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# Harmonic oscillator with power-law increasing time-dependent effective mass\*

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With the aid of the dynamical invariant method, we determine the exact eigenstates of the power-law increasing time-dependent effective mass harmonic oscillator. Duelling on the thermodynamic properties of the system we have determined its time-dependent specific heat, energy fluctuation and entropy. The effect on thermodynamic properties of considering a finite number of quantum states involved with evaluating the time-dependent partition function is analyzed. The measure of quantum decoherence and classical correlation is determined for the system.

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## I. INTRODUCTION

The study of quantum systems with time dependent mass has a wide range of applications in areas like plasma physics, gravitaion and quantum optics. The quantum mechanical time-dependent harmonic oscillator has been solved under various circumstances such as damping and a time-dependent mass or frequency or both Ref. [1 and 2]. This problem has been worked out in terms of time-dependent Green functions using a path integral method Ref. [3]. Other techniques have also been used such as the timespace rescaling or transformation method and the time-dependent invariant method. Ref. [4].

Invariants in mechanical systems with explicitly time-dependent Hamiltonians are constants of motion of central importance in the study of dynamical systems Ref. [5 and 6]. Among the procedures developed for obtaining invariants, a straightforward derivation for the classical time-dependent harmonic oscillator has been presented which leads directly to the orthogonal functions invariant, also called the Lewis invariant Ref. [7 and 8].

Given a time-dependent Hamiltonian, one can obtain the corresponding time-dependent Schrödinger equation. However, there is no certainty as to whether or not this Schrödinger equation represents quantum mechanical dissipative system unless the measure of quantum decoherence and classical correlation tells so. Ref. [9 and 10].

It has been shown that the classical limit of the specific heat of the one dimensional harmonic oscillator is attained at moderate temperatures if a relatively small number of quantum levels ( $n \ll 100$ ) is considered

Ref. [11]. Our interest is centered on analyzing the dissipative properties and the effect of time-dependent effective mass on the specific heat, energy fluctuation, entropy and quantum coherence of the 1D harmonic oscillator.

## II. HAMILTONIAN, INVARIANT OPERATOR AND WAVE FUNCTION

The power law suppressed harmonic oscillator Hamiltonian is given by;

$$\hat{H}(t) = X(t)\hat{p}^2 + Y(t)\hat{x}^2 \quad (1)$$

With

$$X(t) = (1 + st)^{-\gamma} \frac{1}{2m} \quad (2)$$

and

$$Y(t) = (1 + st)^{\gamma} \frac{m\omega^2}{2} \quad (3)$$

It has the following classical equation of motion;

$$\frac{d^2x(t)}{dt^2} + \frac{\gamma s}{(1 + st)} \frac{dx(t)}{dt} + \omega^2 x(t) = 0 \quad (4)$$

We assume that the quantum theory of this system is prescribed by the time-dependent Schrödinger equation;

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = \hat{H}(t) \psi(x, t) \quad (5)$$

Lewis and Riesenfeld showed that the invariant operator satisfying the quantum Liouville-Von Neumann equation;

$$i\hbar \frac{\partial}{\partial t} \hat{I}(t) + [\hat{I}(t), \hat{H}(t)] = 0 \quad (6)$$

provides the exact quantum states of the Schrödinger equation as its eigenstates up to time-dependent phase

\* This is the Caldirola-Kanai model in which the exponential function is replaced by a power-law function.

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factors. Proceeding as in [9 and 10] we introduce a pair of first order invariant operators

$$\hat{a}_{u(t)} = \frac{i}{\sqrt{\hbar}} \left[ u^*(t)\hat{p} - \frac{1}{X(t)} \left( \frac{d}{dt} u^*(t) \right) \hat{x} \right] \quad (7)$$

And

$$\hat{a}_u^\dagger(t) = -\frac{i}{\sqrt{\hbar}} \left[ u(t)\hat{p} - \frac{1}{X(t)} \left( \frac{d}{dt} u(t) \right) \hat{x} \right] \quad (8)$$

We require these operators to obey the quantum Liouville-Von Neumann equations;

$$i\hbar \frac{\partial}{\partial t} \hat{a}_{u(t)} + [\hat{a}_{u(t)}, \hat{H}(t)] = 0 \quad (9)$$

and

$$i\hbar \frac{\partial}{\partial t} \hat{a}_u^\dagger(t) + [\hat{a}_u^\dagger(t), \hat{H}(t)] = 0 \quad (10)$$

Where  $u(t)$  is a complex solution to the classical equation of motion eq. (4) The Invariant  $\hat{I}(t)$  is given by

$$\hat{I}(t) = \hbar\Omega \left[ \hat{a}_u^\dagger(t)\hat{a}_u(t) + \frac{1}{2} \right] \quad (11)$$

with

$$\Omega = \frac{X(t)}{2r(t)^2 k^2} \quad (12)$$

Where  $k$  is a constant that normalizes the complex solution  $u(t)$  to satisfy the Wronskian condition eq. (17) and  $r(t)$  is the amplitude of the solution. The solution to eq. (4) is found in the form of first and second kind Bessel functions;

$$x(t) = C_1 t^{\frac{1-\gamma}{2}} J_{\frac{\gamma-1}{2}}(t\omega) + C_2 t^{\frac{1-\gamma}{2}} Y_{\frac{\gamma-1}{2}}(t\omega) \quad (13)$$

By selecting the arbitrary constants  $C_1$  and  $C_2$  as 1 and  $i$  respectively, and using the substitutions  $f(t) = t^{\frac{1-\gamma}{2}} J_{\frac{\gamma-1}{2}}(t\omega)$  and  $g(t) = t^{\frac{1-\gamma}{2}} Y_{\frac{\gamma-1}{2}}(t\omega)$ , we express the complex solution  $u(t)$  as;

$$u(t) = kr(t) \exp(i\theta(t)) \quad (14)$$

Where;

$$\theta(t) = \arctan\left(\frac{g(t)}{f(t)}\right) \quad (15)$$

And

$$r(t) = [f(t)^2 + g(t)^2]^{\frac{1}{2}} \quad (16)$$

By normalizing the complex solution to satisfy the Wronskian condition;

$$W_r(u, u^*) = \frac{1}{X} \left[ u(t) \frac{d}{dt} u^*(t) - u^*(t) \frac{d}{dt} u(t) \right] = i \quad (17)$$

We find the integral constant  $k$ ;

$$k = \frac{i}{r(t)} \sqrt{\frac{X}{2\dot{\theta}(t)}} \quad (18)$$

With eq. (17) satisfied, it turns out that the standard commutation relation for the first order operators

$$[\hat{a}_u(t), \hat{a}_u^\dagger(t)] = 1 \quad (19)$$

Is guaranteed for all times. The number operator defined as;

$$\hat{N}(t) = \hat{a}_u^\dagger(t)\hat{a}_u(t) \quad (20)$$

Also satisfies eq. (9) and eq. (10) and yields the number state as an exact quantum state;

$$\hat{N}(t)|n, t\rangle = n|n, t\rangle \quad (21)$$

Operating eq. (7) to the ground eigenstate  $\phi_0(x, t)$  yields the ground state wave function;

$$\phi_0(x, t) = \left[ \frac{\dot{\theta}(t)}{\pi\hbar X} \right]^{\frac{1}{4}} \exp \left[ -\frac{x^2}{4X(t)\hbar} \left( \dot{\theta} + i \frac{\dot{r}(t)}{r(t)} \right) \right] \quad (22)$$

The  $n^{th}$  eigenstate is obtained by acting the raising operator eq. (8)  $n$  times to the ground state.

$$\phi_n(x, t) = \frac{1}{\sqrt{n!}} \hat{a}_u^\dagger(t)^n \phi_0(x, t) \quad (23)$$

We obtain;

$$\phi_n(x, t) = \frac{1}{\sqrt{2^n n!}} \left[ \frac{\dot{\theta}(t)}{\pi\hbar X(t)} \right]^{\frac{1}{4}} H_n \left[ \left( \frac{\dot{\theta}(t)}{4X(t)\hbar} \right)^{\frac{1}{2}} x \right] \exp \left[ -\frac{x^2}{4X(t)\hbar} \left( \dot{\theta} + i \frac{\dot{r}(t)}{r(t)} \right) \right] \quad (24)$$

The eigen states of the invariant operator differ from

the solution to the Schrodinger equation only by a time

dependent phase factor  $e^{i\eta(t)}$  such that the solution to the Schrodinger equation is given by;

$$\psi_n(x, t) = \phi_n(x, t)e^{i\eta(t)} \quad (25)$$

substituting eq.(25) into the Schrodinger equation, we obtain;

$$\frac{d}{dt}\eta(t) = \left\langle n, t \left| i\frac{\partial}{\partial t} - \frac{1}{\hbar}\hat{H}(t) \right| n, t \right\rangle \quad (26)$$

From the real part, the time dependent phase factor is obtained;

$$\eta(t) = -\frac{1}{\hbar} \int E_n(t) dt \quad (27)$$

$$\langle \hat{p}^2 \rangle = \frac{\hbar}{X^2} \dot{u}^*(t)\dot{u}(t)(2n+1) = \frac{\hbar}{X} \dot{\theta}(t) \left( n + \frac{1}{2} \right) \left[ 1 + \frac{\pi 2^{\gamma-2} \dot{r}(t)^2 (st+1)^\gamma}{s\dot{\theta}(t)} \right] \quad (31)$$

From eq. (30 and 31). the uncertainty relation follows;

$$\Delta x \Delta p = \hbar \left( n + \frac{1}{2} \right) \left[ 1 + \frac{\pi 2^{\gamma-2} \dot{r}(t)^2 (st+1)^\gamma}{s\dot{\theta}(t)} \right]^{\frac{1}{2}} \quad (32)$$

It can be seen that;

$$\Delta x \Delta p \geq \frac{\hbar}{2} \quad (33)$$

### III. SPECIFIC HEAT

To determine the partition function, we re-write eq. (35) as follows;

$$E_n = \hbar\omega \left( n + \frac{1}{2} \right) \zeta(t) \quad (36)$$

Where  $E_n(t)$  is the expectation value of the hamiltonian operator determined later.

Using eq. (7,8 and 17), the position and momentum operators are expressed respectively as;

$$\hat{x} = \sqrt{\hbar} [\hat{a}_u^\dagger(t)u^*(t) + \hat{a}_u(t)u(t)] \quad (28)$$

And

$$\hat{p} = \frac{\sqrt{\hbar}}{X(t)} [\hat{a}_u^\dagger(t)\dot{u}^*(t) + \hat{a}_u(t)\dot{u}(t)] \quad (29)$$

It follows that the average momentum and position of the particle are both zero. The wave function has the dispersion relations;

$$\langle \hat{x}^2 \rangle = \hbar u^*(t)u(t)(2n+1) = \frac{\hbar X(t)}{\theta(t)} \left( n + \frac{1}{2} \right) \quad (30)$$

And

The expectation value for the mechanical energy of the system is given by;

$$\langle n, t | E | n, t \rangle = X \langle n, t | p^2 | n, t \rangle + Y \langle n, t | x^2 | n, t \rangle \quad (34)$$

We obtain

$$\langle n, t | E | n, t \rangle = \hbar\omega \left( n + \frac{1}{2} \right) \left[ \frac{\pi 2^{\gamma-2} \dot{r}(t)^2 (st+1)^\gamma}{\omega s} + \frac{\dot{\theta}(t)}{\omega} + \frac{\omega}{4\dot{\theta}(t)} \right] \quad (35)$$

Where

$$\zeta(t) = \left[ \frac{\pi 2^{\gamma-2} \dot{r}(t)^2 (st+1)^\gamma}{\omega s} + \frac{\dot{\theta}(t)}{\omega} + \frac{\omega}{4\dot{\theta}(t)} \right] \quad (37)$$

The partition function is given by;

$$Z = \sum_{n=0}^{\infty} \exp \left[ -\frac{E_n}{K_B T} \right] = \sum_{n=0}^{\infty} \exp \left[ -(n + \frac{1}{2})\zeta(t)y \right] \quad (38)$$

Where

$$y = \frac{\hbar\omega}{K_B T} \quad (39)$$

The sum to infinity of eq. (38) yields;

$$Z = \frac{1}{2} \csc \left[ \zeta(t) \frac{y}{2} \right] \quad (40)$$

The specific heat at constant volume normalized to the Boltzmann constant  $K_B$  is related to the partition function by;

$$\frac{C_v}{K_B} = y^2 \frac{\partial^2 \ln(Z)}{\partial y^2} \quad (41)$$

We find;

$$\frac{C_v}{K_B} = \frac{\zeta(t)^2 y^2}{4} \csc^2 \left[ \zeta(t) \frac{y}{2} \right] \quad (42)$$

For a finite number of quantum states, we consider the sum in eq.(38) to the first  $c$  quantum states. The resulting normalized specific heat is eq.(43).

$$\frac{C_c}{K_B} = \frac{y^2 \zeta(t)^2}{(e^{y\zeta(t)} - 1)^2 (e^{(c+1)y\zeta(t)} - 1)^2} \times \left[ (c+1)^2 \left( -e^{(c+1)y\zeta(t)} \right) - (c+1)^2 e^{(c+3)y\zeta(t)} + 2c(c+2)e^{(c+2)y\zeta(t)} + e^{(2c+3)y\zeta(t)} + e^{y\zeta(t)} \right] \quad (43)$$

Figure 1 shows a plot of  $\frac{C_c}{K_B}$  against  $\frac{K_B T}{\hbar\omega}$ . We find that at  $c = 120$ , the curve does not fully trace the path of  $c = \infty$ . In contrast, it is reported for the case of quantum harmonic oscillator of constant mass Ref. [11] that the  $c = 50$  curve coincides with that of  $c = \infty$ . Figure 2 shows a plot of  $\frac{C_c}{K_B}$  against  $t$ . All three curves in Figure 2 coincide. This is indicative of the fact that the number of quantum states considered has no effect on the time evolution of specific heat.

#### IV. ENERGY FLUCTUATION

The root mean square energy fluctuation of the system  $f$ , defined as;

$$f = \frac{\sqrt{\langle E^2 \rangle - \langle E \rangle^2}}{\langle E \rangle} \quad (44)$$

measures the thermal stability of the system.  $f$  is related to the specific heat by;

$$f = \frac{K_B T \sqrt{C_v}}{\langle E \rangle} \quad (45)$$

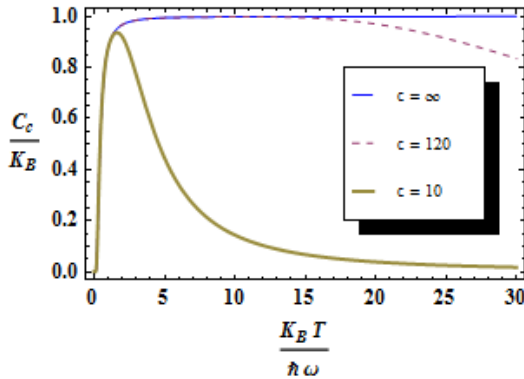


FIG. 1.  $\frac{C_c}{K_B}$  vs  $\frac{K_B T}{\hbar\omega}$  at  $t = 5$ . We used  $\gamma = 2$ ,  $s = \hbar = m = \omega = 1$

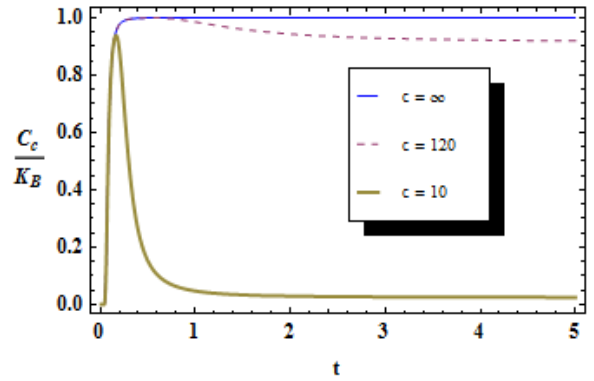


FIG. 2.  $\frac{C_c}{K_B}$  vs  $t$  at  $\frac{K_B T}{\hbar\omega} = 5$ . We used  $\gamma = 2$ ,  $s = \hbar = m = \omega = 1$

Substituting eq. (36), eq. (39) and eq. (42) into eq. (45) yields;

$$f = \frac{\hbar\omega}{y} \frac{1}{(n + \frac{1}{2})\zeta(t)\hbar\omega} \left[ \frac{y^2\zeta(t)^2}{4\sinh^2(\zeta(t)\frac{r}{2})} \right]^{\frac{1}{2}} \quad (46)$$

eq. (46) simplifies to;

$$f = \frac{1}{(n + \frac{1}{2})} \left[ \frac{1}{\exp(\zeta(t)y) + \exp(-\zeta(t)y) - 2} \right] \quad (47)$$

Figure 3 shows a 3D surface plot of eq(47)

## V. ENTROPY

The entropy normalized to Boltzmann constant is related to the partition function as follows;

$$\frac{S}{K_B} = \ln(Z) + y \frac{\partial}{\partial y} \ln(Z) \quad (48)$$

Substituting eq. (40) in eq. (48), we have;

$$\frac{S}{K_B} = \zeta(t)\frac{y}{2} \coth(\zeta(t)\frac{y}{2}) - \ln \left[ 2\sinh(\zeta(t)\frac{y}{2}) \right] \quad (49)$$

The high temperature limit of eq. (49), corresponds to  $y \rightarrow 0$ , yields;

$$\frac{S}{K_B} \rightarrow 1 - \ln(\zeta(t)y) = S_h \quad (50)$$

The low temperature limit of eq. (49) corresponding to  $y \rightarrow \infty$ , yields;

$$\frac{S}{K_B} \rightarrow \zeta(t)y \exp(\zeta(t)y) = S_l \quad (51)$$

Figure 4 shows a 3D surface plot of eq. (51)

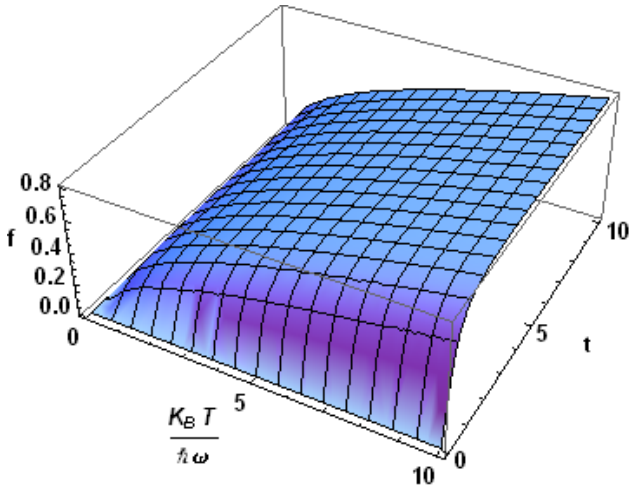


FIG. 3. r.m.s energy fluctuation  $f$  against  $\frac{K_B T}{\hbar\omega}$  and  $t$ , i.e. eq. (47). Parameters used are  $s = 1$ ,  $\gamma = 2$ ,  $\hbar = m = \omega = 1$

## VI. DECOHERENCE

The probability density matrix  $\rho_n(x', x, t)$  of the Gaussian wave function eq. (24) is given by;

$$\rho_n(x', x, t) = \psi_n(x', t)\psi_n^*(x, t) \quad (52)$$

For the ground state  $n = 0$ , we obtain

$$\rho_0(x', x, t) = C \exp \left[ -\frac{b}{2}(x'^2 + x^2) - i\frac{d}{2}(x'^2 - x^2) \right] \quad (53)$$

Where

$$C = \sqrt{\frac{\dot{\theta}(t)}{\pi\hbar X}}, \quad b = \frac{\dot{\theta}(t)}{\hbar X}, \quad d = \frac{1}{\hbar X} \dot{r}(t) \quad (54)$$

Using the substitution;

$$x_c = \frac{1}{\sqrt{2}}(x' + x) \quad (55)$$

And

$$x_\delta = \frac{1}{\sqrt{2}}(x' - x) \quad (56)$$

Such that

$$xx' = \frac{1}{2}(x_c^2 - x_\delta^2) \quad (57)$$

eq. (53) is written in the form

$$\rho_0(x', x, t) = C e^{-\Gamma_\mu x_c x_\delta - \Gamma_c x_c^2 - \Gamma_\delta x_\delta^2} \quad (58)$$

With  $\Gamma_c$ ,  $\Gamma_\delta$  and  $\Gamma_\mu$  given respectively as;

$$\Gamma_c = \frac{b}{2}, \quad \Gamma_\delta = \frac{b}{2}, \quad \Gamma_\mu = -id \quad (59)$$

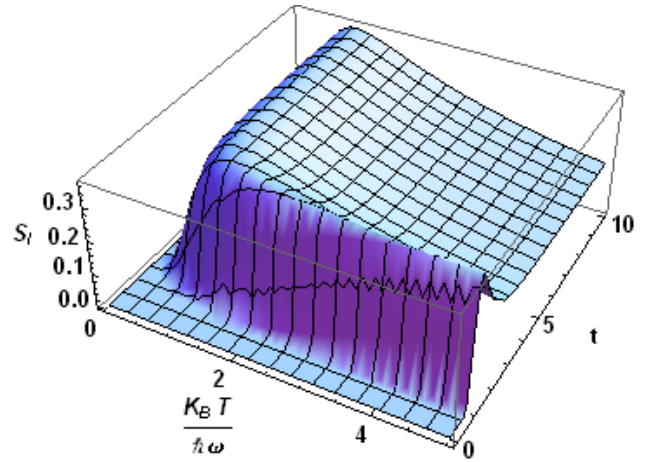


FIG. 4. Low temperature approximation of entropy  $\frac{S_l}{K_B}$  against  $\frac{K_B T}{\hbar\omega}$  and  $t$ , i.e. eq(48). We have used  $s = 1$ ,  $\gamma = 2$ ,  $\hbar = m = \omega = 1$

The coefficient of  $x_\delta^2$  gives the degree of quantum decoherence with the representation-independent measure of quantum decoherence given by;

$$\delta_{QD} = \frac{1}{2} \sqrt{\frac{\Gamma_c}{\Gamma_\delta}} = \frac{1}{2} \quad (60)$$

It is clear that the condition for quantum decoherence  $\delta_{CC} \ll 1$  is not satisfied for this system. The measure of classical correlation is given by;

$$\delta_{CC} = \sqrt{\frac{\Gamma_c^2 \Gamma_\delta^2}{\Gamma_\mu^* \Gamma_\mu}} = \frac{t^{-2\alpha}}{\pi^2 \hbar X} \frac{1}{r(t)^3 \dot{r}(t)} \quad (61)$$

The system shows classical correlation conditioned by ( $\delta_{CC} \ll 1$ ) only for ( $\gamma < 0$ )

## VII. CONCLUDING REMARKS

The following conclusions are drawn;

1) The specific heat initially increases rapidly to its high temperature limit of 1 then remains constant. When a finite number of quantum states is considered

at moderate temperature, the specific heat peaks, then drops to a limiting value  $\neq 0$

2) The classical limit of specific heat for this system is attained for a larger number of quantum states ( $n > 100$ ) as opposed to ( $n \ll 100$ ) for an oscillator of constant mass.

3) The evolution of thermal capacity with time at a given temperature is independent of the number of quantum states considered. Fig. (2)

4) Thermal instability of the system increases with temperature and with time but levels out at a maximum after a period of time for a given temperature. Fig. (3) shows that the system spends more time in the unstable region at any given temperature

5) The entropy of the system increases to a maximum with time then decreases to a limiting value but does not attain zero

6) The oscillator with time-dependent effective mass as power-law function preserves quantum coherence, however, classical correlation is satisfied only for ( $\gamma < 0$ ) which corresponds to decreasing mass or to an amplified oscillator.

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