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Abstract. Let Γ be a topological semigroup acting on a topological space X, and let Γ_0 be a subsemigroup of Γ. We give general conditions ensuring that Γ and Γ_0 have the same transitive points.

1. Introduction

In this paper, we consider the following problem. Let Γ be a topological semigroup acting continuously on a topological space X, and let Γ_0 be a sub-semigroup of Γ. Assume that Γ has a transitive point $x \in X$, i.e. the orbit $\Gamma \cdot x$ is dense in X. When is it possible to conclude that x is also a transitive point for the sub-semigroup Γ_0?

As stated, this is a problem in topological dynamics. However, our motivation comes from **linear dynamics**, i.e. the dynamics of linear operators. More precisely, our starting examples are the following three interesting results due to S. Ansari [**1**], F. León Saavedra-V. Müller [**16**] and A. Conejero-V. Müller-A. Peris [**7**].

(1) *Powers of hypercyclic operators are hypercyclic.*

(2) *Rotations of hypercyclic operators are hypercyclic.*

(3) *Every single operator in a hypercyclic* 1*-parameter semigroup* $(T_t)_{t\geq 0}$ *is hypercyclic.*

Moreover, in each case the hypercyclic vectors are the same. (Here we use the terminology prevailing in linear dynamics: an operator or a semigroup of operators is hypercyclic if it has a transitive point, and a hypercyclic vector is any such transitive point).

Besides the formal similarity of these results, the proofs of (2) and (3) given in [**16**] and [**7**] are quite similar too, and it is possible to give also a proof of (1) along the same lines. This was pointed out in Chapter 3 of [**4**], which was an attempt to push the analogy beyond this mere observation. However, at that time there was still something missing, namely some general statement of the form "two semigroups share the same transitive points" having (1), (2) and (3) as reasonably straightforward consequences.

On the other hand, such a statement was found by S. Shkarin in [**21**]. The main result of [**21**] is in fact purely topological (see Section 2), and it has the following consequence: *if T is a hypercyclic operator acting on a topological vector space X and if g is a topological generator of a compact group G then, for any T-hypercyclic vector $x \in X$, the set $\{(g^n, T^n x);\ n \geq 1\}$ is dense in $G \times X$. In other words, the point $(\mathbf{1}_G, x) \in G \times X$ is $(\tau_g \times T)$-transitive, where $\tau_g : G \to G$ is the (left) translation by g and $(\tau_g \times T)(h, z) = (gh, Tz)$.*

It is not hard to see that (1) and (2) above follow quite easily from Shkarin's result. (The inference of (3) is not that trivial, but this seems to be inevitable; see Section 2). Moreover, this is indeed a statement of the form "some semigroup Γ_0 has the same transitive points as some larger semigroup Γ": just let the semigroup $\Gamma := G \times \mathbb{N}$ act on $G \times X$ in the obvious way, i.e. $(\xi, n) \cdot (h, x) = (\xi h, T^n x)$, and put $\Gamma_0 := \{(g^n, n);\ n \geq 1\}$.

In this paper, our aim is to prove two general results of this type. That is, we give some conditions ensuring that a topological semigroup Γ (acting on some topological space X) and a sub-semigroup $\Gamma_0 \subset \Gamma$ have the same transitive points. Our first theorem is purely linear and can be used to recover the aforementioned results of Ansari, León-Müller and Conejero-Müller-Peris, whereas our second theorem is a generalization of Shkarin's theorem.

In both cases, a key role will be played by the *quotient space* Γ/Γ_0. Since we are dealing with semigroups and not groups, something is needed regarding the mere existence of this quotient. We shall say that $\mathbf{\Gamma/\Gamma_0}$ **is well-defined** if there is a topological *group* G and a continous and open surjective homomorphism $\pi_0 : \Gamma \to G$ such that $\Gamma_0 = \ker(\pi_0)$. Of course, we define the quotient group Γ/Γ_0 to be the group G, the obvious uniqueness question being easily settled (see Lemma 3.1 below).

Before stating the results, let us introduce some terminology. All topological spaces under consideration are assumed to be Hausdorff.

By a **dynamical system**, we mean a pair (X, Γ) where X is a topological space and Γ is a topological semigroup acting continuously on X. That is, we are given a jointly continuous map $(\gamma, x) \mapsto \gamma \cdot x$ from $\Gamma \times X$ into X such that $\gamma_1 \cdot (\gamma_2 \cdot x) = (\gamma_1 \gamma_2) \cdot x$ for any x, γ_1, γ_2. When $\Gamma = \mathbb{N} = \{1, 2, \dots\}$, i.e. when the action is given by the iterates of a single continuous map $T : X \to X$, we write (X, T) in place of (X, Γ).

The dynamical system (X, Γ) is said to be **point transitive** if there is some $x \in X$ such that $\Gamma \cdot x := \{\gamma \cdot x;\ \gamma \in \Gamma\}$ is dense in X. Any such point x is called a *transitive point* for Γ, and the set of all transitive points for Γ is denoted by $Trans(\Gamma)$. When $\Gamma = \mathbb{N} = \{1, 2, ...\}$, $(X, \Gamma) = (X, T)$, we write of course $Trans(T)$ in place of $Trans(\Gamma)$.

If (X, Γ) is a dynamical system then, for each $\gamma \in \Gamma$, we denote by $T_\gamma : X \to X$ the continuous map defined by $T_\gamma(x) = \gamma \cdot x$. When Γ has a unit (denoted by $\mathbf{1}$), it is assumed that $T_{\mathbf{1}}$ is the identity map.

The dynamical system is (X, Γ) is said to be **linear** if X is a topological vector space and every T_γ is a linear operator. In other words, the map $\gamma \mapsto T_\gamma$ is a *linear representation* of the topological semigroup Γ. In that case, we use the linear terminology and notation. Thus, we say "hypercyclic" instead of "transitive", and we write $HC(\Gamma)$ instead of $Trans(\Gamma)$.

Our first result reads as follows.

THEOREM 1.1. *Let (X, Γ) be a hypercyclic linear dynamical system with a completely metrizable acting semigroup Γ, and let Γ_0 be a subsemigroup of Γ such that Γ/Γ_0 is well-defined. Assume that the following hold:*

(a) *$\Gamma_0 \cdot X$ is dense in X;*

(b) *Γ/Γ_0 is compact and* abelian*;*

(c) *$HC(\Gamma)$ is Γ-invariant and there is at least one $\gamma \in \Gamma$ such that the operator T_γ is hypercyclic.*

Then (X, Γ_0) is hypercyclic, with the same hypercyclic vectors as (X, Γ).

Remark. When the semigroup Γ is abelian, it is easily seen that $HC(\Gamma)$ is Γ-invariant if and only if all operators T_γ have dense range. Hence, the assumptions of Theorem 1.1 are fulfilled if Γ is abelian, all operators T_γ have dense range, Γ/Γ_0 is compact and some T_γ is hypercyclic.

For the sake of illustration, we point out the following immediate consequence.

COROLLARY 1.1. *Let $\mathcal{T} = (T_\gamma)_{\gamma \in \mathbb{R}^n}$ be a (jointly continuous) group of linear operators such that at least one operator T_γ is hypercyclic. Then the group generated by any basis $(\gamma_1, \dots, \gamma_n)$ of $\mathbb{R}^n$ is hypercyclic, with the same hypercyclic vectors as $\mathcal{T}$.*

The nice thing with Theorem 1.1 is that it is rather general and very simply stated. Yet, it is is not completely satisfactory. Rather unexpectedly, we have been unable to deduce directly from it the Conejero-Müller-Peris theorem about 1-parameter hypercyclic semigroups in full generality, because we don't know how to prove directly that some operator T_a is hypercyclic if the semigroup $(T_t)_{t \geq 0}$ is, without assuming that the underlying topological vector space X is *metrizable*; but perhaps Theorem 1.1 is not responsible for that. However, there is a general version of the León-Müller theorem dealing with rotations of an arbitrary semigroup rather than rotations of a single operator (see Theorem 2.1), and this result does not follow either from Theorem 1.1. Finally, Theorem 1.1 is a linear statement from which Shkarin's theorem can certainly not be recovered.

Theorem 1.2 below is a kind of answer to these objections. Shkarin's theorem follows very easily from it, and it can also be used to prove directly the Conejero-Müller-Peris theorem with no metrizability assumption as well as the general version of the León-Müller theorem. These two results do not appear to follow from Shkarin's theorem itself, even though Theorem 1.2 is quite reminiscent from [**21**].

To formulate Theorem 1.2 properly, we need an additional definition. Recall first that if $T : Z \to Z$ is a continuous self-map of a topological space Z, a point $z \in Z$ is said to be **T-recurrent** if z is a cluster point of the sequence $(T^n(z))_{n\in\mathbb{N}}$. Since $\mathbb{N}$ starts with 1, it is equivalent to say that z is in the closure of the set $\{T^n(z);\ n \in \mathbb{N}\}$ (recall that Z is Hausdorff); in particular, any transitive point is recurrent. Moreover, we shall say that a topological space B is locally path-connected *at some point* $z \in B$ if z has a neighbourhood basis consisting of path-connected sets. Finally, recall that a topological space B is said to be *simply path-connected* if B is path-connected and any closed path in B is null-homotopic.

Definition. We shall say that a dynamical system (Z, T) has **property (S)** if there is a point $z \in Z$ with the following properties:

(i) z is T-recurrent;

(ii) one can find two sets $A, B \subset Z$ such that

- $z \in A \subset B$,
- A is T-invariant and path-connected,
- B is simply path-connected and locally path-connected at z.

For example, (Z, T) has property (S) provided Z has a T-invariant, simply path-connected and locally path-connected subset containing a T-recurrent point. In particular, if T is a hypercyclic linear operator then $(HC(T), T)$ has property (S); see Corollary 2.1 below. The letter "S" refers to Shkarin's paper [**21**].

Our second result reads as follows. Recall that a **character** of a topological semigroup Γ is a continuous homomorphism $\chi : \Gamma \to \mathbb{T}$, where $\mathbb{T}$ is the circle group. A character is *nontrivial* if it is not identically **1**. Throughout the paper, we denote by $\widehat{\Gamma}$ the character group of Γ.

Theorem 1.2. *Let (X, Γ) be a point transitive dynamical system, with a completely metrizable acting semigroup Γ. Let also Γ_0 be a sub-semigroup of Γ such that Γ/Γ_0 is well-defined. Assume that the following hold:*

(a) $\Gamma_0 \cdot X$ *is dense in* X*;*

(b) Γ/Γ_0 *is compact and* abelian*;*

(c) *for any nontrivial character $\chi \in \widehat{\Gamma}$ such that $\Gamma_0 \subset \ker(\chi)$, one can find $\gamma \in \Gamma$ such that $\chi(\gamma) \neq$ **1** and a T_γ-invariant set $Z \subset Trans(\Gamma)$ such that the dynamical system (Z, T_γ) has property (S).*

Then (X, Γ_0) is point transitive, with the same transitive points as (X, Γ).

Remark. Let $\pi_0 : \Gamma \to \Gamma/\Gamma_0$ be the quotient map, and let us denote by $\mathbf{S}$ the set of all $\gamma \in \Gamma$ for which one can find a T_γ-invariant set $Z \subset Trans(\Gamma)$ such that the dynamical system (Z, T_γ) has property (S). Then condition (c) above may be formulated as follows: *the subgroup generated by $\pi_0(\mathbf{S})$ is dense in Γ/Γ_0.*

It will be clear from the proofs that Theorem 1.1 is essentially a special case of Theorem 1.2, if not a formal consequence. However, since the former has a much simpler formulation, it seemed better to state it separately, at least for the sake of readability. One can formulate an artificial statement having both Theorems 1.1 and 1.2 as immediate consequences (see Section 6), but this seems to add nothing.

The general ideas needed for proving Theorems 1.1 and 1.2 are the same as in [**16**], [**7**] and [**21**]. As pointed out in [**21**], these ideas go back in fact to the influential paper [**9**] by H. Furstenberg. Actually, in the case of a *compact* ground space X, Shkarin's theorem is essentially proved in [**9**] and also in W. Parry's paper [**18**], albeit not stated explicitely in this form.

However, the examples we have in mind come from linear dynamics, where the space X is of course highly non-compact. Moreover, since we are dealing with a general semigroup Γ, some preliminary work is required to make the "usual" ideas work. Finally, the main difference with Chapter 3 of [**4**] are the following: (i) one of the assumptions made in [**4**] happens to be superfluous; (ii) when writing [**4**], the authors were not aware of Shkarin's theorem; (iii) while the final parts in the proofs of the results of Ansari, León-Müller and Conejero-Müller-Peris are treated separately in [**4**], with an *ad hoc* connectedness argument in each case, this is no longer the case in the present paper.

The paper is organized as follows. In Section 2, we explain how Theorems 1.1 and 1.2 can be used to recover the results of Ansari, León-Müller, Conejero-Müller-Peris and Shkarin mentioned at the beginning of this introduction. Section 3 contains some preliminary results about compact quotients of semigroups. The proofs of Theorems 1.1 and 1.2 are given in Section 4. The key steps are a general "abstract" result characterizing the non-transitivity of a sub-semigroup (Theorem 4.1), and a lemma showing that dynamical systems with property (S) have no nonconstant eigenfunction (Lemma 4.1). Section 5 contains some additional results. In particular, we prove there a "supercyclic" version of Theorem 1.1. Finally, we conclude the paper with a few remarks and some possibly interesting questions.

Notation. As already indicated, we denote by $\mathbb{N}$ the set of all *positive* integers; that is, $\mathbb{N}$ starts with 1. The set of all nonnegative integers is denoted by $\mathbb{Z}_+$, and the set of all nonegative real numbers by $\mathbb{R}_+$. Unless otherwise specified, $(0, \infty)$ will be considered as an *additive* semigroup. As a rule, we use the multiplicative notation for the law of a semi-group Γ, even if Γ is abelian. Accordingly, the unit element (if there is any) is denoted by the symbol $\mathbf{1}$. This has an obvious drawback when Γ is e.g. $\mathbb{R}_+$ or $\mathbb{Z}_+$: the unit element is $\mathbf{1} = 0$ and should not be confused with 1. Finally, if (X, Γ) is a dynamical system and Γ has a unit, it is assumed that the map $T_{\mathbf{1}} : X \to X$ is the identity map.

2. Applications of the main results

2.1. *The ALMCMP Theorem.* The following theorem summarizes the results of Ansari [**1**], León-Müller [**16**] and Conejero-Müller-Peris [**7**] mentioned in the introduction.

THEOREM 2.1. *Let X be a real or complex infinite-dimensional topological vector space.*

(1) *If T is a hypercyclic operator on X then T^p is hypercyclic for any positive integer p, with the same hypercyclic vectors.*

(2) *Assume that X is a complex vector space.*

  (a) *If T is a hypercyclic operator on X then ωT is hypercyclic for any $\omega \in \mathbb{T}$, with the same hypercyclic vectors.*

  (b) *More generally, let $\mathcal{S}$ be a multiplicative semigroup of operators on X, and assume that there exists an operator $R \in \mathcal{L}(X)$ commuting with $\mathcal{S}$ such that $R - \mu I$ has dense range for any $\mu \in \mathbb{C}$. If the semigroup $\mathbb{T} \cdot \mathcal{S} := \{\xi S;\ \xi \in \mathbb{T}, S \in \mathcal{S}\}$ is hypercyclic then so is $\mathcal{S}$, with the same hypercyclic vectors.*

(3) *If $\mathcal{T} = (T_t)_{t \geq 0}$ is jointly continuous hypercyclic semigroup of operators on X then each operator T_a, $a > 0$ is hypercyclic, with the same hypercyclic vectors as $\mathcal{T}$.*

Remark 1. Part (2a) follows indeed from (2b) by considering the semigroup $\mathcal{S}$ generated by ωT and putting $R := T$, since it is well-known that $P(T)$ has dense range for every nonzero polynomial P if T is hypercyclic (see e.g. [**4**], Chapter 1).

Remark 2. Part (2b) has interesting applications. In particular, it implies the so-called **positive supercyclicity theorem**: *If T is a* supercyclic *operator such that $T - \mu I$ has dense range for every $\mu \in \mathbb{C}$, then T is positively supercyclic, i.e. there is some $x \in X$ such that the set $\{rT^n(x);\ r > 0, n \in \mathbb{N}\}$ is dense in X. In fact, any supercyclic vector for T is positively supercyclic.* Positively supercyclic operators have been completely characterized in [**21**].

Remark 3. The joint continuity assumption in (3) (i.e. the continuity of the map $(t, x) \mapsto T_t(x)$) is easily seen to be equivalent to the *local equicontinuity* of the semigroup (T_t), i.e. the equicontinuity of $(T_t)_{t \in K}$ for any compact set $K \subset \mathbb{R}_+$. Moreover, if the Uniform Boundedness Principle is available (e.g. if the topological vector space X is barreled and locally convex or is a Baire space) then every C_0-semigroup on X is locally equicontinuous.

In this sub-section, we show how to deduce (1), (2a) and (3) from Theorem 1.1 assuming that the topological vector space X is *metrizable* for part (3), and the full Theorem 2.1 from Theorem 1.2.

2.1.1. *Using Theorem 1.1.* Parts (1) and (2a) follow immediately from the remark just after Theorem 1.1. In (1), the semigroup Γ is $\mathbb{N}$, $T_n = T^n$ and $\Gamma_0 = p\mathbb{N}$. The quotient Γ/Γ_0 is finite and $T_1 = T$ is hypercyclic. In (2a), Γ is $\mathbb{T}\times\mathbb{N}$, $T_{(\xi,n)} = \xi(\omega T)^n$ and $\Gamma_0 = \{\mathbf{1}\} \times \mathbb{N}$. The quotient Γ/Γ_0 is isomorphic to $\mathbb{T}$ and $T_{(\omega^{-1},1)} = T$ is hypercyclic. In both cases Γ is abelian and all operators T_γ, $\gamma \in \Gamma$ has dense range.

The deduction of (3) from Theorem 1.1 is less straightforward. We take as Γ the (additive) semigroup $(0,\infty)$ and $\Gamma_0 = a\mathbb{N}$, for some fixed $a > 0$. Then Γ/Γ_0 is well-defined and isomorphic to the circle group $\mathbb{T}$, thanks to the canonical quotient map $\pi_0 : (0,\infty) \to \mathbb{T}$ defined by $\pi_0(t) = e^{2i\pi t/a}$ (this map is indeed open since we are considering $\Gamma = (0,\infty)$ rather than $\mathbb{R}_+$). Moreover, if $z_0 \in HC(\Gamma)$ then, for any $A > 0$, the set $(A,\infty)\cdot z_0 := \{T_s(z_0);\ s > A\}$ is dense in X because the compact set $\{T_s(z_0);\ s \in [0,A]\}$ is nowhere dense. It follows that each operator T_t, $t > 0$ has dense range, since $(t,\infty)\cdot z_0 \subset \mathrm{ran}(T_t)$ by the semigroup property. What remains to be shown is that *some* operator T_t, $t > 0$ is hypercyclic. This is in fact an old result of Oxtoby and Ulam [**17**], which has nothing to do with linearity. Assuming that the topological vector space X is metrizable, one can prove it by a Baire category argument, as follows.

Since $(0,\infty)$ is separable, the space X is separable and metrizable, so it has a countable basis of open sets. Let $z_0 \in HC(\Gamma)$. We show that $z_0 \in HC(T_t)$ for comeager many $t \in (0,\infty)$. By the Baire category Theorem and since X is second-countable, it is enough to show that for every fixed open set $V \subset X$ and any nontrivial interval $(a,b) \subset (0,\infty)$, one can find $t \in (a,b)$ and $n \in \mathbb{N}$ such that $T_{nt}(z_0) \in V$. Now, it is easy to check that $\bigcup_{n\in\mathbb{N}}(na,nb)$ contains an interval (A,∞). Since (as observed above) the set $\{T_s(z_0);\ s > A\}$ is dense in X, one can find $s > A$ such that $T_s(z_0) \in V$; and by we have just said this s may be written as $s = nt$ with $t \in (a,b)$. □

Remark. Once (3) is known to hold in the metrizable case, one can deduce the result for a general topological vector space X by an argument due to K.-G. Grosse-Erdmann and A. Peris [**13**]. The trick is the following: *if $\Gamma = (T_t)_{t\geq 0}$ is a 1-parameter locally equicontinuous semigroup of operators on X then, for any neighbourhood W of 0 in X, one can find a Γ-invariant subspace $N \subset X$ such that $N \subset W$ and a metrizable vector space topology τ on X/N, coarser than the ususal quotient topology, such that the induced quotient semigroup $\Gamma_{X/N}$ is locally equicontinuous on $(X/N,\tau)$.* Taking this temporarily for granted, let us fix $a > 0$, and let $x \in X$ be any hypercyclic vector for Γ. It has to be shown that for any $z \in X$ and any neighbourhood O of 0 in X, one can find $n \in \mathbb{N}$ such that $T_{na}(x) \in z + O$. Choose a neighbourhood W of 0 such that $W + W \subset O$, and let N be as above. Then $[x]_{X/N}$ is a hypercyclic vector for the quotient semigroup $\Gamma_{X/N}$ because the canonical quotient map is continuous from X onto $(X/N,\tau)$. By the metrizable case, it follows that one can find $n \in \mathbb{N}$ such that $T_{na}(x) \in z + W + N \subset z + O$.

To find the subspace N and the topology τ as above, we may assume that W is balanced. Put $W_0 := W$ and use the local equicontinuity of Γ to get a decreasing sequence $(W_n)_{n\geq 0}$ of balanced neighbourhoods of 0 such that $W_{n+1} + W_{n+1} \subset W_n$ and $T_t(W_{n+1}) \subset W_n$ for each n and every $t \in [0, n]$. Then $N := \bigcap_{n\geq 0} W_n$ is a balanced additive subgroup of X, whence a linear subspace, and it is clearly T_t-invariant for every $t \in [0, \infty[$. The vector space topology τ on X/N is defined by declaring that $([W_n]_{X/N})_{n\geq 0}$ is a neighbourhood basis at 0. This is a Hausdorff topology since $\bigcap_n [W_n]_{X/N} = \{0\}$, so $(X/N, \tau)$ is metrizable since there is a countable neighbourhood basis at 0. Finally, the quotient semigroup $\Gamma_{X/N}$ is locally equicontinuous on $(X/N, \tau)$ because for any $K \in \mathbb{R}_+$ and any $n \geq 0$, one can find p such that $T_t(W_p) \subset W_n$ for all $t \in [0, K]$ (e.g. $p := 1 + \max(n, K)$).

2.1.2. *Using Theorem 1.2.* We now proceed to explain how to deduce Theorem 2.1 directly from Theorem 1.2. The first thing to do is of course to find a way of detecting property (S) inside a linear dynamical system. This is the content of the next lemma, where $\mathcal{L}(X)$ is equipped with the strong operator topology.

LEMMA 2.1. *Let X be a topological vector space, let $T \in \mathcal{L}(X)$, and let $Z \subset X$ be T-invariant. Assume that we have at hand at multiplicative semigroup $\mathcal{M} \subset \mathcal{L}(X)$ containing I and T and an operator $R \in \mathcal{L}(X)$ such that Z is invariant under every operator of the form $\alpha R + \beta M$, where $M \in \mathcal{M}$ and $\alpha, \beta \geq 0$ are not both 0. Then the dynamical system (Z, T) has property (S) provided one of the following holds:*

- *$\mathcal{M}$ is path-connected and T has a recurrent point $z \in Z$ such that $R(z) = z$;*
- *$\mathcal{M}$ is compact, path-connected and locally path-connected, and T has a recurrent point in Z.*

Proof. We first recall that a set $C \subset X$ is said to be *star-shaped* at some point $c \in C$ if $[c, v] \subset C$ for any $v \in C$. Clearly, any star-shaped set is simply path-connected. Moreover, any point $c \in X$ has a neighbourhood basis consisting of sets which are star-shaped at c (even though the topological vector space X is not assumed to be locally convex); and since the property of being star-shaped at c is preserved under intersections, it follows that if a set $C \subset X$ is star-shaped at c, then C is locally path-connected at c.

Now, let us fix a T-recurrent point $z \in Z$. We put $A := \mathcal{M}\cdot z = \{M(z);\ M \in \mathcal{M}\}$ and $B := \{sR(z) + (1-s)M(z);\ s \in [0, 1], M \in \mathcal{M}\}$. Then $z \in A \subset B \subset Z$ and A is T-invariant. Moreover, in both cases A is path-connected and $B = \bigcup_{v\in A}[R(z), v]$ is star-shaped at $R(z)$, hence simply path-connected. If $R(z) = z$, then B is star-shaped at z, hence locally path-connected at z. If $\mathcal{M}$ is compact and locally path-connected then B is locally path-connected, being a continuous image of the compact locally path-connected space $[0, 1] \times \mathcal{M}$.

□

Corollary 2.1. *Let X be a topological vector space, let $T \in \mathcal{L}(X)$, and let $Z \subset X$ be T-invariant. In each of the following 3 cases, the dynamical system (Z, T) has property (S).*

(i) *T is hypercyclic and $Z = HC(T)$;*

(ii) *$T = \lambda_0 I$ for some $\lambda \in \mathbb{T}$ and there is an operator $R \in \mathcal{L}(X)$ such that Z is invariant under $\alpha R + \beta' I$ whenever $(\alpha, \beta') \neq (0, 0)$.*

(iii) *$Z = HC(\mathcal{T})$ and $T = T_a$ for some $a > 0$, where $\mathcal{T} = (T_t)_{t \geq 0}$ is a 1-parameter hypercyclic semigroup on X.*

Proof. (i) Recall that $P(T)$ has dense range for any nonzero polynomial P, so that $Z = HC(T)$ is invariant under $P(T)$. Moreover, any $z \in Z$ is obviously T-recurrent. Denoting by $\mathcal{P}_+$ the set of all nonzero polynomials with nonnegative coefficients and applying Lemma 2.1 with $R = I$ and the *convex* multiplicative semigroup $\mathcal{M} := \{P(T);\ P \in \mathcal{P}_+\}$, the result follows.

(ii) Since $\lambda_0 \in \mathbb{T}$, any $z \in Z$ is T-recurrent. Apply Lemma 2.1 with the semigroup $\mathcal{M} := \{\lambda I;\ \lambda \in \mathbb{T}\}$.

(iii) Note that $Z = HC(\mathcal{T})$ is indeed T_a-invariant because T_a has dense range (as already observed) and commutes with $\mathcal{T}$. By Corollary 3.1 below, T_a has a recurrent point in Z. Moreover, $T_t + \mu I$ has dense range for any $t > 0$ and every $\mu \in \mathbb{K}$ by [**7**] Lemma 2.1, so that Z is invariant under any operator of the form $\alpha I + \beta T_t$, $t, \alpha, \beta \geq 0$, $(\alpha, \beta) \neq (0, 0)$. Applying Lemma 2.1 with $\mathcal{M} = \mathcal{T}$ and $R = I$, the result follows. □

We can now give the

Proof of Theorem 2.1. We take $\Gamma = \mathbb{N}$, $\Gamma_0 = p\mathbb{N}$ in case (1), $\Gamma = \mathbb{T} \times \mathcal{S}$ (where $\mathcal{S}$ has the discrete topology), $\Gamma_0 = \{\mathbf{1}\} \times \mathcal{S}$ in case (2b), and $\Gamma = (0, \infty)$, $\Gamma_0 = a\mathbb{N}$ in case (3). Put $\Gamma^* := \{\gamma \in \Gamma;\ HC(\Gamma) \text{ is } T_\gamma\text{−invariant}\}$. In each case, we just have to show that if $\chi \in \widehat{\Gamma}$ is a nontrivial character such that $\Gamma_0 \subset \ker(\chi)$, then one can find $\gamma \in \Gamma^*$ such that $\chi(\gamma) \neq \mathbf{1}$ and the dynamical system $(HC(\Gamma), T_\gamma)$ has property (S). We fix the character χ, and we put $Z := HC(\Gamma)$.

In case (1), $\Gamma^* = \Gamma$ since Γ is abelian and T^n has dense range for all $n \in \mathbb{N}$. Since χ is nontrivial and 1 generates $\Gamma = \mathbb{N}$, we have $\chi(1) \neq \mathbf{1}$; and since the operator $T_1 = T$ is hypercyclic, the dynamical system $(HC(\Gamma), T_1) = (HC(T), T)$ has property (S) by Corollary 2.1 (i).

In case (2b), we may assume that $\mathbf{1} = I \in \mathcal{S}$ since $HC(\mathcal{S} \cup \{I\}) = HC(\mathcal{S})$. Then one can find $\lambda_0 \in \mathbb{T}$ such that $\chi(\lambda_0, I) \neq \mathbf{1}$, since otherwise $\chi(\lambda, S) = \chi(\lambda, I)\chi(\mathbf{1}, S) = \mathbf{1}$ for every $(\lambda, S) \in \Gamma$. Since $T_{(\lambda_0, I)} = \lambda_0 I$, we have $(\lambda_0, I) \in \Gamma^*$. If R is the operator appearing in (2b), then $Z = HC(\Gamma)$ is invariant under $\alpha R + \beta' I$ for every $(\alpha, \beta') \neq (0, 0)$ since $\alpha R + \beta' I$ has dense range and commutes with Γ. By Corollary 2.1 (ii), the dynamical system $(HC(\Gamma), T_{(\lambda_0, I)})$ has property (S).

In case (3), $\Gamma^* = \Gamma$ because Γ is abelian and all operators T_t, $t > 0$ have dense range. By Corollary 2.1 (iii), the dynamical system (Z, T_a) has property (S) for every $a \in \Gamma$. Thus, we may pick any $a \in \Gamma$ such that $\chi(a) \neq \mathbf{1}$. □

2.2. *Shkarin's Theorem.* The following theorem is the main result of [**21**].

THEOREM 2.2. *Let $T : X \to X$ be a continuous point transitive map on a topological space X. Assume that one can find a nonempty set $Y \subset Trans(T)$ which is T-invariant, simply path-connected and locally path-connected. Let also G be a monothetic compact group, and let g be a topological generator of G. Then, for any $x \in Trans(T)$, the set $\{(g^n, T^n(x));\ n \geq 1\}$ is dense in $G \times X$.*

Remark 1. The "linear" consequence quoted in the introduction is obtained by taking e.g. $Y := \{P(T)x;\ P$ polynomial $\neq 0\}$, for some T-hypercyclic vector x. Parts (1) and (2a) in Theorem 2.1 follow quite easily from this result, but part (2b) does not. The deduction of (3) seems to require a metrizability assumption, just like in the first proof of Theorem 2.1 given above.

Remark 2. When the space X is *compact* and metrizable, Shkarin's theorem is essentially proved (but not stated) in [**9**] and [**18**]. It is not clear that the proofs given there can be adapted to give the result for an arbitrary topological space X.

Let us see how Shkarin's theorem can be deduced from Theorem 1.2.

Proof of Theorem 2.2. We first note the following slight subtlety: to show that the set $\{(g^n, T^n(x));\ n \geq 1\}$ is dense in $G \times X$ for any $x \in Trans(T)$, it is in fact enough to show that $\{(g^n, T^n(x));\ n \geq 0\}$ is dense. Indeed, since the set Y is connected and dense in X, the space X has no isolated points (unless it is reduced to a single point, in which case there is nothing to prove). Hence $G \times X$ has no isolated points either and we may replace "$n \geq 1$" by "$n \geq 0$".

Accordingly, we take $\Gamma = G \times \mathbb{Z}_+$ (and not $G \times \mathbb{N}$), $T_{(\xi,n)}(h, x) = (\xi h, T^n(x))$, and $\Gamma_0 = \{(g^n, n);\ n \in \mathbb{Z}_+\}$. Since G is abelian (being monothetic), the map $\pi_0 : \Gamma \to G$ defined by $\pi_0(\xi, n) = g^{-n}\xi$ is a continuous homomorphism from Γ onto G with kernel Γ_0, and π_0 is easily seen to be open. Thus, $\Gamma/\Gamma_0 \simeq G$ is well-defined, compact and abelian. Moreover, $\Gamma_0 \cdot X$ is dense in X and $Trans(\Gamma)$ is Γ-invariant because Γ is abelian and every T_γ has dense range. Finally, it is clear that $Trans(\Gamma) = G \times Trans(T)$.

Now, let $\chi \in \widehat{\Gamma}$ be a nontrivial character such that $\Gamma_0 \subset \ker(\chi)$. Then we must have $\chi(\mathbf{1}_G, 1) \neq \mathbf{1}$. Indeed, otherwise $\chi(\mathbf{1}_G, n) = \chi(\mathbf{1}_G, 1)^n = \mathbf{1}$ for every $n \in \mathbb{Z}_+$, and hence $\chi(\xi, n) = \chi(\xi, 0)\chi(\mathbf{1}_G, n)$ depends only on $\xi \in G$. But since $\chi(g, 1) = \mathbf{1}$, it follows that $\chi(g^k, n) = \chi(g^k, k) = \mathbf{1}$ for all $k \in \mathbb{N}$ and all $n \in \mathbb{Z}_+$, a contradiction since χ is nontrivial and g is a topological generator of G. Now, it follows at once from the assumptions of Shkarin's theorem that the dynamical system $(Trans(\Gamma), T_{(\mathbf{1}_G,1)}) = (G \times Trans(T), I_G \times T)$ has property (S): just put $A = B := \{\mathbf{1}_G\} \times Y$. □

3. Compact quotients of semigroups

3.1. *Definition of the quotient.* By a **quotient map** of a topological semigroup Γ, we mean any continous and open homomorphism $\pi : \Gamma \to G$ from Γ onto some topological *group* G. As the following trivial lemma shows, quotient maps are well suited to define ... quotients.

LEMMA 3.1. *Let* Γ *be a topological semigroup, and let* $\pi:\Gamma\to G$ *be a quotient map of* Γ*. Let also* $\phi:\Gamma\to H$ *be a continuous homomorphism from* Γ *into a topological group* H*, and assume that* $\ker(\pi)\subset\ker(\phi)$*. Then* ϕ *factors through the quotient map* $\pi:\Gamma\to G$*: there is a unique continuous homomorphism* $\widetilde{\phi}:G\to H$ *such that* $\phi=\widetilde{\phi}\circ\pi$*.*

Proof. The only thing we have to check is that if $\gamma_1,\gamma_2\in\Gamma$ satisfy $\pi(\gamma_1)=\pi(\gamma_2)$ then $\phi(\gamma_1)=\phi(\gamma_2)$. Once this is done, one can unambiguously define $\widetilde{\phi}:G\to H$ by the requirement $\phi=\widetilde{\phi}\circ\pi$ (the continuity of $\widetilde{\phi}$ coming from the open-ness of π). Since π is onto, one can find $\xi\in\Gamma$ such that $\pi(\xi)=g^{-1}$, where $\pi(\gamma_1)=g=\pi(\gamma_2)$. Then $\pi(\gamma_1\xi)=\mathbf{1}_G=\pi(\gamma_2\xi)$, i.e. $\gamma_1\xi\in\ker(\pi)$ and $\gamma_2\xi\in\ker(\pi)$. Hence $\phi(\gamma_1\xi)=\mathbf{1}_G=\phi(\gamma_2\xi)$, so that $\phi(\gamma_1)=\phi(\xi)^{-1}=\phi(\gamma_2)$. □

It follows from this lemma that if $\pi:\Gamma\to G$ and $\pi':\Gamma\to G'$ are two quotient maps of Γ then the topological groups G and G' are isomorphic, and in fact there is unique topological isomorphism $J:G\to G'$ such that $J\circ\pi=\pi'$. Thus, given a subsemigroup $\Gamma_0\subset\Gamma$, it makes sense to speak of the quotient topological group Γ/Γ_0 provided there is a quotient map of Γ with kernel Γ_0, and one can even speak of "the canonical quotient map" $\pi_0:\Gamma\to\Gamma/\Gamma_0$.

3.2. *Fundamental domains.* It is well-known that any continous and open map $\pi:E\to F$ from a completely metrizable space E onto a topological space F is *compact covering*, i.e. each compact set $L\subset F$ is the image of some compact set $K\subset E$ (see e.g. [**6**] IX.2, Proposition 18). In the context of compact quotients of semigroups, we have the following more precise result, which will be essential for our purpose.

LEMMA 3.2. *Let* Γ *be a completely metrizable topological semigroup, and let* Γ_0 *be a subsemigroup of* Γ*. Assume that* Γ/Γ_0 *is well-defined and compact. Then, for any* $\gamma_0\in\Gamma_0$*, there exists a compact set* $K_0\subset\Gamma$ *such that the following properties hold :*

- $K_0\cap\Gamma_0=\{\gamma_0\}$*;*
- *for any* $\gamma\in\Gamma$*, one can find* $k\in K_0$ *such that* $\gamma k\in\Gamma_0$ *and* $k\gamma\in\Gamma_0$*.*

Proof. Let d be a compatible complete metric on Γ, and let $\pi_0:\Gamma\to\Gamma/\Gamma_0$ be the canonical quotient map. Let us also put $A=\{\gamma_0\}\cup(\Gamma\setminus\Gamma_0)$. Finally, let $(\varepsilon_n)_{n\geq 1}$ be a decreasing sequence of positive numbers tending to 0.

For any point $\gamma\in A$, choose an open set V^1_γ such that $\gamma\in V^1_\gamma$ and $\mathrm{diam}(V^1_\gamma)\leq\varepsilon_1$, and moreover $\overline{V^1_\gamma}\cap\Gamma_0=\emptyset$ if $\gamma\neq\gamma_0$ (this can be done since Γ_0 is closed in Γ). The sets $\pi_0(V^1_\gamma)$, $\gamma\in A$ obviously cover Γ/Γ_0. Since the quotient map $\pi_0:\Gamma\to\Gamma/\Gamma_0$ is open and $G=\Gamma/\Gamma_0$ is compact, one can find a finite set $I_1\subset A$ such that

$\pi_0(W_1) = G$, where $W_1 = \bigcup_{\gamma\in I_1} V^1_\gamma$. Note that $W_1 \cap \Gamma_0 \subset V^1_{\gamma_0}$, so $W_1 \cap \Gamma_0$ has diameter at most ε_1; and of course $\gamma_0 \in W_1$.

Now repeat the construction with A replaced by $A_1 := (W_1 \setminus \Gamma_0) \cup \{\gamma_0\}$, choosing open sets V^2_γ with diameter at most ε_2 and such that $\overline{V_\gamma} \cap \Gamma_0 = \emptyset$ if $\gamma \neq \gamma_0$. We also require that each $\overline{V^2_\gamma}$ is contained in $V^1_{\gamma'}$ for some $\gamma' \in I_1$. This produces a finite set $I_2 \subset \Gamma$ and an open set W_2. Proceeding inductively, we construct a sequence of open sets $W_n \subset \Gamma$ and a sequence of finite sets $I_n \subset \Gamma$ such that the following properties hold :

(i) W_n has the form $W_n = \bigcup_{\gamma\in I_n} V^n_\gamma$, where $\mathrm{diam}(V^n_\gamma) \le \varepsilon_n$;

(ii) each $\overline{V^{n+1}_\gamma}$ is contained in $V^n_{\gamma'}$ for some $\gamma' \in I_n$;

(iii) $\pi_0(W_n) = G$;

(iv) $\gamma_0 \in W_n$ and $\mathrm{diam}(W_n \cap \Gamma_0) \le \varepsilon_n$.

Now, put $K_0 = \bigcap_{n\ge 1} \overline{W_n}$. Then K_0 is a compact subset of Γ by (i), and $K_0 \cap \Gamma_0 = \{\gamma_0\}$ by (iv). Moreover, it follows from (ii) and (iii) that $\pi_0(K_0) = G$. Indeed, let us fix $g \in G$. Consider the set $\mathcal{T}_g$ made up of of all finite sequences of the form $(\gamma_1, \dots, \gamma_n)$, where $\gamma_k \in I_k$, $g \in \pi_0(V^k_\gamma)$ for all k, and $\overline{V^{k'}_{\gamma_{k'}}} \subset V^k_{\gamma_k}$ whenever $k' > k$. Then $\mathcal{T}_g$ is a finitely branching tree (with respect to the extension ordering), which contains arbitrarily long finite sequences by (ii) and (iii). By König's infinity Lemma, the tree $\mathcal{T}_g$ has an infinite branch. In other words, one can find an infinite sequence $(\gamma_n) \in \prod_{n\ge 1} I_n$ such that $\overline{V^{n+1}_{\gamma_{n+1}}} \subset V^n_{\gamma_n}$ and $g \in \pi_0(V^n_{\gamma_n})$ for all $n \ge 1$. For each n, pick a point $\xi_n \in V^n_{\gamma_n}$ such that $\pi_0(\xi_n) = g$. Then (ξ_n) is a Cauchy sequence in Γ whose limit ξ belongs to $\bigcap_{n\ge 1} \overline{V^n_{\gamma_n}} \subset K_0$, and $\pi_0(\xi) = g$ by the continuity of π_0.

To conclude the proof, let $\gamma \in \Gamma$ be arbitray. Pick a point $k \in K_0$ such that $\pi_0(k) = \pi_0(\gamma)^{-1}$. Then $\pi_0(\gamma k) = \mathbf{1}_G = \pi_0(k\gamma)$, so that $\gamma k \in \Gamma_0$ and $k\gamma \in \Gamma_0$. □

Remark 1. Any compact set $K_0 \subset \Gamma$ satisfying the conclusion of Lemma 3.2 wil be called a **fundamental domain for** $(\Gamma/\Gamma_0, \gamma_0)$.

Remark 2. In all the "concrete" applications we have in mind, the existence of a fundamental domain is obvious. For example, when $\Gamma = \mathbb{N}$, $\Gamma = p\mathbb{N}$ and $\gamma_0 = p$ we may take $K_0 = \{p\}$; when $\Gamma = (0, \infty)$, $\Gamma_0 = a\mathbb{N}$ and $\gamma_0 = a$ we may take $K_0 = [a/2, 3a/2]$; and when $\Gamma = G \times \Gamma_0$ where G is a compact group we may take $K_0 = G \times \{\gamma_0\}$.

3.3. *Recurrence.* As an application of Lemma 3.2, we now prove a result concerning recurrent points. We have already used a very special case of it in Section 2 (see the second proof of Theorem 2.1). However, this particular case does not appear to be very much easier to prove than the general result.

Let us first recall some well-known terminology. If I is any set, we denote by 2^I the power set of I. A family $\mathcal{F} \subset$ is said to be *co-hereditary* if it is upward closed under inclusion, i.e. $\mathcal{F} \ni F \subset F'$ implies $F' \in \mathcal{F}$. A co-hereditary $\mathcal{F} \subset 2^I$ is *proper* if $\mathcal{F} \neq \emptyset$ and all sets $F \in \mathcal{F}$ are nonempty. A *filter* of subsets of I is a proper co-hereditary family $\mathcal{F} \subset 2^I$ which is closed under finite intersections.

Let (X, Γ) be a dynamical system, and let $\mathcal{F}$ be a proper co-hereditary family of subsets of Γ. A point $x \in X$ is said to be **$\mathcal{F}$-recurrent** if the set

$$\mathbf{N}(x, V) := \{\gamma \in \Gamma;\ \gamma \cdot x \in V\}$$

belongs to $\mathcal{F}$ for any neighbourhood V of x. When Γ is (infinite and) discrete and $\mathcal{F} = \mathcal{F}_\infty$, the family of all infinite subsets of Γ, this yields the usual notion of recurrence. In particular, when $\Gamma = \mathbb{N}$, $(X, \Gamma) = (X, T)$, the $\mathcal{F}_\infty$-recurrent points are simply the T-recurrent points.

If $\mathcal{F} \subset 2^\Gamma$ is a proper co-hereditary family, the *dual family* $\mathcal{F}^*$ is the family of all sets $F^* \subset \Gamma$ such that $F^* \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. Clearly $\mathcal{F}^*$ is also proper and co-hereditary, and it is not hard to check that $(\mathcal{F}^*)^* = \mathcal{F}$. Moreover, a point $x \in X$ is $\mathcal{F}^*$-recurrent if and only if $x \in \overline{F \cdot x}$ for every $F \in \mathcal{F}$; and when $\mathcal{F}$ is a filter, this means that one can find a net $(\gamma_j) \subset \Gamma$ such that $\gamma_j \cdot x \to x$ and $\gamma_j \in F$ eventually for any $F \in \mathcal{F}$.

An natural example is obtained by considering the family $\mathcal{F}_\leftarrow$ of all "terminal" subsets of Γ: a set F is in $\mathcal{F}_\leftarrow$ iff it contains $\Gamma\tau$ for some $\tau \in \Gamma$. When $\Gamma = \mathbb{N}$, this is the family of cofinite sets, so $(\mathcal{F}_\leftarrow)^*$-recurrence is just T-recurrence; and when $\Gamma = (0, \infty)$, a point $x \in$ is $(\mathcal{F}_\leftarrow)^*$-recurrent iff there is some net $(t_i) \subset \Gamma$ tending to $+\infty$ such that $T_{t_i}(x) \to x$. Another natural example is when Γ is locally compact and non-compact, and $\mathcal{F}$ is the family of all (punctured) neighbourhoods of ∞. In this case, a point $x \in X$ is $\mathcal{F}^*$-recurrent iff there is a net $(t_i) \subset \Gamma$ tending to ∞ (which does not mean that $t_i \to +\infty$ in the case $\Gamma = (0, \infty)$) such that $T_{t_i}(x) \to x$.

If Γ_0 is a subset of Γ and $\mathcal{F} \subset 2^\Gamma$, we put $\mathcal{F} \cap \Gamma_0 := \{F \cap \Gamma_0;\ F \in \mathcal{F}\}$. Of course, $\mathcal{F} \cap \Gamma_0$ is co-hereditary if $\mathcal{F}$ is, but $\mathcal{F} \cap \Gamma_0$ may not be proper (if $\mathcal{F}$ is), i.e. it may contain $\emptyset$.

LEMMA 3.3. *Let (X, Γ) be a dynamical system with a completeley metrizable acting semigroup Γ, and let Γ_0 be a subsemigroup of Γ. Assume that Γ/Γ_0 is well-defined and compact. Let also $\mathcal{F} \subset 2^\Gamma$ be a* filter *of subsets of Γ, invariant under right-translations, i.e. $F \in \mathcal{F}$ implies $F\tau \in \mathcal{F}$ for every $\tau \in \Gamma$. Finally, let $\gamma_0 \in \Gamma_0$. Assume that there exists a fundamental domain K_0 for $(\Gamma/\Gamma_0, \gamma_0)$ with the following property: for any $\alpha \in \Gamma$ and every $F \in \mathcal{F}$, the set $\{\xi \in \Gamma;\ K_0\alpha\,\xi \subset F\}$ belongs to $\mathcal{F}$. Then ($\mathcal{F} \cap \Gamma_0$ is proper and) for any $\mathcal{F}^*$-recurrent $x \in X$, the point $T_{\gamma_0}(x)$ is $(\mathcal{F} \cap \Gamma_0)^*$-recurrent.*

Proof. Let $x \in X$ be Γ-recurrent. We have to show that $\gamma_0 x \in \overline{(F \cap \Gamma_0) \cdot (\gamma_0 x)}$ for every $F \in \mathcal{F}$; so we fix $F \in \mathcal{F}$ and we put $F_0 = F \cap \Gamma_0$. We are looking for a net $(\tau_i) \subset F \cap \Gamma_0$ such that $\tau_i \gamma_0 x \to \gamma_0 x$.

Put $G := \Gamma/\Gamma_0$ and let $\pi_0 : \Gamma \to G$ be the canonical quotient map. We first show that one can find a net $(\gamma_i) \subset \Gamma$ such that $K_0\gamma_i \subset F$, $\gamma_i\gamma_0 \cdot x \to x$ and

$\pi_0(\gamma_i\gamma_0) \to \mathbf{1}_G$. To do this, we have to check that for each neighbourhood V of x and each neighbourhood O of $\mathbf{1}_G$, one can find $\gamma \in \Gamma$ such that $K_0\gamma \subset F$, $\pi_0(\gamma\gamma_0) \in O$ and $(\gamma\gamma_0) \cdot x \in V$.

Since x is $\mathcal{F}^*$- recurrent and $\mathcal{F}$ is a filter, one can find a net $(\lambda_j) \subset \Gamma$ such that $\lambda_j \cdot x \to x$ and $\lambda_j \in B$ eventually for any $B \in \mathcal{F}$. By compactness, we may assume that $\pi_0(\lambda_j) \to g \in G$. By compactness of G again, we may choose a positive integer N such that $g^N \in O$, and then a neighbourhood W of g such that $\prod_{i=1}^N g_i \in O$ whenever $g_1, \dots, g_N \in W$. Now pick j_1 such that $\lambda_{j_1} \cdot x \in V$ and $\pi_0(\lambda_{j_1}) \in W$, then j_2 such that $(\lambda_{j_1}\lambda_{j_2}) \cdot x \in V$ (i.e. $\lambda_{j_2} \cdot x \in T_{\lambda_1}^{-1}(V)$) and $\pi_0(\lambda_{j_2}) \in W$, ..., and finally j_N such that $(\lambda_{j_1} \cdots \lambda_{j_N}) \cdot x \in V$, $\pi_0(\lambda_{j_N}) \in W$ *and* λ_{j_N} may be written as $\lambda_{j_N} = \xi\gamma_0$ for some $\xi \in \gamma$ such that $(K_0\lambda_{j_1} \cdots \lambda_{j_{N-1}}) \cdot \xi \subset F$. This can be done since by assumption the set $A := \{\xi \in \Gamma;\ (K_0\lambda_{j_1} \cdots \lambda_{j_{N-1}}) \cdot \xi \subset F\}$ is in $\mathcal{F}$, and so is $B = A \cdot \gamma_0$ since $\mathcal{F}$ is invariant under right-translations. Then $\lambda := \lambda_{j_1} \cdots \lambda_{j_N}$ may be written as $\lambda = \gamma\gamma_0$ where γ has the required properties.

Having our net (γ_i) at hand, we may pick $k_i \in K_0$ such that $\tau_i := k_i\gamma_i \in \Gamma_0$ for each i. Then $\tau_i \in F \cap \Gamma_0$ by the choice of γ_i. By compactness, we may assume that $k_i \to k \in K_0$. Then $(k_i\gamma_i\gamma_0) \cdot x \to k \cdot x$, by the joint continuity of the map $(\gamma, z) \mapsto \gamma \cdot z$. Now, $\pi_0(k_i\gamma_i\gamma_0) = \mathbf{1}_G = \pi_0(\gamma_0)$ and $\pi_0(\gamma_i\gamma_0) \to \mathbf{1}_G$, hence $\pi_0(k_i) \to \mathbf{1}_G$. It follows that $\pi_0(k) = \mathbf{1}_G$, so that $k \in K_0 \cap \Gamma_0$ and hence $k = \gamma_0$. Thus, we have found a net $(\tau_i) \subset F \cap \Gamma_0$ such that $\tau_i\gamma_0 \cdot x \to \gamma_0 \cdot x$, as required. □

The particular case of Lemma 3.3 that was used in Section 2 reads as follows:

Corollary 3.1. *Let $(T_t)_{t\geq 0}$ be a locally equicontinuous semigroup of operators on a topological vector space X, and let $x \in X$. Assume that there is a net (t_i) tending to $+\infty$ such that $T_{t_i}(x) \to x$. Then, for each $a > 0$, the point $T_a(x)$ is T_a-recurrent.*

Proof. Apply Lemma 3.3 with $\Gamma = (0, \infty)$, $\Gamma_0 = a\mathbb{N}$, $\gamma_0 = a$ and the family $\mathcal{F}_\leftarrow$ of all terminal subsets of Γ. The family $\mathcal{F}_\leftarrow$ is indeed a filter because Γ is abelian. □

4. **Proofs of Theorems 1.1 and 1.2**

In this section, we prove Theorems 1.1 and 1.2. The proofs rely on a general abstract result about non-transitive subsemigroups (Theorem 4.1) and a lemma concerning dynamical systems with property (S) (Lemma 4.1). For the sake of readability, we first state Theorem 4.1, then state and prove Lemma 4.1, then give the proofs of Theorems 1.1 and 1.2, and finally prove the key Theorem 4.1. The following definition will be useful throughout.

Definition. Let (X, Γ) and (X', Γ) be two dynamical system (with the same acting semigroup Γ). Let also $Z \subset X$. Then the dynamical system (X', Γ) is said to be a **pseudo-factor of** (Z, Γ) if there is a continuous map $p : Z \to X'$ such that $p(\gamma \cdot z) = \gamma \cdot p(z)$ whenever $(z, \gamma) \in Z \times \Gamma$ and $\gamma \cdot z \in Z$.

When Z is Γ-invariant and the above map $p : Z \to X'$ is onto, the dynamical system (X', Γ) is a **factor** of the (well-defined) dynamical system (Z, Γ). This is a basic notion in topological dynamics, which justifies somehow the terminology "pseudo-factor". However, in the present case the set Z is not assumed to be Γ-invariant (so that, strictly speaking, there is no dynamical system (Z,Γ)), and the pseudo-factoring map p is not even assumed to have dense range.

4.1. *Eigencharacters and eigenfunctions.* Let (X,Γ) be a dynamical system, and let $Z\subset X$. A character $\chi\in\widehat{\Gamma}$ is an **eigencharacter for** (Z,Γ) if there exists a continuous function $f:Z\to\mathbb{T}$ such that $f(\gamma\cdot z)=\chi(\gamma)f(z)$ whenever $(z,\gamma)\in Z\times\Gamma$ and $\gamma\cdot z\in Z$. Such a function f is called an **eigenfunction** associated with χ. This terminology calls for some comments.

Remark 1. As in the above definition of pseudo-factors, the set Z is not assumed to be Γ-invariant.

Remark 2. One may call "eigenfunction for (Z,Γ)" any map $f:Z\to\mathbb{T}$ such that $f(\gamma\cdot z)=\chi(\gamma)f(z)$ whenever $(z,\gamma)\in Z\times\Gamma$ and $\gamma\cdot z\in Z$, for some map $\chi\to\mathbb{C}$. Putting $\Gamma_Z:=\{\gamma\in\Gamma;\ \gamma\cdot Z\subset Z\}$, it is easily checked that χ induces a homomorphism from the semigroup Γ_Z into the circle group $\mathbb{T}$, hence a character of Γ_Z if f is continuous. However, it is a priori unclear whether χ can be extended to a character of Γ, i.e. to an eigencharacter for (Z,Γ).

Remark 3. An eigenfunction for a dynamical system $(Z,\mathbb{N})=(Z,T)$ is nothing else but a pseudo-factoring maps from (Z,T) into a dynamical system of the form $(\mathbb{T},\tau_g)$, where $g\in\mathbb{T}$ and τ_g is the (left) translation by g. Such dynamical systems are ususally called *Kronecker systems*; see below.

The following result is the key to the proofs of Theorems 1.1 and 1.2.

Theorem 4.1. *Let (X,Γ) be a point transitive dynamical system, with a completely metrizable acting semigroup Γ. Let also Γ_0 be a sub-semigroup of Γ such that $G=\Gamma/\Gamma_0$ is well-defined, compact and* abelian. *Finally, assume that $\Gamma_0\cdot X$ is dense in X. If $Trans(\Gamma)\neq Trans(\Gamma_0)$, then there is a nontrivial character $\chi\in\widehat{\Gamma}$ which is an eigencharacter for $(Trans(\Gamma),\Gamma)$ and such that $\Gamma_0\subset\ker(\chi)$.*

There seems to be no hope of reversing the implication in Theorem 4.1: after all, there may be no pair $(z,\gamma)\in Trans(\Gamma)\times\Gamma$ such that $\gamma\cdot z\in Trans(\Gamma)$. However, if $Trans(\Gamma)$ is Γ-invariant then we do get a rather intuitive characterization of the non-transitivity of Γ_0. This holds in particular if Γ is abelian and all maps T_γ, $\gamma\in\Gamma$ have dense range.

Corollary 4.1. *Let (X,Γ) be a point transitive dynamical system, with a completely metrizable acting semigroup Γ, and let Γ_0 be a sub-semigroup of Γ such that $G=\Gamma/\Gamma_0$ is well-defined, compact and abelian. Assume that $\Gamma_0\cdot X$ is dense in X and that $Trans(\Gamma)$ is Γ-invariant. Then the following are equivalent:*

(i) *there is some point $x\in X$ which is Γ-transitive but not Γ_0-transitive;*

(ii) *the dynamical system* $(Trans(\Gamma), \Gamma)$ *admits a nontrivial eigencharacter which is trivial on* Γ_0*;*

(iii) *the dynamical system* $(Trans(\Gamma), \Gamma)$ *admits a continous, non constant eigenfunction which is constant on every* Γ_0*-orbit;*

(iv) *the sub-semigroup* Γ_0 *has no transitive points.*

Proof. That (i) implies (ii) is the content of Theorem 4.1, and the implications (ii) $\implies$ (iii) $\implies$ (iv) $\implies$ (i) are obvious. □

Thus, we see that (under the assumptions of Corollary 4.1) Γ and the sub-semigroup Γ_0 have the same transitive points provided that Γ_0 is already known to be point transitive. It would be nice to have a simple direct proof of this result.

4.2. *Property (S) and Kronecker systems.* In this sub-section, we prove that dynamical systems with property (S) have no nontrivial Kronecker pseudo-factors. Let us first recall the relevant definition.

Definition. A **Kronecker system** is a dynamical system of the form (K, τ_g), where K is a compact abelian group and τ_g is the translation by some fixed element $g \in K$. It is **nontrivial** if $g \neq \mathbf{1}_K$.

This definition is slightly nonstandard: usually, it is required that g is a topological generator of K. This does not really matter since one can consider instead of K the closed subgroup of K generated by g. On the other hand, it is important to note that the compact group K is assumed to be abelian. It follows that if a Kronecker system (K, τ_g) is nontrivial then it has a nontrivial Kronecker pseudo-factor of the form $(\mathbb{T}, \tau_h)$. Indeed, if $\chi : K \to \mathbb{T}$ is any character of K such that $\chi(g) \neq \mathbf{1}$ then χ is a pseudo-factoring map from (K, τ_g) into the nontrivial system $(\mathbb{T}, \tau_{\chi(g)})$.

For the sake of brevity, we shall say that a dynamical system (X, T) is **anti-Kronecker** if it has no nontrivial Kronecker pseudo-factor. The proof of the next lemma is greatly inspired from that of Lemma 2.7 in [**21**].

Lemma 4.1. *Dynamical systems with property (S) are anti-Kronecker.*

Proof. Let (Z, T) have property (S), and assume that (Z, T) has a nontrivial Kronecker pseudo-factor (K, τ_g), with witness $p : Z \to K$. As noticed a few lines above, we may assume that $K = \mathbb{T}$.

Choose z, A, B according to he definition of property (S). Since the set B is simply path-connected, one can define a map $f : B \to \mathbb{R}$ as follows: for any $x \in B$, $f(x)$ is the winding number $\mathbf{w}(p \circ \alpha_x)$ of $p \circ \alpha_x$, for any path α_x inside B starting at z and ending up at x. Moreover, this map is *continuous* at z because B is locally path-connected at z. Hence, we can choose an open neighbourhood V of z such that $|\mathbf{w}(p \circ \alpha)| < 1$ for any path α inside B with endpoints in V.

Since z is T-recurrent, we can find a large positive integer N such that $T^N(z) \in V$. Now, let α_0 be a continuous path inside A with initial point z and terminal point $T(z)$, and for any $i \in \{0, \dots, N-1\}$, put $\alpha_i := T^i \circ \alpha_0$, so that α_i is a path inside A with initial point $T^i(z)$ and terminal point $T^{i+1}(z)$. Now let α be the concatenation of $\alpha_0, \dots, \alpha_{N-1}$. This is a path inside B with endpoints in V, so $|\mathbf{w}(p \circ \alpha)| < 1$. However, $\mathbf{w}(p \circ \alpha) = \sum_{i=0}^{N-1} \mathbf{w}(p \circ \alpha_i)$. Since $p(T(x)) = gp(x)$ for any $x \in Z$, we see that $p \circ \alpha_i = g^i \cdot (p \circ \alpha_0)$ for every $i \in \{0, \dots, N-1\}$, so that $\mathbf{w}(p \circ \alpha_i) = \mathbf{w}(p \circ \alpha_0)$. Hence, we get $\mathbf{w}(p \circ \alpha) = N\,\mathbf{w}(p \circ \alpha_0)$. Finally, $\mathbf{w}_0 := \mathbf{w}(p \circ \alpha_0)$ is nonzero since $p \circ \alpha_0$ has endpoints $p(z)$ and $gp(z) \neq p(z)$ (we are assuming that $g \neq \mathbf{1}$). Thus, we get $|\mathbf{w}(p \circ \alpha)| \geq 1$ if N is large enough, since $\mathbf{w}_0$ does not depend on N. This is a contradiction. □

Remark 1. It follows from Lemma 4.1 that if (X, T) is a dynamical system with at least one recurrent point and X is simply path-connected and locally path connected, then (X, T) is anti-Kronecker. In particular, if X is *compact*, simply path-connected and locally path-connected, then any dynamical system (X, T) is anti-Kronecker. This was proved by Furstenberg in [**9**].

Remark 2. Lemma 4.1 would no longer be true if the compact group K were allowed to be non-abelian in the definition of a Kronecker system. Indeed, let K be any simply path-connected and locally path connected compact group (e.g. $K := SU(2)$, the group of all unitary (complex) 2×2 matrices M with $\det(M) = 1$). Then, for any $g \in K$, the "Kronecker" dynamical system (K, τ_g) has property (S).

However, what Lemma 4.1 really says is that if a dynamical system of the form (K, τ_g) happens to be a pseudo-factor of some dynamical system (Z, T) with property (S), then g belongs to the closed subgroup K' generated by the commutators of K (i.e. all $h \in K$ of the form $aba^{-1}b^{-1}$). Indeed, K' is a closed normal subgroup of K and the quotient group K/K' is abelian, so the dynamical system $(K/K', \tau_{[g]})$ is a Kronecker pseudo-factor of (Z, T) and hence $[g] = \mathbf{1}$ in K/K'.

Remark 3. When the ground space X is compact and metrizable, a minimal dynamical system (X, T) is anti-Kronecker if and only if the continuous map T is **weakly mixing**, i.e. $T \times T$ is point transitive on $X \times X$. This is a well-known result due independently to Keynes-Robertson [**15**] and Petersen [**19**] (see [**11**], Theorem 2.3). When X is not compact, this needs not be true. For example, if T is a hypercyclic operator then $(HC(T), T)$ is anti-Kronecker (by Corollary 2.1), but T needs not be weakly mixing by [**20**].

4.3. *Proofs of Theorems 1.1 and 1.2.*

4.3.1. *Proof of Theorem 1.2.* Towards a contradiction, assume that there exists some Γ-transitive point $x \in X$ which is not Γ_0-transitive. By Theorem 4.1, there is a nontrivial eigencharacter $\chi \in \widehat{\Gamma}$ for $(Trans(\Gamma), \Gamma)$ such that $\Gamma_0 \subset \ker(\chi)$. Let $f : Trans(\Gamma) \to \mathbb{T}$ be an associated eigenfunction, i.e. $f(\gamma \cdot z) = \chi(\gamma) f(z)$ whenever $(z, \gamma) \in Trans(\Gamma) \times \Gamma$ and $\gamma \cdot z \in Trans(\Gamma)$.

By assumption, one can find $\gamma \in \Gamma$ such that $g := \chi(\gamma) \neq \mathbf{1}$ and a T_γ-invariant set $Z \subset Trans(\Gamma)$ such that the dynamical system (Z, T_γ) has property (S). But since $f(T_\gamma(z)) = gf(z)$ for all $z \in Z$, the nontrivial Kronecker system $(\mathbb{T}, \tau_g)$ is a pseudo-factor of (Z, T_γ), which contradicts Lemma 4.1.

□

4.3.2. *Proof of Theorem 1.1.* By Theorem 4.1, it is enough to prove that there is no nontrivial eigencharacter for $(HC(\Gamma), \Gamma)$ whose kernel contain Γ_0. We show that in fact *any* eigencharacter for $(HC(\Gamma), \Gamma)$ is trivial. Let χ be such a character, and let $f : HC(\Gamma) \to \mathbb{T}$ be an associated eigenfunction. By assumption, there is at least one $\gamma \in \Gamma$ such that $T := T_\gamma$ is hypercyclic; pick any $z \in HC(T)$. Then $T^n(z) \in HC(T) \subset HC(\Gamma)$ and $f(T^n(z)) = \chi(\gamma)^n f(z)$ for all $n \in \mathbb{N}$. If $\chi(\gamma) = \mathbf{1}$ then, since $\{T^n(z);\ n \in \mathbb{N}\}$ is dense in X, it follows that f is constant, and we are done since $\chi(\gamma') = f(\gamma' \cdot z)/f(z)$ for every $\gamma' \in \Gamma$. Here, the Γ-invariance of $HC(\Gamma)$ was used since we needed $f(\gamma' \cdot z)$ to be well-defined for *every* $\gamma' \in \Gamma$. If $g := \chi(\gamma) \neq \mathbf{1}$, then the nontrivial Kronecker system $(\mathbb{T}, \tau_g)$ is pseudo-factor of $(HC(T), T)$, a contradiction since the latter is anti-Kronecker by Corollary 2.1. □

4.4. *Products, quotients, and proof of Theorem 4.1.* This sub-section is devoted to the proof of Theorem 4.1. The following notation will be useful.

Notation. Let (Z, Λ) be a dynamical system. Given $z, z' \in Z$, we write $z \xrightarrow{\Lambda} z'$ if $z' \in \overline{\Lambda \cdot z}$; and we write $z \xleftrightarrow{\Lambda} z'$ when both $z \xrightarrow{\Lambda} z'$ and $z' \xrightarrow{\Lambda} z$.

Let (X, Γ) be a dynamical system, and let Γ_0 is a subsemigroup of Γ such that $G = \Gamma/\Gamma_0$ is well-defined. Then Γ acts in a natural way on the product space $G \times X$:

$$\gamma \cdot (g, x) = (\pi_0(\gamma)g, \gamma \cdot x)\,,$$

where $\pi_0 : \Gamma \to G$ is the quotient map. The dynamical system $(G \times X, \Gamma)$ may be called the *diagonal product extension* of (X, Γ) via the quotient map π_0. To emphasize the dependence on the sub-semigroup Γ_0, we write $(G\times X, \pi_0\times\Gamma)$ instead of $(G \times X, \Gamma)$. The next lemma (and its corollary below) relates the transitivity of Γ_0 to that of $\pi_0 \times \Gamma$.

Lemma 4.2. *Let (X, Γ) be a dynamical system, with a completely metrizable acting semigroup Γ, and let Γ_0 be a sub-semigroup of Γ. Let also $x \in X$. Assume that $G = \Gamma/\Gamma_0$ is well-defined and compact, and that $\overline{\Gamma_0 \cdot \overline{\Gamma x}} = \overline{\Gamma \cdot x}$. If $(\mathbf{1}_G, x) \xrightarrow{\pi_0\times\Gamma} (g, x)$ for every $g \in G$ then $\overline{\Gamma_0 x} = \overline{\Gamma x}$.*

Proof. Assume that $(\mathbf{1}_G, x) \xrightarrow{\pi_0\times\Gamma} (g, x)$ for every $g \in G$. Since $(\Gamma_0\Gamma)x$ is known to be dense in Γx, it is enough to show that $\gamma_0 \cdot z \in \overline{\Gamma_0 x}$ for any $\gamma_0 \in \Gamma_0$ and every $z \in \Gamma x$. (This fact was implicit in the proof of Lemma 3.3.) Let us fix γ_0 and $z = \xi \cdot x$, and let K_0 be a fundamental domain for $(\Gamma/\Gamma_0, \gamma_0)$. By assumption, there is a net $(\xi_i)\subset\Gamma$ such that $\pi_0(\xi_i)\to\pi_0(\xi)^{-1}$ and $\xi_i\cdot x\to x$. Choose $k_i\in K_0$ such that $\gamma_i:=k_i(\xi\xi_i)\in\Gamma_0$. By compactness, we may assume that the net (k_i) is convergent, $k_i\to k\in K_0$. Since $\pi_0(k_i\xi_i\xi)=\mathbf{1}_G$ and $\pi_0(\xi\xi_i)\to\mathbf{1}_G$, we see that $\pi_0(k)=\mathbf{1}_G$, so that $k\in\Gamma_0\cap K_0$ and hence $k=\gamma_0$. Now, $\gamma_i\cdot x\to(k\xi)\cdot x=\gamma_0\cdot z$, by the joint continuity of the map $(\gamma,u)\mapsto\gamma\cdot u$. Thus, we have shown that $\gamma_0\cdot z\in\overline{\Gamma_0\cdot x}$, as required.

□

COROLLARY 4.2. *Under the hypotheses of Lemma 4.2, assume additionally that T_ξ has dense range for a dense set of $\xi\in\Gamma$. Then x is Γ_0-transitive iff $(\mathbf{1}_G,x)$ is $(\pi_0\times\Gamma)$-transitive in $G\times X$.*

Proof. Assume first that $x\in Trans(\Gamma_0)$. Then, obviously, $(\mathbf{1}_G,x)\xrightarrow{\pi_0\times\Gamma}(\mathbf{1}_G,z)$ for any $z\in X$. Hence, $(\mathbf{1}_G,x)\xrightarrow{\pi_0\times\Gamma}(\pi_0(\xi),\xi\cdot z)$ for all $(\xi,z)\in\Gamma\times X$; and since T_ξ has dense range for a dense set of ξ, it follows that $(\mathbf{1}_G,x)$ is $(\pi_0\times\Gamma)$-transitive. Conversely, if $(\mathbf{1}_G,x)$ is $(\pi_0\times\Gamma)$-transitive then $\overline{\Gamma_0 x}=\overline{\Gamma x}=X$, by the lemma.

□

We now use Lemma 4.2 to relate the Γ_0-orbits with the actions of Γ on the *coset spaces* associated with subgroups of $G=\Gamma/\Gamma_0$. If H is a closed subgroup of G then Γ acts in a natural way on G/H, the space of left cosets defined by H; namely, if $\gamma\in\Gamma$ and $gH\in G/H$ then $\gamma\cdot(gH)=(\pi_0(\gamma)g)H$. We shall refer to the dynamical system $(G/H,\Gamma)$ defined by this action as *the canonical action* $(G/H,\Gamma)$.

PROPOSITION 4.1. *Let (X,Γ) be a dynamical system, with a completely metrizable acting semigroup Γ, and let Γ_0 be a sub-semigroup of Γ such that that $G=\Gamma/\Gamma_0$ is well-defined and compact. Let also $x\in X$ and assume that $x\xrightarrow{\Gamma}x$. Finally, put*

$$H(x):=\{g\in G;\ (\mathbf{1}_G,x)\xrightarrow{\pi_0\times\Gamma}(g,x)\}\,.$$

- (a) *The set $H(x)$ is a closed subgroup of G.*
- (b) *Assume that $\overline{\Gamma_0\cdot\overline{\Gamma x}}=\overline{\Gamma\cdot x}$. If $\overline{\Gamma_0\cdot x}\neq\overline{\Gamma\cdot x}$ then $H(x)$ is a* proper *subgroup of G.*
- (c) *Put $E(x):=\{y\in X;\ x\overset{\Gamma}{\longleftrightarrow}y\}$. Then the canonical action $(G/H(x),\Gamma)$ is a pseudo-factor of $(E(x),\Gamma)$.*

Proof. For any $x,y\in X$, let us put

$$H_{y,x}:=\{g\in G;\ (\mathbf{1}_G,x)\xrightarrow{\pi_0\times\Gamma}(g,y)\}\,.$$

Thus, $g\in H_{y,x}$ iff there is a net $(\gamma_i)\subset\Gamma$ such that $\pi_0(\gamma_i)\to g$ and $\gamma_i\cdot x\to y$. It follows at once from the compactness of G that $H_{y,x}$ is nonempty if and only if $x\xrightarrow{\Gamma}y$. Moreover, it is an elementary exercise to show that

$$H_{z,y}\cdot H_{y,x}\subset H_{z,x}$$

for any $x, y, z \in X$. Since $x \xrightarrow{\Gamma} x$, it follows that $H(x) = H_{x,x}$ is a nonempty closed sub-semigroup of the compact group G, and hence in fact a closed *subgroup* of G, by a well-known (and easy) argument. This proves (a).

Part (b) follows at once from Lemma 4.2.

Let us now prove (c). We put $H := H(x)$, and for any set $A \subset G$, we denote by AH the image of A in the coset space G/H.

We first claim that if $y \in E(x)$, then $H_{y,x}H$ is reduced to a single point. Indeed, since $y \xrightarrow{\Gamma} x$ we may pick $v_0 \in H_{x,y}$. If u is any point in $H_{y,x}$, then $v_0 u \in H_{x,y} \cdot H_{y,x} \subset H_{x,x} = H$, so that $uH = v_0^{-1}H$. Thus, $H_{y,x}H$ contains at most one point, hence exactly one point since $H_{y,x} \neq \emptyset$.

Now, we define a map $p : E(x) \to G/H$ as follows: if $y \in E(x)$ then $\{p(y)\} = H_{y,x}H$. This map p is easily seen to be continuous: indeed, if C is any closed subset of G/H, then $p^{-1}(C) = \{y \in E(x);\ \exists g \in G\ :\ (g, y) \in \overline{\Gamma \cdot (\mathbf{1}_G, x)} \text{ and } gH \in C\}$ is closed in $E(x)$ because G is compact and the relation $R(g, y)$ appearing after the existential quantifier is closed in $G \times E(x)$. Moreover, if $y \in E(x)$ and $\gamma \in \Gamma$ then $\pi_0(\gamma) \cdot H_{y,x} \subset H_{\gamma \cdot y,x}$. It follows that $\gamma \cdot (H_{y,x}H) = (\pi_0(\gamma) \cdot H_{y,x})H \subset H_{\gamma \cdot y,x}H$, so that $p(\gamma \cdot y) = \gamma \cdot p(y)$ if $y \in E(x)$ and $\gamma \cdot y \in E(x)$. This shows that p is a pseudo-factoring map from $(E(x), \Gamma)$ into $(G/H, \Gamma)$. □

Remark. With the notation of the above proof, we see that (being a subgroup of G) $H_{x,x}$ contains $\mathbf{1}_G$ as soon as $x \xrightarrow{\Gamma} x$. When $\Gamma = \mathbb{N}$, i.e. $(X, \Gamma) = (X, T)$ for some continuous map $T : X \to X$, it follows that if $x \in X$ is a recurrent point for T then, for any compact group G and every $g \in G$, the point $(\mathbf{1}_G, x)$ is a recurrent point for $\tau_g \times T : G \times X \to G \times X$. This is quite a well-known result (see e.g. [**10**]). The point in Shkarin's theorem 2.2 is that with some additional assumptions, one can replace "recurrent" by "transitive".

Corollary 4.3. *Let (X, Γ) be a dynamical system, with a completely metrizable acting semigroup Γ, and let Γ_0 be a sub-semigroup of Γ such that that $G = \Gamma/\Gamma_0$ is well-defined and compact and $\Gamma_0 \cdot X$ is dense in X. If $Trans(\Gamma) \neq Trans(\Gamma_0)$, then there is a proper closed subgroup $H \subset G$ such that the canonical action $(G/H, \Gamma)$ is a pseudo-factor of $(Trans(\Gamma), \Gamma)$.*

Proof. Assume that $Trans(\Gamma) \neq Trans(\Gamma_0)$ and let us pick any point $x \in Trans(\Gamma) \setminus Trans(\Gamma_0)$. Since $\Gamma_0 \cdot X$ is dense in X and $x \in Trans(\Gamma)$, we have $\overline{(\Gamma_0\Gamma) \cdot x} = X = \overline{\Gamma \cdot x}$. Moreover, $E(x) = Trans(\Gamma)$. By Proposition 4.1, the result follows. □

It is now a very short step to the

Proof of Theorem 4.1. Assume that $Trans(\Gamma) \neq Trans(\Gamma_0)$. Then, Corollary 4.3 provides us with a proper closed subgroup $H \subset G$ such that the canonical action $(G/H, \Gamma)$ is a pseudo-factor of $(Trans(\Gamma), \Gamma)$, with witness $p : Trans(\Gamma) \to G/H$. Since the compact group G is abelian, one can find a nontrivial character $\phi \in \widehat{G}$ such that $H \subset \ker(\phi)$. Let us denote by $[\phi]$ the character of G/H induced by ϕ. If we put $\chi := \phi\circ\pi_0$ (where $\pi_0 : \Gamma \to G$ is the canonical quotient map) and $f := [\phi]\circ p$, then χ is a nontrivial character of Γ such that $\Gamma_0 \subset \ker(\chi)$, and $f(\gamma\cdot z) = \chi(\gamma)f(z)$ whenever $(z,\gamma) \in Trans(\Gamma)\times\Gamma$ and $\gamma\cdot z \in Trans(\Gamma)$. This concludes the proof. □

5. **Further results**

5.1. *Supercyclic semigroups.* Let X be a topological vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A linear dynamical system (X,Γ) is **supercyclic** if there is some $x \in X$ whose projective Γ-orbit $\mathbb{K}\Gamma\cdot x := \{\lambda T_\gamma(x);\ \lambda \in \mathbb{K},\ \gamma \in \Gamma\}$ is dense in X; equivalently, if the dynamical system $(X, \mathbb{K}\times\Gamma)$ is hypercyclic, where $\mathbb{K}$ is considered as a multiplicative semigroup and $\mathbb{K}\times\Gamma$ acts on X in the obvious way. The set of all supercyclic vectors for Γ is denoted by $SC(\Gamma)$. When $\Gamma = \mathbb{N}$, $(X,\Gamma) = (X,T)$, one says that the *operator* T is supercyclic. More generally, a dynamical system (X,Γ) is **Λ-supercyclic** for a given multiplicative semigroup $\Lambda \subset \mathbb{K}$, if there is some $x \in X$ such that $\Lambda\Gamma\cdot x$ is dense in X. The following result is a supercyclic version of Theorem 1.1. As shown in [**5**], it holds on *complex* topological vector spaces only.

Proposition 5.1. *Let (X,Γ) be a supercyclic linear dynamical system, where X is a* complex *topological vector space and the acting semigroup Γ is completely metrizable and abelian. Let Γ_0 be a subsemigroup of Γ such that Γ/Γ_0 is well-defined and compact. Assume that all operators T_γ have dense range and that at least one T_γ is supercyclic. Then (X,Γ_0) is supercyclic, with the same supercyclic vectors as (X,Γ).*

For the proof, it is convenient to make use of the *projective space* $\mathbb{P}X$ associated with X. Recall that $\mathbb{P}X$ is the quotient space $(X\setminus\{0\})/_\equiv$, where $u\equiv v$ iff u and v are colinear. The space $\mathbb{P}X$ is equipped with the quotient topology, and we denote by $\mathbb{P} : X\setminus\{0\} \to \mathbb{P}X$ the natural quotient map. This map is continuous (by definition), and it is easily seen to be also open.

Any operator $T \in \mathcal{L}(X)$ respects the colinearity relation $\equiv$, and hence T induces in a natural way a continuous map $\mathbb{P}T : \mathbb{P}X \to \mathbb{P}X$. Moreover, it is easily checked (using the open-ness of the quotient map $\mathbb{P}$) that T is supercyclic if and only if $\mathbb{P}T$ is point transitive on $\mathbb{P}X$, and that $Trans(\mathbb{P}T) = \mathbb{P}(SC(T))$. In particular, $Trans(\mathbb{P}T)$ is $\mathbb{P}T$-invariant. The proof of Proposition 5.1 relies on the following analogue of Corollary 2.1.

Lemma 5.1. *Let X be a complex topological vector space. If $T \in \mathcal{L}(X)$ is supercyclic, then the dynamical system $(Trans(\mathbb{P}T), \mathbb{P}T)$ has property (S).*

Proof. We may clearly assume that $\dim(X) > 1$. Then X is in fact infinite-dimensional since otherwise there are no supercyclic operators on X. We start with the following

Fact. Let Z be a linear subspace of X, and let H be a closed, affine subspace of Z. Then $\mathbb{P}(H\setminus\{0\})$ is locally path-connected and simply path-connected.

Proof of Fact. Replacing Z by the linear span of H, we may assume that either $H = Z$ or H is a closed hyperplane in Z.

Since any point of H has an open neighbourhood basis consisting of star-shaped sets, $H \setminus \{0\}$ is locally path-connected. Moreover, since we are considering complex spaces, $H \setminus \{0\}$ is also simply path-connected except when $H = Z$ and $\dim(Z) = 1$, in which case there is nothing to prove since $\mathbb{P}(M \setminus \{0\})$ is reduced to a single point. So it is enough to show that

(1) the restriction of $\mathbb{P}$ to $H \setminus \{0\}$ is open from $H \setminus \{0\}$ onto $\mathbb{P}(H \setminus \{0\})$;

(2) any closed path in $\mathbb{P}(H \setminus \{0\})$ can be lifted to a closed path in $H \setminus \{0\}$.

Assume first that $H = Z$, i.e. H is a linear subspace of X. Then (1) is clear because $H \setminus \{0\}$ is $\equiv$-saturated, and (2) is also clear for the same reason since it is well-known that any closed path in $\mathbb{P}X$ can be lifted to a closed path in $X \setminus \{0\}$ (see e.g. [**21**] Lemma A.3).

Assume now that H is a closed hyperplane in Z, i.e. $H = \{h \in Z;\ \phi(h) = 1\}$ for some continuous linear functional $\phi : Z \to \mathbb{C}$. If V is an open set in H, then its $\equiv$-saturation $\widetilde{V}$ is open in $Z \setminus \{0\}$ since $\widetilde{V} = \{z \in Z;\ \phi(z) \neq 0 \text{ and } \frac{z}{\phi(z)} \in V\}$; so (1) follows from the previous case. Similarly, (2) follows from the previous case since if $\gamma : [0,1] \to Z \setminus \{0\}$ is a closed path in $Z \setminus \{0\}$ such that $\mathbb{P}(\gamma(t)) \in \mathbb{P}(H \setminus \{0\})$ for all t, then the formula $\gamma_H(t) := \frac{\gamma(t)}{\phi(\gamma(t))}$ makes sense and defines a closed path in $H \setminus \{0\}$ such that $\mathbb{P} \circ \gamma_H = \mathbb{P} \circ \gamma$. □

Let us now fix a supercyclic operator $T \in \mathcal{L}(X)$, and let z be any supercyclic vector for T. Then $\mathbb{P}(z)$ is a transitive point for $\mathbb{P}T$ and hence a recurrent point. So it is enough to find a T-invariant set $M \subset X \setminus \{0\}$ such that $z \in M \subset SC(T)$ and $\mathbb{P}(M)$ is locally path-connected and simply path-connected. As is well-known (see e.g. [**24**]), two cases may occur.

Case 1. $P(T)$ has dense range for every polynomial $P \neq 0$.

In this case, we put $H = Z := \operatorname{span}\{T^n(z);\ n \geq 0\}$. Then $M := H \setminus \{0\}$ has the required properties by the above Fact.

Case 2. There is a complex number $\lambda_0 \neq 0$ such that $\overline{(T - \lambda_0 I)(X)}$ has codimension 1 in X and $P(T)$ has dense range for every polynomial P with $P(\lambda_0) \neq 0$.

Replacing T by $\lambda_0^{-1}T$, we may in fact assume that $\lambda_0 = 1$ (notice that $\mathbb{P}(\mu T) = \mathbb{P}T$ for any $\mu \in \mathbb{C}^*$). We put $Z := \operatorname{span}\{T^n(z);\ n \geq 0\}$ and $M = H := \{P(T)z;\ P \text{ polynomial},\ P(1) = 1\}$. By the above fact, we just have to check that H is *closed* in Z. Let ϕ be a continuous linear functional on X such that $\ker(\phi) = \overline{(T - I)(X)}$. Then $T^*(\phi) = \phi$ (where T^* is the adjoint operator), so we have $\phi(P(T)x) = P(1)\phi(x)$ for any $x \in X$ and every polynomial P. Since $\phi(z) \neq 0$, we may assume that $\phi(z) = 1$, and it follows that $H = \{h \in Z;\ \phi(h) = 1\}$. This shows that H is indeed closed in Z, and the proof is complete.

□

Proof of Proposition 5.1. Let us denote by the symbols $\mathbb{P}\Gamma$ and $\mathbb{P}\Gamma_0$ the semigroups Γ and Γ_0 acting on the projective space $\mathbb{P}X$. Then $\mathbb{P}\Gamma$ is point transitive (with $Trans(\mathbb{P}\Gamma) = \mathbb{P}(SC(\Gamma))$) and we have to show that $\mathbb{P}\Gamma_0$ is also transitive, with the same transitive points. By Theorem 4.1, it is enough to check that any eigencharacter χ for $(Trans(\mathbb{P}\Gamma), \mathbb{P}\Gamma)$ is trivial. Let us fix such a character χ, and let $p : Trans(\mathbb{P}\Gamma) \to \mathbb{T}$ be an associated eigenfunction. Towards a contradiction, we assume that χ is nontrivial, so that p is non-constant.

By assumption, one can pick $\gamma \in \Gamma$ such that the operator $T = T_\gamma$ is supercyclic. Then the induced map $\mathbb{P}T$ is transitive on $\mathbb{P}X$. Since $Trans(\mathbb{P}T)$ is contained in $Trans(\mathbb{P}\Gamma)$, we have $p((\mathbb{P}T)^n(z)) = g^n p(z)$ for any $z \in Trans(\mathbb{P}T)$ and every $n \in \mathbb{N}$, where $g := \chi(\gamma)$. Since p is non-constant and $Trans(\mathbb{P}T) = \mathbb{P}(SC(T))$ is dense in $\mathbb{P}X$, it follows that $g \neq \mathbf{1}$. Thus, we see that the nontrivial Kronecker system $(\mathbb{T}, \tau_g)$ is a pseudo-factor of the dynamical system $(Trans(\mathbb{P}T), \mathbb{P}T)$. This contradicts Lemma 5.1.

□

As in the hypercyclic case, one can easily deduce from Proposition 5.1 the supercyclic versions of Ansari's and Conejero-Müller-Peris' theorems. The Ansari case (powers) goes back to [**1**]. The Conejero-Müller-Peris case (1-parameter semigroups) was obtained recently by S. Shkarin in [**22**]. Shkarin's proof is quite interesting and rather different from the one we are about to give. Unlike the one in [**22**], our proof works in the metrizable case only, but it can be adapted to give the result without additional assumption on X; see below.

Corollary 5.1. *Let X be a complex topological vector space.*

(1) *If $T \in \mathcal{L}(X)$ is supercyclic then so is T^p for any positive integer p, with the same supercyclic vectors.*

(2) *Assume that X is metrizable. If $(T_t)_{t\geq 0}$ is a jointly continuous supercyclic semigroup on X, then every operator T_a, $a > 0$ is supercyclic, with the same supercyclic vectors as the semigroup (T_t).*

Proof. Part (1) is immediate. For Part (2), the only thing to check is that some operator T_t is supercyclic. Let x be any supercyclic vector for the semigroup $(T_s)_{s\geq 0}$. Then the set $\{\lambda T_s(x);\ \lambda \in \mathbb{C}, s \geq A\}$ is dense in X for any $A > 0$, because T_A has dense range (see below) and commutes with every T_s. Using this and the metrizability of X, a simple Baire category argument shows that x is T_t-supercyclic for a comeager set of t (see the first proof of Theorem 2.1).

To show that T_A has dense range for every $A > 0$, we may assume that $\dim(X) > 1$. It is in fact enough to show that T_ε has dense range for *some* $\varepsilon > 0$, since $n\varepsilon \geq A$ for some $n \in \mathbb{N}$ and hence $\mathrm{Ran}(T_A) \supset \mathrm{Ran}(T_\varepsilon^n)$ by the semigroup property. Otherwise (taking again a supercyclic vector x for (T_s)), the set $\{\lambda T_s(x);\ \lambda \in \mathbb{C}, s \geq \varepsilon\}$ is nowhere dense for any $\varepsilon > 0$ since it is contained in the nowhere dense subspace $\mathrm{Ran}(T_\varepsilon)$, and hence the set $\{\lambda T_s(x);\ \lambda \in \mathbb{C}, s < \varepsilon\}$ is dense in X. It follows that for any $z \in X$, one can find a net $(\lambda_i, \varepsilon_i) \subset \mathbb{C} \times \mathbb{R}_+$ with

$\varepsilon_i \to 0$ such that $\lambda_i T_{\varepsilon_i}(x) \to z$. If $|\lambda_i| \to \infty$ then $T_{\varepsilon_i}(x) \to 0$, a contradiction since $T_{\varepsilon_i}(x) \to x$. So the net (λ_i) has a convergent subnet, and it follows that $z \in \mathbb{C}x$ for any $z \in X$, a contradiction since $\dim(X) > 1$. □

Remark. Part (2) holds in fact without any metrizability assumption on X. Indeed, the general result may be deduced from the metrizable case exactly as for hypercyclic semigroups (see the remark just after the first proof of Theorem 2.1). Alternatively, one may use a variant of Proposition 5.1 where the assumption "some T_γ is supercyclic" is replaced by "$(Trans(\mathbb{P}\Gamma), \mathbb{P}T_\gamma)$ is anti-Kronecker for every $\gamma \in \Gamma$". If one proceeds in this way, the key point is to show that if $\Gamma = (T_s)_{s\geq 0}$ is a supercyclic semigroup on X then, for any $t > 0$, the dynamical system $(Trans(\mathbb{P}\Gamma), \mathbb{P}T_t)$ has property (S). This, in turn, is proved exactly as Lemma 5.1 once the following two facts are established.

(i) one can find a supercylic vector z for the semigroup $(T_s)_{s\geq 0}$ and a net $(\lambda_i, n_i) \subset \mathbb{C} \times \mathbb{N}$ with $n_i \to \infty$ such that $\lambda_i T_{n_i t}(z) \to z$.

(ii) either $P(T_t)$ has dense range for every polynomial $P \neq 0$, or there is a complex number $\lambda_0 \neq 0$ such that $\overline{(T_t - \lambda_0 I)(X)}$ has codimension 1 in X and $P(T_t)$ has dense range for any polynomial P with $P(\lambda_0) \neq 0$.

Indeed, (i) ensures that $\mathbb{P}z \in Trans(\mathbb{P}\Gamma)$ and that $\mathbb{P}z$ is $\mathbb{P}T_t$-recurrent, whereas (ii) is just what is needed for imitating the proof of Lemma 5.1.

The proof of (i) is essentially the same as that of Corollary 3.1. To prove (ii), one key fact is that if H is any closed, Γ-invariant subspace of X then H has infinite codimension or codimension at most 1. Since Γ induces a 1-parameter supercyclic semigroup on the quotient space X/H, this follows because there are no supercyclic 1-parameter semigroups on a complex finite-dimensional space Z unless $\dim(Z) \in \{0, 1\}$ ([**23**] Lemma 5.1). The second key fact ([**22**] Lemma 2.5) is that λI is never supercyclic on a space with dimension > 1, so that $H_\lambda := \overline{(T_t - \lambda I)(X)}$ has codimension at most 1 for any $\lambda \in \mathbb{C}$. Now, assume that H_{λ_0} has codimension 1 for some $\lambda_0 \in \mathbb{C}$. Then $\lambda_0 \neq 0$ (because T_t has dense range). If $\lambda \neq \lambda_0$, then the Γ-invariant supspace $H_\lambda \cap H_{\lambda_0}$ has codimension at most 2, so in fact at most 1, and hence we have either $H_\lambda = X$ or $H_\lambda = H_{\lambda_0}$. The latter is impossible since T_t would then act as both λI and $\lambda_0 I$ on the corresponding (nontrivial) quotient space. Thus, we see that $H_\lambda = X$, i.e. $T_t - \lambda I$ has dense range for every $\lambda \neq \lambda_0$. This concludes the proof.

Proposition 5.1 can also be used in tandem with the León-Müller theorem to get the following result about *positively* supercyclic semigroups.

Corollary 5.2. *Let (X, Γ) be a supercyclic linear dynamical system, where X is a complex topological vector space and the acting semigroup Γ is completely metrizable and abelian. Let Γ_0 be a subsemigroup of Γ such that Γ/Γ_0 is well-defined. Assume that Γ/Γ_0 is compact and that at least one operator T_γ is supercyclic. Moreover, assume that one can find an operator R commuting with all T_γ, $\gamma \in \Gamma_0$ such that*

$R-\mu I$ has dense range for every $\mu\in\mathbb{C}$. Then (X,Γ_0) is $\mathbb{R}_+$-supercyclic, and every supercyclic vector for Γ is in fact $\mathbb{R}_+$-supercyclic for Γ_0.

Proof. Applying the León-Müller theorem (Theorem 2.1 (2b)) to the semigroup $\mathcal{S}=\{rT_\gamma;\ r>0,\gamma\in\Gamma_0\}$, we see that any supercyclic vector for Γ_0 is in fact $\mathbb{R}_+$-supercyclic. Hence, the result follows at once from Proposition 5.1. □

Corollary 5.3. *Let $(T_t)_{t\geq 0}$ be a jointly continuous supercyclic semigroup on a complex topological vector space. Assume that $T_t-\mu I$ has dense range for some $t>0$ and all $\mu\in\mathbb{C}$. Then each operator T_a, $a>0$ is positively supercyclic and every supercyclic vector for the semigroup (T_t) is in fact positively supercyclic for T_a.*

5.2. *Other variations on the main results.* The following results can be easily deduced from the proof of Theorem 1.2. Recall that a dynamical system (Z,Γ) is said to be **minimal** if every Γ-orbit is dense.

Proposition 5.2. *Let (Z,Γ) be a minimal dynamical system with a completely metrizable acting semigroup Γ, and let Γ_0 be a sub-semigroup of Γ such that Γ/Γ_0 is well-defined and $\Gamma_0\cdot Z$ is dense in Z. Then (Z,Γ_0) is also minimal provided one of the following holds:*

(1) *Γ/Γ_0 is finite and Z is connected;*

(2) *Γ/Γ_0 is compact and abelian, every T_γ has a recurrent point, and Z is simply path-connected and locally path-connected.*

(3) *Γ/Γ_0 is compact and abelian, and there is at least one $\gamma\in\Gamma$ such that T_γ is weakly mixing.*

Proof. Part (2) follows from Theorem 1.2 as stated, since $Trans(\Gamma)=Z$.

To prove (1), we use Corollary 4.3. If $\overline{\Gamma_0\cdot x}\neq Z$ for some $x\in Z$ then one can find a proper closed subgroup $H\subset G:=\Gamma/\Gamma_0$ such that the canonical action $(G/H,\Gamma)$ is a pseudo-factor of (Z,Γ). However, G/H is finite and Z is connected, so any pseudo-factoring map from (Z,Γ) into $(G/H,\Gamma)$ must be constant. This is a contradiction.

To prove (3), it is enough to show that if $T:Z\to Z$ is weakly mixing, then the dynamical system (Z,T) is anti-Kronecker. This is quite well-known and easy to check, as follows. Towards a contradiction, assume that (Z,T) has a nontrivial Kronecker pseudo-factor (K,τ_g), and let $f:(Z,T)\to(K,\tau_g)$ be a pseudo-factoring map. If (z_0,z_0') is a $(T\times T)$-transitive point in $Z\times Z$ then putting $a_0:=f(z_0)$ and $a_0':=f(z_0')$, the set $\{(g^na_0,g^na_0');\ n\in\mathbb{N}\}$ is dense in $f(Z)\times f(Z)$. In particular, this set contains (a_0,a_0) in its closure, which is clearly not possible unless $a_0=a_0'$. Since f is non-constant, this is a contradiction. □

Remark. One can use (1) to prove Ansari's theorem as well as its supercyclic version. In fact, if Λ is any multiplicative sub-semigroup of $\mathbb{C}^*$ and if T is a Λ-supercyclic operator on a (complex) topological vector space X, then T^p is Λ-supercyclic for any positive integer p, with the same Λ-supercyclic vectors. To see this, apply (1) with $\Gamma = \Lambda \times \mathbb{N}$ and $\Gamma_0 = \Lambda \times p\mathbb{N}$. Denoting by Z the set of all Λ-supercyclic vectors for T, the dynamical system (Z, Γ) is minimal. Hence, it is enough to check that Z is connected. Now, the operator T is supercyclic, so there is a complex number $\lambda_0 \neq 0$ such that $P(T)$ has dense range for every polynomial P with $P(\lambda_0) \neq 0$. Then, for any $z \in Z$, the set $\{P(T)z;\ P(\lambda_0) \neq 0\}$ is contained in Z. Since this set is connected and dense in X, this concludes the proof.

Another related result is the following proposition, which should be compared with Shkarin's theorem. In the case of a compact ground space X, this result can be extracted from [**18**].

Proposition 5.3. *Let (X, T) be a point transitive dynamical system, and let $x \in Trans(T)$. Let also G be a compact metrizable abelian group. Moreover, assume that G is* connected. *Then the set of all $g \in G$ such that $\{(g^n, T^n(x));\ n \in \mathbb{N}\}$ is dense in $G \times X$ is a residual subset of G.*

Proof. Let $\Gamma := G \times \mathbb{Z}_+$ act on $G \times X$ as expected, $T_{(\xi,n)}(h, z) = (\xi h, T^n(z))$. Then $Trans(\Gamma) = G \times Trans(T)$ and $Trans(\Gamma)$ is Γ-invariant.

Let us denote by M the set of all $g \in G$ such that the set $\{(g^n, T^n(x));\ n \in \mathbb{N}\}$ is not dense in $G \times X$. If $g \in M$, then $\{(g^n, T^n(x));\ n \in \mathbb{Z}_+\}$ is not dense either because $(\mathbf{1}_G, x)$ is a recurrent point of $\tau_g \times T$ (see the remark after Proposition 4.1). By Theorem 4.1 applied with $\Gamma_0 = \Gamma_g := \{(g^n, n);\ n \in \mathbb{Z}_+\}$, one can find a nontrivial character $\phi_g \in \widehat{\Gamma}$ and a (nonconstant) continuous function $f_g : G \times Trans(T) \to \mathbb{T}$ such that $\phi_g(g, 1) = \mathbf{1}$ and $f_g(\xi, T^n(x)) = \phi_g(\xi, n) f_g(\mathbf{1}_G, x)$ for all $(\xi, n) \in \Gamma$. Putting $\chi_g(\xi) := \phi_g(\xi, 0)$ and $\alpha_g := \phi_g(\mathbf{1}_G, 1)$, this becomes $f_g(\xi, T^n(x)) = \chi_g(\xi)\alpha_g^n f_g(\mathbf{1}_G, x)$. Moreover, we have $\alpha_g = \chi_g(g)^{-1}$ since $\phi_g(g, 1) = \mathbf{1}$, hence we get $f_g(\xi, T^n(x)) = \chi_g(\xi g^{-n}) f_g(\mathbf{1}_G, x)$. Since f_g is nonconstant and $x \in Trans(T)$, it follows in particular that the character χ_g is nontrivial. Moreover, it is apparent that the function f_g is in fact uniquely determined by the character χ_g, up to a multiplicative constant. Hence, if $g_1, g_2 \in M$ may be associated with the same character $\chi \in \widehat{G}$, then $\chi(g_1) = \chi(g_2)$. Thus, denoting by $\widehat{G}^*$ the set of all nontrivial characters of G, we have arrived at the following conclusion: there is a family of complex numbers $(\alpha_\chi)_{\chi \in \widehat{G}^*} \subset \mathbb{T}$ such that $M \subset \bigcup_{\chi \in \widehat{G}^*} \chi^{-1}(\alpha_\chi)$. Now, since G is connected, every nontrivial character χ has a nowhere dense kernel, and hence nowhere dense level sets. Since $\widehat{G}$ is countable (because G is metrizable), it follows that M is a set of the first Baire category, which concludes the proof. □

Remark 1. Taking a trivial space $X = \{x_0\}$, it follows that any compact, connected metrizable abelian group is monothetic, with a residual set of topological generators. This is, of course, quite well-known. Actually, this is a characterization of connectedness within the class of compact metrizable abelian groups. Indeed, if

G is not connected, then it has a proper clopen subgroup H, and no $h \in H$ can be a topological generator of G. On the other hand, there are compact metrizable monothetic groups which are not connected, e.g. $G = \mathbb{T} \times \mathbb{Z}_2$.

Remark 2. It could seem natural to expect that $\{(g^n, T^n(x));\ n \in \mathbb{N}\}$ is dense in $G \times X$ for any topological generator g of G. However, this needs not be true. Consider for example $X = G = \mathbb{T}$ and $T = \tau_g : \mathbb{T} \to \mathbb{T}$, where g is a topological generator of $\mathbb{T}$.

5.3. *Anti-Kronecker systems.* The following proposition gives a characterization of anti-Kronecker systems. This result is implicit in [**21**], and also in [**18**] when the space X is compact.

PROPOSITION 5.4. *For a minimal dynamical system (X, T), the following are equivalent.*

(i) *(X, T) is anti-Kronecker;*

(ii) *$(K \times X, \tau_g \times T)$ is minimal for every minimal Kronecker system (K, τ_g);*

(iii) *$(K \times X, \tau_g \times T)$ is point transitive for every minimal Kronecker system (K, τ_g).*

Proof. If (ii) fails to hold for some minimal Kronecker system (K, τ_g) then, as in the proof of Proposition 5.3, one can find a continuous function $f : K \times X \to \mathbb{T}$ and a unimodular complex number $\alpha \neq \mathbf{1}$ such that $f(k, T(x)) = \alpha f(k, x)$ for all $(k, x) \in K \times X$ (we have indeed $\alpha \neq \mathbf{1}$ because $\alpha = \chi(g)^{-1}$ for some nontrivial character $\chi \in \widehat{K}$ and g is a topological generator of K). Then the map $p : X \to \mathbb{T}$ defined by $p(x) := f(\mathbf{1}, x)$ is a pseudo-factoring map from (X, T) into the nontrivial Kronecker system $(\mathbb{T}, \tau_\alpha)$. This shows that (i) implies (ii).

That (ii) implies (iii) is trivial. Finally, assume that (X, T) has a nontrivial Kronecker pseudo-factor (K, τ_g) with pseudo-factoring map $p : X \to K$. Let $\widetilde{K}$ be the closed subgroup of K generated by g, so that the Kronecker dynamical system $(\widetilde{K}, \tau_g)$ is minimal. Replacing $p(x)$ by $p(x_0)^{-1}p(x)$, we may assume that $p(x_0) = \mathbf{1}_K$ for some $x_0 \in X$. Then $p(T^n(x_0)) = g^n$ for all $n \in \mathbb{N}$, and since x_0 is T-transitive it follows that $p(X) \subset \widetilde{K}$. Therefore, the dynamical system $(\widetilde{K} \times \widetilde{K}, \tau_g \times \tau_g)$ is a pseudo-factor of $(\widetilde{K} \times X, \tau_g \times T)$ with pseudo-factoring map $q : \widetilde{K} \times X \to \widetilde{K} \times \widetilde{K}$ defined by $q(k, x) = (k, p(x))$. If $(\widetilde{K} \times X, \tau_g \times T)$ is point transitive, then so is $(\widetilde{K} \times \widetilde{K}, \tau_g \times \tau_g)$ because q has dense range, which is clearly not possible unless $\widetilde{K} = \{\mathbf{1}\}$, i.e. $g = \mathbf{1}$. This shows that (iii) implies (i). □

COROLLARY 5.4. *If X is a simply path-connected and locally path-connected compact metric space, then any minimal dynamical system (X, T) is weakly mixing.*

Proof. By Shkarin's theorem and Proposition 5.4, the minimal dynamical system (X, T) is anti-Kronecker, and hence T is weakly mixing because X is compact (see Remark 3 after Lemma 4.1). □

5.4. *Group topologies on orbits.* Let (X, Γ) be a dynamical system, and assume that the acting semigroup Γ is a group. If $x \in X$ has trivial stabilizer (i.e. $\gamma \cdot x = x$ only if $\gamma = \mathbf{1}$) then the orbit $\Gamma \cdot x$ may be identified (as a set) with Γ and hence it is canonically equipped with a group structure, obtained by transfering the group operation of Γ. The original topology on $\Gamma \cdot x$ (i.e. its topology as a subspace of X) is in general strictly coarser than the group topology induced by Γ, and it has in fact no reason for being a group topology. Now, the following simple remark shows that group topologies on $\Gamma \cdot x$ are closely related to eigencharacters for the dynamical system $(\Gamma \cdot x, \Gamma)$ (which is, of course, not surprising).

Remark 5.1. *Let (X, Γ) be a dynamical system, where Γ is a completely metrizable abelian group, and let $x \in X$ have trivial stabilizer. Let also Γ_0 be a co-compact subgroup of Γ. Then the following are equivalent:*

(i) *there is a nontrivial eigencharacter for the dynamical system $(\Gamma \cdot x, \Gamma)$ which is trivial on Γ_0;*

(ii) *there is a (perhaps not Hausdorff) group topology σ on $\Gamma \cdot x$ which is coarser than the original topology and such that $\Gamma_0 \cdot x$ is not dense in $(\Gamma \cdot x, \sigma)$.*

Proof. Identifying $\Gamma \cdot x$ with Γ, let us denote by σ_x the topology on $\Gamma \cdot x$ generated by all eigencharacters for the dynamical system $(\Gamma \cdot x, \Gamma)$. Equivalently, σ_x is the topology generated by all continuous eigenfunctions f for $(\Gamma \cdot x, \Gamma)$ such that $f(x) = \mathbf{1}$. By its very definition, σ_x is a group topology on $\Gamma \cdot x$ coarser than the original topology, and the eigencharacters of $(\Gamma \cdot x, \Gamma)$ are characters of the topological group $(\Gamma \cdot x, \sigma)$. This shows that (i) implies (ii).

Conversely, assume that there is a coarser group topology σ on $\Gamma \cdot x$ such that $\Gamma_0 \cdot x$ is not dense in $(\Gamma \cdot x, \sigma)$. Let us endow the quotient group $\Gamma \cdot x / \Gamma_0 \cdot x \simeq \Gamma/\Gamma_0$ with the quotient topology induced by the group topology σ. Since σ is coarser than the original topological topology on $\Gamma \cdot x$, which is in turn coarser than the group topology induced by Γ, this topology is coarser than the quotient topology of Γ/Γ_0, whence $\Gamma \cdot x / \Gamma_0 \cdot x$ is *compact* (perhaps not Hausdorff). Since $\Gamma_0 \cdot x$ is not dense in $(\Gamma \cdot x, \sigma)$, it follows that there is a nontrivial character on $\Gamma \cdot x / \Gamma_0 \cdot x$, and hence a nontrivial character f on $(\Gamma \cdot x, \sigma)$ such that $\Gamma_0 \cdot x \subset \ker(f)$. Since σ is coarser than the original topology on $\Gamma \cdot x$, the map f is continuous with respect to this topology. Thus, we have found a nonconstant eigenfunction for the dynamical system $(\Gamma \cdot x, \Gamma)$ which is constant on $\Gamma_0 \cdot x$. □

Applying Theorem 4.1, we immediately deduce

Proposition 5.5. *Let (X, Γ) be a dynamical system, where Γ is a completely metrizable abelian group, and let $x \in X$ have trivial stabilizer. Let also Γ_0 be a co-compact subgroup of Γ. Then $\overline{\Gamma_0 \cdot x} \neq \overline{\Gamma \cdot x}$ if and only if there is a group topology σ on $\Gamma \cdot x$ which is coarser than the original topology and such that $\Gamma_0 \cdot x$ is not dense in $(\Gamma \cdot x, \sigma)$.*

6. Concluding remarks

We conclude the paper with some additional remarks and questions.

1. The following statement is a direct generalization of both Theorems 1.1 and 1.2. *Assume that Γ/Γ_0 is compact and abelian, and that $\Gamma_0\cdot X$ is dense in X. Then Γ and Γ_0 have the same transitive points provided the following holds: for each nontrivial eigencharacter χ for $(Trans(\Gamma),\Gamma)$ such that $\ker(\chi)\supset\Gamma_0$, one can find $\gamma\in\Gamma$ such that $\chi(\gamma)\neq\mathbf{1}$ and a T_γ-invariant set $Z\subset Trans(\Gamma)$ such that the dynamical system (Z,T_γ) has property (S).* However, this statement looks quite artificial. Indeed, in view of Theorem 4.1 and Lemma 4.1, it can be formulated as follows: *If one can find a character $\chi\in\widehat{\Gamma}$ witnessing that $Trans(\Gamma)\neq Trans(\Gamma_0)$ then one can find $\gamma\in\Gamma$ witnessing that χ cannot exist* (!)

2. Compactness of the quotient group Γ/Γ_0 is essential in the proofs of Theorems 1.1 and 1.2. The following remark shows that this is not due to a defect in the proofs. Recall that a topological space is said to be **Polish** if it is separable and completely metrizable. Recall also that a **representation** of a topological group Γ is just a linear dynamical system (X,Γ), where X is a topological vector space (more accurately, the representation is the homomorphism $\gamma\mapsto T_\gamma$ from Γ into the linear group $GL(X)$).

Remark 6.1. *Let Γ be a Polish locally compact abelian group, and let Γ_0 be a closed subgroup of Γ. If Γ/Γ_0 is non-compact then one can find a hypercyclic representation $(\mathcal{H},\Gamma)$ on a separable infinite-dimensional Hilbert space $\mathcal{H}$ such that $(\mathcal{H},\Gamma_0)$ is not hypercyclic. Moreover, this representation has the following property: T_γ is hypercyclic for any $\gamma\in\Gamma$ such that $\pi_0(\gamma)^k\to\infty$ as $k\to\infty$. (Such a γ can always be found if Γ is compactly generated).*

This remark is an immediate consequence of the next lemma.

Lemma 6.1. *If G is a Polish, locally compact and non-compact abelian group then G admits a hypercyclic representation $(\mathcal{H},G)$ on some (separable) infinite-dimensional Hilbert space $\mathcal{H}$. Moreover, the representation may be chosen in such a way that T_g is hypercyclic for every $g\in G$ such that $g^k\to\infty$ as $k\to\infty$ (if there is any).*

Proof of Remark 6.1. Apply the lemma with $G:=\Gamma/\Gamma_0$ and put $T_\gamma:=T_{\pi_0(\gamma)}$, $\gamma\in\Gamma$, where $\pi_0:\Gamma\to G$ is the canonical quotient map. If Γ is compactly generated then, by the "structure theorem" for locally compact abelian groups (see [**14**], Theorem 9.8), the compactly generated group $G=\Gamma/\Gamma_0$ has the form $\mathbb{R}^n\times\mathbb{Z}^m\times K$, where K is a compact group; and either n or m is nonzero since G is non-compact. So one can indeed find $\gamma\in\Gamma$ such that $\pi_0(\gamma)^k\to\infty$ as $k\to\infty$.

□

Proof of Lemma 6.1. The proof is the same as in the well-known cases $G=\mathbb{Z}$ and $G=\mathbb{R}$. Consider the weighted L^2-space $\mathcal{H}:=L^2_w(G)$, where $w:G\to(0,\infty)$ is a positive continuous function such that $w\in C_0(G)$ and $\sup_{g\in G}\frac{w(\xi^{-1}g)}{w(g)}<\infty$ for any

$\xi \in G$, and let G act on $\mathcal{H}$ by translations, i.e. $T_\xi f(g) = f(\xi g)$. Then the dynamical system $(\mathcal{H}, G)$ is easily seen to satisfy the "group version" of Kitai's Criterion for hypercyclicity: there is a dense set $\mathcal{D} \subset \mathcal{H}$ (namely $\mathcal{D} := C_{00}(G)$, the set of all compactly supported continuous functions on G) and a group $(S_\xi)_{\xi\in G}$ defined on $\mathcal{D}$ (namely $S_\xi := T_\xi^{-1}$) such that $T_\xi S_\xi(u) \to u$ and both $T_\xi(u)$, $S_\xi(u)$ tend to 0 as $\xi \to \infty$, for any $u \in \mathcal{D}$.

To construct the "weight" $w : G \to (0,\infty)$, one may proceed as follows. Write $G = \bigcup_{i\geq 0} E_i$, where (E_i) is an increasing sequence of compact sets such that $E_0 = \{\mathbf{1}\}$ and every compact subset of G is contained in the interior of some E_i. Put $C_0 := E_0$, and define inductively a sequence of compact sets $(C_i)_{i\geq 0} \subset G$ as follows: $C_{i+1} = E_{n_{i+1}}$, where n_{i+1} is the smallest n such that $C_i \cup \bigcup_{j,j'\leq i} C_j C_{j'} \subset \mathring{E}_n$. Then $\bigcup_{i\geq 0} C_i = G$, $C_i \subset \mathring{C}_{i+1}$ for all i and $C_i C_j \subset C_{i+j}$ for all $i, j \geq 0$. Now, use the Tietze extension theorem to find a continuous function $\mathfrak{i} : G \to [0,\infty[$ such that $\mathfrak{i}(g) \equiv 0$ on C_0, $\mathfrak{i}(g) \equiv 2i+2$ on $C_{2i+2} \setminus \mathring{C}_{2i+1}$ for all $i \geq 0$ and $2i \leq \mathfrak{i}(g) \leq 2i+2$ on $\mathring{C}_{2i+1} \setminus C_{2i}$. Define the weight w by $w(g) := 2^{-\mathfrak{i}(g)}$. Then $w \in C_0(G)$ because $\mathfrak{i}(g) \geq 2i$ whenever $g \in G \setminus C_{2i}$ (so that $\mathfrak{i}(g) \to \infty$ as $g \to \infty$). Moreover, since $g \in C_{\mathfrak{i}(g)+2}$ for all g and $\mathfrak{i}(g) \leq k+1$ whenever $g \in C_k$, we see that $\mathfrak{i}(gg') \leq \mathfrak{i}(g) + \mathfrak{i}(g') + 5$ for all $g, g' \in G$, and hence $w(gg') \geq 2^{-5} w(g) w(g')$. It follows that $\inf_{g\in G} \frac{w(\xi g)}{w(g)} > 0$ for all $\xi \in G$, so that $\sup_{g\in G} \frac{w(\xi^{-1}g)}{w(g)} < \infty$, as required.

□

Incidentally, the following question may be interesting. Let Γ be a non-compact Polish group. Is it always possible to find a hypercyclic Hilbert space representation of Γ, or at least a hypercyclic representation on some Banach space X? Otherwise, for which Polish groups Γ is it possible to find such a representation?

3. In Theorem 1.1 or Corollary 1.1, the assumption that some operator T_γ is hypercyclic may look unnecessarily strong and not very natural. However, the following example (essentially taken from [**21**]) shows that this assumption cannot be simply removed.

Example 6.1. *Let X be a complex separable (infinite-dimensional) Banach space, and let $(T_t)_{t\in\mathbb{R}}$ be any 1-parameter hypercyclic C_0-group on X. Let α be any real number such that $\alpha/\pi \notin \mathbb{Q}$, choose $C > 1$, and let $(T_{(s,t)})_{(s,t)\in\mathbb{R}\times\mathbb{R}}$ be the 2-parameters group defined on $X \oplus \mathbb{C}$ by $T_{(s,t)} := T_t \oplus C^{s-t} e^{i\alpha t} I$. Then the group $(T_{(s,t)})_{(s,t)\in\mathbb{R}\times\mathbb{R}}$ is hypercyclic but the subgroup generated by $T_{(1,0)}$ and $T_{(0,1)}$ is not.*

Proof. Since the set $\{C^{m-n} e^{i\alpha n};\ (m,n) \in \mathbb{Z}\times\mathbb{Z}\}$ is nowhere dense in $\mathbb{R}$, it is clear that the group generated by $T_{(1,0)}$ and $T_{(0,1)}$ is not hypercyclic. Now, let x be any hypercyclic vector for the group $(T_t)_{t\in\mathbb{R}}$. Then either $\{T_t(x);\ t \leq 0\}$ or $\{T_t(x);\ t \geq 0\}$ is somewhere dense in X, and hence everywhere dense by the Costakis-Peris theorem [**8**]. Thus, we may assume that x is in fact hypercyclic for the semigroup $(T_t)_{t\geq 0}$. By the Conejero-Müller-Peris theorem and Shkarin's theorem (!), it follows that the set $\{(T_n(x), e^{i\alpha n});\ n \in \mathbb{N}\}$ is dense in $X \times \mathbb{T}$. Thus, given any vector $z = u \oplus re^{i\theta} \in X \oplus \mathbb{C}$, one can first find $n \in \mathbb{N}$ such that $C^{-n} < r$

and $T_n(x) \oplus e^{in\alpha}$ is close to $u \oplus e^{i\theta}$, and then $s \in \mathbb{R}_+$ such that $C^{s-n} = r$ to get that $T_{(s,n)}(x,1) = T_n(x) \oplus C^{s-n}e^{in\alpha}$ is close to z. This shows that the group $(T_{s,t})_{(s,t)\in\mathbb{R}\times\mathbb{R}}$ is hypercyclic, with hypercyclic vector $x \oplus 1$. □

Nevertheless, the assumption that some T_γ is hypercyclic should not be considered as "necessary". Indeed, as shown by F. Bayart in [**2**], there exist hypercyclic *holomorphic groups* $(T_z)_{z\in\mathbb{C}}$ such that no single operator T_z is hypercyclic, and yet the subgroup generated by any basis (z_1, z_2) of $\mathbb{C} = \mathbb{R}^2$ is hypercyclic.

4. Still regarding the same assumption, it would be nice to have a simple direct proof (without any metrizability assumption) of the fact that if a 1-parameter semigroup of operators $(T_t)_{t\geq 0}$ is hypercyclic, then there is at least one hypercyclic operator in it.

5. Even if all operators in a linear dynamical system (X, Γ) are hypercyclic, they may not have the same hypercyclic vectors, and in fact they may have no common hypercyclic vector at all. As shown in [**2**], this can even happen with a holomorphic group $(T_z)_{z\in\mathbb{C}}$.

6. Shkarin's theorem and Proposition 5.4 suggest the following question: is it possible to characterize "intrinsically" the anti-Kroneckerness of a dynamical system (X, T)? As already said, when X is *compact* and metrizable, anti-Kroneckerness is equivalent to weak mixing (i.e. point transitivity of $T \times T$), and this can be characterized in terms of the sets $\mathbf{N}(U, V) := \{n \in \mathbb{N};\ T^n(U) \cap V \neq \emptyset\}$: a dynamical system (X, T) is weakly mixing iff each set $\mathbf{N}(U, V)$ contains arbitrarily long intervals. A similar question may be asked for **mildly mixing** systems. A dynamical system (X, T) is said to be mildly mixing if, for any point transitive compact dynamical system (K, S), the dynamical system $(K \times X, S \times T)$ is point transitive. In the compact case, mild mixing can also be characterized in terms of the sets $\mathbf{N}(U, V)$; see [**11**]. The linear version of this problem is the following: is it possible to characterize the linear operators T such that $S \times T$ is hypercyclic for any hypercyclic operator S?

7. It follows easily from Shkarin's theorem that if T is a hypercyclic operator on some topological vector space X and if $x \in HC(T)$ then, for each nonempty open set $W \subset X$, the set $\mathbf{N}(x, W) = \{n \in \mathbb{N};\ T^n(x) \in W\}$ is dense in $b\mathbb{Z}$, the **Bohr compactification** of $\mathbb{Z}$. However, T may be non-weakly mixing by [**20**]. In this case, by the characterization of weakly mixing systems mentioned above, one can find W such that the difference set $\mathbf{N}(x, W) - \mathbf{N}(x, W)$ (which is equal to $\mathbf{N}(W, W)$) does not contain arbitrarily long intervals, because each set $\mathbf{N}(U, V)$ contains a translate of some $\mathbf{N}(W, W)$. Moreover, one can also find W such that $\mathbf{N}(x, W) - \mathbf{N}(x, W)$ does not have bounded gaps (see [**12**]). Thus, we have examples of sets of integers which are dense in $b\mathbb{Z}$ but with some smallness property. It may be quite interesting to investigate this further.

8. A **group extension** of a dynamical system (X, T) is dynamical system of the form $(G \times X, \widetilde{T})$, where G is a topological group and $\widetilde{T}(\xi, x) = (g(x)\xi, T(x))$ for some continuous map $g : X \to G$. If g is a constant map, one gets the dynamical system $(G \times X, \tau_g \times T)$. It is well-known that if $x \in X$ is a recurrent point for T, then $(\mathbf{1}_G, x)$ is a recurrent point for any compact group extension of (X, T) (see [**10**]). This leads to the following question: is there a Shkarin's theorem for general compact group extensions?

8'. Group extensions are particular cases of **skew products**. Skew products of a different type directly connected to linear dynamics are studied in [**3**].

9. As already mentionned, the general ideas involved in the proofs of Theorems 1.1 and 1.2 go back to the paper [**9**] by H. Furstenberg. The setting of [**9**] is that of a group Γ acting on a compact metric space (X, d), and the purpose is to get a structure theorem for **distal** dynamical systems (X, Γ). A dynamical system is said to be distal if $\inf_{\gamma \in \Gamma} d(\gamma \cdot x, \gamma \cdot y) > 0$ whenever $x \neq y$. One notable consequence of the work done in [**9**] is that every nontrivial minimal distal system (X, Γ) has a nontrivial eigencharacter. Comparing this with Theorem 4.1, we see that very loosely speaking, the distality condition is replaced in our setting by the assumption "$Trans(\Gamma_0) \neq Trans(\Gamma)$", and the compactness of the ground space X is replaced by that of the quotient group Γ/Γ_0. All of this is of course quite vague, but it looks plausible that some more general theorem is hiding somewhere.

10. It would be interesting to know if something general can still be said if the quotient group Γ/Γ_0 is not assumed to be abelian. In particular, is Theorem 1.1 still true without this assumption?

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