

HYPERCYCLIC ABELIAN SEMIGROUPS OF AFFINE MAPS ON \mathbb{C}^n

YAHYA N'DAO

ABSTRACT. We give a characterization of hypercyclic abelian semigroup \mathcal{G} of affine maps on \mathbb{C}^n . If \mathcal{G} is finitely generated, this characterization is explicit. We prove in particular that no abelian group generated by n affine maps on \mathbb{C}^n has a dense orbit.

1. INTRODUCTION

Let $M_n(\mathbb{C})$ be the set of all square matrices of order $n \geq 1$ with entries in \mathbb{C} and $GL(n, \mathbb{C})$ be the group of all invertible matrices of $M_n(\mathbb{C})$. A map $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is called an affine map if there exist $A \in M_n(\mathbb{C})$ and $a \in \mathbb{C}^n$ such that $f(x) = Ax + a$, $x \in \mathbb{C}^n$. We denote $f = (A, a)$, we call A the *linear part* of f . The map f is invertible if $A \in GL(n, \mathbb{C})$. Denote by $MA(n, \mathbb{C})$ the vector space of all affine maps on \mathbb{C}^n and $GA(n, \mathbb{C})$ the group of all invertible affine maps of $MA(n, \mathbb{C})$.

Let \mathcal{G} be an abelian affine sub-semigroup of $MA(n, \mathbb{C})$. For a vector $v \in \mathbb{C}^n$, we consider the orbit of \mathcal{G} through v : $\mathcal{G}(v) = \{f(v) : f \in \mathcal{G}\} \subset \mathbb{C}^n$. Denote by \overline{E} the closure of a subset $E \subset \mathbb{C}^n$. The group \mathcal{G} is called *hypercyclic* if there exists a vector $v \in \mathbb{C}^n$ such that $\overline{\mathcal{G}(v)} = \mathbb{C}^n$. For an account of results and bibliography on hypercyclicity, we refer to the book [5] by Bayart and Matheron.

Let $n \in \mathbb{N}_0$ be fixed, denote by:

- $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$.
 - $\mathcal{B}_0 = (e_1, \dots, e_{n+1})$ the canonical basis of \mathbb{C}^{n+1} and I_{n+1} the identity matrix of $GL(n+1, \mathbb{C})$.
- For each $m = 1, 2, \dots, n+1$, denote by:
- $\mathbb{T}_m(\mathbb{C})$ the set of matrices over \mathbb{C} of the form

$$(1.1) \quad \begin{bmatrix} \mu & & & 0 \\ a_{2,1} & \mu & & \\ \vdots & \ddots & \ddots & \\ a_{m,1} & \dots & a_{m,m-1} & \mu \end{bmatrix}$$

- $\mathbb{T}_m^*(\mathbb{C})$ the group of matrices of the form (1.1) with $\mu \neq 0$.

Let $r \in \mathbb{N}$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$ such that $n_1 + \dots + n_r = n+1$. In particular, $r \leq n+1$. Write

- $\mathcal{K}_{\eta,r}(\mathbb{C}) := \mathbb{T}_{n_1}(\mathbb{C}) \oplus \dots \oplus \mathbb{T}_{n_r}(\mathbb{C})$. In particular if $r = 1$, then $\mathcal{K}_{\eta,1}(\mathbb{C}) = \mathbb{T}_{n+1}(\mathbb{C})$ and $\eta = (n+1)$.
- $\mathcal{K}_{\eta,r}^*(\mathbb{C}) := \mathcal{K}_{\eta,r}(\mathbb{C}) \cap GL(n+1, \mathbb{C})$.
- $u_0 = (e_{1,1}, \dots, e_{r,1}) \in \mathbb{C}^{n+1}$ where $e_{k,1} = (1, 0, \dots, 0) \in \mathbb{C}^{n_k}$, for $k = 1, \dots, r$. So $u_0 \in \{1\} \times \mathbb{C}^n$.
- $p_2 : \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ the second projection defined by $p_2(x_1, \dots, x_{n+1}) = (x_2, \dots, x_{n+1})$.

2000 *Mathematics Subject Classification.* 37C85, 47A16.

Key words and phrases. affine, hypercyclic, dense, orbit, abelian semigroup.

- $e^{(k)} = (e_1^{(k)}, \dots, e_r^{(k)}) \in \mathbb{C}^{n+1}$ where

$$e_j^{(k)} = \begin{cases} 0 \in \mathbb{C}^{n_j} & \text{if } j \neq k \\ e_{k,1} & \text{if } j = k \end{cases} \quad \text{for every } 1 \leq j, k \leq r.$$

- $\exp : M_{n+1}(\mathbb{C}) \longrightarrow GL(n+1, \mathbb{C})$ is the matrix exponential map; set $\exp(M) = e^M$, $M \in M_{n+1}(\mathbb{C})$.
- Define the map $\Phi : GA(n, \mathbb{C}) \longrightarrow GL(n+1, \mathbb{C})$

$$f = (A, a) \longmapsto \begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix}$$

We have the following composition formula

$$\begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Ab + a & AB \end{bmatrix}.$$

Then Φ is an injective homomorphism of groups. Write

- $G = \Phi(\mathcal{G})$, it is an abelian sub-semigroup of $GL(n+1, \mathbb{C})$.
- Define the map $\Psi : MA(n, \mathbb{C}) \longrightarrow M_{n+1}(\mathbb{C})$

$$f = (A, a) \longmapsto \begin{bmatrix} 0 & 0 \\ a & A \end{bmatrix}$$

We can see that Ψ is injective and linear. Hence $\Psi(MA(n, \mathbb{C}))$ is a vector subspace of $M_{n+1}(\mathbb{C})$. We prove (see Lemma 2.5) that Φ and Ψ are related by the following property

$$\exp(\Psi(MA(n, \mathbb{C}))) = \Phi(GA(n, \mathbb{C})).$$

Let consider the normal form of \mathcal{G} : By Proposition 2.1, there exists a $P \in \Phi(GA(n, \mathbb{C}))$ and a partition η of $(n+1)$ such that $G' = P^{-1}GP \subset \mathcal{K}_{\eta,r}^*(\mathbb{C}) \cap \Phi(MA(n, \mathbb{C}))$. For such a choice of matrix P , we let

- $v_0 = Pu_0$. So $v_0 \in \{1\} \times \mathbb{C}^n$, since $P \in \Phi(GA(n, \mathbb{C}))$.
- $w_0 = p_2(v_0) \in \mathbb{C}^n$. We have $v_0 = (1, w_0)$.
- $\varphi = \Phi^{-1}(P) \in MA(n, \mathbb{C})$.
- $g = \exp^{-1}(G) \cap (P(\mathcal{K}_{\eta,r}^*(\mathbb{C}))P^{-1})$. If $G \subset \mathcal{K}_{\eta,r}^*(\mathbb{C})$, we have $P = I_{n+1}$ and $g = \exp^{-1}(G) \cap \mathcal{K}_{\eta,r}^*(\mathbb{C})$.
- $g^1 = g \cap \Psi(MA(n, \mathbb{C}))$. It is an additive sub-semigroup of $M_{n+1}(\mathbb{C})$ (because by Lemma 3.2, g is an additive sub-semigroup of $M_{n+1}(\mathbb{C})$).
- $g_u^1 = \{Bu : B \in g^1\} \subset \mathbb{C}^{n+1}$, $u \in \mathbb{C}^{n+1}$.
- $\mathfrak{q} = \Psi^{-1}(g^1) \subset MA(n, \mathbb{C})$. Then \mathfrak{q} is an additive sub-semigroup of $MA(n, \mathbb{C})$ and we have $\Psi(\mathfrak{q}) = g^1$. By Corollary 2.9, we have $\exp(\Psi(\mathfrak{q})) = \Phi(\mathcal{G})$.
- $\mathfrak{q}_v = \{f(v), f \in \mathfrak{q}\} \subset \mathbb{C}^n$, $v \in \mathbb{C}^n$.

For groups of affine maps on \mathbb{K}^n ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}), their dynamics were recently initiated for some classes in different point of view, (see for instance, [6], [7], [8], [4]). The purpose here is to give analogous results as for linear abelian sub-semigroup of $GL(n, \mathbb{C})$ ([2], Theorem 1.1).

Our main results are the following:

Theorem 1.1. *Let \mathcal{G} be an abelian sub-semigroup of $MA(n, \mathbb{C})$. Then the following are equivalent:*

- (i) \mathcal{G} is hypercyclic.
- (ii) the orbit $\mathcal{G}(w_0)$ is dense in \mathbb{C}^n .
- (iii) \mathfrak{q}_{w_0} is an additive sub-semigroup dense in \mathbb{C}^n .

For a *finitely generated* abelian sub-semigroup $\mathcal{G} \subset MA(n, \mathbb{R})$, let introduce the following property: Consider the following rank condition on a collection of affine maps $f_1, \dots, f_p \in \mathcal{G}$. Let $f'_1, \dots, f'_p \in \mathfrak{q}$ be such that $e^{\Psi(f'_k)} = \Phi(f_k)$, $k = 1, \dots, p$. We say that f_1, \dots, f_p satisfy the *property \mathcal{D}* if for every $(s_1, \dots, s_p; t_2, \dots, t_r) \in \mathbb{Z}^{p+r-1} \setminus \{0\}$:

$$\text{rank} \begin{bmatrix} \text{Re}(f'_1(w_0)) & \dots & \text{Re}(f'_p(w_0)) & 0 & \dots & 0 \\ \text{Im}(f'_1(w_0)) & \dots & \text{Im}(f'_p(w_0)) & 2\pi p_2(e^{(2)}) & \dots & 2\pi p_2(e^{(r)}) \\ s_1 & \dots & s_p & t_2 & \dots & t_r \end{bmatrix} = 2n + 1.$$

For $r = 1$, this means that for every $(s_1, \dots, s_p) \in \mathbb{Z}^p \setminus \{0\}$:

$$\text{rank} \begin{bmatrix} \text{Re}(f'_1(w_0)) & \dots & \text{Re}(f'_p(w_0)) \\ \text{Im}(f'_1(w_0)) & \dots & \text{Im}(f'_p(w_0)) \\ s_1 & \dots & s_p \end{bmatrix} = 2n + 1.$$

For a vector $v \in \mathbb{C}^n$, we write $v = \text{Re}(v) + i\text{Im}(v)$ where $\text{Re}(v)$ and $\text{Im}(v) \in \mathbb{R}^n$. The next result can be stated as follows:

Theorem 1.2. *Let \mathcal{G} be an abelian sub-semigroup of $MA(n, \mathbb{C})$ and let $f_1, \dots, f_p \in \mathcal{G}$ generating \mathcal{G}^* and let $f'_1, \dots, f'_p \in \mathfrak{q}$ be such that $e^{\Psi(f'_1)} = \Phi(f_1), \dots, e^{\Psi(f'_p)} = \Phi(f_p)$. Then the following are equivalent:*

- (i) \mathcal{G} is hypercyclic.
- (ii) the maps $\varphi^{-1} \circ f_1 \circ \varphi, \dots, \varphi^{-1} \circ f_p \circ \varphi$ in $MA(n, \mathbb{C})$ satisfy the property \mathcal{D} .
- (iii) $\mathfrak{q}_{w_0} = \begin{cases} \sum_{k=1}^p \mathbb{N}f'_k(w_0) + 2i\pi \sum_{k=2}^r \mathbb{Z}(p_2(Pe^{(k)})), & \text{if } r \geq 2 \\ \sum_{k=1}^p \mathbb{N}f'_k(w_0), & \text{if } r = 1 \end{cases}$
is an additive sub-semigroup dense in \mathbb{C}^n .

Corollary 1.3. *Let \mathcal{G} be an abelian sub-semigroup of $MA(n, \mathbb{C})$ and $G = \Phi(\mathcal{G})$. Let $P \in \Phi(GA(n, \mathbb{C}))$ such that $P^{-1}GP \subset \mathcal{K}_{\eta, r}(\mathbb{C})$ where $1 \leq r \leq n + 1$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$. If \mathcal{G} is generated by $2n - r + 1$ commuting invertible affine maps, then it has no dense orbit.*

Corollary 1.4. *Let \mathcal{G} be an abelian sub-semigroup of $MA(n, \mathbb{C})$. If \mathcal{G} is generated by n commuting invertible affine maps, then it has no dense orbit.*

2. NORMAL FORM OF ABELIAN AFFINE GROUPS

Proposition 2.1. ([1], Proposition 2.1) *Let \mathcal{G} be an abelian subgroup of $GA(n, \mathbb{C})$ and $G = \Phi(\mathcal{G})$. Then there exists $P \in \Phi(GA(n, \mathbb{C}))$ such that $P^{-1}GP$ is a subgroup of $\mathcal{K}_{\eta, r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$, for some $r \leq n + 1$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$.*

This proposition can be generalized for any abelian affine semigroup as follow:

Proposition 2.2. *Let \mathcal{G} be an abelian sub-semigroup of $MA(n, \mathbb{C})$ and $G = \Phi(\mathcal{G})$. Then there exists $P \in \Phi(GA(n, \mathbb{C}))$ such that $P^{-1}GP$ is a subgroup of $\mathcal{K}_{\eta, r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$, for some $r \leq n + 1$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$.*

The same proof of Proposition 2.1 remained valid for Proposition 2.2.

The group $G' = P^{-1}GP$ is called the *normal form* of G . In particular we have $Pu_0 = v_0 \in \{1\} \times \mathbb{C}^n$. To prove Proposition 2.1, we need the following results:

Denote by \mathcal{L}_G the set of the linear parts of all elements of \mathcal{G} . Then \mathcal{L}_G is an abelian sub-semigroup of $M_n(\mathbb{C})$. A subset $F \subset \mathbb{C}^n$ is called *G-invariant* (resp. *\mathcal{L}_G -invariant*) if $A(F) \subset F$ for any $A \in G$ (resp. $A \in \mathcal{L}_G$).

Proposition 2.3. ([3]) *Let G' be an abelian sub-semigroup of $M_m(\mathbb{C})$, $m \geq 1$. Then there exists $P \in GL(m, \mathbb{C})$ such that $P^{-1}G'P$ is a sub-semigroup of $\mathcal{K}_{\eta', r'}(\mathbb{C})$, for some $r' \leq m$ and $\eta' = (n'_1, \dots, n'_{r'}) \in \mathbb{N}_0^r$.*

Lemma 2.4. ([2], Proposition 3.2) $\exp(\mathcal{K}_{\eta, r}(\mathbb{C})) = \mathcal{K}_{\eta, r}^*(\mathbb{C})$.

Lemma 2.5. ([1], Lemma 2.8) $\exp(\Psi(MA(n, \mathbb{C}))) = GA(n, \mathbb{C})$.

Lemma 2.6. ([1], Lemma 2.9) *If $N \in PK_{\eta, r}(\mathbb{C})P^{-1}$ such that $e^N \in \Phi(GA(n, \mathbb{C}))$, then there exists $k \in \mathbb{Z}$ such that $N - 2ik\pi I_{n+1} \in \Psi(MA(n, \mathbb{C}))$.*

Denote by $G^* = G \cap GL(n+1, \mathbb{C})$.

Lemma 2.7. ([2], Lemma 4.2) *One has $\exp(\mathfrak{g}) = G^*$.*

Corollary 2.8. ([1], Corollary 2.11) *Let $G = \Phi(\mathcal{G})$. We have $\mathfrak{g} = \mathfrak{g}^1 + 2i\pi\mathbb{Z}I_{n+1}$.*

Denote by $\mathcal{G}^* = \mathcal{G} \cap GA(n, \mathbb{C})$.

Corollary 2.9. *We have $\exp(\Psi(\mathfrak{q})) = \Phi(\mathcal{G}^*)$.*

Proof. By Lemmas 2.7 and 2.8, We have $G = \exp(\mathfrak{g}) = \exp(\mathfrak{g}^1 + 2i\pi\mathbb{Z}I_{n+1}) = \exp(\mathfrak{g}^1)$. Since $\mathfrak{g}^1 = \Psi(\mathfrak{q})$, we get $\exp(\Psi(\mathfrak{q})) = \Phi(\mathcal{G})$. \square

3. PROOF OF THEOREM 1.1

Let \tilde{G} be the semigroup generated by G and $\mathbb{C}I_{n+1} = \{\lambda I_{n+1} : \lambda \in \mathbb{C}\}$. Then \tilde{G} is an abelian sub-semigroup of $GL(n+1, \mathbb{C})$. By Proposition 2.1, there exists $P \in \Phi(GA(n, \mathbb{C}))$ such that $P^{-1}\tilde{G}P$ is a sub-semigroup of $\mathcal{K}_{\eta, r}^*(\mathbb{C})$ for some $r \leq n+1$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$ and this also implies that $P^{-1}\tilde{G}P$ is a sub-semigroup of $\mathcal{K}_{\eta, r}^*(\mathbb{C})$. Set $\tilde{\mathfrak{g}} = \exp^{-1}(\tilde{G}) \cap (PK_{\eta, r}(\mathbb{C})P^{-1})$ and $\tilde{\mathfrak{g}}_{v_0} = \{Bv_0 : B \in \tilde{\mathfrak{g}}\}$. Then we have the following theorem, applied to \tilde{G} :

Theorem 3.1. ([3], Theorem 1.1) *Under the notations above, the following properties are equivalent:*

- (i) \tilde{G} has a dense orbit in \mathbb{C}^{n+1} .
- (ii) the orbit $\tilde{G}(v_0)$ is dense in \mathbb{C}^{n+1} .
- (iii) $\tilde{\mathfrak{g}}_{v_0}$ is an additive sub-semigroup dense in \mathbb{C}^{n+1} .

Lemma 3.2. ([2], Lemma 4.1) *The sets \mathfrak{g} and $\tilde{\mathfrak{g}}$ are additive subgroups of $M_{n+1}(\mathbb{C})$. In particular, \mathfrak{g}_{v_0} and $\tilde{\mathfrak{g}}_{v_0}$ are additive subgroups of \mathbb{C}^{n+1} .*

Recall that $g^1 = g \cap \Psi(MA(n, \mathbb{C}))$ and $\mathfrak{q} = \Psi^{-1}(g^1) \subset MA(n, \mathbb{C})$.

Lemma 3.3. *Under the notations above, one has:*

- (i) $\tilde{g} = g^1 + \mathbb{C}I_{n+1}$.
- (ii) $\{0\} \times \mathfrak{q}_{w_0} = g_{v_0}^1$.

Proof. (i) Let $B \in \tilde{g}$, then $e^B \in \tilde{G}$. One can write $e^B = \lambda A$ for some $\lambda \in \mathbb{C}^*$ and $A \in G$. Let $\mu \in \mathbb{C}$ such that $e^\mu = \lambda$, then $e^{B-\mu I_{n+1}} = A$. Since $B - \mu I_{n+1} \in PK_{\eta,r}(\mathbb{C})P^{-1}$, so $B - \mu I_{n+1} \in \exp^{-1}(G) \cap PK_{\eta,r}(\mathbb{C})P^{-1} = g$. By Corollary 2.8, there exists $k \in \mathbb{Z}$ such that $B' := B - \mu I_{n+1} + 2ik\pi I_{n+1} \in g^1$. Then $B \in g^1 + \mathbb{C}I_{n+1}$ and hence $\tilde{g} \subset g^1 + \mathbb{C}I_{n+1}$. Since $g^1 \subset \tilde{g}$ and $\mathbb{C}I_{n+1} \subset \tilde{g}$, it follows that $g^1 + \mathbb{C}I_{n+1} \subset \tilde{g}$ (since \tilde{g} is an additive group, by Lemma 3.2). This proves (i).

(ii) Since $\Psi(\mathfrak{q}) = g^1$ and $v_0 = (1, w_0)$, we obtain for every $f = (B, b) \in \mathfrak{q}$,

$$\begin{aligned} \Psi(f)v_0 &= \begin{bmatrix} 0 & 0 \\ b & B \end{bmatrix} \begin{bmatrix} 1 \\ w_0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ b + Bw_0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ f(w_0) \end{bmatrix}. \end{aligned}$$

Hence $g_{v_0}^1 = \{0\} \times \mathfrak{q}_{w_0}$. □

Lemma 3.4. *The following assertions are equivalent:*

- (i) $\overline{\mathfrak{q}_{w_0}} = \mathbb{C}^n$.
- (ii) $\overline{g_{v_0}^1} = \{0\} \times \mathbb{C}^n$.
- (iii) $\overline{\tilde{g}_{v_0}} = \mathbb{C}^{n+1}$.

Proof. (i) \iff (ii) follows from the fact that $\{0\} \times \mathfrak{q}_{w_0} = g_{v_0}^1$ (Lemma 3.3,(ii)).

(ii) \implies (iii) : By Lemma 3.3,(ii), $\tilde{g}_{v_0} = g_{v_0}^1 + \mathbb{C}v_0$. Since $v_0 = (1, w_0) \notin \{0\} \times \mathbb{C}^n$ and $\mathbb{C}I_{n+1} \subset \tilde{g}$, we obtain $\mathbb{C}v_0 \subset \tilde{g}_{v_0}$ and so $\mathbb{C}v_0 \subset \overline{\tilde{g}_{v_0}}$. Therefore $\mathbb{C}^{n+1} = \{0\} \times \mathbb{C}^n \oplus \mathbb{C}v_0 = \overline{g_{v_0}^1} \oplus \mathbb{C}v_0 \subset \overline{\tilde{g}_{v_0}}$ (since, by Lemma 3.2, \tilde{g}_{v_0} is an additive sub-semigroup of \mathbb{C}^{n+1}). Thus $\overline{\tilde{g}_{v_0}} = \mathbb{C}^{n+1}$.

(iii) \implies (ii) : Let $x \in \mathbb{C}^n$, then $(0, x) \in \overline{\tilde{g}_{v_0}}$ and there exists a sequence $(A_m)_{m \in \mathbb{N}} \subset \tilde{g}$ such that $\lim_{m \rightarrow +\infty} A_m v_0 = (0, x)$. By Lemma 3.3, we can write $A_m v_0 = \lambda_m v_0 + B_m v_0$ with $\lambda_m \in \mathbb{C}$ and $B_m = \begin{bmatrix} 0 & 0 \\ b_m & B_m^1 \end{bmatrix} \in g^1$ for every $m \in \mathbb{N}$. Since $B_m v_0 \in \{0\} \times \mathbb{C}^n$ for every $m \in \mathbb{N}$ then $A_m v_0 = (\lambda_m, b_m + B_m^1 w_0 + \lambda_m w_0)$. It follows that $\lim_{m \rightarrow +\infty} \lambda_m = 0$ and $\lim_{m \rightarrow +\infty} A_m v_0 = \lim_{m \rightarrow +\infty} B_m v_0 = (0, x)$, thus $(0, x) \in \overline{g_{v_0}^1}$. Hence $\{0\} \times \mathbb{C}^n \subset \overline{g_{v_0}^1}$. Since $g^1 \subset \Psi(MA(n, \mathbb{C}))$, $g_{v_0}^1 \subset \{0\} \times \mathbb{C}^n$ then we conclude that $\overline{g_{v_0}^1} = \{0\} \times \mathbb{C}^n$. □

Lemma 3.5. *Let $x \in \mathbb{C}^n$ and $G = \Phi(\mathcal{G})$. The following are equivalent:*

- (i) $\overline{\mathcal{G}(x)} = \mathbb{C}^n$.
- (ii) $\overline{G(1, x)} = \{1\} \times \mathbb{C}^n$.
- (iii) $\overline{\tilde{G}(1, x)} = \mathbb{C}^{n+1}$.

Proof. (i) \iff (ii) : is obvious since $\{1\} \times \mathcal{G}(x) = G(1, x)$ by construction.

(iii) \implies (ii) : Let $y \in \mathbb{C}^n$ and $(B_m)_m$ be a sequence in \tilde{G} such that $\lim_{m \rightarrow +\infty} B_m(1, x) = (1, y)$. One can write $B_m = \lambda_m \Phi(f_m)$, with $f_m \in \mathcal{G}$ and $\lambda_m \in \mathbb{C}^*$, thus $B_m(1, x) = (\lambda_m, \lambda_m f_m(x))$, so $\lim_{m \rightarrow +\infty} \lambda_m = 1$. Therefore, $\lim_{m \rightarrow +\infty} \Phi(f_m)(1, x) = \lim_{m \rightarrow +\infty} \frac{1}{\lambda_m} B_m(1, x) = (1, y)$. Hence, $(1, y) \in \overline{G(1, x)}$.

(ii) \implies (iii) : Since $\mathbb{C}^{n+1} \setminus (\{0\} \times \mathbb{C}^n) = \bigcup_{\lambda \in \mathbb{C}^*} \lambda(\{1\} \times \mathbb{C}^n)$ and for every $\lambda \in \mathbb{C}^*$, $\lambda G(1, x) \subset \tilde{G}(1, x)$, we get

$$\begin{aligned} \mathbb{C}^{n+1} &= \overline{\mathbb{C}^{n+1} \setminus (\{0\} \times \mathbb{C}^n)} \\ &= \overline{\bigcup_{\lambda \in \mathbb{C}^*} \lambda(\{1\} \times \mathbb{C}^n)} \\ &= \overline{\bigcup_{\lambda \in \mathbb{C}^*} \lambda \overline{G(1, x)}} \subset \overline{\tilde{G}(1, x)} \end{aligned}$$

Hence $\mathbb{C}^{n+1} = \overline{\tilde{G}(1, x)}$. □

Proposition 3.6. *Let G be an abelian subsemigroup of $M_n(\mathbb{C})$ and $G^* = G \cap GL(n, \mathbb{C})$. Then G is locally hypercyclic (resp. hypercyclic) if and only if so is G^* .*

Proof. Suppose that $\overline{G^*(u)} \neq \emptyset$, for some $u \in \mathbb{K}^n$. Then $\emptyset \neq \overline{G^*(u)} \subset \overline{G(u)}$ and so $\overline{G(u)} \neq \emptyset$.

Conversely, suppose that $\overline{G(u)} \neq \emptyset$, for some $u \in \mathbb{C}^n$. By proposition 2.1, one can suppose that G is an abelian sub-semigroup of $\mathcal{K}_{\eta, r}(\mathbb{C})$. Write $G' := (G \setminus G^*) \cup \{I_n\}$. then G' is a sub-semigroup of G .

- If $G' = \{I_n\}$ then $G = G^*$ and so G^* is locally hypercyclic.
- If $G' \neq \{I_n\}$ then

$$G(u) \subset \left(\bigcup_{A \in (G' \setminus \{I_n\})} \text{Im}(A) \right) \cup G^*(u).$$

As every $A \in (G' \setminus \{I_n\})$, is non invertible, then $\text{Im}(A) \subset \bigcup_{k=1}^r H_k$ where

$$H_k := \left\{ u = [u_1, \dots, u_r]^T \in \mathbb{C}^n, u_j \in \mathbb{C}^{n_j}, u_k \in \{0\} \times \mathbb{C}^{n_{k-1}} \quad \begin{matrix} 1 \leq j \leq r, \\ j \neq k \end{matrix} \right\}.$$

It follows that

$$G(u) \subset \left(\bigcup_{k=1}^r H_k \right) \cup G^*(u),$$

and so

$$\overline{G(u)} \subset \left(\bigcup_{k=1}^r H_k \right) \cup \overline{G^*(u)}.$$

Since $\dim H_k = n - 1$, $\overline{H_k} = \emptyset$, for every $1 \leq k \leq r$ and therefore $\overline{G^*(u)} \neq \emptyset$. □

Lemma 3.7. *Let G be an abelian subsemigroup of $\mathcal{K}_{\eta, r}(\mathbb{C})$, $G^* = G \cap GL(n, \mathbb{C})$ and $g^* = \exp^{-1}(G^*) \cap \mathcal{K}_{\eta, r}(\mathbb{C})$. Then $g = g^*$.*

Proof. Let $G' = G \setminus G^*$. Since $e^A \in GL(n, \mathbb{C})$ for every $A \in M_n(\mathbb{C})$ and $G' \subset M_n(\mathbb{C}) \setminus GL(n, \mathbb{C})$ then $\exp^{-1}(G^*) = \emptyset$. As $g = (\exp^{-1}(G') \cap \mathcal{K}_{\eta,r}(\mathbb{C})) \cup g^*$ then $g = g^*$. \square

Proof of Theorem 1.1.

By Proposition 2.1, one can suppose that G is an abelian subsemigroup of $\mathcal{K}_{\eta,r}(\mathbb{C})$. We let $G^* = G \cap GL(n, \mathbb{C})$, $g^* = \exp^{-1}(G^*) \cap \mathcal{K}_{\eta,r}(\mathbb{C})$ and $(g^*)_{u_0} = \{Bu_0, B \in g^*\}$. By applying Proposition 2.7 on G^* , the orbit $G^*(u_0)$ is dense in \mathbb{C}^n if and only if $(g^*)_{u_0}$ is an additive subsemigroup, locally dense in \mathbb{C}^n . By Proposition 4.2, Lemma 4.3 and Proposition 4.6, Theorem 1.1 is proved.

.....
 (ii) \implies (i) : is obvious.

(i) \implies (ii) : Suppose that \mathcal{G} is hypercyclic, so $\overline{\mathcal{G}(x)} = \mathbb{C}^n$ for some $x \in \mathbb{C}^n$. By Lemma 3.5,(iii), $\overline{\mathcal{G}(1, x)} = \mathbb{C}^{n+1}$ and by Theorem 3.1, $\overline{\mathcal{G}(v_0)} = \mathbb{C}^{n+1}$. Then by Lemma 3.5, $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$, since $v_0 = (1, w_0)$.

(ii) \implies (iii) : Suppose that $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$. By Lemma 3.5, $\overline{\mathcal{G}(v_0)} = \mathbb{C}^{n+1}$ and by Theorem 3.1, $\overline{\mathfrak{g}_{v_0}} = \mathbb{C}^{n+1}$. Then by Lemma 3.4, $\overline{\mathfrak{q}_{w_0}} = \mathbb{C}^n$.

(iii) \implies (ii) : Suppose that $\overline{\mathfrak{q}_{w_0}} = \mathbb{C}^n$. By Lemma 3.4, $\overline{\mathfrak{g}_{v_0}} = \mathbb{C}^{n+1}$ and by Theorem 3.1, $\overline{\mathcal{G}(v_0)} = \mathbb{C}^{n+1}$. Then by Lemma 3.5, $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$. \square

Proof of Corollary ??. Assume that $\mathcal{G} \subset GL(n, \mathbb{C})$ then take $P = \text{diag}(1, Q)$ and $G = \Phi(\mathcal{G})$, then $P^{-1}GP \subset \mathcal{K}_{\eta,r'+1}(\mathbb{C})$ where $\eta = (1, n'_1, \dots, n'_{r'})$. Hence $u_0 = (1, u'_0)$, $v_0 = Pu_0 = (1, Qu'_0)$ and thus $w_0 = Qu'_0 = v'_0$. Every $f = (A, 0) \in \mathcal{G}$ is simply noted A . Then for every $A \in \mathcal{G}$, $\Phi(A) = \text{diag}(1, A)$. We can verify that $g^1 = \{\text{diag}(0, B) : B \in g'\}$ where $g' = \exp^{-1}(\mathcal{G}) \cap Q(\mathcal{K}_{\eta',r'}(\mathbb{C}))Q^{-1}$ and so $\mathfrak{q} = \Psi^{-1}(g^1) = g'$. Hence the proof of Corollary ?? follows directly from Theorem 1.1. \square .

4. FINITELY GENERATED SUBGROUPS

Recall the following result proved in [3] which applied to G can be stated as following:

Proposition 4.1. ([3], Proposition 5.1) *Let G be an abelian sub-semigroup of $M_n(\mathbb{C})$ such that G^* is generated by A_1, \dots, A_p and let $B_1, \dots, B_p \in \mathfrak{g}$ such that $A_k = e^{B_k}$, $k = 1, \dots, p$ and $P \in GL(n+1, \mathbb{C})$ satisfying $P^{-1}GP \subset \mathcal{K}_{\eta,r}(\mathbb{C})$. Then:*

$$\mathfrak{g} = \sum_{k=1}^p \mathbb{N}B_k + 2i\pi \sum_{k=1}^r \mathbb{Z}PJ_kP^{-1} \quad \text{and} \quad \mathfrak{g}_{v_0} = \sum_{k=1}^p \mathbb{N}B_kv_0 + \sum_{k=1}^r 2i\pi \mathbb{Z}Pe^{(k)},$$

where $J_k = \text{diag}(J_{k,1}, \dots, J_{k,r})$ with $J_{k,i} = 0 \in \mathbb{T}_{n_i}(\mathbb{C})$ if $i \neq k$ and $J_{k,k} = I_{n_k}$.

Proposition 4.2. *Let \mathcal{G} be an abelian sub-semigroup of $GA(n, \mathbb{C})$ such that \mathcal{G}^* is generated by f_1, \dots, f_p and let $f'_1, \dots, f'_p \in \mathfrak{q}$ such that $e^{\Psi(f'_k)} = \Phi(f_k)$, $k = 1, \dots, p$. Let P be as in Proposition 2.1. Then:*

$$\mathfrak{q}_{w_0} = \begin{cases} \sum_{k=1}^p \mathbb{N}f'_k(w_0) + \sum_{k=2}^r 2i\pi \mathbb{Z}p_2(Pe^{(k)}), & \text{if } r \geq 2 \\ \sum_{k=1}^p \mathbb{N}f'_k(w_0), & \text{if } r = 1 \end{cases}$$

Proof. Let $G = \Phi(\mathcal{G})$. Then G is generated by $\Phi(f_1), \dots, \Phi(f_p)$. Apply Proposition 4.1 to G , $A_k = \Phi(f_k)$, $B_k = \Psi(f'_k) \in \mathfrak{g}^1$, then we have

$$\mathfrak{g} = \sum_{k=1}^p \mathbb{Z}\Psi(f'_k) + 2i\pi\mathbb{Z}\sum_{k=1}^r PJ_kP^{-1}.$$

We have $\sum_{k=1}^p \mathbb{Z}\Psi(f'_k) \subset \Psi(MA(n, \mathbb{C}))$. Moreover, for every $k = 2, \dots, r$, $J_k \in \Psi(MA(n, \mathbb{C}))$, hence $PJ_kP^{-1} \in \Psi(MA(n, \mathbb{C}))$, since $P \in \Phi(GA(n, \mathbb{C}))$. However, $mPJ_1P^{-1} \notin \Psi(MA(n, \mathbb{C}))$ for every $m \in \mathbb{Z} \setminus \{0\}$, since J_1 has the form $J_1 = \text{diag}(1, J')$ where $J' \in M_n(\mathbb{C})$. As $\mathfrak{g}^1 = \mathfrak{g} \cap \Psi(MA(n, \mathbb{C}))$, then $mPJ_1P^{-1} \notin \mathfrak{g}^1$ for every $m \in \mathbb{Z} \setminus \{0\}$. Hence we obtain:

$$\mathfrak{g}^1 = \begin{cases} \sum_{k=1}^p \mathbb{N}\Psi(f'_k) + \sum_{k=2}^r 2i\pi\mathbb{Z}PJ_kP^{-1}, & \text{if } r \geq 2 \\ \sum_{k=1}^p \mathbb{N}\Psi(f'_k), & \text{if } r = 1 \end{cases}$$

Since $J_k u_0 = e^{(k)}$, we get

$$\mathfrak{g}_{v_0}^1 = \begin{cases} \sum_{k=1}^p \mathbb{N}\Psi(f'_k)v_0 + \sum_{k=2}^r 2i\pi\mathbb{Z}Pe^{(k)}, & \text{if } r \geq 2 \\ \sum_{k=1}^p \mathbb{N}\Psi(f'_k)v_0, & \text{if } r = 1 \end{cases}$$

By Lemma 3.3,(iii), one has $\{0\} \times \mathfrak{q}_{w_0} = \mathfrak{g}_{v_0}^1$ and $\Psi(f'_k)v_0 = (0, f'_k(w_0))$, so $\mathfrak{q}_{w_0} = p_2(\mathfrak{g}_{v_0}^1)$. It follows that

$$\mathfrak{q}_{w_0} = \begin{cases} \sum_{k=1}^p \mathbb{N}f'_k(w_0) + \sum_{k=2}^r 2i\pi\mathbb{Z}p_2(Pe^{(k)}), & \text{if } r \geq 2 \\ \sum_{k=1}^p \mathbb{N}f'_k(w_0), & \text{if } r = 1 \end{cases}$$

The proof is completed. \square

Recall the following proposition which was proven in [9]:

Proposition 4.3. (cf. [9], page 35). *Let $F = \mathbb{Z}u_1 + \dots + \mathbb{Z}u_p$ with $u_k = \text{Re}(u_k) + i\text{Im}(u_k)$, where $\text{Re}(u_k), \text{Im}(u_k) \in \mathbb{R}^n$, $k = 1, \dots, p$. Then F is dense in \mathbb{C}^n if and only if for every $(s_1, \dots, s_p) \in \mathbb{Z}^p \setminus \{0\}$:*

$$\text{rank} \begin{bmatrix} \text{Re}(u_1) & \dots & \dots & \text{Re}(u_p) \\ \text{Im}(u_1) & \dots & \dots & \text{Im}(u_p) \\ s_1 & \dots & \dots & s_p \end{bmatrix} = 2n + 1.$$

Proof of Theorem 1.2: This follows directly from Theorem 1.1, Propositions 4.2 and 4.3.

Proof of Corollary 1.3: First, by Proposition 4.3, if $F = \mathbb{Z}u_1 + \dots + \mathbb{Z}u_m$, $u_k \in \mathbb{C}^n$ with $m \leq 2n$, then F cannot be dense in \mathbb{C}^n . Now, by the form of \mathfrak{q}_{w_0} in Proposition 4.2, \mathfrak{q}_{w_0} cannot be dense in \mathbb{C}^n and so Corollary 1.3 follows by Theorem 1.2. \square

Proof of Corollary 1.4: Since $n \leq 2n - r + 1$ (because $r \leq n + 1$), Corollary 1.4 follows from Corollary 1.3. \square

5. EXAMPLE

Example 5.1. Let \mathcal{G} the sub-semigroup of $GA(2, \mathbb{C})$ generated by $f_1 = (A_1, a_1)$, $f_2 = (A_2, a_2)$, $f_3 = (A_3, a_3)$ and $f_4 = (A_4, a_4)$ where $A_1 = I_2$, $a_1 = (1 + i, 0)$,

$$A_2 = \text{diag}(1, e^{-2+i}), \quad a_2 = (0, 0), \quad A_3 = \text{diag}\left(1, e^{\frac{-\sqrt{2}}{\pi} + i\left(\frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2}\right)}\right),$$

$$a_3 = \left(\frac{-\sqrt{3}}{2\pi} + i\left(\frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi}\right), 0\right), \quad A_4 = I_2, \quad a_4 = (2i\pi, 0).$$

Then \mathcal{G} is hypercyclic.

Proof. First one can check that \mathcal{G} is abelian: $f_i f_j = f_j f_i$ for every $i, j = 1, 2, 3, 4$. Denote by $G = \Phi(\mathcal{G})$. Then G is generated by

$$\Phi(f_1) = \begin{bmatrix} 1 & 0 & 0 \\ 1+i & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Phi(f_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2+i} \end{bmatrix},$$

$$\Phi(f_3) = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-\sqrt{3}}{2\pi} + i\left(\frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi}\right) & 1 & 0 \\ 0 & 0 & e^{\frac{-\sqrt{2}}{\pi} + i\left(\frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2}\right)} \end{bmatrix}, \quad \Phi(f_4) = \begin{bmatrix} 1 & 0 & 0 \\ 2i\pi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $f'_i = (B_i, b_i)$, $i = 1, 2, 3, 4$ where

$$B_1 = \text{diag}(0, 0) = 0, \quad b_1 = (1 + i, 0),$$

$$B_2 = \text{diag}(0, -2 + i), \quad b_2 = (0, 0),$$

$$B_3 = \text{diag}\left(0, \frac{-\sqrt{2}}{\pi} + i\left(\frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2}\right)\right), \quad b_3 = \left(\frac{-\sqrt{3}}{2\pi} + i\left(\frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi}\right), 0\right),$$

$$B_4 = \text{diag}(0, 0) = 0, \quad b_4 = (2i\pi, 0).$$

Then we have $e^{\Psi(f'_i)} = \Phi(f_i)$, $i = 1, 2, 3, 4$.

Here $r = 2$, $\eta = (2, 1)$, G is an abelian sub-semigroup of $\mathcal{K}_{(2,1),2}^*(\mathbb{C})$. We have $P = I_3$, $\varphi = (I_2, 0)$,

$u_0 = v_0 = (1, 0, 1)$, $e^{(2)} = (0, 0, 1)$ and $w_0 = (0, 1)$. By Proposition 4.2, $\mathfrak{q}_{w_0} = \sum_{k=1}^4 \mathbb{Z} f'_k(w_0) + 2i\pi \mathbb{Z} p_2(e^{(2)})$. On the other hand, for every $(s_1, s_2, s_3, s_4, t_2) \in \mathbb{Z}^5 \setminus \{0\}$, write

$$M_{(s_1, s_2, s_3, s_4, t_2)} = \begin{bmatrix} \text{Re}(B_1 w_0 + b_1) & \text{Re}(B_2 w_0 + b_2) & \text{Re}(B_3 w_0 + b_3) & \text{Re}(B_4 w_0 + b_4) & 0 \\ \text{Im}(B_1 w_0 + b_1) & \text{Im}(B_2 w_0 + b_2) & \text{Im}(B_3 w_0 + b_3) & \text{Im}(B_4 w_0 + b_4) & 2\pi e^{(2)} \\ s_1 & s_2 & s_3 & s_4 & t_2 \end{bmatrix}.$$

Then the determinant:

$$\begin{aligned} \Delta &= \det(M_{(s_1, s_2, s_3, s_4, t_2)}) \\ &= \begin{vmatrix} 1 & 0 & -\frac{\sqrt{3}}{2\pi} & 0 & 0 \\ 0 & -2 & -\frac{\sqrt{2}}{\pi} & 0 & 0 \\ 1 & 0 & \frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2\pi} & 2\pi & 0 \\ 0 & 1 & \frac{\sqrt{2}}{2\pi} - \frac{\sqrt{7}}{2} & 0 & 2\pi \\ s_1 & s_2 & s_3 & s_4 & t_2 \end{vmatrix} \\ &= 2\pi \left(-s_1 \sqrt{3} + 2s_2 \sqrt{2} - 4s_3 \pi + s_4 \sqrt{5} - t_2 \sqrt{7}\right). \end{aligned}$$

Since π , $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{7}$ are rationally independent, $\Delta \neq 0$ for every $(s_1, s_2, s_3, s_4, t_2) \in \mathbb{Z}^5 \setminus \{0\}$. It follows that $\text{rank}(M_{(s_1, s_2, s_3, s_4, t_2)}) = 5$. Hence f_1, \dots, f_4 satisfy the property \mathcal{D} . By Theorem 1.2, \mathcal{G} is hypercyclic. \square

REFERENCES

1. A. Ayadi, *Hypercyclic abelian groups of affine maps on \mathbb{C}^n* , Canad. Math. Bull. Vol. 56 (3), (2013) pp. 477-490.
2. A. Ayadi and H. Marzougui, *Dense orbits for abelian subgroups of $GL(n, \mathbb{C})$* , Foliations 2005: World Scientific, Hackensack, NJ, (2006), 47–69.
3. A. Ayadi and H. Marzougui *Abelian semigroups of matrices on \mathbb{C}^n and hypercyclicity*, Proceedings of the Edinburgh Mathematical Society, available on CJO. doi:10.1017/S0013091513000539.
4. A. Ayadi, H. Marzougui and Y. N'dao, *On the dynamic of abelian groups of affine maps on \mathbb{C}^n and \mathbb{R}^n* , Preprint ICTP, IC /2009/062 (2009).
5. F. Bayart, E. Matheron, *Dynamics of Linear Operators*, Cambridge Tracts in Math., 179, Cambridge University Press, (2009).
6. M. Javaheri, *A generalization of Dirichlet approximation theorem for the affine actions on real line*, Journal of Number Theory **128** (2008) 1146-1156.
7. R.S. Kulkarni, *Dynamics of linear and affine maps*, Asian J.Math. **12**, no 3, (2008), 321-344.
8. V. Bergelson, M. Misiurewicz and S. Senti, *Affine actions of a free semigroup on the real line* Ergod. Th. and Dynam. Sys. **26**, (2006), 1285–1305.
9. Waldschmidt.M, *Topologie des points rationnels*, Cours de troisième Cycle, Université P. et M. Curie (Paris VI), (1994/95).

YAHYA N'DAO, UNIVERSITY OF MONCTON, DEPARTMENT OF MATHEMATICS AND STATISTICS, CANADA
E-mail address: yahiaandao@yahoo.fr ; yahiaandao@voila.fr