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# Essential norms of weighted composition operators on the space $\mathcal{H}^\infty$ of Dirichlet series.

Pascal Lefèvre

July 12, 2008

## Abstract

We estimate the essential norm of a weighted composition operator relatively to the class of Dunford-Pettis operators or the class of weakly compact operators, on the space  $\mathcal{H}^\infty$  of Dirichlet series. As particular cases, we obtain the precise value of the generalized essential norm of a composition operator and of a multiplication operator.

**Key Words** : composition operator, essential norm, Dirichlet series, weakly compact operator, Dunford-Pettis operators.

**AMS classification** : 30B50, 46E15, 47B10, 47B33.

## 0 Introduction

The aim of this paper is to investigate the complete continuity and weak compactness of weighted composition operators on the space  $\mathcal{H}^\infty$  of Dirichlet series. The composition operators were investigated so far in many papers. The monographs [CmC] and [S] are very good survey on this topic. The composition operators are very often investigated on  $H^p$  spaces ( $1 < p < \infty$ ), but, on  $H^p$  spaces, their weak compactness and complete continuity are trivial problems (because of reflexivity). The investigation in the setting of Dirichlet series is more recent: see, for example, [B2], [GH] and [Q2].

Let us recall some terminology. We are going to work on some half planes  $\mathbb{C}_\theta$ , with  $\theta \geq 0$ :

$$\mathbb{C}_\theta = \{s \in \mathbb{C} \mid \operatorname{Re}(s) > \theta\}.$$

The space  $\mathcal{H}(\mathbb{C}_0)$  denotes the space of analytic functions on  $\mathbb{C}_0$ .

The space  $\mathcal{H}^\infty$  of Dirichlet series is

$$\mathcal{H}^\infty = \{f \in \mathcal{H}(\mathbb{C}_0) \mid f \text{ bounded, } f(s) = \sum_{n \geq 1} a_n n^{-s} \text{ on some half plane } \mathbb{C}_\varepsilon \text{ with } \varepsilon > 0\}.$$

(In fact, a result of Bohr [Bo] implies that any  $\varepsilon > 0$  works in the definition)

The space  $\mathcal{H}^\infty$  is the version of the classical Hardy space  $H^\infty$  in the setting of Dirichlet series.

It is natural to introduce the equivalent to the disk algebra

$$\mathcal{A} = \{f \in \mathcal{H}^\infty \mid f \text{ continuous on } \overline{\mathbb{C}_0}\}.$$

Both spaces  $\mathcal{H}^\infty$  and  $\mathcal{A}$  are normed by  $\|f\|_\infty = \sup \{|f(s)|; s \in \mathbb{C}_0\}$ .

Before getting interested in some special properties of composition operators on  $\mathcal{H}^\infty$ , we have to know when they are defined. Actually, the case  $\mathcal{H}^\infty$  is less complicated than the case of general  $\mathcal{H}^p$  spaces: *An analytic function  $\varphi : \mathbb{C}_0 \rightarrow \mathbb{C}_0$  defines a bounded composition operator  $C_\varphi : f \mapsto f \circ \varphi$  on  $\mathcal{H}^\infty$  if and only if  $\varphi(s) = \alpha_0 s + \sum_{n \geq 1} \alpha_n n^{-s}$ , with  $\alpha_0 \in \mathbb{N}$  (see [B1] after Cor2. p. 217, or [B2] p.65). In the sequel, we shall always assume that  $\varphi$  fulfills this condition. We then have  $\|C_\varphi\| = 1$ .*

The characterization of the compactness for composition operators the space  $\mathcal{H}^\infty$  of Dirichlet series is due to Bayart [B1]Th.18. Actually, Bayart estimates the (classical) essential norm of a composition operator on  $\mathcal{H}^\infty$ . Let us recall his result:

**Theorem.** [B1],[B2] *Let  $C_\varphi$  be a composition operator on  $\mathcal{H}^\infty$ .  $C_\varphi$  is compact if and only if  $\varphi(\mathbb{C}_0) \subset \mathbb{C}_\varepsilon$  for some  $\varepsilon > 0$ .*

The compactness of weighted composition operators was studied in the classical frame of the disk algebra in [K]. Some extensions of this results are studied in [L], where generalized essentials norms are computed.

We are going to use rather elementary techniques, adapted from [L], to estimate the essential norm, relatively to Dunford-Pettis operators and weakly compact operators, of weighted composition operators on  $\mathcal{H}^\infty$ .

We first precise some terminology:

**Definition 0.1** *Let  $X, Y$  be Banach spaces and  $\mathcal{I}$  a closed subspace of operator of the space  $B(X, Y)$  of bounded operators from  $X$  to  $Y$ . The essential norm of  $T \in B(X, Y)$  relatively to  $\mathcal{I}$  is the distance from  $T$  to  $\mathcal{I}$ :*

$$\|T\|_{e, \mathcal{I}} = \inf\{\|T + S\|; S \in \mathcal{I}\}.$$

*This is the canonical norm on the quotient space  $B(X, Y)/\mathcal{I}$ .*

*If moreover  $\mathcal{I}$  is an ideal of the space  $B(X)$  then  $B(X)/\mathcal{I}$  is an algebra.*

The classical case corresponds to the case of compact operators  $\mathcal{I} = \mathcal{K}(X, Y)$  (in this case, the preceding quotient space is the Calkin algebra). In the sequel, we get interested in the case of weakly compact operators:  $\mathcal{I} = \mathcal{W}(X, Y)$ ; and in the case of completely continuous operators (= Dunford-Pettis operators):  $\mathcal{I} = \mathcal{DP}(X, Y)$ . Compact operators are both weakly compact and completely continuous.

Recall that a Banach space  $X$  has the *Dunford-Pettis property* if, for every Banach space  $Y$  and every operator  $T : X \rightarrow Y$  which is weakly compact,  $T$  maps a weakly Cauchy sequence in  $X$  into a norm Cauchy sequence. A good survey on the subject (until the early eighties) is the paper of Diestel [D]. A Banach space  $X$  has the *property (V) of Pelczyński* if, for every Banach space  $Y$  and every operator  $T : X \rightarrow Y$  which is not weakly compact, there exists a subspace  $X_0$  of  $X$  isomorphic to  $c_0$  such that  $T|_{X_0}$  is an isomorphic embedding.

If the space of Dirichlet series  $\mathcal{H}^\infty$  had both property (V) and the Dunford-Pettis property, then the two ideals  $\mathcal{W}(\mathcal{H}^\infty, Y)$  and  $\mathcal{DP}(\mathcal{H}^\infty, Y)$  would coincide, for every Banach  $Y$ . It turns out that  $\mathcal{H}^\infty$  does not have property (V) and it is unknown whether  $\mathcal{H}^\infty$  has the Dunford-Pettis property.

**Claim.**  $\mathcal{H}^\infty$  does not have property (V) (we have no reference for this remark). We are going to see this as a consequence of the Bohr inequality. This reads as follows (see [Q1]):

$$\sum_{p \in \mathcal{P}} |a_p| \leq \|f\|_\infty \quad \text{for every } f \in \mathcal{H}^\infty$$

where  $\mathcal{P}$  stands for the set of prime numbers.

This means that the space  $\{f \in \mathcal{H}^\infty \mid f(s) = \sum_{p \in \mathcal{P}} a_p p^{-s}\}$  is a complemented subspace of  $\mathcal{H}^\infty$ , isomorphic to  $\ell^1$ . Thus, the underlying projection can neither be weakly compact, nor fix a copy of  $c_0$ . This proves the claim.

Let us point out too that the same remark implies that the space  $\mathcal{H}^\infty$  does not verify the Grothendieck Theorem: the projection (given by the Bohr inequality) from  $\mathcal{H}^\infty$  to  $\ell^1$  is bounded and cannot be 2-summing.

Given  $u \in \mathcal{H}^\infty$  and an analytic function  $\varphi$  from  $\mathbb{C}_0$  to  $\mathbb{C}_0$  defining a composition operator, we shall study in the paper the (generalized) essential norm of the weighted composition operator  $T_{u,\varphi}$ :

$$T_{u,\varphi}(f) = u.(f \circ \varphi) \quad \text{where } f \in \mathcal{H}^\infty.$$

Of course, when  $u = \mathbb{1}$ , this operator is the classical composition operator and simply denoted by  $C_\varphi$ . When  $\varphi = Id_{\mathbb{C}_0}$ , this operator is the multiplication operator  $M_u$  by  $u$ .

Observe that  $T_{u,\varphi}$  is always bounded from  $\mathcal{H}^\infty$  to  $\mathcal{H}^\infty$ , with  $\|T_{u,\varphi}\| = \|u\|_\infty$ , where  $\|u\|_\infty = \sup\{|u(s)|; s \in \mathbb{C}_0\}$ .

The following quantity plays a crucial role in the estimate of the essential norm: we define

$$n_\varphi(u) = \limsup_{r \rightarrow 0^+} \sup\{|u(s)|; s \in \mathbb{C}_0, \operatorname{Re}(\varphi(s)) \leq r\}$$

which defines a finite number since  $u$  is bounded.

If  $\inf \operatorname{Re}(\varphi) > 0$  then  $n_\varphi(u) = 0$  (i.e. the supremum over the empty set is taken as 0).

## 1 Characterization of weak compactness and complete continuity.

We first need the following lemma.

**Lemma 1.1** *Let  $(h_n)_{n \geq 0}$  be a sequence in the disk algebra  $A(\mathbb{D})$ , to which we associate the sequence in  $\mathcal{A}$  defined by  $H_n(s) = h_n(2^{-s})$ .*

*If  $(h_n)_{n \geq 0}$  is weakly Cauchy in  $A(\mathbb{D})$ , then  $(H_n)_{n \geq 0}$  is weakly Cauchy in  $\mathcal{A}$ .*

Moreover

*i)  $(H_n)_{n \geq 0}$  is weakly null if and only if  $H_n(ix) \rightarrow 0$ , for every  $x \in \mathbb{R}$ .*

*ii)  $(H_n)_{n \geq 0}$  is weakly Cauchy if and only if  $(H_n(ix))$  is convergent, for every  $x \in \mathbb{R}$ .*

**Proof.** First notice that in i) and ii) the “only if” part is obvious since  $H \mapsto H(ix)$  clearly defines a linear functional on  $\mathcal{A}$  for each  $x \in \mathbb{R}$ .

Observe that, for every  $h \in A(\mathbb{D})$ ,  $H(s) = h(2^{-s})$  defines a function in  $\mathcal{A}$ . Indeed, if  $h(z) = \sum c_j z^j$  for  $z$  in the open unit disk  $\mathbb{D}$ , then  $H(s) = \sum c_j 2^{-js}$  is convergent for  $s \in \mathbb{C}_0$ . Moreover  $H$  is continuous on  $i\mathbb{R}$ .

Now, let  $\xi$  be a linear functional on  $\mathcal{A}$ . We can define a linear functional on  $A(\mathbb{D})$  in the following way:  $\chi(h) = \xi(H)$ , with  $H(s) = h(2^{-s})$ . The first part of the lemma easily follows:  $\xi(H_n)$  converges.

Thus, there is a Borel measure  $\mu$  on  $\mathbb{T}$  such that  $\xi(H) = \int_{\mathbb{T}} h d\mu$ . We can conclude the “if” part in i) and ii) because  $H_n(ix) = h_n(2^{-ix})$  and the dominated convergence Theorem applies.  $\blacksquare$

Now, we can establish the following characterization, which is a generalization of Th.18 [B1].

**Theorem 1.2** *With the previous notations, the following assertions are equivalent*

- 1)  $T_{u,\varphi} : \mathcal{A} \rightarrow \mathcal{H}^\infty$  is completely continuous.
- 2)  $T_{u,\varphi} : \mathcal{A} \rightarrow \mathcal{H}^\infty$  is weakly compact.
- 3)  $n_\varphi(u) = 0$ .
- 4)  $T_{u,\varphi} : \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty$  is compact.

**Proof.** Obviously 4 implies 1 and 2.

1  $\Rightarrow$  3. Assume that  $\inf \operatorname{Re}(\varphi) = 0$  and  $n_\varphi(u) > \varepsilon_0 > 0$ .

Choose any sequence  $s_j \in \mathbb{C}_0$  such that  $\operatorname{Re}(\varphi(s_j))$  converges to 0 and  $|u(s_j)| \geq \varepsilon_0$ . Extracting a subsequence if necessary, we may suppose that  $2^{-\varphi(s_j)}$  converges to some  $a$ , belonging to the torus. We shall write  $a = 2^{-i\alpha}$  where  $\alpha \in \mathbb{R}$ .

Now, we consider the sequence of functions  $F_n(s) = f_n(2^{-s})$  where  $f_n(z) = 2^{-n}(\bar{a}z+1)^n$  lies in the unit ball of the disk algebra.  $(F_n)$  is clearly a weakly Cauchy sequence in  $\mathcal{A}$  thanks to Lemma 1.1.ii. Actually  $F_n(s) \rightarrow 0$  for every  $s \in \overline{\mathbb{C}_0} \setminus \{i\alpha\}$  and  $F_n(i\alpha) = 1$ .

The operator  $T_{u,\varphi}$  being a Dunford-Pettis operator, the sequence  $(u.F_n \circ \varphi)_{n \in \mathbb{N}}$  is norm-Cauchy, hence converging to some  $\sigma \in \mathcal{H}^\infty$ . But for every fixed  $s \in \mathbb{C}_0$ ,  $u(s).F_n \circ \varphi(s)$  converges both to 0 and  $\sigma(s)$ , so that  $\sigma = 0$ .

Fixing  $\varepsilon > 0$ , there exists  $n_0$  such that  $\sup_{s \in \mathbb{C}_0} |u(s)F_{n_0} \circ \varphi(s)| \leq \varepsilon$ . Choosing  $s = s_{j_0}$  with  $j_0$  large enough to realize  $|F_{n_0} \circ \varphi(s_{j_0})| \geq 1 - \varepsilon$ , we have:

$$\varepsilon \geq |u(s_{j_0})|(1 - \varepsilon) \geq (1 - \varepsilon)\varepsilon_0.$$

As  $\varepsilon$  is arbitrary, this gives a contradiction.

2  $\Rightarrow$  3. Assume that  $\inf \operatorname{Re}(\varphi) = 0$  and  $n_\varphi(u) > \varepsilon_0 > 0$ . In the same way, choose any sequence  $s_j \in \mathbb{C}_0$  such that  $\operatorname{Re}(\varphi(s_j))$  converges to 0 and  $|u(s_j)| \geq \varepsilon_0$ . We may assume that  $2^{-\varphi(s_j)}$  converges to some  $a = 2^{-i\alpha} \in \mathbb{T}$  and we consider the same sequence of functions  $F_n$ . The operator  $T_{u,\varphi}$  being a weakly compact operator, there exists a sequence on integers  $(n_k)$  such that  $(u.F_{n_k} \circ \varphi)_{k \in \mathbb{N}}$  is weakly convergent to some  $\sigma \in \mathcal{H}^\infty$ . Testing

the weak convergence on the point evaluation  $\delta_s \in (\mathcal{H}^\infty)^*$ , for each  $s \in \mathbb{C}_0$ , we obtain that  $\sigma = 0$ .

By the Mazur Theorem, there exists a convex combination of these functions which is norm convergent to 0:

$$\sum_{k \in I_m} c_k u.(F_{n_k} \circ \varphi) \longrightarrow 0$$

where  $c_k \geq 0$  and  $\sum_{k \in I_m} c_k = 1$ .

Now, fixing  $\varepsilon \in (0, \varepsilon_0/2)$ , we have for a suitable  $m_0$

$$\sup_{s \in \mathbb{C}_0} \left| \sum_{k \in I_{m_0}} c_k u(s).F_{n_k}(\varphi(s)) \right| \leq \varepsilon.$$

So, for every  $j$

$$\varepsilon_0 \left| \sum_{k \in I_{m_0}} c_k.F_{n_k}(\varphi(s_j)) \right| \leq \left| \sum_{k \in I_{m_0}} c_k u(s_j).F_{n_k}(\varphi(s_j)) \right| \leq \varepsilon.$$

Letting  $j$  tends to infinity, we have  $F_{n_k}(\varphi(s_j)) \rightarrow F_{n_k}(i\alpha) = 1$ , for each  $k \in I_{m_0}$ , so that

$$\varepsilon_0 = \varepsilon_0 \left| \sum_{k \in I_{m_0}} c_k \right| \leq \varepsilon.$$

This gives a contradiction.

3  $\Rightarrow$  4. Note that  $T_{u,\varphi} = M_u \circ C_\varphi$ .

If  $\inf \operatorname{Re}(\varphi) > 0$  then  $\varphi(\mathbb{C}_0) \subset \mathbb{C}_\varepsilon$  for some  $\varepsilon > 0$  and  $C_\varphi$  is compact thanks to Bayart's Theorem, recalled in the introduction.

If  $\inf \operatorname{Re}(\varphi) = 0$  and  $\limsup_{r \rightarrow 0^+} \{|u(s)|; s \in \mathbb{C}_0, \operatorname{Re}(\varphi(s)) \leq r\} = 0$  then  $T_{u,\varphi}$  is compact.

Indeed, given a sequence in the unit ball of  $\mathcal{H}^\infty$ , we can extract a subsequence  $(f_n)_n$  uniformly converging on every half plane  $\mathbb{C}_\theta$ , with  $\theta > 0$ . This is due to a version for Dirichlet series of the classical Montel Theorem, proved by Bayart (see [B1] Lemma 18 or Lemme 5.2 [B2]). Hence, given  $\varepsilon > 0$ , we choose  $\theta > 0$  such that, when  $\operatorname{Re}(\varphi(s)) \leq \theta$ ,  $|u(s)| \leq \varepsilon$ . Then we have

$$\|u.(f_n - f_m) \circ \varphi\|_\infty \leq \max \left\{ \|u\|_\infty \cdot \sup_{\varphi(s) \in \mathbb{C}_\theta} |(f_n - f_m) \circ \varphi(s)|; 2\varepsilon \right\}$$

which is less than  $2\varepsilon$ , when  $n, m$  are large enough. ■

**Corollary 1.3** *Let  $C_\varphi$  be a composition operator on  $\mathcal{H}^\infty$ . The following assertions are equivalent.*

- i)  $C_\varphi$  is completely continuous.
- ii)  $C_\varphi$  is weakly compact.
- iii)  $C_\varphi$  is compact.
- iv)  $\inf \operatorname{Re}(\varphi) > 0$ .

**Proof.** If  $\inf \operatorname{Re}(\varphi) > 0$ , it is actually compact. If  $C_\varphi$  is completely continuous (resp. weakly compact) on  $\mathcal{H}^\infty$  then its restriction to  $\mathcal{A}$  is as well. The result follows from the preceding theorem in the case  $u = \mathbf{1}$ . ■

**Remark.** We have the same results when the operators act from  $\mathcal{A}$  into itself (under the extra assumption that  $\varphi \in \mathcal{A}$ ).

We recall that  $\varphi^*$  denotes the (non tangential) boundary values of  $\varphi$ : it is defined almost everywhere on the imaginary axis. From Theorem 1.2, we can deduce

**Corollary 1.4** *Let  $u \in \mathcal{H}^\infty$ .*

- 1) *Assume that  $E = \{y \in \mathbb{R} \mid \operatorname{Re}(\varphi^*(iy)) = 0\}$  has positive Lebesgue measure. Then  $T_{u,\varphi}$  is weakly compact or completely continuous if and only if  $u = 0$ .*
- 2)  *$M_u : \mathcal{A} \rightarrow \mathcal{H}^\infty$  is weakly compact or completely continuous if and only if  $u = 0$ .*

**Proof.** Under the hypothesis of weak compactness or complete continuity of  $T_{u,\varphi}$ , we have  $n_\varphi(u) = 0$ , due to Theorem 1.2. Let us fix  $\varepsilon > 0$  and take  $r > 0$  such that for every  $s \in \mathbb{C}_0$ :

$$\operatorname{Re}(\varphi(s)) < r \implies |u(s)| \leq \varepsilon.$$

Now fix  $\varepsilon > 0$ . The hypothesis on  $\varphi$  implies that, for almost every  $y \in E$  and any sequence  $(x_n)_n$  in  $\mathbb{R}^{*+}$ :

$$\operatorname{Re}(\varphi(s_n)) \longrightarrow \operatorname{Re}(\varphi^*(iy)) = 0$$

where  $s_n = x_n + iy$  (actually, we could replace  $s_n$  by any sequence in  $\mathbb{C}_0$  non tangentially converging to  $iy$ ).

But for almost every  $y \in E$  (let's say for  $y \in E_0$  where  $E_0 \subset E$  has positive Lebesgue measure):  $u(s_n) \longrightarrow u^*(iy)$ , the boundary value of  $u$ , defined almost everywhere on the imaginary axis.

Therefore, for every  $y \in E_0$  and  $n$  large enough, we have  $\operatorname{Re}(\varphi(s_n)) < r$ , hence  $|u^*(iy)| \leq \varepsilon$ . Since  $\varepsilon > 0$  is arbitrary the boundary value of  $u$  vanishes on a set of positive Lebesgue measure, so  $u = 0$  everywhere on  $\mathbb{C}_0$ .

The second point is an immediate consequence of the first one. ■

## 2 Essential norms.

In the sequel,  $X$  denotes either  $\mathcal{A}$  or  $\mathcal{H}^\infty$ . We shall adapt techniques of section 1 to compute essential norms. We get a generalization of the Theorem of Bayart in several directions. We first need the following lower estimate

**Lemma 2.1** *Let  $u \in \mathcal{H}^\infty$  and  $\varphi : \mathbb{C}_0 \rightarrow \mathbb{C}_0$  defining a composition operator. We assume that  $\mathcal{I} \subset \mathcal{W}(X, \mathcal{H}^\infty) \oplus \mathcal{DP}(X, \mathcal{H}^\infty)$ .*

*Then*

$$n_\varphi(u) \leq \|T_{u,\varphi}\|_{e,\mathcal{I}}.$$

**Proof.** The proof mixes the one of Theorem 1.2 with the one of [B1] (relying on an idea due to Zheng [Z]) and is very similar to the one given in [L] in the frame of classical Hardy spaces. For sake of completeness, we give a detailed proof. We already know that  $\|T_{u,\varphi}\|_{e,\mathcal{I}} = 0$  if and only if  $T_{u,\varphi}$  is completely continuous if and only if  $n_\varphi(u) = 0$  if and

only if  $T_{u,\varphi}$  is compact. We assume now that  $T_{u,\varphi}$  is not compact and this implies that  $\inf \operatorname{Re}(\varphi) = 0$ .

We choose a sequence  $s_j \in \mathbb{C}_0$  such that  $\operatorname{Re}(\varphi(s_j))$  converges to 0 and  $|u(s_j)|$  converges to  $n_\varphi(u)$ . We may assume that  $2^{-\varphi(s_j)}$  converges to some  $a = 2^{-i\alpha}$ .

We introduce the sequence of functions (where  $n \geq 2$ )

$$H_n(s) = \frac{n\bar{a}2^{-s} - (n-1)}{n - (n-1)\bar{a}2^{-s}},$$

which lies in the unit ball of  $\mathcal{A}$ .

Obviously,  $H_n(s) = h_n(2^{-s})$  where  $h_n$  lies in the unit ball of the disk algebra, with  $h_n(z) \rightarrow -1$  for every  $z \in \mathbb{D} \setminus \{a\}$  and  $h_n(a) = 1$ . So,  $H_n(s) \rightarrow -1$  for every  $s \in \overline{\mathbb{C}_0} \setminus \{i\alpha\}$  and  $H_n(i\alpha) = 1$ .

Now, let  $S \in \mathcal{I}$ . Let us write  $S = D + W$ , where  $W$  is weakly compact and  $D$  is Dunford-Pettis.

As  $D \in \mathcal{DP}(X, \mathcal{H}^\infty)$  and the sequence  $(H_n)_n$  is a weakly Cauchy sequence by Lemma 1.1., the sequence  $(D(H_n))_n$  is a Cauchy sequence, hence convergent to some  $\Delta \in \mathcal{H}^\infty$ .

As  $W \in \mathcal{W}(X, \mathcal{H}^\infty)$ , up to an extraction, the sequence  $(S(H_n))_n$  is weakly convergent to some  $w \in \mathcal{H}^\infty$ . By the Mazur Theorem, we can find some  $c_k \geq 0$  with  $\sum_{k \in I_m} c_k = 1$ , where  $I_m \subset \mathbb{N}$ ; and  $\sum_{k \in I_m} c_k W(H_k) \rightarrow w$ . Moreover, we can assume that  $\sup I_m < \inf I_{m+1}$ .

Introducing  $\tilde{H}_m = \sum_{k \in I_m} c_k H_k$ , we have: for every  $s \in \mathbb{C}_0$ ,  $\tilde{H}_m(s) \rightarrow -1$  and for every  $m$ ,  $\tilde{H}_m(\varphi(s_j)) \rightarrow 1$ . Clearly,  $(D(\tilde{H}_n))_n$  is norm convergent to  $\Delta$ , so  $(S(\tilde{H}_n))_n$  is norm convergent to  $\sigma = \Delta + w$ .

For every integer  $n$ :

$$\|(T_{u,\varphi} - S)(\tilde{H}_n)\|_\infty \geq \|T_{u,\varphi}(\tilde{H}_n) - \sigma\|_\infty - \|S(\tilde{H}_n) - \sigma\|_\infty$$

and we already know that  $\|S(\tilde{H}_n) - \sigma\|_\infty \rightarrow 0$ .

For every  $s \in \overline{\mathbb{C}_0} \setminus \{i\alpha\}$ , we have  $|u(s) \cdot \tilde{H}_n \circ \varphi(s) - \sigma(s)| \rightarrow |\sigma(s) + u(s)|$ .

If  $|\sigma(s_0) + u(s_0)| > n_\varphi(u)$  for some  $s_0 \in \mathbb{C}_0$ , then

$$\|T_{u,\varphi} - S\| \geq \overline{\lim} \|(T_{u,\varphi} - S)(\tilde{H}_n)\|_\infty \geq \overline{\lim} |u(s_0) \cdot \tilde{H}_n \circ \varphi(s_0) - \sigma(s_0)| = |\sigma(s_0) + u(s_0)| \geq n_\varphi(u).$$

If not, then  $\|\sigma + u\|_\infty \leq n_\varphi(u)$  and for every  $s \in \mathbb{C}_0$ ,  $|\sigma(s) - u(s)| \geq 2|u(s)| - n_\varphi(u)$ . We have for every  $n \geq 2$  and every integer  $j$ :

$$\begin{aligned} \|T_{u,\varphi} - S\| &\geq |u(s_j) \cdot \tilde{H}_n \circ \varphi(s_j) - \sigma(s_j)| - \|S(\tilde{H}_n) - \sigma\|_\infty \\ &\geq 2|u(s_j)| - n_\varphi(u) - |u(s_j)| \cdot |\tilde{H}_n \circ \varphi(s_j) - 1| - \|S(\tilde{H}_n) - \sigma\|_\infty. \end{aligned}$$

Letting first  $j$  tend to infinity, we obtain  $\|T_{u,\varphi} - S\| \geq n_\varphi(u) - \|S(\tilde{H}_n) - \sigma\|_\infty$ .

At last, letting  $n$  tend to infinity, we have  $\|T_{u,\varphi} - S\| \geq n_\varphi(u)$ .

The conclusion follows. ■



For the upper estimate, we have

**Lemma 2.2** *Let  $u \in \mathcal{H}^\infty$  and  $\varphi : \mathbb{C}_0 \rightarrow \mathbb{C}_0$  defining a composition operator. Then*

$$\|T_{u,\varphi}\|_e \leq \inf\{2n_\varphi(u), \|u\|_\infty\}.$$

**Proof.** Fix  $\varepsilon > 0$ . There exists  $r \in (0, 1)$  such that for every  $s \in \mathbb{C}_0$ ,

$$\operatorname{Re}(\varphi(s)) \leq r \implies |u(s)| \leq n_\varphi(u) + \varepsilon.$$

Now, fixing  $\rho > 0$  for a while, we introduce the operator defined for  $s \in \mathbb{C}_0$  by

$$S(f)(s) = u(s).f(\varphi(s) + \rho).$$

In other words,  $S = T_{u,\varphi_\rho}$  with  $\varphi_\rho = \varphi + \rho$ . By the Theorem of Bayart,  $S$  is a compact operator since  $\varphi_\rho(\mathbb{C}_0) \subset \mathbb{C}_\rho$ . We have

$$\|T_{u,\varphi} - S\| = \sup_{\substack{f \in \mathcal{H}^\infty \\ \|f\|_\infty \leq 1}} \sup_{\operatorname{Re}(\varphi(s)) > 0} |u(s)| \cdot |f \circ \varphi(s) - f \circ (\varphi(s) + \rho)|.$$

First observe that

$$\sup_{\substack{f \in \mathcal{H}^\infty \\ \|f\|_\infty \leq 1}} \sup_{\operatorname{Re}(\varphi(s)) \leq r} |u(s)| \cdot |f \circ \varphi(s) - f \circ (\varphi(s) + \rho)| \leq 2(n_\varphi(u) + \varepsilon).$$

On the other hand, we claim that

$$\sup_{\substack{f \in \mathcal{H}^\infty \\ \|f\|_\infty \leq 1}} \sup_{\operatorname{Re}(\varphi(s)) > r} |f \circ \varphi(s) - f \circ (\varphi(s) + \rho)| \xrightarrow{\rho \rightarrow 0^+} 0$$

Indeed,

$$\sup_{\substack{f \in \mathcal{H}^\infty \\ \|f\|_\infty \leq 1}} \sup_{\operatorname{Re}(\varphi(s)) > r} |f \circ \varphi(s) - f \circ (\varphi(s) + \rho)| \leq \sup_{\substack{f \in \mathcal{H}^\infty \\ \|f\|_\infty \leq 1}} \sup_{\operatorname{Re}(w) > r} |f(w) - f(w + \rho)|$$

and using the analogue for Dirichlet series of the Montel Theorem (cited above), it is easy to see that

$$\lim_{\rho \rightarrow 0^+} \sup_{\operatorname{Re}(w) > r} \sup_{\substack{f \in \mathcal{H}^\infty \\ \|f\|_\infty \leq 1}} |f(w) - f(w + \rho)| = 0.$$

So we can choose  $\rho > 0$  such that

$$\sup_{\substack{f \in \mathcal{H}^\infty \\ \|f\|_\infty \leq 1}} \sup_{\operatorname{Re}(\varphi(s)) > r} |f \circ \varphi(s) - f \circ (\varphi(s) + \rho)| \leq \varepsilon.$$

Finally,

$$\|T_{u,\varphi} - S\| \leq \max\{\varepsilon \|u\|_\infty; 2(n_\varphi(u) + \varepsilon)\}.$$

As  $\varepsilon > 0$  is arbitrary, we conclude  $\|T_{u,\varphi}\|_e \leq 2n_\varphi(u)$ . This gives the result.  $\blacksquare$

We summarize our results in the following theorem.

**Theorem 2.3** *Let  $u \in \mathcal{H}^\infty$  and  $\varphi : \mathbb{C}_0 \rightarrow \mathbb{C}_0$  defining a composition operator. We assume that  $\mathcal{K}(X, \mathcal{H}^\infty) \subset \mathcal{I} \subset \mathcal{W}(X, \mathcal{H}^\infty) \oplus \mathcal{DP}(X, \mathcal{H}^\infty)$ .*

*Then*

$$\|T_{u,\varphi}\|_{e,\mathcal{I}} \approx n_\varphi(u).$$

*More precisely*

$$n_\varphi(u) \leq \|T_{u,\varphi}\|_{e,\mathcal{I}} \leq \inf\{2n_\varphi(u), \|u\|_\infty\}.$$

*As a particular case, when  $n_\varphi(u) = \|u\|_\infty$ , the equality holds:  $\|T_{u,\varphi}\|_{e,\mathcal{I}} = \|T_{u,\varphi}\|_e = \|u\|_\infty$ .*

We specify two particular cases.

**Corollary 2.4** *Let  $u \in \mathcal{H}^\infty$  and  $\varphi : \mathbb{C}_0 \rightarrow \mathbb{C}_0$  defining a composition operator. We assume that  $\mathcal{K}(X, \mathcal{H}^\infty) \subset \mathcal{I} \subset \mathcal{W}(X, \mathcal{H}^\infty) \oplus \mathcal{DP}(X, \mathcal{H}^\infty)$ .*

- 1)  $\|M_u\|_{e,\mathcal{I}} = \|M_u\|_e = \|u\|_\infty$ .
- 2)  $\|C_\varphi\|_{e,\mathcal{I}} = 1$  if  $\inf \operatorname{Re}(\varphi) = 0$     and     $\|C_\varphi\|_{e,\mathcal{I}} = 0$  if  $\inf \operatorname{Re}(\varphi) > 0$

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