



HAL
open science

Exponential relaxation to self-similarity for the superquadratic fragmentation equation

Pierre Gabriel, Francesco Salvarani

► **To cite this version:**

Pierre Gabriel, Francesco Salvarani. Exponential relaxation to self-similarity for the superquadratic fragmentation equation. *Applied Mathematics Letters*, 2014, 27, pp.74–78. 10.1016/j.aml.2013.08.001 . hal-00860231v2

HAL Id: hal-00860231

<https://hal.science/hal-00860231v2>

Submitted on 23 Oct 2014 (v2), last revised 12 Apr 2016 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Exponential relaxation to self-similarity for the superquadratic fragmentation equation

P. Gabriel* F. Salvarani †

Abstract

We consider the self-similar fragmentation equation with a superquadratic fragmentation rate and provide a quantitative estimate of the spectral gap.

Keywords: fragmentation equation, self-similarity, exponential convergence, spectral gap, long-time behavior

AMS Class. No. Primary: 35B40, 35P15, 45K05; Secondary: 82D60, 92D25

1 Introduction

The *fragmentation equation*

$$\begin{cases} \partial_t f(t, x) = \mathcal{F}f(t, x), & t \geq 0, x > 0, \\ f(0, x) = f_{\text{in}}(x), & x > 0 \end{cases} \quad (1)$$

is a model that describes the time evolution of a population structured with respect to the size x of the individuals.

The key term of the model is the fragmentation operator \mathcal{F} , defined as

$$\mathcal{F}f(x) := \int_x^\infty b(y, x)f(y) dy - B(x)f(x). \quad (2)$$

The fragmentation operator quantifies the generation of smaller individuals from a member of the population of size x : the individuals split with a rate $B(x)$ and generate smaller individuals of size $y \in (0, x)$, whose distribution is governed by the kernel $b(x, y)$.

Models involving the fragmentation operator appear in various applications. Among them we can mention crushing of rocks, droplet breakup or combustion [?] which are pure fragmentation phenomena, but also cell division [?], protein polymerization [?] or data transmission protocols on the web [?], for which the fragmentation process occurs together with some “growth” phenomenon.

*University of Versailles St-Quentin-en-Yvelines, Laboratoire de Mathématiques de Versailles, CNRS UMR 8100, 45 Avenue de États-Unis, 78035 Versailles cedex, France. Email: pierre.gabriel@uvsq.fr

†Dipartimento di Matematica F. Casorati, Università degli Studi di Pavia, Via Ferrata 1, 27100 Pavia, Italy. Email: francesco.salvarani@unipv.it

In order to ensure the conservation of the total mass of particles which may occur during the fragmentation process, the coefficients $B(x)$ and $b(y, x)$ must be linked through the relation

$$\int_0^y xb(y, x) dx = yB(y). \quad (3)$$

This assumption ensures, at least formally, the mass conservation law

$$\forall t > 0, \quad \int_0^\infty xf(t, x) dx = \int_0^\infty xf_{\text{in}}(x) dx := \rho_{\text{in}}. \quad (4)$$

Moreover, it is well known that $xf(t, x)$ converges to a Dirac mass at $x = 0$ when $t \rightarrow +\infty$. Usually, the various contributions that are available in the literature restrict their attention to coefficients which satisfy the homogeneous assumptions (see [?] for instance)

$$B(x) = x^\gamma, \quad \gamma > 0, \quad \text{and} \quad b(y, x) = y^{\gamma-1}p\left(\frac{x}{y}\right), \quad (5)$$

where $d\mu(z) := p(z) dz$ is a positive measure supported on $[0, 1]$ which satisfies

$$\int_0^1 z d\mu(z) = 1.$$

These hypotheses guarantee that the relation (3) is verified.

From a mathematical point of view, it is convenient to perform the (mass preserving) self-similar change of variable

$$f(t, x) = (1+t)^{2/\gamma}g\left(\frac{1}{\gamma}\log(1+t), (1+t)^{1/\gamma}x\right),$$

or, by writing g in terms of f ,

$$g(t, x) := e^{-2t}f(e^{\gamma t} - 1, e^{-t}x).$$

It allows to deduce that $g(t, x)$ satisfies the *self-similar fragmentation equation*

$$\begin{cases} \partial_t g + \partial_x(xg) + g = \gamma\mathcal{F}g, & t \geq 0, x > 0, \\ g(0, x) = f_{\text{in}}(x), & x > 0. \end{cases} \quad (6)$$

Equation (6) belongs to the class of *growth-fragmentation equations* and it admits – unlike Equation (1) – positive steady-states [?, ?, ?].

Denote by G the unique positive steady-state of Equation (6) with normalized mass, i.e. the solution of

$$(xG)' + G = \gamma\mathcal{F}G, \quad G > 0, \quad \int_0^\infty xG(x) dx = 1.$$

Then it has been proved (see [?, ?]) that the solution $g(t, x)$ of the self-similar fragmentation equation (6) converges to $\rho_{\text{in}}G(x)$ when $t \rightarrow +\infty$.

Coming back to the fragmentation equation (1), this result implies the convergence of $f(t, x)$ to the self-similar solution $(t, x) \mapsto \rho_{\text{in}}(1+t)^{2/\gamma}G((1+t)^{1/\gamma}x)$ and hence the convergence of $xf(t, x)$ to a Dirac mass δ_0 .

In order to obtain more precise quantitative properties of the previous equation, one can wonder about the rate of convergence of $g(t, x)$ to the asymptotic profile $G(x)$. Many recent

works are dedicated to this question and prove, under different assumptions and with different techniques, an exponential rate of convergence for growth-fragmentation equations [?, ?, ?, ?, ?, ?, ?, ?].

Nevertheless, to our knowledge the only results about the specific case of the self-similar fragmentation equation are those provided by Cáceres, Cañizo and Mischler [?, ?]. They prove exponential convergence in the Hilbert space $L^2((x + x^k) dx)$ for a sufficiently large exponent k in [?], and in the Banach space $L^1((x^m + x^M) dx)$ for suitable exponents $1/2 < m < 1 < M < 2$ in [?]. For proving their results, the authors of the aforementioned articles require the measure p to be a bounded function (from above and below) and the power γ of the fragmentation rate to be less than 2.

The current paper aims to obtain a convergence result for super-quadratic rates, namely when $\gamma \geq 2$. We obtain exponential convergence to the asymptotic state by working in the weighted Hilbert space $L^2(x dx)$, under the following assumptions:

$$\gamma \geq 2 \quad \text{and} \quad p(z) \equiv 2. \quad (7)$$

The fact that $p(z)$ is a constant means that the distribution of the fragments is uniform: the probability to get a fragment of size x or x' from a particle of size $y > x, x'$ is exactly the same. Then the condition $\int_0^1 zp(z) dz = 1$ imposes this constant to be equal to 2, meaning that the fragmentation is necessarily binary. Our assumption on p is more restrictive than in [?, ?], but in return we get a stronger result in the sense that we obtain an estimate of the exponential rate. Now we state the main theorem of this paper.

Theorem 1.1. *Let $g_{\text{in}} \in L^1(x dx) \cap L^2(x dx)$ and let $g \in C([0, \infty), L^1(x dx))$ be the unique solution of the self-similar fragmentation equation (6) with initial condition g_{in} and with fragmentation coefficients satisfying (5) and (7), that is*

$$B(x) = x^\gamma, \quad \gamma \geq 2 \quad \text{and} \quad b(y, x) = 2y^{\gamma-1}.$$

Then the following estimate holds:

$$\|g(t, \cdot) - \rho_{\text{in}} G\|_{L^2(x dx)} \leq \|g_{\text{in}} - \rho_{\text{in}} G\|_{L^2(x dx)} e^{-t}, \quad t \geq 0.$$

2 Preliminaries

Define the suitable weighted spaces

$$\dot{L}_k^p := L^p(\mathbb{R}^+, x^k dx) \quad \text{for } p \geq 1, k \in \mathbb{R}, \quad \text{and} \quad \dot{W}_1^{1,1} := W^{1,1}(\mathbb{R}^+, x dx).$$

For $u \in \dot{L}_1^1$ we denote moreover by

$$M(x) := \int_0^x yu(y) dy$$

the primitive of $xu(x)$ which vanishes at $x = 0$.

Now we recall the following existence and uniqueness result of a solution to the fragmentation equation, easily deduced from [?], Theorems 3.1-3.2 and Lemma 3.4:

Theorem [?]. *If*

$$B(x) = x^\gamma, \quad \gamma \geq 2 \quad \text{and} \quad b(y, x) = 2y^{\gamma-1},$$

for any $f_{\text{in}} \in \dot{L}_m^1 \cap \dot{L}_1^1$ with $m < 1$, there exists a unique solution $f \in C([0, \infty); \dot{L}_1^1)$ to the fragmentation equation (1) such that the mass conservation (4) is satisfied. If, moreover, $f_{\text{in}} \in \Xi := L^1 \cap \dot{L}_\gamma^1 \cap \dot{W}_1^{1,1}$, the associated solution g to the self-similar fragmentation equation is such that

$$(g(t, \cdot))_{t \geq 0} \text{ is uniformly bounded in } \Xi.$$

In the following lemma we give some useful properties of the set

$$\Xi = L^1 \cap \dot{L}_\gamma^1 \cap \dot{W}_1^{1,1}$$

and of the subset

$$\Xi_0 := \left\{ u \in \Xi, \int_0^\infty xu(x) \, dx = 0 \right\}.$$

Lemma 2.1. *The set $\Xi = L^1 \cap \dot{L}_\gamma^1 \cap \dot{W}_1^{1,1}$ satisfies*

$$G \in \Xi, \quad \Xi \subset C(0, \infty) \cap \dot{L}_1^2 \cap \dot{L}_{\gamma+1}^2 \quad \text{and}$$

$$\forall u \in \Xi, \quad \lim_{x \rightarrow 0} xu(x) = \lim_{x \rightarrow +\infty} xu(x) = 0.$$

Moreover, if $\gamma \geq 1$, for any function $u \in \Xi_0$ the following inequality holds:

$$\forall x > 0, \quad |M(x)| \leq x^{1-\gamma} \|u\|_{\dot{L}_\gamma^1}.$$

Proof. The fact that the steady-state G belongs to Ξ is a consequence of the estimates in [?]. In the case when $p(z) \equiv 2$ it can also be deduced from the explicit formula (see [?])

$$G(x) = \frac{\gamma}{\Gamma(2/\gamma)} e^{-x^\gamma}$$

where Γ is the Euler Gamma function.

For $\gamma \geq 1$ and $u \in \Xi$ such that $\int_0^\infty xu(x) \, dx = 0$ we can write for $x > 0$

$$|M(x)| = \left| - \int_x^\infty yu(y) \, dy \right| \leq x^{1-\gamma} \int_0^\infty y^\gamma |u(y)| \, dy.$$

□

3 Proof of the main theorem

Define the self-similar fragmentation operator $\mathcal{L}u := -(xu)' - u + \gamma \mathcal{F}u$ and denote by

$$(u, v) := \int_0^\infty xu(x)v(x) \, dx$$

the canonical scalar product in \dot{L}_1^2 . Theorem 1.1 is a consequence of the following result.

Theorem 3.1. *Under Assumptions (5) and (7), i.e. for*

$$B(x) = x^\gamma, \quad \gamma \geq 2 \quad \text{and} \quad b(y, x) = 2y^{\gamma-1},$$

we have

$$\forall u \in \Xi_0, \quad (u, \mathcal{L}u) \leq -\|u\|_{\dot{L}_1^2}^2.$$

Proof. Using Lemma 2.1 we can deduce, for $u \in \Xi_0$,

$$(u, (xu)') = \int_0^\infty xu(x)(xu(x))' dx = \frac{1}{2} \int_0^\infty ((xu(x))^2)' dx = 0$$

and

$$\begin{aligned} (u, \mathcal{F}u) &= 2 \int_0^\infty xu(x) \int_x^\infty y^{\gamma-1} u(y) dy dx - \int_0^\infty x^{\gamma+1} u^2(x) dx \\ &= 2 \int_0^\infty x^{\gamma-1} u(x) \int_0^x yu(y) dy dx - \int_0^\infty x^{\gamma+1} u^2(x) dx \\ &= 2 \int_0^\infty x^{\gamma-2} M'(x)M(x) dx - \int_0^\infty x^{\gamma+1} u^2(x) dx \\ &= -(\gamma-2) \int_0^\infty x^{\gamma-3} M^2(x) dx - \int_0^\infty x^{\gamma+1} u^2(x) dx \leq 0. \end{aligned}$$

□

Proof of Theorem 1.1. Assume first that $g_{\text{in}} \in \Xi$. From Theorem 3.1 we obtain the differential inequality

$$\frac{d}{dt} \|g(t, \cdot) - \rho_{\text{in}} G\|_{\dot{L}_1^2} \leq -\|g(t, \cdot) - \rho_{\text{in}} G\|_{\dot{L}_1^2}$$

which gives the result. Then we may remove the additional assumption $g_{\text{in}} \in \Xi$. □

4 Conclusion

We have proved a spectral gap result for the self-similar fragmentation operator \mathcal{L} with a superquadratic fragmentation rate $B(x)$. More precisely we have obtained that this spectral gap is larger than 1. This is a new result concerning the long-time behaviour of the fragmentation equation (1). It also allows to extend the results obtained in [?] for non-linear growth-fragmentation equations.

Acknowledgement

This work has been performed in the framework of the ANR project KIBORD, headed by Laurent Desvillettes.

References