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Exponential relaxation to self-similarity for the superquadratic fragmentation equation

P. Gabriel* F. Salvarani †

Abstract

We consider the self-similar fragmentation equation with a superquadratic fragmentation rate and provide a quantitative estimate of the spectral gap.

Keywords: fragmentation equation, self-similarity, exponential convergence, spectral gap, long-time behavior

AMS Class. No. Primary: 35B40, 35P15, 45K05; Secondary: 82D60, 92D25

1 Introduction

The *fragmentation equation*

$$\begin{cases} \partial_t f(t, x) = \mathcal{F}f(t, x), & t \geq 0, x > 0, \\ f(0, x) = f_{\text{in}}(x), & x > 0 \end{cases} \quad (1)$$

is a model that describes the time evolution of a population structured with respect to the size x of the individuals.

The key term of the model is the fragmentation operator \mathcal{F} , defined as

$$\mathcal{F}f(x) := \int_x^\infty b(y, x)f(y) dy - B(x)f(x). \quad (2)$$

The fragmentation operator quantifies the generation of smaller individuals from a member of the population of size x : the individuals split with a rate $B(x)$ and generate smaller individuals of size $y \in (0, x)$, whose distribution is governed by the kernel $b(x, y)$.

Models involving the fragmentation operator appear in various applications. Among them we can mention crushing of rocks, droplet breakup or combustion [2] which are pure fragmentation phenomena, but also cell division [14], protein polymerization [9] or data transmission protocols on the web [3], for which the fragmentation process occurs together with some “growth” phenomenon.

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In order to ensure the conservation of the total mass of particles which may occur during the fragmentation process, the coefficients $B(x)$ and $b(y, x)$ must be linked through the relation

$$\int_0^y xb(y, x) dx = yB(y). \quad (3)$$

This assumption ensures, at least formally, the mass conservation law

$$\forall t > 0, \quad \int_0^\infty xf(t, x) dx = \int_0^\infty xf_{\text{in}}(x) dx := \rho_{\text{in}}. \quad (4)$$

Moreover, it is well known that $xf(t, x)$ converges to a Dirac mass at $x = 0$ when $t \rightarrow +\infty$. Usually, the various contributions that are available in the literature restrict their attention to coefficients which satisfy the homogeneous assumptions (see [8] for instance)

$$B(x) = x^\gamma, \quad \gamma > 0, \quad \text{and} \quad b(y, x) = y^{\gamma-1}p\left(\frac{x}{y}\right), \quad (5)$$

where $d\mu(z) := p(z) dz$ is a positive measure supported on $[0, 1]$ which satisfies

$$\int_0^1 z d\mu(z) = 1.$$

These hypotheses guarantee that the relation (3) is verified.

From a mathematical point of view, it is convenient to perform the (mass preserving) self-similar change of variable

$$f(t, x) = (1+t)^{2/\gamma}g\left(\frac{1}{\gamma}\log(1+t), (1+t)^{1/\gamma}x\right),$$

or, by writing g in terms of f ,

$$g(t, x) := e^{-2t}f(e^{\gamma t} - 1, e^{-t}x).$$

It allows to deduce that $g(t, x)$ satisfies the *self-similar fragmentation equation*

$$\begin{cases} \partial_t g + \partial_x(xg) + g = \gamma\mathcal{F}g, & t \geq 0, x > 0, \\ g(0, x) = f_{\text{in}}(x), & x > 0. \end{cases} \quad (6)$$

Equation (6) belongs to the class of *growth-fragmentation equations* and it admits – unlike Equation (1) – positive steady-states [7, 8, 15].

Denote by G the unique positive steady-state of Equation (6) with normalized mass, i.e. the solution of

$$(xG)' + G = \gamma\mathcal{F}G, \quad G > 0, \quad \int_0^\infty xG(x) dx = 1.$$

Then it has been proved (see [8, 12]) that the solution $g(t, x)$ of the self-similar fragmentation equation (6) converges to $\rho_{\text{in}}G(x)$ when $t \rightarrow +\infty$.

Coming back to the fragmentation equation (1), this result implies the convergence of $f(t, x)$ to the self-similar solution $(t, x) \mapsto \rho_{\text{in}}(1+t)^{2/\gamma}G((1+t)^{1/\gamma}x)$ and hence the convergence of $xf(t, x)$ to a Dirac mass δ_0 .

In order to obtain more precise quantitative properties of the previous equation, one can wonder about the rate of convergence of $g(t, x)$ to the asymptotic profile $G(x)$. Many recent

works are dedicated to this question and prove, under different assumptions and with different techniques, an exponential rate of convergence for growth-fragmentation equations [1, 3, 4, 5, 6, 11, 13, 15].

Nevertheless, to our knowledge the only results about the specific case of the self-similar fragmentation equation are those provided by Cáceres, Cañizo and Mischler [4, 5]. They prove exponential convergence in the Hilbert space $L^2((x + x^k) dx)$ for a sufficiently large exponent k in [5], and in the Banach space $L^1((x^m + x^M) dx)$ for suitable exponents $1/2 < m < 1 < M < 2$ in [4]. For proving their results, the authors of the aforementioned articles require the measure p to be a bounded function (from above and below) and the power γ of the fragmentation rate to be less than 2.

The current paper aims to obtain a convergence result for super-quadratic rates, namely when $\gamma \geq 2$. We obtain exponential convergence to the asymptotic state by working in the weighted Hilbert space $L^2(x dx)$, under the following assumptions:

$$\gamma \geq 2 \quad \text{and} \quad p(z) \equiv 2. \quad (7)$$

The fact that $p(z)$ is a constant means that the distribution of the fragments is uniform: the probability to get a fragment of size x or x' from a particle of size $y > x, x'$ is exactly the same. Then the condition $\int_0^1 zp(z) dz = 1$ imposes this constant to be equal to 2, meaning that the fragmentation is necessarily binary. Our assumption on p is more restrictive than in [4, 5], but in return we get a stronger result in the sense that we obtain an estimate of the exponential rate. Now we state the main theorem of this paper.

Theorem 1.1. *Let $g_{\text{in}} \in L^1(x dx) \cap L^2(x dx)$ and let $g \in C([0, \infty), L^1(x dx))$ be the unique solution of the self-similar fragmentation equation (6) with initial condition g_{in} and with fragmentation coefficients satisfying (5) and (7), that is*

$$B(x) = x^\gamma, \quad \gamma \geq 2 \quad \text{and} \quad b(y, x) = 2y^{\gamma-1}.$$

Then the following estimate holds:

$$\|g(t, \cdot) - \rho_{\text{in}} G\|_{L^2(x dx)} \leq \|g_{\text{in}} - \rho_{\text{in}} G\|_{L^2(x dx)} e^{-t}, \quad t \geq 0.$$

2 Preliminaries

Define the suitable weighted spaces

$$\dot{L}_k^p := L^p(\mathbb{R}^+, x^k dx) \quad \text{for } p \geq 1, k \in \mathbb{R}, \quad \text{and} \quad \dot{W}_1^{1,1} := W^{1,1}(\mathbb{R}^+, x dx).$$

For $u \in \dot{L}_1^1$ we denote moreover by

$$M(x) := \int_0^x yu(y) dy$$

the primitive of $xu(x)$ which vanishes at $x = 0$.

Now we recall the following existence and uniqueness result of a solution to the fragmentation equation, easily deduced from [8], Theorems 3.1-3.2 and Lemma 3.4:

Theorem [8]. *If*

$$B(x) = x^\gamma, \quad \gamma \geq 2 \quad \text{and} \quad b(y, x) = 2y^{\gamma-1},$$

for any $f_{\text{in}} \in \dot{L}_m^1 \cap \dot{L}_1^1$ with $m < 1$, there exists a unique solution $f \in C([0, \infty); \dot{L}_1^1)$ to the fragmentation equation (1) such that the mass conservation (4) is satisfied. If, moreover, $f_{\text{in}} \in \Xi := L^1 \cap \dot{L}_\gamma^1 \cap \dot{W}_1^{1,1}$, the associated solution g to the self-similar fragmentation equation is such that

$$(g(t, \cdot))_{t \geq 0} \text{ is uniformly bounded in } \Xi.$$

In the following lemma we give some useful properties of the set

$$\Xi = L^1 \cap \dot{L}_\gamma^1 \cap \dot{W}_1^{1,1}$$

and of the subset

$$\Xi_0 := \left\{ u \in \Xi, \int_0^\infty xu(x) \, dx = 0 \right\}.$$

Lemma 2.1. *The set $\Xi = L^1 \cap \dot{L}_\gamma^1 \cap \dot{W}_1^{1,1}$ satisfies*

$$G \in \Xi, \quad \Xi \subset C(0, \infty) \cap \dot{L}_1^2 \cap \dot{L}_{\gamma+1}^2 \quad \text{and}$$

$$\forall u \in \Xi, \quad \lim_{x \rightarrow 0} xu(x) = \lim_{x \rightarrow +\infty} xu(x) = 0.$$

Moreover, if $\gamma \geq 1$, for any function $u \in \Xi_0$ the following inequality holds:

$$\forall x > 0, \quad |M(x)| \leq x^{1-\gamma} \|u\|_{\dot{L}_\gamma^1}.$$

Proof. The fact that the steady-state G belongs to Ξ is a consequence of the estimates in [1]. In the case when $p(z) \equiv 2$ it can also be deduced from the explicit formula (see [7])

$$G(x) = \frac{\gamma}{\Gamma(2/\gamma)} e^{-x^\gamma}$$

where Γ is the Euler Gamma function.

For $\gamma \geq 1$ and $u \in \Xi$ such that $\int_0^\infty xu(x) \, dx = 0$ we can write for $x > 0$

$$|M(x)| = \left| - \int_x^\infty yu(y) \, dy \right| \leq x^{1-\gamma} \int_0^\infty y^\gamma |u(y)| \, dy.$$

□

3 Proof of the main theorem

Define the self-similar fragmentation operator $\mathcal{L}u := -(xu)' - u + \gamma \mathcal{F}u$ and denote by

$$(u, v) := \int_0^\infty xu(x)v(x) \, dx$$

the canonical scalar product in \dot{L}_1^2 . Theorem 1.1 is a consequence of the following result.

Theorem 3.1. *Under Assumptions (5) and (7), i.e. for*

$$B(x) = x^\gamma, \quad \gamma \geq 2 \quad \text{and} \quad b(y, x) = 2y^{\gamma-1},$$

we have

$$\forall u \in \Xi_0, \quad (u, \mathcal{L}u) \leq -\|u\|_{L^2_1}^2.$$

Proof. Using Lemma 2.1 we can deduce, for $u \in \Xi_0$,

$$(u, (xu)') = \int_0^\infty xu(x)(xu(x))' dx = \frac{1}{2} \int_0^\infty ((xu(x))^2)' dx = 0$$

and

$$\begin{aligned} (u, \mathcal{F}u) &= 2 \int_0^\infty xu(x) \int_x^\infty y^{\gamma-1} u(y) dy dx - \int_0^\infty x^{\gamma+1} u^2(x) dx \\ &= 2 \int_0^\infty x^{\gamma-1} u(x) \int_0^x yu(y) dy dx - \int_0^\infty x^{\gamma+1} u^2(x) dx \\ &= 2 \int_0^\infty x^{\gamma-2} M'(x)M(x) dx - \int_0^\infty x^{\gamma+1} u^2(x) dx \\ &= -(\gamma-2) \int_0^\infty x^{\gamma-3} M^2(x) dx - \int_0^\infty x^{\gamma+1} u^2(x) dx \leq 0. \end{aligned}$$

□

Proof of Theorem 1.1. Assume first that $g_{\text{in}} \in \Xi$. From Theorem 3.1 we obtain the differential inequality

$$\frac{d}{dt} \|g(t, \cdot) - \rho_{\text{in}} G\|_{L^2_1} \leq -\|g(t, \cdot) - \rho_{\text{in}} G\|_{L^2_1}$$

which gives the result. Then we may remove the additional assumption $g_{\text{in}} \in \Xi$. □

4 Conclusion

We have proved a spectral gap result for the self-similar fragmentation operator \mathcal{L} with a superquadratic fragmentation rate $B(x)$. More precisely we have obtained that this spectral gap is larger than 1. This is a new result concerning the long-time behaviour of the fragmentation equation (1). It also allows to extend the results obtained in [10] for non-linear growth-fragmentation equations.

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