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# Exponential relaxation to self-similarity for the superquadratic fragmentation equation

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## Abstract

We consider the self-similar fragmentation equation with a superquadratic fragmentation rate and provide a quantitative estimate of the spectral gap.

**Keywords:** fragmentation equation, self-similarity, exponential convergence, spectral gap, long-time behavior

**AMS Class. No.** Primary: 35B40, 35P15, 45K05; Secondary: 82D60, 92D25

## 1 Introduction

The *fragmentation equation*

$$\begin{cases} \partial_t f(t, x) = \mathcal{F}f(t, x), & t \geq 0, x > 0, \\ f(0, x) = f_{\text{in}}(x), & x > 0 \end{cases} \quad (1)$$

is a model that describes the time evolution of a population structured with respect to the size  $x$  of the individuals.

The key term of the model is the fragmentation operator  $\mathcal{F}$ , defined as

$$\mathcal{F}f(x) := \int_x^\infty b(y, x)f(y) dy - B(x)f(x). \quad (2)$$

The fragmentation operator quantifies the generation of smaller individuals from a member of the population of size  $x$ : the individuals split with a rate  $B(x)$  and generate smaller individuals of size  $y \in (0, x)$ , whose distribution is governed by the kernel  $b(x, y)$ .

Models involving the fragmentation operator appear in various applications. Among them we can mention crushing of rocks, droplet breakup or combustion [2] which are pure fragmentation phenomena, but also cell division [14], protein polymerization [9] or data transmission protocols on the web [3], for which the fragmentation process occurs together with some “growth” phenomenon.

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In order to ensure the conservation of the total mass of particles which may occur during the fragmentation process, the coefficients  $B(x)$  and  $b(y, x)$  must be linked through the relation

$$\int_0^y xb(y, x) dx = yB(y). \quad (3)$$

This assumption ensures, at least formally, the mass conservation law

$$\forall t > 0, \quad \int_0^\infty xf(t, x) dx = \int_0^\infty xf_{\text{in}}(x) dx := \rho_{\text{in}}. \quad (4)$$

Moreover, it is well known that  $xf(t, x)$  converges to a Dirac mass at  $x = 0$  when  $t \rightarrow +\infty$ . Usually, the various contributions that are available in the literature restrict their attention to coefficients which satisfy the homogeneous assumptions (see [8] for instance)

$$B(x) = x^\gamma, \quad \gamma > 0, \quad \text{and} \quad b(y, x) = y^{\gamma-1}p\left(\frac{x}{y}\right), \quad (5)$$

where  $d\mu(z) := p(z) dz$  is a positive measure supported on  $[0, 1]$  which satisfies

$$\int_0^1 z d\mu(z) = 1.$$

These hypotheses guarantee that the relation (3) is verified.

From a mathematical point of view, it is convenient to perform the (mass preserving) self-similar change of variable

$$f(t, x) = (1+t)^{2/\gamma}g\left(\frac{1}{\gamma}\log(1+t), (1+t)^{1/\gamma}x\right),$$

or, by writing  $g$  in terms of  $f$ ,

$$g(t, x) := e^{-2t}f(e^{\gamma t} - 1, e^{-t}x).$$

It allows to deduce that  $g(t, x)$  satisfies the *self-similar fragmentation equation*

$$\begin{cases} \partial_t g + \partial_x(xg) + g = \gamma\mathcal{F}g, & t \geq 0, x > 0, \\ g(0, x) = f_{\text{in}}(x), & x > 0. \end{cases} \quad (6)$$

Equation (6) belongs to the class of *growth-fragmentation equations* and it admits – unlike Equation (1) – positive steady-states [7, 8, 15].

Denote by  $G$  the unique positive steady-state of Equation (6) with normalized mass, i.e. the solution of

$$(xG)' + G = \gamma\mathcal{F}G, \quad G > 0, \quad \int_0^\infty xG(x) dx = 1.$$

Then it has been proved (see [8, 12]) that the solution  $g(t, x)$  of the self-similar fragmentation equation (6) converges to  $\rho_{\text{in}}G(x)$  when  $t \rightarrow +\infty$ .

Coming back to the fragmentation equation (1), this result implies the convergence of  $f(t, x)$  to the self-similar solution  $(t, x) \mapsto \rho_{\text{in}}(1+t)^{2/\gamma}G((1+t)^{1/\gamma}x)$  and hence the convergence of  $xf(t, x)$  to a Dirac mass  $\delta_0$ .

In order to obtain more precise quantitative properties of the previous equation, one can wonder about the rate of convergence of  $g(t, x)$  to the asymptotic profile  $G(x)$ . Many recent

works are dedicated to this question and prove, under different assumptions and with different techniques, an exponential rate of convergence for growth-fragmentation equations [1, 3, 4, 5, 6, 11, 13, 15].

Nevertheless, to our knowledge the only results about the specific case of the self-similar fragmentation equation are those provided by Cáceres, Cañizo and Mischler [4, 5]. They prove exponential convergence in the Hilbert space  $L^2((x + x^k) dx)$  for a sufficiently large exponent  $k$  in [5], and in the Banach space  $L^1((x^m + x^M) dx)$  for suitable exponents  $1/2 < m < 1 < M < 2$  in [4]. For proving their results, the authors of the aforementioned articles require the measure  $p$  to be a bounded function (from above and below) and the power  $\gamma$  of the fragmentation rate to be less than 2.

The current paper aims to obtain a convergence result for super-quadratic rates, namely when  $\gamma \geq 2$ . We obtain exponential convergence to the asymptotic state by working in the weighted Hilbert space  $L^2(x dx)$ , under the following assumptions:

$$\gamma \geq 2 \quad \text{and} \quad p(z) \equiv 2. \quad (7)$$

The fact that  $p(z)$  is a constant means that the distribution of the fragments is uniform: the probability to get a fragment of size  $x$  or  $x'$  from a particle of size  $y > x, x'$  is exactly the same. Then the condition  $\int_0^1 zp(z) dz = 1$  imposes this constant to be equal to 2, meaning that the fragmentation is necessarily binary. Our assumption on  $p$  is more restrictive than in [4, 5], but in return we get a stronger result in the sense that we obtain an estimate of the exponential rate. Now we state the main theorem of this paper.

**Theorem 1.1.** *Let  $g_{\text{in}} \in L^1(x dx) \cap L^2(x dx)$  and let  $g \in C([0, \infty), L^1(x dx))$  be the unique solution of the self-similar fragmentation equation (6) with initial condition  $g_{\text{in}}$  and with fragmentation coefficients satisfying (5) and (7), that is*

$$B(x) = x^\gamma, \quad \gamma \geq 2 \quad \text{and} \quad b(y, x) = 2y^{\gamma-1}.$$

*Then the following estimate holds:*

$$\|g(t, \cdot) - \rho_{\text{in}} G\|_{L^2(x dx)} \leq \|g_{\text{in}} - \rho_{\text{in}} G\|_{L^2(x dx)} e^{-t}, \quad t \geq 0.$$

## 2 Preliminaries

Define the suitable weighted spaces

$$\dot{L}_k^p := L^p(\mathbb{R}^+, x^k dx) \quad \text{for } p \geq 1, k \in \mathbb{R}, \quad \text{and} \quad \dot{W}_1^{1,1} := W^{1,1}(\mathbb{R}^+, x dx).$$

For  $u \in \dot{L}_1^1$  we denote moreover by

$$M(x) := \int_0^x yu(y) dy$$

the primitive of  $xu(x)$  which vanishes at  $x = 0$ .

Now we recall the following existence and uniqueness result of a solution to the fragmentation equation, easily deduced from [8], Theorems 3.1-3.2 and Lemma 3.4:

**Theorem [8].** *If*

$$B(x) = x^\gamma, \quad \gamma \geq 2 \quad \text{and} \quad b(y, x) = 2y^{\gamma-1},$$

for any  $f_{\text{in}} \in \dot{L}_1^2 \cap \dot{L}_1^1$ , there exists a unique solution  $f \in C([0, \infty); \dot{L}_1^1) \cap L_{loc}^1([0, \infty); \dot{L}_{1+\gamma}^1)$  to the fragmentation equation (1) such that the mass conservation (4) is satisfied. If, moreover,  $f_{\text{in}} \in \Xi := \dot{L}_{1+\gamma}^1 \cap \dot{W}_1^{1,1}$ , the associated solution  $g$  to the self-similar fragmentation equation is such that

$$(g(t, \cdot))_{t \geq 0} \text{ is uniformly bounded in } \Xi.$$

In the following lemma we give some useful properties of the set

$$\Xi = \dot{L}_{1+\gamma}^1 \cap \dot{W}_1^{1,1}$$

and of the subset

$$\Xi_0 := \left\{ u \in \Xi, \int_0^\infty xu(x) \, dx = 0 \right\}.$$

**Lemma 2.1.** *The set  $\Xi = \dot{L}_{1+\gamma}^1 \cap \dot{W}_1^{1,1}$  satisfies*

$$G \in \Xi, \quad \Xi \subset C(0, \infty) \cap L^1 \cap \dot{L}_1^2 \cap \dot{L}_{1+\gamma}^2 \quad \text{and}$$

$$\forall u \in \Xi, \quad \lim_{x \rightarrow 0} xu(x) = \lim_{x \rightarrow +\infty} xu(x) = 0.$$

Moreover for any function  $u \in \Xi_0$  the following inequality holds:

$$\forall x > 0, \quad |M(x)| \leq x^{-\gamma} \|u\|_{\dot{L}_{1+\gamma}^1}.$$

*Proof.* The fact that the steady-state  $G$  belongs to  $\Xi$  is a consequence of the estimates in [1]. In the case when  $p(z) \equiv 2$  it can also be deduced from the explicit formula (see [7])

$$G(x) = \frac{\gamma}{\Gamma(2/\gamma)} e^{-x^\gamma}$$

where  $\Gamma$  is the Euler Gamma function.

For  $\gamma \geq 0$  and  $u \in \Xi$  such that  $\int_0^\infty xu(x) \, dx = 0$  we can write for  $x > 0$

$$|M(x)| = \left| - \int_x^\infty yu(y) \, dy \right| \leq x^{-\gamma} \int_0^\infty y^{1+\gamma} |u(y)| \, dy.$$

□

### 3 Proof of the main theorem

Define the self-similar fragmentation operator  $\mathcal{L}u := -(xu)' - u + \gamma \mathcal{F}u$  and denote by

$$(u, v) := \int_0^\infty xu(x)v(x) \, dx$$

the canonical scalar product in  $\dot{L}_1^2$ . Theorem 1.1 is a consequence of the following result.

**Theorem 3.1.** *Under Assumptions (5) and (7), i.e. for*

$$B(x) = x^\gamma, \quad \gamma \geq 2 \quad \text{and} \quad b(y, x) = 2y^{\gamma-1},$$

*we have*

$$\forall u \in \Xi_0, \quad (u, \mathcal{L}u) \leq -\|u\|_{L_1^2}^2.$$

*Proof.* Using Lemma 2.1 we can deduce, for  $u \in \Xi_0$ ,

$$(u, (xu)') = \int_0^\infty xu(x)(xu(x))' dx = \frac{1}{2} \int_0^\infty ((xu(x))^2)' dx = 0$$

and

$$\begin{aligned} (u, \mathcal{F}u) &= 2 \int_0^\infty xu(x) \int_x^\infty y^{\gamma-1} u(y) dy dx - \int_0^\infty x^{\gamma+1} u^2(x) dx \\ &= 2 \int_0^\infty x^{\gamma-1} u(x) \int_0^x yu(y) dy dx - \int_0^\infty x^{\gamma+1} u^2(x) dx \\ &= 2 \int_0^\infty x^{\gamma-2} M'(x)M(x) dx - \int_0^\infty x^{\gamma+1} u^2(x) dx \\ &= -(\gamma-2) \int_0^\infty x^{\gamma-3} M^2(x) dx - \int_0^\infty x^{\gamma+1} u^2(x) dx \leq 0. \end{aligned}$$

□

*Proof of Theorem 1.1.* Assume first that  $g_{\text{in}} \in \Xi$ . From Theorem 3.1 we obtain the differential inequality

$$\frac{d}{dt} \|g(t, \cdot) - \rho_{\text{in}} G\|_{L_1^2} \leq -\|g(t, \cdot) - \rho_{\text{in}} G\|_{L_1^2}$$

which gives the result. Then we may remove the additional assumption  $g_{\text{in}} \in \Xi$ . □

## 4 Conclusion

We have proved a spectral gap result for the self-similar fragmentation operator  $\mathcal{L}$  with a superquadratic fragmentation rate  $B(x)$ . More precisely we have obtained that this spectral gap is larger than 1. This is a new result concerning the long-time behaviour of the fragmentation equation (1). It also allows to extend the results obtained in [10] for non-linear growth-fragmentation equations.

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