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# ON THE HOCHSCHILD HOMOLOGY OF OPEN FROBENIUS ALGEBRAS

HOSSEIN ABBASPOUR

ABSTRACT. We prove that the Hochschild homology (and cohomology) of a symmetric open Frobenius algebra  $A$  has a natural coBV and BV structure. The underlying coalgebra and algebra structure may not be resp. counital and unital. Moreover we prove that the product and coproduct satisfy the Frobenius compatibility condition i.e. the coproduct on  $HH_*(A)$  is a map of left and right  $HH_*(A)$ -modules. If  $A$  is commutative, we also introduced a natural BV structure on the relative Hochschild homology  $\widehat{HH}_*(A)$  after a shift in degree. We anticipate that the product of this BV structure to be related to the Goresky-Hingston product on the cohomology of free loop spaces.

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## 1. INTRODUCTION

There have been many attempts in finding an algebraic model for the string topology operations introduced by Chas-Sullivan ([CS, CS04]) and Cohen-Godin ([CG04]). One approach is to use the Hochschild cohomology of (closed) Frobenius algebras [CJ02, Mer04, Tra08, TZ06, Kau07, Kau08, Men09, WW]. In particular Félix-Thomas [FT08] proved that over rationals and for any closed simply connected manifold  $M$  the Chas-Sullivan BV-algebra  $H_*(LM)$  is isomorphic to  $HH^*(A, A^\vee)$  where  $A$  is a finite dimensional model (i.e. Frobenius algebra) for the cochains algebra  $M$ .

In [KS09] Kontsevich-Soibelman constructed an action of the chains of moduli spaces of Riemann surfaces on the Hochschild complex of a Frobenius algebras. This is a special of Constello's theorem [Cos07] for Calabi-Yau categories and induces a natural BV and coBV structure on the the Hochschild homology and the Frobenius compatibility between the BV product and coBV coproduct.

In this paper we assume that  $A$  is a symmetric open Frobenius algebra (unital) therefore  $A$  is not endowed with a (homotopic) non-degenerate scalar product. Instead  $A$  is equipped with a compatible pair of product and coproduct. We prove that the Hochschild homology  $HH_*(A, A)$  is a coBV (Section 3) and BV algebra (Section 4). As a result, the Hochschild cohomology  $HH^*(A, A^\vee)$  is also BV and coBV. Moreover, in Section 6 we prove that the product and coproduct satisfy the Frobenius compatibility as well. Such a compatibility was expected in the light of Cohen-Godin work for free loop spaces (of not necessarily closed manifolds).

It is worth mentioning that  $HH_*(A, A)$  is generally not a unital algebra (or equivalently  $HH^*(A, A^\vee)$  is not counital), reflecting the fact (in the geometric side) the free loop spaces are infinite dimensional manifolds thus their homology are not counital. Similarly  $HH_*(A, A)$  is not counital (or equivalently  $HH^*(A, A^\vee)$  is not unital) because the underlying manifold of the cochain complex  $A$  may not be a closed compact one. We recall that a unit for the Chas-Sullivan algebra on  $H_*(LM)$  exist if and only if the underlying manifold is closed manifold in which case the cycle consisting of constant loops is the unit.

We also identify a natural BV-product on the relative Hochschild homology after a shift in degree. We believe that this product should be an algebraic model of the Goresky-Hingston [GH09] product on the relative cohomology  $H^*(LM, M)$ . As far as we know this product was not known even for (closed) Frobenius algebras.

Our results can potentially be used to give an algebraic model for the string topology of not necessarily closed manifolds. That would require generalizing our results to an appropriate homotopic setting.

**Convention:** Here, for the sake of simplicity, we work over  $\mathbb{Z}_2$ . Readers interested in sign discussion are referred to [WW, Tra08]. Throughout this paper  $\mathbf{k}$  is a commutative ring and  $A = \mathbf{k} \oplus \bar{A}$  is an augmented unital differential associative  $\mathbf{k}$ -algebra with  $\deg d_A = +1$ ,  $\bar{A} = A/\mathbf{k}$  i.e.  $\bar{A}$  is the kernel of the augmentation  $\epsilon : A \rightarrow \mathbf{k}$ .

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## 2. OPEN FROBENIUS ALGEBRAS AND BV-ALGEBRAS

**Definition 2.1.** (DG open Frobenius algebra). A *differential graded open Frobenius  $\mathbf{k}$ -algebra* of degree  $m$  is a triple  $(A, \cdot, \delta)$  such that:

- (1)  $(A, \cdot)$  is a unital differential graded associative algebra whose product has degree zero,
- (2)  $(A, \delta)$  is differential graded coassociative coalgebra whose coproduct has degree  $m$ ,
- (3)  $\delta : A \rightarrow A \otimes A$  is a right and left  $A$ -module map which using (simplified) Sweedler' notation reads

$$\sum_{(x,y)} (x.y)' \otimes (xy)'' = \sum_{(y)} x.y' \otimes y'' = \sum_{(x)} (-1)^{m|x|} x' \otimes x''.y$$

Here we have simplified Sweedler's notation for the coproduct  $\delta x = \sum_i x'_i \otimes x''_i$ , to  $\delta x = \sum_{(x)} x' \otimes x''$  where  $(x)$  should be thought of as the index set for  $i$ 's. We shall say  $(A, \cdot, \delta)$  is symmetric if  $\sum_{(1)} 1' \otimes 1'' = \sum_{(1)} 1'' \otimes 1'$ .

We recall that an ordinary (DG) Frobenius algebra, sometimes called *closed Frobenius (DG) algebra*, is a unital associative differential graded algebra equipped with a *nondegenerate symmetric inner product*  $\langle -, - \rangle$  which is *invariant i.e*

$$\langle xy, z \rangle = \langle x, yz \rangle.$$

In particular the inner product allows us to identify  $A$  with its dual  $A^\vee = \text{Hom}_{\mathbf{k}}(A, \mathbf{k})$  (as right and left  $A$ -modules) and defines a coproduct on  $A$  by

$$A \simeq A^\vee \xrightarrow{\text{dual of product}} (A \otimes A)^\vee \simeq A^\vee \otimes A^\vee \simeq A \otimes A.$$

The coproduct is coassociative and satisfies condition (3) of the definition above, in other words a closed Frobenius algebra is also an open Frobenius algebra. Moreover,  $\eta : A \rightarrow \mathbf{k}$  defined by  $\eta(x) = \langle x|1 \rangle$  is a counit (trace), that is

$$(2.1) \quad \sum_{(x)} \eta(x')x'' = x.$$

As a consequence we have the following identities

$$(2.2) \quad \sum_{(x)} \eta(x'y)x'' = \sum_{(x)} (-1)^{|x'|m} \eta(x''y)x' = xy.$$

**Example:** An important example of symmetric open Frobenius algebra is the cohomology with compact support  $H_c^*(M)$  of an oriented  $n$ -dimensional manifold  $M$  (not necessarily closed). Note that  $H_c^*(M)$  is naturally equipped with the usual cup product and using the Poincaré duality (see [Hat02] Theorem 3.35),

$$H_c^*(M) \simeq H_{*-n}(M)$$

one can transfer the natural coproduct  $\Delta$  on  $H_*(M)$  to  $H_c^*(M)$ . Then  $(H_c^*(M), \cup, \Delta)$  is a symmetric open Frobenius algebra whose differential is identically zero. The Frobenius compatibility condition is satisfied because the Poincaré duality isomorphism above is a map of  $H_c^*(M)$ -modules. Over rationals,  $H^*(M) \simeq \text{Hom}(H_c^*(M), \mathbb{Q})$  thus  $H^*(M)$  is an open Frobenius algebra as well. Therefore lifting this open Frobenius algebra structure to the cochain level and a modification of our work to a homotopic setting, would provide a new algebraic model for the string topology operation on  $H_*(LM)$ . Some works in this direction have been done by Wilson [Wil] and Chen [Che12].

If  $M$  is a closed then  $H^*(M) = H_c^*(M)$  is indeed a closed Frobenius manifold since  $H^*(M)$  has a counit given by  $\int : H^*(M) \rightarrow \mathbb{Z}$  be evaluation on the fundamental class of  $M$  and the Poincaré duality is given by capping with the fundamental class. The non-degenerate inner product is defined by  $\langle x, y \rangle := \int_{[M]} x \cup y$ . Over the rationals it is possible to lift this Frobenius algebra structure to the cochains level. By a result of Lambrechts and Stanley [LS08], there is a connected finite dimensional commutative DG algebra  $A$  which is quasi-isomorphic to  $C^*(M)$  the cochains of a given  $n$ -dimensional manifold  $M$  and is equipped with a bimodule isomorphism  $A \rightarrow A^\vee$  inducing the Poincaré duality  $H^*(M) \rightarrow H_{*-n}(M)$ .

**Proposition 2.2.** (1) *A closed Frobenius algebra is finite dimensional and symmetric.*

- (2) An open Frobenius algebra with a counit is indeed a closed Frobenius algebra, in particular it is finite dimensional.
- (3) A symmetric and commutative open Frobenius algebra is cocommutative.
- (4) For any  $z$  in a symmetric open Frobenius  $A$ ,  $\sum_{(z)} z''z'$  belongs to the center of  $A$ .

*Proof.* 1) The finite dimensionality follows from the identity  $x = \sum_{(1)} 1'\langle x|1''\rangle$  where  $\delta(1) = \sum_{(1)} 1' \otimes 1''$ . Being symmetric is a consequence of the inner product being symmetric. We have

$$(2.3) \quad \langle y, z \rangle = \langle 1, yz \rangle = \sum \langle 1', y \rangle \langle 1'', z \rangle,$$

left hand side being symmetric in terms of  $y$  and  $z$  implies that  $\sum_{(1)} 1' \otimes 1'' = \sum_{(1)} 1'' \otimes 1'$ .

2) The inner product can be defined using the counit by setting  $\langle x, y \rangle = \eta(xy)$ . It is clearly invariant and the nondegeneracy follows from the identity (2.2) for  $x = 1$ : If  $\langle y, z \rangle = 0$  for all  $z$ , then

$$y = \sum_{(x)} (-1)^{|x'|m} \eta(1''y) 1' = 0.$$

- 3)  $\sum x' \otimes x'' = \sum x1' \otimes 1'' = \sum x1'' \otimes 1' = \sum 1''x \otimes 1' = \sum x'' \otimes x'$ .
- 4)

$$\begin{aligned} x \sum_{(z)} z''z' &= \sum_{(z)} xz''z' = \sum_{(1)} x1''z1' = \sum_{(1)} x1'z1'' = \sum_{(x)} x'zx'' = \sum_{(1)} 1'z1''x \\ &= \sum_{(1)} 1''z1'x = \sum_{(z)} z''z'x. \end{aligned}$$

□

Let us explain how the Frobenius algebras give rise to BV-algebra, but before we recall the definition of the BV-algebras and the definition of Hochschild homology and cohomology.

**Definition 2.3.** (Batalin-Vilkovisky algebra) A BV-algebra is a Gerstenhaber algebra  $(V^*, \cdot, [-, -])$  with a degree one operator  $\Delta : V^* \rightarrow V^{*+1}$  whose deviation from being a derivation for the product  $\cdot$  is the bracket  $[-, -]$ , *i.e.*

$$[a, b] := (-1)^{|a|} \Delta(ab) - (-1)^{|a|} \Delta(a)b - a\Delta(b),$$

and  $\Delta^2 = 0$ .

There are very interesting examples using the differential forms of Riemannian or symplectic manifolds, which are essentially due to Kozsul [Kos85]. The inspiring example for us is the homology of the free loop space  $LM := C^0(S^1, M)$  of an oriented manifold [CS] for which an algebraic model can be obtained using Hochschild cohomology of cochains algebras of  $M$  [Jon87]. Let us recall the definition of the Hochschild complex.

A differential graded  $(A, d)$ -module, or  $A$ -module for short, is a  $\mathbf{k}$ -complex  $(M, d)$  together with an (left)  $A$ -module structure  $\cdot : A \times M \rightarrow M$  such that  $d_M(am) = d_A(a)m + (-1)^{|a|} ad_M(m)$ . The multiplication map is of degree zero *i.e.*  $\deg(am) = \deg a + \deg m$ . In particular, the identity above implies that the differential of  $M$  has to be of degree 1.

Similarly for a  $(M, d_M)$  a graded differential  $(A, d)$  – *bimodule*, we have

$$d_M(amb) = d_A(a)mb - (-1)^{|a|}ad_M(m)b + (-1)^{|a|+|m|}amd_Ab,$$

or equivalently,  $M$  is a  $(A^e := A \otimes A^{op}, d_A \otimes 1 + 1 \otimes d_A)$  DG-module where  $A^{op}$  is the algebra whose underlying graded vector space is  $A$  with the opposite multiplication of  $A$ , i.e.  $a \cdot^{op} b = (-1)^{|a| \cdot |b|} b \cdot a$ . From now on  $Mod(A)$  denotes the category of (left or right) (differential)  $A$ -modules and  $Mod(A^e)$  denotes the category of differential  $A$ -bimodules. All modules considered in this article are differential modules. We will also drop the indices from the differential when there is no possibility of confusion. The (*normalized*) *Hochschild chain complex* with coefficients in  $M$  is defined to be

$$(2.4) \quad C_*(A, M) := M \otimes T(s\bar{A})$$

and comes equipped with a degree +1 differential  $D = d_0 + d_1$ . We recall that  $TV = \bigoplus_{n \geq 0} V^{\otimes n}$  denotes the tensor algebra of a  $\mathbf{k}$ -module  $V$ .

The internal differential is given by

$$(2.5) \quad \begin{aligned} d_0(m[a_1, \dots, a_n]) &= \sum_{i=1}^{n-1} (-1)^{\epsilon_i} m[a_1, \dots, d_A a_i, \dots, a_n] \\ &\quad - (-1)^{\epsilon_n} d_M m[a_1, \dots, a_n], \end{aligned}$$

and the external differential is

$$(2.6) \quad \begin{aligned} d_1(m[a_1, \dots, a_n]) &= ma_1[a_2, \dots, a_n] + \\ &\quad \sum_{i=1}^{n-1} (-1)^{\epsilon_i} m[a_1, \dots, a_i a_{i+1}, \dots, a_n] \\ &\quad - (-1)^{\epsilon_n} a_n m[a_1, \dots, a_{n-1}], \end{aligned}$$

with  $\epsilon_i = |a_1| + \dots + |a_i| - i$ . Note that the degree of  $m[a_1, \dots, a_n]$  is  $\sum_{i=1}^n |a_i| - n + |m|$ .

When  $M = A$ , by definition  $(C_*(A), D = d_0 + d_1) := (C_*(A, A), D = d_0 + d_1)$  is the *Hochschild chain complex of  $A$*  and  $HH_*(A, A) := \ker D / \text{im } D$  is the Hochschild cohomology of  $A$ .

Similarly we define the  $M$ -valued *Hochschild cochain* of  $A$  to be

$$C^*(A, M) := \text{Hom}_{\mathbf{k}}(T(s\bar{A}), M).$$

For a homogenous cochain complex  $f \in C^n(A, M)$ , the degree  $|f|$  is defined to be the degree of the linear map  $f : (s\bar{A})^{\otimes n} \rightarrow M$ . In the case of Hochschild cochains, the external differential of  $f \in \text{Hom}(s\bar{A}^{\otimes n}, M)$  is

$$(2.7) \quad \begin{aligned} d_1(f)(a_1, \dots, a_n) &= -(-1)^{(|a_1|+1)|f|} a_1 f(a_2, \dots, a_n) + \\ &\quad - \sum_{i=2}^n (-1)^{\epsilon_i} f(a_1, \dots, a_{i-1} a_i, \dots, a_n) + (-1)^{\epsilon_n} f(a_1, \dots, a_{n-1}) a_n, \end{aligned}$$

where  $\epsilon_i = |f| + |a_1| + \dots + |a_{i-1}| - i + 1$ . The internal differential of  $f \in C^*(A, M)$  is

$$(2.8) \quad d_0 f(a_1, \dots, a_n) = d_M f(a_1, \dots, a_n) - \sum_{i=1}^n (-1)^{\epsilon_i} f(a_1, \dots, d_A a_i, \dots, a_n).$$

When  $M = A^\vee$  the  $\mathbf{k}$ -dual of  $A$ , by definition  $(C^*(A), D = d_0 + d_1) := (C_*(A, A^\vee), D = d_0 + d_1)$  is the *Hochschild cochain complex* of  $A$  and  $HH^*(A, A) := \ker D / \text{im } D$  is the Hochschild cohomology of  $A$ . Note that  $C^*(A)$  is  $\mathbf{k}$ -dual of  $C_*(A)$ .

*Gerstenhaber bracket and cup product:* When  $M = A$ , for  $x \in C^m(A, A)$  and  $y \in C^n(A, A)$  one defines the *cup product*  $x \cup y \in C^{m+n}(A, A)$  and the *Gerstenhaber bracket*  $[x, y] \in C^{m+n-1}(A, A)$  by

$$(2.9) \quad (x \cup y)(a_1, \dots, a_{m+n}) := (-1)^{|y|(\sum_{i \leq m} |a_i| + 1)} x(a_1, \dots, a_m) y(a_{m+1}, \dots, a_{m+n}),$$

and

$$(2.10) \quad [x, y] := x \circ y - (-1)^{(|x|+1)(|y|+1)} y \circ x,$$

where

$$(x \circ_j y)(a_1, \dots, a_{m+n-1}) = (-1)^{(|y|+1) \sum_{i \leq j} (|a_i| + 1)} x(a_1, \dots, a_j, y(a_{j+1}, \dots, a_{j+m}), \dots).$$

and

$$(2.11) \quad x \circ y = \sum_j x \circ_j y$$

Note that this is not an associative product. It turns out that the operations  $\cup$  and  $[-, -]$  are chain maps, hence they define two well-defined operations on  $HH^*(A, A)$ . Moreover,

**Theorem 2.4.** (*Gerstenhaber* [Ger63]) *Let  $A$  be a differential graded associative algebra.  $(HH^*(A, A), \cup, [-, -])$  is a Gerstenhaber algebra that is:*

- (1)  $\cup$  is an associative and graded commutative product,
- (2)  $[x, y \cup z] = [x, y] \cup z + (-1)^{(|x|-1)|y|} y \cup [x, z]$  (*Leibniz rule*),
- (3)  $[x, y] = (-1)^{(|x|-1)(|y|-1)} [y, x]$ ,
- (4)  $[[x, y], z] = [x, [y, z]] - (-1)^{(|x|-1)(|y|-1)} [y, [x, z]]$  (*Jacobi identity*).

The homotopy for the commutativity of the cup product  $x \cup y$  is given by  $x \circ y$ .

The following theorem how a Frobenius algebra gives rise to a BV-algebra.

**Theorem 2.5.** (*Tradler* [Tra08]) *The Hochschild cohomology  $HH^*(A, A)$  of a Frobenius algebra  $A$  is naturally a BV-algebra whose underlying Gerstenhaber structure is the standard one [Ger63]. The BV-operator corresponds to Connes' operator  $B^\vee$  using the natural isomorphism  $HH^*(A, A^\vee) \simeq HH^*(A, A)$ . Dually, the Hochschild homology  $HH_*(A, A)$  is a coBV-algebra.*

The main idea here is that we try to identify the (co)BV-structures directly on  $HH_*(A, A)$  and its dual theory  $HH^*(A, A^\vee)$  rather than  $HH^*(A, A)$ .

### 3. COBV STRUCTURE ON HOCHSCHILD HOMOLOGY

In this section we present a natural coBV-structure on the Hochschild homology of a symmetric open Frobenius algebra. The natural candidate for the coBV operator on  $HH_*(A, A)$  the Connes operator  $B$ , so we just need a coproduct on the Hochschild chains  $C_*(A, A)$ . This is given by

$$(3.1) \quad \theta(a_0[a_1, \dots, a_n]) = \sum_{(a_0), 0 \leq i \leq n} +(a'_0[a_1, \dots, a_{i-1}, a_i]) \otimes (a''_0[a_{i+1}, \dots, a_n])$$

This gives rise to a product for Hochschild cochains as follows: For  $f, g \in C^*(A, A^\vee) = \text{Hom}(A \otimes T(s\bar{A}), \mathbf{k})$  we set

$$(f * g)(x) := \mu(f \otimes g)\theta(x)$$

where  $\mu : \mathbf{k} \otimes \mathbf{k} \rightarrow \mathbf{k}$  is the multiplication. More explicitly

$$(f * g)(a_0[a_1, \dots, a_n]) = \sum_{(a_0), 0 \leq i < n} +f(a'_0[a_1, \dots, a_{i-1}, a_i])g(a''_0[a_{i+1}, \dots, a_n]).$$

In the case of a closed Froebnius algebra this product corresponds to the standard cup product on  $HH^*(A, A)$  using the isomorphism

$$HH^*(A, A) \simeq HH^*(A, A^\vee)$$

induced by the inner product on  $A$ . More explicitly we identify  $A$  with  $A^\vee$  using the map  $a \mapsto (a^\vee(x) := \langle a, x \rangle)$ . Therefore to a cochain  $f \in C^*(A, A)$ ,  $f : A^{\otimes n} \rightarrow A$ , corresponds a cochain  $\tilde{f} \in C^*(A) = C^*(A, A^\vee)$  given by  $\tilde{f} \in \text{Hom}(A^{\otimes n}, A^\vee) \simeq \text{Hom}(A^{\otimes(n+1)}, \mathbf{k})$ ,

$$\tilde{f}(a_0, a_1, \dots, a_n) := \langle f(a_1, \dots, a_n), a_0 \rangle.$$

The inverse of this isomorphism is given by

$$f(a_1, \dots, a_n) := \sum_{(1)} \tilde{f}(1', a_1, \dots, a_n)1''.$$

For two cochain  $f : A^{\otimes p} \rightarrow A$  and  $g : A^{\otimes q} \rightarrow A$  in  $C^*(A, A)$  we have

$$\begin{aligned} \widetilde{f \cup g}(a_0, a_1, \dots, a_{p+q}) &= \langle (f \cup g)(a_1, \dots, a_{p+q}), a_0 \rangle = \langle f(a_1, \dots, a_p)g(a_{p+1}, \dots, a_{p+q}), a_0 \rangle \\ &= \langle f(a_1, \dots, a_p), g(a_{p+1}, \dots, a_{p+q})a_0 \rangle \\ &= \sum_{(a_0)} \langle f(a_1, \dots, a_p), \langle g(a_{p+1}, \dots, a_{p+q}), a'_0 \rangle a'_0 \rangle \\ &= \sum_{(a_0)} \langle f(a_1, \dots, a_p), \langle g(a_{p+1}, \dots, a_{p+q}), a''_0 \rangle a'_0 \rangle \\ &= \sum_{(a_0)} \langle f(a_1, \dots, a_p), a'_0 \rangle \langle g(a_{p+1}, \dots, a_{p+q}), a''_0 \rangle \\ &= (\tilde{f} * \tilde{g})(a_0[a_1, \dots, a_{p+q}]) \end{aligned}$$

**Theorem 3.1.** *For a symmetric open Frobenius algebra  $A$ ,  $(HH_*(A, A), \theta, B)$  is a coBV-algebra. As a consequence  $(HH^*(A, A^\vee), \theta^\vee, B^\vee)$  is a BV-algebra. In particular if  $A$  is closed Frobenius algebra, then using the natural isomorphism*

$$(3.2) \quad (CC^*(A, A^\vee), \theta^\vee) \simeq (CC^*(A, A), \cup),$$

*$(HH^*(A, A), \cup)$  is naturally a BV-algebra whose BV-operator is image of the Connes operators  $B$  under the isomorphism (3.2). This recovers Tradler's [Tra08], Theorem 2.5 result for closed Frobenius algebras.*



*Proof.* To prove the theorem we show that  $(C_*(A, A), \theta, B)$  is a homotopy coBV coalgebra. It is a direct check that  $\theta$  is co-associative. Just like the computation above, we transfer the homotopy for commutativity of the cup product to  $C^*(A, A^\vee)$  and then dualize it. It turns out that the obtained formula only depends on the product and coproduct, so it makes also sense for open Frobenius algebras. The homotopy for co-commutativity is given by

$$(3.3) \quad h(a_0[a_1, \dots, a_n]) := \sum_{(1), 0 \leq i < j \leq n+1} (a_0[a_1, \dots, a_i, 1'', a_j, \dots, a_n]) \otimes (1'[a_{i+1}, \dots, a_{j-1}]).$$

where for  $j = n + 1$  and  $i = 0$  the correspondings terms are respectively

$$(a_0[a_1, \dots, a_i, 1'']) \otimes (1'[a_{i+1}, \dots, a_n]).$$

and

$$(a_0[1'', a_j, \dots, a_n]) \otimes (1'[a_1, \dots, a_{j-1}]).$$

It is a direct check that  $hd - (d \otimes 1 + 1 \otimes d)h = \theta - \tau \circ \theta$  where  $\tau : C_*(A) \otimes C_*(A) \rightarrow C_*(A) \otimes C_*(A)$  is given by  $\tau(\alpha_1 \otimes \alpha_2) = +\alpha_2 \otimes \alpha_1$ .

To prove that the 7-term (coBV) relation holds, we use the Chas-Sullivan [CS] idea (see also [Tra08]) in the case of the free loop space adapted to the combinatorial (simplicial) situation. First we identify the Gerstenhaber co-bracket explicitly. Let

$$S := h - \tau \circ h$$

Once proven  $S$  is, up to homotopy, the deviation of  $B$  from being a coderivation for  $\theta$ , the 7-term homotopy coBV relation is equivalent to the homotopy co-Leibniz identity for  $S$ .

*Co-Leibniz identity:* The idea of the proof is identical to Lemma 4.6 [CS]. We prove that up to some homotopy we have

$$(3.4) \quad (\theta \otimes id)S = (id \otimes \tau)(S \otimes id)\theta + (id \otimes S)\theta$$

It is a direct check that

$$(id \otimes \tau)(h \otimes id)\theta + (id \otimes h)\theta = (\theta \otimes id)h,$$

so to prove (3.4) we should prove that up to some homotopy

$$(3.5) \quad (id \otimes \tau)(\tau h \otimes id)\theta + (id \otimes \tau h)\theta = (\theta \otimes id)\tau h.$$

The homotopy is given by  $H : C^*(A) \rightarrow (C^*(A))^{\otimes 3}$

$$H(a_0[a_1, \dots, a_n]) = \sum_{0 \leq l < i \leq j < k} \sum_{(1), (1)} (1''[a_l, \dots, a_{i-1}]) \otimes (1''[a_j, \dots, a_{k-1}]) \otimes a_0[a_1, \dots, a_l, 1', a_i \cdots a_{j-1}, 1', a_k, \dots, a_n].$$

Note that in the sum above the sequence  $a_i \cdots a_{j-1}$  can be empty. The identity

$$(d \otimes id \otimes id + id \otimes d \otimes id + id \otimes id \otimes d)H - Hd = (id \otimes \tau)(\tau h \otimes id)\theta + (id \otimes \tau h)\theta - (\theta \otimes id)\tau h$$

can be checked directly.

*Compatibility of B and S:* The final step is to prove that  $S = \theta B + (B \otimes id + id \otimes B)\theta$  up to homotopy. To that end we prove that  $h$  is homotopic to  $(\theta B)_2 - (B \otimes id)\theta$  and similarly  $\tau h \simeq (\theta B)_1 - (id \otimes B)\theta$  where  $\theta B = (\theta B)_1 + (\theta B)_2$ , with

$$(\theta B)_1(a_0[a_1, \dots, a_n]) = \sum_{0 \leq i \leq j \leq n} \sum_{(1)} (1'[a_i, \dots, a_j]) \otimes (1''[a_{j+1}, \dots, a_n, a_0, \dots, a_{i-1}]).$$

and

$$(\theta B)_2(a_0[a_1, \dots, a_n]) = \sum_{0 < j < i \leq n} \sum_{(1)} (1'[a_i, \dots, a_n, a_0, a_1, \dots, a_j]) \otimes (1''[a_{j+1}, \dots, a_{i-1}]).$$

It can be easily checked that

$$H(a_0[a_1, \dots, a_n]) = \sum_{0 \leq k < j \leq i \leq n} \sum_{(a_i)} (1[a_i, \dots, a_n, a_0, a_1, \dots, a_k, 1', a_j, \dots, a_{i-1}]) \otimes (1''[a_{k+1}, \dots, a_{j-1}]),$$

is a homotopy between  $h$  and  $(\theta B)_2 - (B \otimes id)\theta$ . In the formulae describing  $H$ , the sequence  $a_j, \dots, a_{i-1}$  can be empty.

While computing  $dH$  we encounter, the terms corresponding to  $k = 0$  is exactly

$$(B \otimes 1)\theta(a_0[a_1, \dots, a_n]) = \sum (1[a_j, \dots, a_i, a'_0, a_1, \dots, a_{j-1}]) \otimes (a''_0[a_{i+1}, \dots, a_n]).$$

Similarly one proves that  $\tau h \simeq (\theta B)_1 - (id \otimes B)\theta$ .  $\square$

**Remark 3.2.** By Félix-Thomas [Fél] theorem, this cup product on  $HH^*(A, A^\vee)$  provides an algebraic model for the Chas-Sullivan product on  $H_*(LM)$  the homology of the free loop space of closed oriented manifold  $M$ . Here one must work over a field of characteristic zero and for  $A$  one can take the closed (commutative) Frobenius algebra provided by Lambrechts-Stanley result [LS08] on the existence of an algebraic model with Poincaré duality for the cochains of a closed oriented manifold.

#### 4. BV STRUCTURE ON HOCHSCHILD HOMOLOGY

Although there is no action of the chains of the moduli space of Riemann surfaces on the Hochschild complex of an open Frobenius algebra, some parts of such action in the case of (closed) Frobenius algebras can be formulated using the product and coproduct of the underlying algebra (see also [WW]). Therefore one can write down various operations in the Hochschild homology of an open Frobenius algebra, but the desired identities have to be proved directly by giving explicitly the required homotopies. Here is one example.

**Theorem 4.1.** *For  $A$  a symmetric open Frobenius algebra,  $HH_*(A, A)$  can be naturally equipped with a BV-structure whose BV-operator is Connes' operator and the product at the chain level is given by*

$$(4.1) \quad a_0[a_1, \dots, a_n] \circ b_0[b_1, \dots, b_m] = \begin{cases} 0 & \text{if } n > 0 \\ a''_0 a'_0 b_0[b_1, \dots, b_m] & \end{cases}$$

*Proof.* Using Proposition 2.2 (4), it is easily checked that  $\circ$  is a chain map. The product  $\circ$  is strictly associative. We only have to check this for  $x = a[ \ ]$ ,  $y = b[ \ ]$  and  $z = c[c_1, \dots, c_n]$ . We have

$$(x \circ y) \circ z = \sum (a''_0 a'_0 b_0)'' (a''_0 a'_0 b_0)' c_0 [c_1, \dots, c_n] = \sum b''_0 a''_0 a'_0 b'_0 c_0 [c_1, \dots, c_n]$$

which by Proposition 2.2 (4) is equal to

$$\sum a''_0 a'_0 b''_0 b'_0 c_0 [c_1, \dots, c_n] = x \circ (y \circ z).$$

Next we prove that the product is commutative up to homotopy. Indeed the homotopy for  $x = a_0[a_1, \dots, a_m]$  and  $y = b_0[b_1, \dots, b_n]$  is given by

$$(4.2) \quad K(x, y) = \begin{cases} \sum_{(a)} a'_0 b_0 [b_1, \dots, b_n, a''_0] & \text{if } m = 0 \quad \& \quad n \geq 0 \\ \sum_{(a)} b''_0 [b'_0 a_0, a_1, \dots, a_m, ] & \text{if } m \geq 0 \quad \& \quad n = 0 \\ 0 & \text{otherwise} \end{cases}$$

The proof of this fact relies on the following identities,

$$(4.3) \quad \begin{aligned} \sum a'_0 b_0 [b_1, \dots, b_n, a''_0] &= \sum a_0 1' b_0 [b_1, \dots, b_n, 1''] = \sum a_0 1'' b_0 [b_1, \dots, b_n, 1'] \\ &= \sum a_0 b''_n b_0 [b_1, \dots, b'_n] = \sum a_0 1'' b_n b_0 [b_1, \dots, 1'] \\ &= \sum a_0 1' b_n b_0 [b_1, \dots, 1''] = \sum a'_0 b_n b_0 [b_1, \dots, a''_0]. \end{aligned}$$

In the particular case of  $x = a[ \ ]$  and  $y = b[ \ ]$  and for the external differential we have

$$d_1 K(x, y) = \sum (a' b a'' - a'' a' b) = \sum a 1' b 1'' - x \circ y = \sum a 1'' b 1' - x \circ y = \sum a b'' b' - x \circ y$$

which by Proposition 2.2 (4) is,

$$\sum b'' b' a - x \circ y = y \circ x - x \circ y.$$

Since we have also  $d_0 K(x, y) = K(dx, y) + K(x, dy)$ , therefore

$$dK(x, y) = K(dx, y) + K(x, dy) + y \circ x - x \circ y.$$

The Gerstenhaber bracket is naturally defined to be

$$(4.4) \quad \{x, y\} := K(x, y) - K(y, x).$$

Next we prove that the identity

$$(4.5) \quad \{x, y\} = \Delta(x \circ y) + \Delta x \circ y + x \circ \Delta y$$

holds up to homotopy. If  $n > 0$  and  $m > 0$  then all the involved are homotopically zero and there is nothing to prove. We prove the claim for the case of  $m = 0$  and  $n > 0$ , the other case is similar. Since  $\Delta x \circ y$  and  $x \circ \Delta y$  are respectively zero and homotopique to zero, we only have to show that  $\Delta(x \circ y)$  is homotopic to  $\{x, y\}$ . We have

$$\Delta(x \circ y) = \sum_{i=1}^n \sum_{(a)} 1 [b_i, \dots, b_n, a''_0 a'_0 b_0, b_1, \dots, b_{i-1}] + 1 [a''_0 a'_0 b_0, b_1, \dots, b_n]$$

and

$$\{x, y\} = \sum_{(a)} a'_0 b_0 [b_1, \dots, b_n, a''_0] + \sum_{(a)} a''_0 [a'_0 b_0, b_1, \dots, b_n].$$

The desired homotopy is given by

$$H(x, y) := \sum_i \sum_{(a)} 1[b_i \cdots, a_0'', a_0' b_0, b_1, \cdots, b_{i-1}].$$

Finally we prove the Leibniz identity

$$\{x \circ y, z\} = x \circ \{y, z\} + \{x, z\} \circ z$$

holds up to homotopy. Note that the  $\{y, z\}$  is either zero or it belongs to  $\oplus_{n>1} A^{\otimes n}$  therefore the right hand side is always zero, so we need to prove that the left hand side is always homotopic to zero which is clear if one of  $x$  or  $y$  belongs to  $\oplus_{n>1} A^{\otimes n}$ . Using the identity (up to homotopy) (4.5) we know that  $\{x \circ y, z\}$  is homotopic to  $\Delta(x \circ y \circ z)$  which is homotopic to zero if  $z \in \oplus_{n>1} A^{\otimes n}$ . So we may assume that  $x = a[\ ]$  and  $y = b[\ ]$  and  $z = c[\ ]$ , and we have

$$\Delta(x \circ y \circ z) = \Delta((a''ab)''(a''a'b')c) = \Delta(b''a''a'b'c).$$

Since  $a''a'$  is central, we get  $\Delta(x \circ y \circ z) = 1[a''a'b''b'c]$  which is homotopic to zero. The homotopy is given by  $H(a, b, c) = 1[a''a', b''b'c]$  whose boundary is

$$a''a'[b''b'c] + 1[a''a'b''b'c] + b''b'c[a''a'].$$

In fact  $a''a'[b''b'c] = b''b'c[a''a']$  (up to signs) because

$$\begin{aligned} \sum a''a'[b''b'c] &= \sum 1''a1'[1''b1'c] = \sum 1'a1'[1''b1''c] = \sum c'ab'[b''c''] \\ &= \sum c1'a1'[1''b1''] = \sum c1'ab'[b''1''] = \sum c1'ab1'[1''1''] \\ &= \sum c1''ab1'[1''1'] = \sum ca''b1'[1''a'] = \sum c1''b1'[1''a1'] \\ &= \sum cb''b'[a''a'] = \sum b''bc'[a''a'].. \end{aligned}$$

□

**Corollary 4.2.** *For a cocommutative open Frobenius algebra the Gerstenhaber bracket defined by (4.4) on  $HH_*(A, A)$  is identically zero. In particular for a commutative symmetric open Frobenius algebra the Gerstenhaber bracket defined by (4.4) vanishes.*

*Proof.* We know that at the homology level  $\{x, y\} = \Delta(x \circ y)$ . The latter is zero unless  $x = a[\ ]$  and  $y = b[\ ]$  for some  $a$  and  $b$  in  $A$ , in which case

$$\Delta(x \circ y) = 1[a''a'b] = d(1[a'', a'b]).$$

Note that the two terms  $a''[a'b]$  and  $a'b[a'']$  in  $d(1[a'', a'b])$  cancel each other since

$$a'b[a''] = a''b[a'] = (ab)''[(ab)'] = (ab)'[(ab)'] = a'[a''b] = a''[a'b].$$

□

## 5. FROBENIUS COMPATIBILITY OF THE PRODUCT AND COPRODUCT

As we mentioned previously we are inspired by the algebraic structures of the homology of free loop spaces. Cohen-Godin [CG04] result holds even for the manifolds which are not closed. The only difference with the closed case would be that there won't be a counit for the underlying algebra structure. The coalgebra structure generically has no counit, otherwise the homology of the free loop space would have the homotopy type of a finite dimensional manifold which is true for very

special kind of aspherical manifolds. Therefore it is natural to expect a Frobenius compatibility condition (Definition 2.1, (3)) between the product and coproduct.

**Theorem 5.1.** *Let  $A$  be a symmetric open Frobenius algebra. The product (4.1) and coproduct (3.1) satisfy the Frobenius compatibility condition, Definition 2.1, (3).*

*Proof.* We only have to prove that  $\theta$  is a map of left and right  $HH_*(A)$ -modules. First we consider the right  $HH_*(A)$ -module structure. Let  $x = a_0[a_1, \dots, a_m]$  and  $y = b_0[b_1, \dots, b_n] \in C_*(A, A)$ . If  $m \geq 1$ , then  $\theta(x \circ y) = 0$ , and

$$\begin{aligned}
\theta x \cdot y &= \left( \sum_{(a_0), 1 \leq k \leq m} a'_0[a_1, \dots, a_k] \otimes a''_0[a_{k+1}, \dots, a_m] \right) \cdot y \\
&= \left( \sum_{(a_0), 1 \leq k \leq m} a'_0[a_1, \dots, a_k] \otimes (a''_0[a_{k+1}, \dots, a_m] \circ y) \right) \\
(5.1) \quad &= \sum_{(a_0), (a''_0)} a'_0[a_1, \dots, a_m] \otimes (a''_0)''(a''_0)'b_0[b_1, \dots, b_n] \\
&= \sum_{(a_0), (1)} a'_0[a_1, \dots, a_m] \otimes 1''a'_01'b_0[b_1, \dots, b_n].
\end{aligned}$$

So we have to prove the latter is homotopic to zero, in fact the homotopy is given by

$$\begin{aligned}
H(x, y) &= \sum_{(a_0)} a'_0[a_1, \dots, a_m] \otimes a''_01'b_0[b_1, \dots, b_n, 1''] \\
(5.2) \quad &= \sum_{(1), (1)} 1'[a_1, \dots, a_m] \otimes 1''a_01'b_0[b_1, \dots, b_n, 1''] \\
&= \sum_{(1), (1)} 1''[a_1, \dots, a_m] \otimes 1'a_01'b_0[b_1, \dots, b_n, 1''].
\end{aligned}$$

If  $m = 0$ , then

$$\begin{aligned}
\theta(x \circ y) &= \theta\left(\sum_{(a_0)} a''_0a'_0b_0[b_1, \dots, b_n]\right) \\
&= \sum_{(a_0), (a''_0a'_0b_0)} (a''_0a'_0b_0)'[b_1, \dots, b_k] \otimes (a''_0a'_0b_0)''[b_{k+1}, \dots, b_n] \\
&= \sum_{(a_0), (1)} 1'[b_1, \dots, b_k] \otimes 1''a''_0a'_0b_0[b_{k+1}, \dots, b_n] \\
(5.3) \quad &= \sum_{(1), (1)} 1'[b_1, \dots, b_k] \otimes 1''1''a_01'b_0[b_{k+1}, \dots, b_n] \\
&= \sum_{(1), (1)} a_01'[b_1, \dots, b_k] \otimes 1''1''1'b_0[b_{k+1}, \dots, b_n] \\
&= \sum_{(1), (1)} a_0b_01'[b_1, \dots, b_k] \otimes 1''1''1''[b_{k+1}, \dots, b_n].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\theta(x) \circ y &= \sum_{(a_0)} a'_0[ ] \otimes (a''_0[ ] \circ y) \\
&= \sum_{(a_0), (a'_0)} a'_0[ ] \otimes (a''_0)'' (a''_0)' b_0[b_1, \dots, b_n] \\
(5.4) \quad &= \sum_{(a_0), (1)} a'_0[ ] \otimes 1'' a''_0 1' b_0[b_1, \dots, b_n] \\
&= \sum_{(1), (1)} a_0 1' [ ] \otimes 1'' 1' 1'' b_0[b_1, \dots, b_n] \\
&= \sum_{(1), (1)} a_0 b_0 1' [ ] \otimes 1'' 1' 1'' [b_1, \dots, b_n].
\end{aligned}$$

So it remains to prove that

$$(5.5) \quad \sum_{\substack{(1), (1) \\ 1 \leq k \leq n}} a_0 b_0 1' [b_1, \dots, b_k] \otimes 1'' 1' 1'' [b_{k+1}, \dots, b_n],$$

is homotopic to zero. Here is the desired homotopy:

$$\begin{aligned}
(5.6) \quad K(x, y) &:= \sum_{\substack{(1), (1) \\ 1 \leq k \leq n}} a_0 b_0 1' [b_1, \dots, b_k] \otimes 1' 1'' [b_{k+1}, \dots, b_n, 1''] \\
&+ \sum_{(1), (1)} a_0 b_0 1' [b_1, \dots, b_n, 1''] \otimes 1' 1'' [ ],
\end{aligned}$$

if  $x = a[ ]$  and  $y = b_0[b_1, \dots, b_n]$ ; and  $K(x, y) := 0$  if  $x = a_0[a_1, \dots, a_m]$  with  $m > 0$ .

The left  $HH_*(A)$ -module structure is simpler to prove. The only non trivial case is when  $m = 0$ , otherwise both  $x \cdot \theta(y)$  and  $\theta(x \circ y)$  are zero. For  $x = a_0[ ]$  we have,

$$\begin{aligned}
(5.7) \quad x \cdot \theta(y) &= \sum_{\substack{(b_0) \\ 0 \leq k \leq n}} (x \circ b'_0[b_1, \dots, b_k]) \otimes b''_0[b_{k+1}, \dots, b_n] \\
&= \sum_{\substack{(b_0), (a_0) \\ 0 \leq k \leq n}} a''_0 a'_0 b'_0 [b_1, \dots, b_k] \otimes b''_0 [b_{k+1}, \dots, b_n].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(5.8) \quad \theta(x \circ y) &= \theta \left( \sum_{(a_0)} a''_0 a'_0 b_0 [b_1, \dots, b_n] \right) \\
&= \sum_{\substack{(a_0), (a'_0 a' b_0) \\ 0 \leq k \leq n}} (a''_0 a'_0 b_0)' [b_1, \dots, b_k] \otimes (a''_0 a'_0 b_0)'' [b_{k+1}, \dots, b_n] \\
&= \sum_{\substack{(a_0), (b_0) \\ 0 \leq k \leq n}} a''_0 a'_0 b'_0 [b_1, \dots, b_k] \otimes b''_0 [b_{k+1}, \dots, b_n].
\end{aligned}$$

□

## 6. BV STRUCTURE ON THE RELATIVE HOCHSCHILD HOMOLOGY OF COMMUTATIVE FROBENIUS ALGEBRAS

In this section we exhibit a BV structure on the relative Hochschild homology of a symmetric commutative open Frobenius algebra. In particular we introduce a product on the shifted relative Hochschild homology of symmetric commutative Frobenius algebras whose dual could be an algebraic model for Chas-Sullivan [CS04] / Goresky-Hingston [GH09] coproduct on  $H_*(LM, M)$ . One should mention that Chas-Sullivan worked with an equivariant version of this product on  $H_*^{S^1}(LM, M)$  which is a generalization of Turavev co-Lie bracket [Tur91].

For a commutative DG-algebra  $A$  the relative Hochschild chains are defined to be

$$(6.1) \quad \widetilde{CC}_*(A) = \bigoplus_{n \geq 1} A \otimes \bar{A}^{\otimes n}.$$

equipped with the Hochschild differential. Since  $A$  is commutative,  $\widetilde{CC}_*(A)$  is stable under the Hochschild differential and fits in the split short exact sequence of complexes,

$$(6.2) \quad 0 \longrightarrow (A, d_A) \longrightarrow CC_*(A, A) \longrightarrow \widetilde{CC}_*(A)$$

The homology of  $\widetilde{CC}_*(A)$  is denoted  $\widetilde{HH}_*(A)$  and is called the relative Hochschild homology of  $A$ .

**Theorem 6.1.** *The shifted relative Hochschild homology  $\widetilde{HH}_*(A)[1-m]$  of a degree  $m$  symmetric commutative open Frobenius algebra  $A$  is a BV algebra whose BV-operator is the Connes operator and the product is given by*

$$(6.3) \quad \begin{aligned} x \cdot y &= \sum_{(a_0 b_0)} +(a_0 b_0)'[a_1, \dots, a_m, (a_0 b_0)'', b_1, \dots, b_n] \\ &= \sum_{a_0} +a_0'[a_1, \dots, a_m, a_0'' b_0, b_1, \dots, b_n] \\ &= \sum_{b_0} +a_0 b_0'[a_1, \dots, a_m, b_0'', b_1, \dots, b_n] \end{aligned}$$

for  $x = a_0[a_1, \dots, a_m]$  and  $y = b_0[b_1, \dots, b_n] \in \widetilde{CC}_*(A)$ .

*Proof.* Note that the identities above hold because  $A$  is an open Frobenius algebra. The product defined above is a chain map and strictly associative, but commutative only up to homotopy, and the homotopy being given by

$$(6.4) \quad \begin{aligned} H_1(x, y) &= \sum_{(a_0 b_0)} +1[a_1, \dots, a_n, (a_0 b_0)', b_1, \dots, b_m, (a_0 b_0)''] \\ &\quad + \sum_{i=1}^n \sum_{(a_0 b_0)} +1[a_{i+1}, \dots, a_n, (a_0 b_0)', b_1, \dots, b_m, (a_0 b_0)'', a_1, \dots, a_i]. \end{aligned}$$

To prove that the 7-term relation holds, we adapt once again Chas-Sullivan's [CS] idea to a simplicial situation. First we identify the Gerstenhaber bracket directly. Let

$$(6.5) \quad x \circ y := \sum_{i=0}^m \sum_{(a_0)} b_0 [b_1, \dots, b_i, a'_0, a_1, \dots, a_n, a''_0, b_{i+1}, \dots, b_m],$$

and then define  $\{x, y\} := x \circ y + y \circ x$ . Next we prove that the bracket  $\{-, -\}$  is homotopic to the deviation of the BV-operator from being a derivation. For that we decompose  $\Delta(x \circ y)$  in two pieces:

$$B_1(x, y) := \sum_{j=1}^m \sum_{(a_0 b_0)} +1 [b_{j+1}, \dots, b_m, (a_0 b_0)', a_1, \dots, a_n, (a_0 b_0)'', b_1, \dots, b_j],$$

$$B_2(x, y) := \sum_{j=1}^m \sum_{(a_0 b_0)} +1 [a_{j+1}, \dots, a_n, (a_0 b_0)', b_1, \dots, b_m, (a_0 b_0)'', a_1, \dots, a_j],$$

so that  $B = B_1 + B_2$ . Then  $x \circ y$  is homotopic  $B_1(x, y) - x \cdot B y$ . In fact the homotopy is given by

$$H_2(x, y) = \sum_{0 \leq j \leq i \leq m} \sum_{(a_0)} 1 [b_{j+1}, \dots, b_i, (a_0)', a_1, \dots, a_n, (a_0)'', b_{i+1}, \dots, b_m, b_0, \dots, b_j].$$

Similarly for  $y \circ x$  and  $B_2(x \cdot y) - B x \cdot y$ . Therefore we have proved that on  $HH_*(A, A)$  the following identity holds:

$$\{x, y\} = B(x \cdot y) - B x \cdot y + x \cdot B y.$$

Now proving the 7-term relation is equivalent to prove the Leibniz rule for the bracket and the product, *i.e.*

$$\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\}.$$

It is a direct check that  $x \circ (y \cdot z) = (x \circ y) \cdot z + y \cdot (x \circ z)$ . On the other hand  $(y \cdot z) \circ x$  is homotopic to  $(y \circ x) \cdot z - y \cdot (z \circ x)$  using the homotopy

$$H_3(x, y, z) = \sum a_0 [a_1, \dots, a_i, b'_0, b_1, \dots, b_n, b''_0, a_{i+1}, \dots, a_j, c'_0, c_1, \dots, c_m, c''_0, a_{j+1}, \dots, a_p].$$

Here  $z = c_0 [c_1, \dots, c_p]$ . This proves that the Leibniz rule holds up to homotopy.  $\square$

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