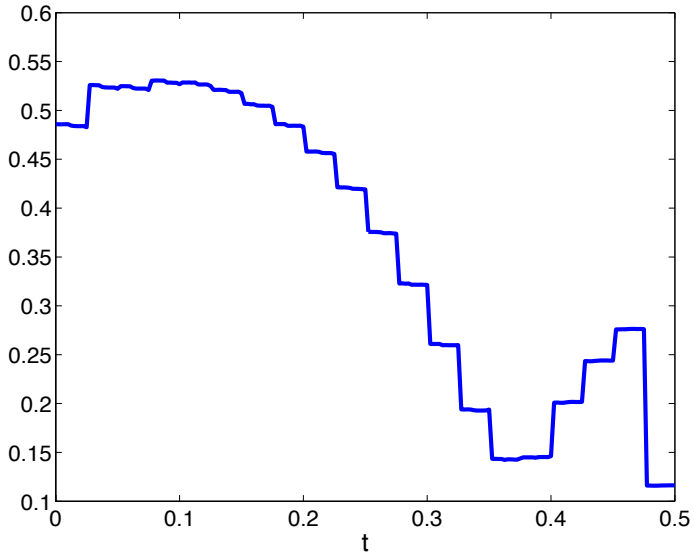

FIGURE 8. **Control problem**; $\nu = 1/40$ - Evolution of $\|\pi_h(\cdot,t)\|_{L^2(\Omega)}$ w.r.t $t \in (0,T)$.

the Laplacian operator $-\partial_{tt} - \Delta$ defining the corrector $\mathbf{v}$ (see (2.3)) does not depend on ν. For $\nu = 1/40\,000$, a similar behavior of the algorithm is observed : however, the norm of the corrector we obtain is about 10 times larger than in the previous case, precisely $\|\mathbf{v}\|_{\mathbf{H}^1(Q_T)} \approx 7.47 \times 10^{-2}$. This is very likely due to the fact that the system (1.1) is not null controllable at the limit when ν equals zero (see [3], p. 195). This can also be seen from the optimality system (2.10): $\nu = 0$ in (2.10) does not imply that the corrector $\boldsymbol{v}$ and the divergence of $\mathbf{y}$ vanish on Q_T so that the extremal points for E are not necessarily controlled solutions for (1.1). Note that we have used a very small but non zero value for ν because the limit case $\nu = 0$ leading to a slightly different functional setting is not covered by Section 2. On the contrary, a larger value of the viscosity increases the dissipation of the solution and leads (in our null controllability context) to a better behavior (in terms of convergence) of the algorithm.

$\mathrm{diam}(\mathcal{T}_h)$	4.29×10^{-2}
$\sharp$ CG iterates	380
$\|\mathbf{y}_h\|_{\mathbf{L}^2(Q_T)}$	3.88×10^{-2}
$\|\mathbf{y}_h\|_{\mathbf{L}^2(0,T;\mathbf{H}^1_0(\Omega))}$	4.86
$\|\mathbf{f}_h\|_{\mathbf{L}^2(q_T)}$	4.64
$\|\pi_h\|_{L^2(Q_T)}$	2.19×10^{-1}
$\|\nabla \cdot \mathbf{y}_h\|_{L^2(Q_T)}$	3.23×10^{-3}
$\|\mathbf{v}_h\|_{\mathbf{H}^1(Q_T)}$	7.47×10^{-2}
$E(\mathbf{y}_h, \pi_h, \mathbf{f}_h)$	1.85×10^{-5}
$\|E'(\mathbf{y}_h, \pi_h, \mathbf{f}_h)\|_{\mathcal{A}_0}$	2.22×10^{-3}
$\hat{E}(\boldsymbol{w}^{\boldsymbol{k}})$	1.34×10^{-8}

TABLE 3. **Control problem**; $\nu = 1/40\,000$; Numerical results for $h := 4.29 \times 10^{-2}$.

The variational approach is also effective to address the controllability of (1.1) with only one control. We refer to [4] where the controllability is proved in that case by the way of Carleman estimates: precisely, it is shown that there exists a constant $C = C(q_T) > 0$ such that the

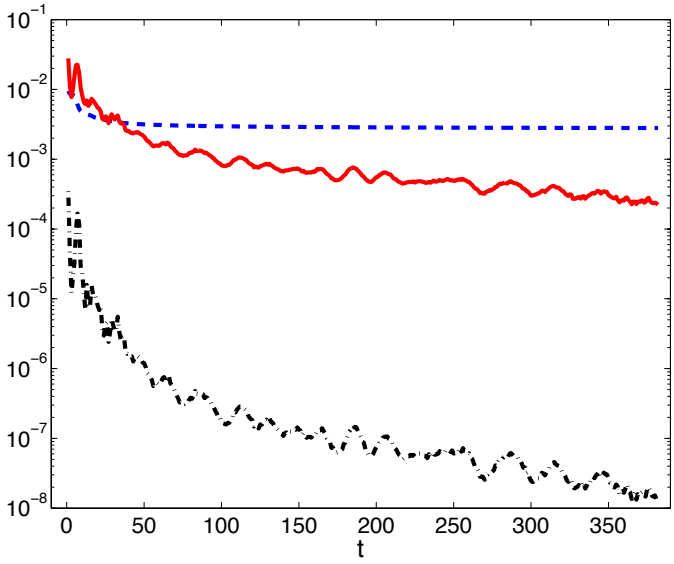

FIGURE 9. **Control problem** ; $\nu = 1/40\,000$; $E(\mathbf{y}_h^k, \pi_h^k, \mathbf{f}_h^k)$ (**dashed line**), $\|E'(\mathbf{y}_h^k, \pi_h^k, \mathbf{f}_h^k)\|_{\mathcal{A}_0}$ (**full line**) and $\hat{E}(\boldsymbol{w_k})$ (**dashed-dotted line**) vs. the iterates k of the CG algorithm.

following inequality holds :

$$\|\boldsymbol{\varphi}(\cdot,0)\|^2_{\mathbf{L}^2(\Omega)} \leq C \iint_{q_T} |\varphi_i|^2 \, dx\, dt \quad (i = 1 \text{ or } i = 2) \tag{5.5}$$

for all $\boldsymbol{\varphi}$ solution of the adjoint system (1.4). The second line of (2.10) is then reduced to $v_i = 0$ in q_T ($i = 1$ or 2) and from (5.5), this is enough to ensure that the corrector $\boldsymbol{v}$ and $\nabla \cdot \mathbf{y}$ vanishes in Q_T. Numerical experiments fully agree with these theoretical results. We observe a similar behavior of the algorithm, except that the norm of the control is larger. Table 4 provides some results when the finer mesh is used.

$\mathrm{diam}(\mathcal{T}_h)$	4.29×10^{-2}
$\sharp$ CG iterates	876
$\|\mathbf{y}_h\|_{\mathbf{L}^2(Q_T)}$	3.43×10^{-1}
$\|\mathbf{y}_h\|_{\mathbf{L}^2(0,T;\mathbf{H}_0^1(\Omega))}$	4.88
$\|\mathbf{f}_h\|_{\mathbf{L}^2(q_T)}$	6.425
$\|\pi_h\|_{L^2(Q_T)}$	2.19×10^{-1}
$\|\nabla \cdot \mathbf{y}_h\|_{L^2(Q_T)}$	5.76×10^{-3}
$\|\mathbf{v}_h\|_{\mathbf{H}^1(Q_T)}$	2.13×10^{-2}
$E(\mathbf{y}_h, \pi_h, \mathbf{f}_h)$	2.45×10^{-4}
$\|E'(\mathbf{y}_h, \pi_h, \mathbf{f}_h)\|_{\mathcal{A}_0}$	1.74×10^{-4}
$\hat{E}(\boldsymbol{w^k})$	8.78×10^{-8}

TABLE 4. **Control problem**; $\nu = 1/40$; One control; Numerical results for $h = 4.29 \times 10^{-2}$.

We end this numerical section and consider the controllability of the Stokes solution to a prescribed trajectory; for any $c > 0$, let us consider the Poiseuille flow

$$\overline{\mathbf{y}} = \left(-\frac{c}{2\nu}\, x_2(1 - x_2), 0 \right), \quad \overline{\pi} = c\, x_1 \tag{5.6}$$

stationary solution of the homogeneous Stokes equation, that is

$$-\nu\Delta\overline{\mathbf{y}}+\nabla\pi=\mathbf{0},\quad \nabla\cdot\overline{\mathbf{y}}=0\quad \text{in}\ \ Q_T. \tag{5.7}$$

We introduce $(\mathbf{z},\sigma)=(\mathbf{y}-\overline{\mathbf{y}},\pi-\overline{\pi})$ where $(\mathbf{y},\pi)$ solves the state equations of (1.1):

$$\mathbf{y}_t-\nu\Delta\mathbf{y}+\nabla\pi=\mathbf{f}\ 1_\omega,\quad \nabla\cdot\mathbf{y}=0\quad \text{in}\ \ Q_T,\quad \mathbf{y}(\cdot,0)=\mathbf{y}_0\quad \text{in}\ \ \Omega \tag{5.8}$$

so that $(\mathbf{z},\sigma)$ solves

$$\mathbf{z}_t-\nu\Delta\mathbf{z}+\nabla\sigma=\mathbf{f}\ 1_\omega,\quad \nabla\cdot\mathbf{z}=0\quad \text{in}\ \ Q_T,\quad \mathbf{z}(\cdot,0)=\mathbf{y}_0-\overline{\mathbf{y}}\quad \text{in}\ \ \Omega. \tag{5.9}$$

We add to this system the boundary condition $\mathbf{z}=\mathbf{0}$ on Σ_T.

For any $\mathbf{y}_0$ such that $\mathbf{y}_0-\overline{\mathbf{y}}\in\mathbf{H}$, we use our approach to determine a function $\mathbf{f}$ such that $\mathbf{z}(\cdot,T)=0$ on Q_T. By definition, $\mathbf{y}:=\mathbf{z}+\overline{\mathbf{y}}$ is then controlled to the trajectory $\overline{\mathbf{y}}$ at time T.

We consider the following data : $\Omega=(0,5)\times(0,1)$, $\omega=(1,2)\times(0,1)$, $T=2$ and $\nu=1/40$ and

$$\mathbf{y}_0=\overline{\mathbf{y}}+\nabla\times\psi,\quad \psi=K(1-x_2)^2x_2^2(5-x_1)^2x_1^2,\quad m\in\mathbb{N} \tag{5.10}$$

We take K such that $\|\nabla\times\psi\|_{L^2(\Omega)}=2$. Note that $\mathbf{y}_0-\overline{\mathbf{y}}$ belongs to $\mathbf{H}$ (in particular vanishes on Σ_T). Moreover, in order to smooth in the time direction the control f (and avoid the oscillations observed in Figure 7, we replace the descent direction of the variable $\mathbf{f}$ simply given by $\mathbf{v}$ (see 2.9) by $\tilde{v}$ solution of :

$$(I-10^{-2}\partial_{tt}^2)\,\tilde{\mathbf{v}}=t(T-t)\mathbf{v}\quad \text{in}\quad Q_T,\qquad \tilde{\mathbf{v}}(\cdot,0)=\tilde{\mathbf{v}}(\cdot,T)=0\quad \text{in}\quad \Omega. \tag{5.11}$$

$\tilde{\mathbf{v}}$ is still a descent direction for E with respect to the variable $\mathbf{f}$. Moreover, its vanishes at the initial and final time; the corresponding control enjoys the same property. We observe in practice that this additional regularization step speeds up the conjugate gradient algorithm. The mesh $\mathcal{Q}_h$ has 239 520 elements and 43 296 vertices. The corresponding mesh size is $h\approx 1.27\times10^{-1}$. For $\epsilon=10^{-6}$, the algorithm converges after 672 iterates. We get $\|E'(\mathbf{y}_h,\pi_h,\mathbf{f}_h)\|_{\mathcal{A}_0}\approx 1.84\times10^{-5}$, $\|\mathbf{v}\|_{\mathbf{H}^1(Q_T)}\approx 2.1\times10^{-2}$, $\|\nabla\cdot\mathbf{y}_h\|_{L^2(Q_T)}\approx 1.47\times10^{-2}$ and $\|\mathbf{f}_h\|_{L^2(Q_T)}\approx 2.58$. Figure 10 depicts the iso-values of the first component $y_{1,h}(\cdot,t)=\overline{y}_1+z_h(\cdot,t)$ of the velocity on Ω for $t=t_i\in\frac{i}{5}T$, $i=0,\cdots,5$ and illustrates the controlled of the stokes solution to the trajectory $\overline{y}$ (in (5.6), c is arbitrarily chosen such that $-c/(2\nu)=12$). We refer to [10] where a similar experiment is described under a different variational approach.

6. Remarks - Perspectives

The approach we have presented is really attractive with many respects. It is as the same time very simple and very general, as it can be applied for any linear (null controllable) equations and systems. On a numerical viewpoint, the approach appears to be very robust, notably avoids the ill-posedness of dual type methods and only requires a finite element code for the laplacien in $\mathbb{R}^{N+1}$. Moreover, the finite element framework used here allows iterative meshes adaptation in order to reduce the computational cost and to capture specific local properties of the solution. Similarly, as mentioned in Remark 2.5, the approach is very appropriate to deal with support of controls which depends on time, a situation which is much more delicate to address within dual classical methods where the time variable is distinguished for the space variables.

The decrease of the cost E toward zero - or equivalently the convergence of the sequence $(\mathbf{y}^k,\pi^k,\mathbf{f}^k)_{k>0}$ toward a solution of the Stokes system - is really fast : this is due to the fact that the two variables π and $\mathbf{f}$ are determined as the same time and can be considered as control functions for the null controllability and null divergence constraint simultaneously. This point of view, which offers a lot of flexibility, seems original: it contrasts with classical method where the control $\mathbf{f}$ are determined for solutions $(\mathbf{y},\pi)$ of the Stokes equation.

Moreover, as for the heat equation considered in [25], the method avoids duality arguments and therefore ill-posedness: on the contrary, the controls obtained from the minimization of E

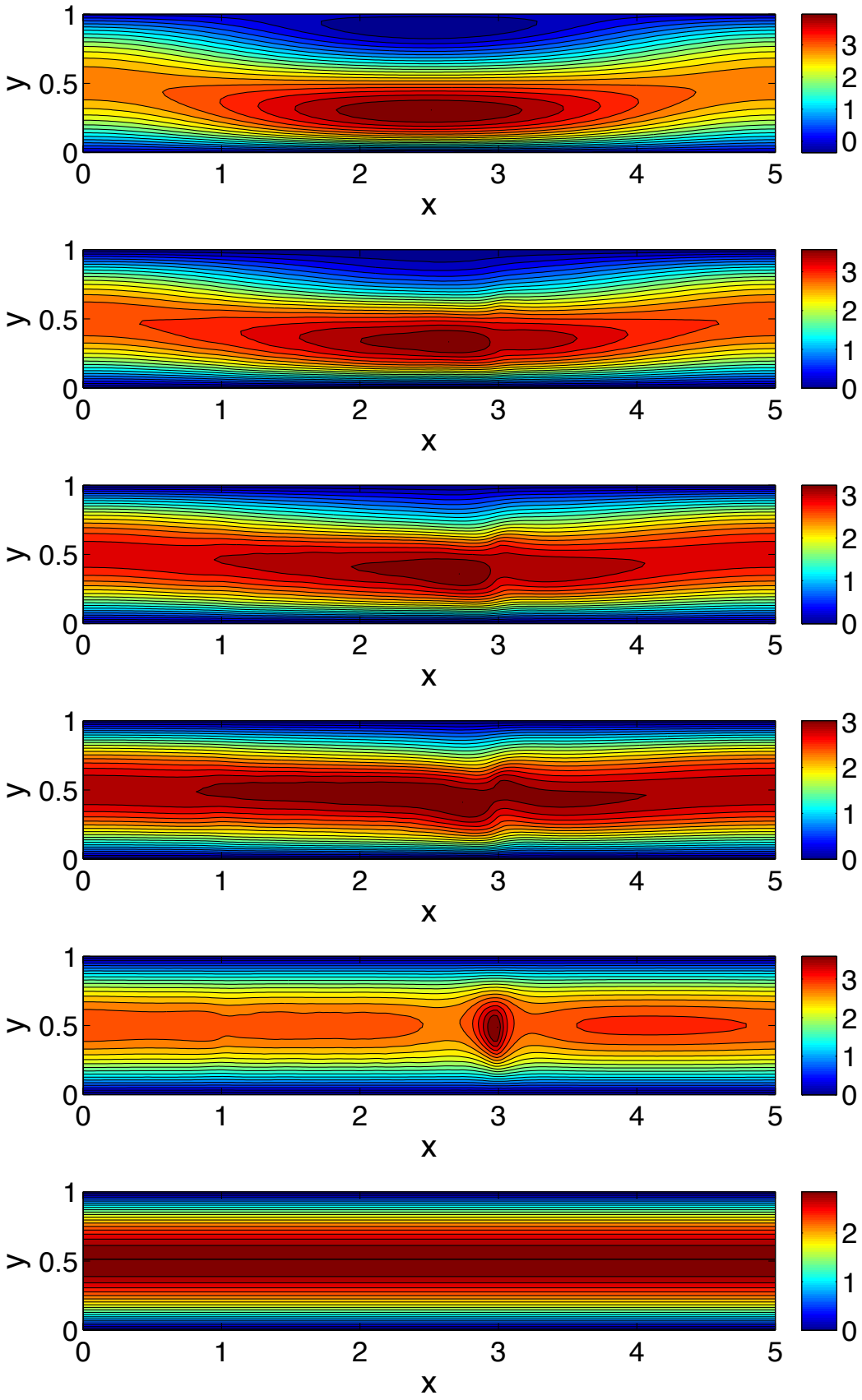

FIGURE 10. **Control problem**; $\nu = 1/40$ - Iso-values of the first component $y_{1,h}(\cdot,t) = \overline{y}_1 + z_h(\cdot,t)$ of the velocity on Ω for $t = t_i \in \frac{i}{5}T$, $i = 0, \cdots, 5$.

does not minimize a priori any particular norm : an additional level of optimization would be needed to get, for instance , the control of minimal square-integrable norm. Precisely, since the set $\{(\mathbf{y},\pi,\mathbf{f})\in\mathcal{A}, E(\mathbf{y},\pi,\mathbf{f})=0\}$ is convex, we may apply an Uzawa type method and find over $\mathcal{A}\times\mathbb{R}$ a saddle point for the Lagrangian

$$\mathcal{L}((\mathbf{y},\pi,\mathbf{f}),\lambda)=\frac{1}{2}\|\mathbf{f}\|^2_{\mathbf{L}^2(q_T)}+\lambda\, E(\mathbf{y},\pi,\mathbf{f}).$$

The real λ is the multiplier corresponding to the constraint $E(\mathbf{y},\pi,\mathbf{f})=0$. Starting from any $\lambda^0\in\mathbb{R}$, the algorithm aims to define a sequence of pair $((\mathbf{y}^k,\pi^k,\mathbf{f}^k),\lambda^k)\in\mathcal{A}\times\mathbb{R}$, $k\geq 0$ according to the following two steps :

- Compute $(\mathbf{y}^k,\pi^k,\mathbf{f}^k)$ such that $\mathcal{L}((\mathbf{y}^k,\pi^k,\mathbf{f}^k),\lambda^k)\leq\mathcal{L}(v,\lambda^k),\quad\forall v\in\mathcal{A}$;
- Compute $\lambda^{k+1}:=\lambda^k+\rho\, E(\mathbf{y}^k,\pi^k,\mathbf{f}^k),\qquad\rho>0.$

The first step is performed using the gradient method developed in Section 3, the functional E being replaced by the functional $\mathcal{L}(\cdot,\lambda^k)$. First experiments lead to satisfactory results and will be given in a distinct work.

We also emphasize that the approach may be used to address the (local) null controllability of the Navier-Stokes system. To our knowledge, the numerical approximation of controls for Navier-Stokes has no been addressed so far. A natural idea consists in linearizing the system and then introducing a constructive fixed point argument. However, as mentioned in [8] for a nonlinear heat equation, the convergence of this strategy depends on the properties of the underlying fixed point operator (namely, its contraction). On the other hand, least-squares approaches have been used since the eighties to solve (direct) non-linear problems. In our context, the procedure is similar: the cost E defined in (2.2) is unchanged (the functional space $\mathcal{A}$ is adapted to the nonlinear setting) and the corrector $\mathbf{v}$ is defined by :

$$\begin{cases} -\mathbf{v}_{tt}-\Delta\mathbf{v}+(\mathbf{y}_t-\nu\Delta\mathbf{y}+(\mathbf{y}\cdot\nabla)\mathbf{y}+\nabla\pi-\mathbf{f}\,1_\omega)=0 \quad\text{in}\quad Q_T,\\ \mathbf{v}=0\quad\text{on}\quad\Sigma_T,\quad \mathbf{v}_t=0\quad\text{on}\quad\Omega\times\{0,T\}.\end{cases}\tag{6.1}$$

Again, from a unique continuation property for the linearized Navier-Stokes system (namely the Oseen equations), one obtain that the extremal point for E are controlled solution for the Navier-Stokes. However, Lemma 3.2 is not valid in this new situation (the underlying operator is no more linear) and must be adapted so as to prove that some minimizing sequences are bounded in $\mathcal{A}$. This challenging issue will be addressed in a future work.

Eventually, as an another non trivial application of this approach, we also mention the controllability of the linear elasticity system in the incompressibility regime.

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