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Abstract. We give a simple proof of the "tree-width duality theorem" of Seymour and Thomas that the tree-width of a finite graph is exactly one less than the largest order of its brambles.

1. Introduction

A *tree-decomposition* $\mathcal{T} = (T, l)$ of a graph $G = (V, E)$ is tree whose nodes are labelled in such a way that

i. $V = \bigcup_{t \in V(T)} l(t)$;
ii. every $e \in E$ is contained in at least one $l(t)$;
iii. for every vertex $v \in V$, the nodes of T whose bags contain v induce a connected subtree of T.

The label of a node is its *bag*. The *width* of $\mathcal{T}$ is $\max\{|l(t)| \; ; \; t \in V(T)\} - 1$, and the *tree-width* $\text{tw}(G)$ of G is the least width of any of its tree-decomposition.

Two subsets X and Y of V *touch* if they meet or if there exists an edge linking them. A set $\mathcal{B}$ of mutually touching connected vertex sets in G is a *bramble*. A *cover* of $\mathcal{B}$ is a set of vertices which meets all its elements, and the *order* of $\mathcal{B}$ is the least size of one of its covers.

In this note, we give a new proof of the following theorem of Seymour and Thomas which Reed [Ree97] calls the "tree-width duality theorem".

Theorem 1 ([ST93]). *Let $k \geq 0$ be an integer. A graph has tree-width $\geq k$ if and only if it contains a bramble of order $> k$.*

Although our proof is quite short, our goal is not to give a shorter proof. The proof in [Die05] is already short enough. Instead, we claim that our proof is much simpler than previous ones. Indeed, the proofs in [ST93, Die05] rely on a reverse induction on the size of a bramble which is not very enlightening. A new conceptually much simpler proof appeared in [LMT10] but this proof is a much more general result on sets of partitions which through a translation process unifies all known duality theorem of this kind such as the branch-width/tangle or the path-width blockade Theorems. We turn this more general proof back into a specific proof for tree-width which we believe is interesting both as an introduction to the framework of [AMNT09, LMT10], and to a reader which does not want to dwell into this framework but still want to have a better understanding of the tree-width duality Theorem.

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2. The proof

So let $G = (V, E)$ be a graph and let k be a fixed integer. A bag of a tree-decomposition of G is *small* if it has size $\leq k$ and is *big* otherwise. A *partial* $(< k)$-*decomposition* is a tree-decomposition $\mathcal{T}$ with no big internal bag and with at least one small bag. Obviously, if all its bags are small, then $\mathcal{T}$ is a tree-decomposition of width $< k$. If not, it contains a big leaf bag and the neighbouring bag $l(u)$ of any such big leaf bag $l(t)$ is small. The nonempty set $l(t) - l(u)$ is a *k-flap* of $\mathcal{T}$.

Now suppose that X and Y are respectively k-flaps of some partial $(< k)$-decompositions (T_X, l_X) and (T_Y, l_Y), and that $S = N(X) \subseteq N(Y)$. Then by identifying the leaves of the two decompositions which respectively contains X and Y and relabelling this node S, then we obtain a new "better" partial $(< k)$-decomposition.

This gluing process is quite powerful. Indeed let $S \subseteq V$ have size $\leq k$ and let $C_1, \ldots, C_p$ be the components of $G - S$. The star whose centre u is labelled $l(u) = S$ and whose p leaves $v_1, \ldots, v_p$ are labelled by $l(v_i) = C_i \cup N(C_i)$ is a partial $(< k)$-decomposition which we call the *star decomposition from S*. It can be shown that if $\text{tw}(G) < k$, then an optimal tree-decomposition can always be obtained by repeatedly applying this gluing process from star decompositions from sets of size $\leq k$. But this process is not powerful enough for our purpose. We need the following lemma.

Lemma 1. *Let X and Y be respectively k-flaps of some partial $(< k)$-decompositions (T_X, l_X) and (T_Y, l_Y) of some graph $G = (V, E)$. If X and Y do not touch, then there exists a partial $(< k)$-decomposition (T, l) whose k-flaps are subsets of k-flaps of (T_X, l_X) and (T_Y, l_Y) other than X and Y.*

Proof. Since, X and Y no not touch, there exists $S \subseteq V$ such that no component of $G - S$ meet both X and Y (for example $N(X)$). Choose such an S with $|S|$ minimal. Note that $|S| \leq |N(X)| \leq k$. Let A contain S and all the components of $G - S$ which meet X, and let $B = (V - A) \cup S$.

Claim 1. *There exists a partial $(< k)$-decomposition of $G[B]$ with S as a leaf and whose k-flaps are subsets of the k-flaps of (T_X, l_X) other than X.*

Let x be the leaf of T_X whose bag contains X. Since $|S|$ is minimum, there exists $|S|$ vertex disjoint paths P_s from X to S ($s \in S$). Note that P_s only meets B in s. For each $s \in S$, pick a node t_s in T_X with $s \in l_X(t_s)$, and let $l'_X(t) = (l_X(t) \cap B) \cup \{s | t \in \text{path from } x \text{ to } t_s\}$ for all $t \in T$. Then (T_X, l'_X) is the tree-decomposition of $G[B]$. Indeed, since we removed only vertices not in B, every vertex and every edge of $G[B]$ is contained in some bag $l'_X(t)$. Moreover, for any $v \notin S$, $l'_X(t)$ contains v if and only if $l_X(t)$ does. And $l'_X(t)$ contains $s \in S$ if $l_X(t)$ does or if t is on the path from x to t_s. In either cases, the vertices $t \in V(T_X)$ whose bag $l'_X(t)$ contain a given vertex induce a subtree of T_X.

Now the size of a bag $l'_X(t)$ is at most $|l_X(t)|$. Indeed, since P_s is a connected subgraph of G, it induces a connected subtree of T_X, and this subtree contains the path from x to t_s. So for every vertex $s \in l'_X(t) \setminus l_X(t)$, there exists at least one other vertex of P_s which as been removed. The decomposition (T_X, l'_X) is thus indeed a partial $(< k)$-decomposition of $G[B]$. It remains to prove that the k-flaps of (T_X, l'_X) are contained in the k-flaps of (T_X, l_X) other than X. But by construction, the only leaf whose bag received new vertices is x and $l'_X(x) = S$ which is small. This finishes the proof of the claim.

Let (T_Y, l'_Y) be obtains in the same way for $G[A]$. By identifying the leaves x and y of T_X and T_Y, we obtain a partial $(< k)$-decomposition which satisfies the conditions of the lemma. □

We are now ready to prove the tree-width duality Theorem.

Proof. For the backward implication, let $\mathcal{B}$ be a bramble of order $> k$ in a graph G. We show that every tree-decomposition (T, l) of G has a part that covers $\mathcal{B}$, and thus $\mathcal{T}$ has width $\geq k$.

We start by orienting the edges t_1t_2 of T. Let T_i be the component of $T \setminus t_1t_2$ which contains t_i and let $V_i = \cup_{t \in V(T_i)} l(t)$. If $X := l(t_1) \cap l(t_2)$ covers $\mathcal{B}$, we are done. If not, then because they are connected, each $B \in \mathcal{B}$ disjoint from X in contained is some $B \subseteq V_i$. This i is the same for all such B, because they touch. We now orient the edge t_1t_2 towards t_i. If every edge of T is oriented in this way and t is the last vertex of a maximal directed path in T, then $l(t)$ covers $\mathcal{B}$.

To prove the forward direction, we now assume that G has tree-width $\geq k$, then any partial $(< k)$-decomposition contains a k-flap. There thus exists a set $\mathcal{B}$ of k-flaps such that

(i) $\mathcal{B}$ contains a flap of every partial $(< k)$-decomposition;
(ii) $\mathcal{B}$ is upward closed, that is if $C \in \mathcal{B}$ and $D \supseteq C$ is a k-flap, then $D \in \mathcal{B}$.

So far, the set of all k-flaps satisfies (i) and (ii).

(iii) Subject to (i) and (ii), $\mathcal{B}$ is inclusion-wise minimal.

The set $\mathcal{B}$ may not be a bramble because it may contain non-connected elements but we claim that the set $\mathcal{B}'$ which contains the connected elements of $\mathcal{B}$ is a bramble of order $\geq k$. Obviously, its elements are connected. To see that its order is $> k$, let $S \subseteq V$ have size $\leq k$. Then $\mathcal{B}'$ contains a k-flap of the star-decomposition from S, and S is thus not a covering of $\mathcal{B}'$.

We now prove that the elements of $\mathcal{B}$ pairwise touch, which finishes the proof that $\mathcal{B}'$ is a bramble. Suppose not, then let X and $Y \in \mathcal{B}$ witness this. Obviously, no subsets of X and Y can touch so let us suppose that they are inclusion-wise minimal in $\mathcal{B}$. The set X being minimal, $\mathcal{B} \setminus \{X\}$ is still upward closed and is a strict subset of $\mathcal{B}$. There thus exists at least one partial $(< k)$-decomposition (T_X, l_X) whose only flap in $\mathcal{B}$ is X. Likewise, let (T_Y, l_Y) have only Y as a flap in $\mathcal{B}$. Let (T, l) be the partial $(< k)$-decomposition satisfying the conditions of Lemma 1. Since $\mathcal{B}$ is upward closed and contains no k-flap of (T_X, l_X) and (T_Y, l_Y) other than X and Y, it contains no k-flap of (T, l), a contradiction. □

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