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ABOUT BREZIS-MERLE PROBLEM WITH HOLDERIAN CONDITION: THE CASE OF THREE BLOW-UP POINTS.

SAMY SKANDER BAHOURA

ABSTRACT. We consider the following problem on open set Ω of \mathbb{R}^2 :

$$\begin{cases} -\Delta u_i = V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^3, \\ u_i = 0 & \text{on } \partial\Omega. \end{cases}$$

We assume that :

$$\int_{\Omega} e^{u_i} dy \leq C,$$

and,

$$0 \leq V_i \leq b < +\infty$$

On the other hand, if we assume that V_i s -holderian with $1/2 < s \leq 1$, and,

$$\int_{\Omega} V_i e^{u_i} dy \leq 32\pi - \epsilon, \quad \epsilon > 0$$

then we have a compactness result, namely:

$$\sup_{\Omega} u_i \leq c = c(b, C, A, s, \Omega).$$

where A is the holderian constant of V_i .

1. INTRODUCTION AND MAIN RESULTS

We set $\Delta = \partial_{11} + \partial_{22}$ on open set Ω of \mathbb{R}^2 with a smooth boundary.

We consider the following problem on $\Omega \subset \mathbb{R}^2$:

$$(P) \begin{cases} -\Delta u_i = V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^3, \\ u_i = 0 & \text{in } \partial\Omega. \end{cases}$$

We assume that,

$$\int_{\Omega} e^{u_i} dy \leq C,$$

and,

$$0 \leq V_i \leq b < +\infty$$

The previous equation is called, the Prescribed Scalar Curvature equation, in relation with conformal change of metrics. The function V_i is the prescribed curvature.

Here, we try to find some a priori estimates for sequences of the previous problem.

Equations of this type were studied by many authors, see [2, 3, 4, 6, 7, 8, 10, 11, 12, 13, 16]. We can see in [4], different results for the solutions of those type of equations with or without boundaries conditions and, with minimal conditions on V , for example we suppose $V_i \geq 0$ and $V_i \in L^p(\Omega)$ or $V_i e^{u_i} \in L^p(\Omega)$ with $p \in [1, +\infty]$.

Among other results, we can see in [4], the following important Theorem,

Theorem A (Brezis-Merle [4]). *If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the previous problem (P) with, $0 < a \leq V_i \leq b < +\infty$, and without the boundary condition, then, for all compact set K of Ω ,*

$$\sup_K u_i \leq c = c(a, b, m, K, \Omega) \text{ if } \inf_{\Omega} u_i \geq m.$$

A simple consequence of this theorem is that, if we assume $u_i = 0$ on $\partial\Omega$ then, the sequence $(u_i)_i$ is locally uniformly bounded. We can find in [4] an interior estimate if we assume $a = 0$, but we need an assumption on the integral of e^{u_i} . We have in [4]:

Theorem B (Brezis-Merle [4]). *If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the previous problem (P) with, $0 \leq V_i \leq b < +\infty$, and,*

$$\int_{\Omega} e^{u_i} dy \leq C,$$

then, for all compact set K of Ω ,

$$\sup_K u_i \leq c = c(b, C, K, \Omega).$$

If, we assume V with more regularity, we can have another type of estimates, $\sup + \inf$. It was proved, by Shafrir, see [13], that, if $(u_i)_i, (V_i)_i$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C \left(\frac{a}{b} \right) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

We can see in [7], an explicit value of $C \left(\frac{a}{b} \right) = \sqrt{\frac{a}{b}}$. In his proof, Shafrir has used the Stokes formula and an isoperimetric inequality. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with A the Lipschitz constant, then, $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$, see Brézis-Li-Shafrir [3]. This result was extended for Hölderian sequences $(V_i)_i$ by Chen-Lin, see [7]. Also, we can see in [10], an extension of the Brezis-Li-Shafrir to compact Riemann surface without boundary. We can see in [11] explicit form, $(8\pi m, m \in \mathbb{N}^*$ exactly), for the numbers in front of the Dirac masses, when the solutions blow-up. Here, the notion of isolated blow-up point is used. Also, we can see in [16] refined estimates near the isolated blow-up points and the bubbling behavior of the blow-up sequences.

In [4], Brezis and Merle proposed the following Problem:

Problem (Brezis-Merle [4]). *If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the previous problem (P) with,*

$$0 \leq V_i \rightarrow V \text{ in } C^0(\Omega).$$

$$\int_{\Omega} e^{u_i} dy \leq C,$$

Is it possible to prove that:

$$\sup_{\Omega} u_i \leq c = c(C, V, \Omega) ?$$

Here, we assume more regularity on V_i , we suppose that $V_i \geq 0$ is C^s (s -holderian) ($1/2 < s \leq 1$). We give the answer where $bC < 32\pi$.

In the similar way, we have in dimension $n \geq 3$, with different methods, some a priori estimates of the type $\sup \times \inf$ for equation of the type:

$$-\Delta u + \frac{n-2}{4(n-1)} R_g(x)u = V(x)u^{(n+2)/(n-2)} \text{ on } M.$$

where R_g is the scalar curvature of a riemannian manifold M , and V is a function. The operator $\Delta = \nabla^i(\nabla_i)$ is the Laplace-Beltrami operator on M .

When $V \equiv 1$ and M compact, the previous equation is the Yamabe equation. T. Aubin and R. Schoen solved the Yamabe problem, see for example [1]. If V is not a constant function, the previous equation is called a prescribing curvature equation, we have many existence results see also [1], for a detailed summary.

We can see in [3], [5], [9], some results for elliptic equations of this type, and, some application of the method of moving-plane to obtain uniform estimates and estimates of type $\sup \times \inf$. See also, [14], [15] for other $\sup + \inf$ inequalities.

In [5], we have a classification result for singular and non-singular solution of the Yamabe equation on open set of \mathbb{R}^n and on \mathbb{R}^n . The method used is of moving-plane and some other estimates.

In [9], we have a basic description of the method of moving-plane, and, in [3], we have an application of this method, namely; inequality of type $\sup + \inf$ on a bounded domain of \mathbb{R}^2 .

Returning to our work, we give a compactness result for the Brezis-Merle Problem when the energy is less than $32\pi - \epsilon$, $\epsilon > 0$. In fact, we extend the result of the author, see [2]. We argue by contradiction and we use some asymptotic estimates for the blow-up functions. Also, we use a term of the Pohozaev identity to conclude to a contradiction.

Our main result is:

Theorem. *Assume that, V_i is uniformly s -holderian with $1/2 < s \leq 1$, and,*

$$\int_{B_1(0)} V_i e^{u_i} dy \leq 32\pi - \epsilon, \quad \epsilon > 0,$$

then we have:

$$\sup_{\Omega} u_i \leq c = c(b, C, A, s, \Omega).$$

for solutions of the problem (P), here A is the holderian constant of V_i .

2. PROOF OF THE RESULT:

Proof of the theorem:

Without loss of generality, we can assume that $\Omega = B_1(0)$ the unit ball centered on the origin.

Here, G is the Green function of the Laplacian with Dirichlet condition on $B_1(0)$. We have (in complex notation):

$$G(x, y) = \frac{1}{2\pi} \log \frac{|1 - \bar{x}y|}{|x - y|},$$

we can write:

$$u_i(x) = \int_{B_1(0)} G(x, y) V_i(y) e^{u_i(y)} dy,$$

We assume that we are in the case of one blow-up point. Following the notation of a previous paper, see [2], we have:

$$\max_{\Omega} u_i = u_i(x_i) \rightarrow +\infty,$$

$$\delta_i = d(x_i, \partial\Omega) \rightarrow 0,$$

for $\epsilon > 0$ small enough, and $|y| = \epsilon$,

$$u_i(x_i + \delta_i y) \leq C_{\epsilon},$$

and, the $\sup + \inf$ inequality gives:

$$u_i(x_i) + 4 \log \delta_i \leq C.$$

Also, we have the following estimates which imply the smallness for a term of the Pohozaev identity:

$$\|\nabla u_i\|_{L^q(B(x_i, \delta_i \epsilon'))} = o(1), \quad \forall 1 \leq q < 2.$$

We have; because V_i is s -holderian with $1/2 < s \leq 1$, the following term of the Pohozaev identity tends to 0

$$J_i = \int_{B(x_i, \delta_i \epsilon')} < x_1^i | \nabla(u_i - u) > (V_i - V_i(x_i)) e^{u_i} dy = o(1).$$

Now, we set:

$$r_i = e^{-u_i(x_i)/2},$$

we write, for $|\theta| \leq \frac{\delta_i \epsilon'}{r_i}$, $0 < \epsilon' < \frac{1}{4}$,

$$\begin{aligned} u_i(x_i + r_i \theta) &= \int_{\Omega} G(x_i + r_i \theta, y) V_i(y) e^{u_i(y)} dx = \\ &= \int_{\Omega - B(x_i, 2\delta_i \epsilon')} G(x_i, y) V_i e^{u_i(y)} dy + \int_{B(x_i, 2\delta_i \epsilon')} G(x_i + r_i \theta, y) V_i e^{u_i(y)} dy = \end{aligned}$$

We write, $y = x_i + r_i \tilde{\theta}$, with $|\tilde{\theta}| \leq 2 \frac{\delta_i \epsilon'}{r_i}$,

$$\begin{aligned} u_i(x_i + r_i \theta) &= \int_{B(0, 2 \frac{\delta_i \epsilon'}{r_i})} \frac{1}{2\pi} \log \frac{|1 - (\bar{x}_i + r_i \tilde{\theta})(x_i + r_i \tilde{\theta})|}{r_i |\theta - \tilde{\theta}|} V_i e^{u_i(y)} r_i^2 dy + \\ &\quad + \int_{\Omega - B(x_i, 2\delta_i \epsilon')} G(x_i + r_i \theta, y) V_i e^{u_i(y)} dy \\ u_i(x_i) &= \int_{\Omega - B(x_i, 2\delta_i \epsilon')} G(x_i, y) V_i e^{u_i(y)} dy + \int_{B(x_i, 2\delta_i \epsilon')} G(x_i, y) V_i e^{u_i(y)} dy \end{aligned}$$

Hence,

$$\begin{aligned} u_i(x_i) &= \int_{B(0, 2 \frac{\delta_i \epsilon'}{r_i})} \frac{1}{2\pi} \log \frac{|1 - \bar{x}_i(x_i + r_i \tilde{\theta})|}{r_i |\tilde{\theta}|} V_i e^{u_i(y)} r_i^2 dy + \\ &\quad + \int_{\Omega - B(x_i, 2\delta_i \epsilon')} G(x_i, y) V_i e^{u_i(y)} dy \end{aligned}$$

We look to the difference,

$$v_i(\theta) = u_i(x_i + r_i \theta) - u_i(x_i) = \int_{B(0, 2 \frac{\delta_i \epsilon'}{r_i})} \frac{1}{2\pi} \log \frac{|\tilde{\theta}|}{|\theta - \tilde{\theta}|} V_i e^{u_i(y)} r_i^2 dy + h_1 + h_2,$$

where,

$$h_1(\theta) = \int_{\Omega - B(x_i, 2\delta_i \epsilon')} G(x_i + r_i \theta, y) V_i e^{u_i(y)} dy - \int_{\Omega - B(x_i, 2\delta_i \epsilon')} G(x_i, y) V_i e^{u_i(y)} dy,$$

and,

$$h_2(\theta) = \int_{B(0, 2\delta_i \epsilon')} \frac{1}{2\pi} \log \frac{|1 - (\bar{x}_i + r_i \tilde{\theta})y|}{|1 - \bar{x}_i y|} V_i e^{u_i(y)} dy.$$

Remark that, h_1 and h_2 are two harmonic functions, uniformly bounded.

According to the maximum principle, the harmonic function $G(x_i + r_i \theta, \cdot)$ on $\Omega - B(x_i, 2\delta_i \epsilon')$ take its maximum on the boundary of $B(x_i, 2\delta_i \epsilon')$, we can compute this maximum:

$$G(x_i + r_i \theta, y_i) = \frac{1}{2\pi} \log \frac{|1 - (\bar{x}_i + r_i \tilde{\theta})y_i|}{|x_i + r_i \theta - y_i|} \simeq \frac{1}{2\pi} \log \frac{(|1 + |x_i||\delta_i - \delta_i(3\epsilon' + o(1))|}{\delta_i \epsilon'} \leq C_{\epsilon'} < +\infty$$

with $y_i = x_i + 2\delta_i\theta_i\epsilon'$, $|\theta_i| = 1$, and $|r_i\theta| \leq \delta_i\epsilon'$.

We can remark, for $|\theta| \leq \frac{\delta_i\epsilon'}{r_i}$, that v_i is such that:

$$v_i = h_1 + h_2 + \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon')} \frac{1}{2\pi} \log \frac{|\tilde{\theta}|}{|\theta - \tilde{\theta}|} V_i e^{u_i(y)} r_i^2 dy,$$

$$v_i = h_1 + h_2 + \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon')} \frac{1}{2\pi} \log \frac{|\tilde{\theta}|}{|\theta - \tilde{\theta}|} V_i(x_i + r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta},$$

with h_1 and h_2 , the two uniformly bounded harmonic functions.

Remark: In the case of 2 or 3 blow-up points, and if we consider the half ball, we have supplementary terms, around the 2 other blow-up terms. Note that the Green function of the half ball is quasi-similar to the one of the unit ball and our computations are the same if we consider the half ball.

We assume that, the blow-up limit is 0 and we take:

$$G(x, y) = \frac{1}{2\pi} \log \frac{|1 - \bar{x}y|}{|x - y|} - \frac{1}{2\pi} \log \frac{|1 - xy|}{|\bar{x} - y|},$$

Asymptotic estimates and the case of one, two and three blow-up points :

By the asymptotic estimates of Cheng-Lin see [8], we can see that, we have the following uniform estimates at infinity:

Lemma 2.1.

$\forall \epsilon, \epsilon' > 0$, $\exists k_{\epsilon, \epsilon'} \in \mathbb{R}_+$, $i_{\epsilon, \epsilon'} \in \mathbb{N}$ and $C_{\epsilon, \epsilon'} > 0$, such that for $i \geq i_{\epsilon, \epsilon'}$ and $k_{\epsilon, \epsilon'} \leq |\theta| \leq \frac{\delta_i\epsilon'}{r_i}$

$$(-4 - \epsilon) \log |\theta| - C_{\epsilon, \epsilon'} \leq v_i(\theta) \leq (-4 + \epsilon) \log |\theta| + C_{\epsilon, \epsilon'},$$

For the proof, we consider the three following sets:

$$A_1 = \{\tilde{\theta}, |\tilde{\theta}| \leq k_\epsilon\}, \quad A_2 = \{\tilde{\theta}, |\theta - \tilde{\theta}| \leq \frac{|\theta|}{2}, |\tilde{\theta}| \geq k_\epsilon\},$$

and,

$$A_3 = \{\tilde{\theta}, |\theta - \tilde{\theta}| \geq \frac{|\theta|}{2}, |\tilde{\theta}| \geq k_\epsilon\}.$$

where k_ϵ is such that;

$$8\pi(1 - \epsilon) \leq \int_{B(0, k_\epsilon)} V_i(x_i + r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} = \int_{B(x_i, k_\epsilon e^{-u_i(x_i)/2})} V_i e^{u_i(y)} dy \leq 8\pi(1 + \epsilon).$$

In fact, if we assume that we have one blow-up point:

$$\int_{B(0, \frac{\delta_i}{2r_i})} V_i(x_i + r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} = \int_{B(x_i, \frac{\delta_i}{2})} V_i e^{u_i(y)} dy \rightarrow 8\pi,$$

To have the uniform bounds $C_\epsilon > 0$, we need to bound uniformly the following quantity:

$$A_i = \int_{B(0, \frac{\delta_i}{2r_i})} \frac{1}{2\pi} \log |\tilde{\theta}| V_i e^{u_i(y)} r_i^2 dy = \int_{B(0, \frac{\delta_i}{2r_i})} \frac{1}{2\pi} \log |\tilde{\theta}| V_i e^{v_i(\tilde{\theta})} d\tilde{\theta}.$$

To obtain this uniform bound, we use the CC.Chen and C.S. Lin computations, see [7], to have the existence of a sequence $l_i \rightarrow +\infty$ such that:

$$\int_{B(0, l_i)} \frac{1}{2\pi} \log |\tilde{\theta}| V_i e^{v_i(\tilde{\theta})} d\tilde{\theta} \leq C,$$

and, on the other hand, the computations of YY.Li and I. Shafir, see [11], to have, for $l_i \leq |\tilde{\theta}| \leq \frac{\delta_i}{2r_i}$:

$$e^{v_i(\tilde{\theta})} \leq \frac{C}{|\tilde{\theta}|^{2\beta+2}},$$

for some $0 < \beta < 1$.

Finally,

$$A_i \leq C.$$

Remark that, in the estimate of CC.Chen and C.S Lin, see [7], we need the assumption that V_i is s -holderian with $0 < s \leq 1$.

To explain more the previous lemma, we write:

$$\begin{aligned} -2\pi v_i + 2\pi h_1 + 2\pi h_2 &= - \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon') \cap A_1} \frac{1}{2\pi} \log \frac{|\tilde{\theta}|}{|\theta - \tilde{\theta}|} V_i(x_i + r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} +, \\ &- \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon') \cap A_2} \frac{1}{2\pi} \log \frac{|\tilde{\theta}|}{|\theta - \tilde{\theta}|} V_i(x_i + r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} + \\ &- \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon') \cap A_3} \frac{1}{2\pi} \log \frac{|\tilde{\theta}|}{|\theta - \tilde{\theta}|} V_i(x_i + r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} = \\ &= -I_1 - I_2 - I_3. \end{aligned}$$

For I_2 , we have: $|\theta - \tilde{\theta}| \leq |\tilde{\theta}|$, hence,

$$-I_2 \leq 0.$$

For I_1 , it is easy to see that:

$$-I_1 \leq \log |\theta| \int_{A_1} V_i(x_i + r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} + C,$$

with C a constant independant of x and i . Here we use the estimates of Chen-Lin.

Since, $|\theta - \tilde{\theta}| \leq |\tilde{\theta}| + |\theta| \leq |\tilde{\theta}||\theta|$ for $|\theta|, |\tilde{\theta}| \geq 1$, we have:

$$-I_3 \leq \log |\theta| \int_{A_3} V_i(x_i + r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta},$$

Thus,

$$-2\pi v_i + 2\pi h_1 + 2\pi h_2 \leq \log |\theta| \int_{A_1 \cup A_3} V_i(x_i + r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} + C,$$

Hence,

$$-2\pi v_i + 2\pi h_1 + 2\pi h_2 \leq (8\pi + \tilde{\epsilon}) \log |\theta| + C,$$

Thus,

$$v_i - h_1 - h_2 \geq (-4 - \epsilon) \log |\theta| - C.$$

For the rest of the proof, we use the same argument as in Cheng-Lin, see [8].

We write:

$$\begin{aligned} v_i - h_1 - h_2 &= \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon') \cap A_1} \frac{1}{2\pi} \log \frac{|\tilde{\theta}|}{|\theta - \tilde{\theta}|} V_i(x_i + r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} +, \\ &+ \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon') \cap A_2} \frac{1}{2\pi} \log \frac{|\tilde{\theta}|}{|\theta - \tilde{\theta}|} V_i(x_i + r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} + \end{aligned}$$

$$\begin{aligned}
& + \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon') \cap A_3} \frac{1}{2\pi} \log \frac{|\tilde{\theta}|}{|\theta - \tilde{\theta}|} V_i(x_i + r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} = \\
& = I_1 + I_2 + I_3.
\end{aligned}$$

We have:

$$I_1 \leq -\log |\theta| \int_{A_1} \frac{1}{2\pi} V_i(x_i + r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} + C,$$

with C a constant independant of x and i . Here we use the estimates of Chen-Lin. For I_3 , we have:

$$I_3 \leq 1.$$

For I_2 , we have:

$$I_2 \leq \frac{1}{2\pi} \int_{\{|\tilde{\theta}-\theta| \leq |\theta|^{-\sigma}\}} \log \frac{1}{|\theta - \tilde{\theta}|} V_i(x_i + r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} + \frac{\epsilon}{2} \log |\theta|,$$

Hence,

$$v_i - h_1 - h_2 \leq (-4 + \epsilon) \log |\theta| + \frac{1}{2\pi} \int_{\{|\tilde{\theta}-\theta| \leq |\theta|^{-\sigma}\}} \log \frac{1}{|\theta - \tilde{\theta}|} V_i(x_i + r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta}.$$

As in [8], we can prove that, $(h_1$ and h_2 are harmonic and satisfy the mean value theorem):

$$\begin{aligned}
v_i - h_1 - h_2 - \int_{\{|\tilde{\theta}-\theta|=r=|\theta|^{-\sigma}\}} (v_i - h_1 - h_2) &= \frac{1}{2\pi} \int_{B_r(x)} \log \frac{r}{|\theta - \tilde{\theta}|} V_i(x_i + r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} \\
\int_{\{|\tilde{\theta}-\theta|=r=|\theta|^{-\sigma}\}} (v_i - h_1 - h_2) &\leq (-4 + \epsilon) \log |\theta|.
\end{aligned}$$

As in the proof of the theorem 1.1 of [8], we use the Brezis-Merle estimate and the two previous estimates to prove that for θ large enough, we have:

$$v_i - h_1 - h_2 \leq (-4 + \epsilon) \log |\theta| + C.$$

To see this : (We write $v_i - h_1 - h_2 = k + q$, with q harmonic with the same boundary value as v_i , we use Brezis-Merle estimate). Note that, h_1 and h_2 are uniformly bounded. Let $\Omega = B_r(\theta)$, where $r = 2|\theta|^{-\sigma}$ we have:

$$\begin{cases} -\Delta k = V_i e^{k+q} & \text{in } \Omega, \\ k = 0 & \text{on } \partial\Omega. \end{cases}$$

By the Brezis-Merle estimate:

$$\int_{\Omega} e^{2k} \leq C_1 |\theta|^{-2\sigma}.$$

We use the fact that q is harmonic to have:

$$q(\theta) \leq Cq(0) + (C - 1)(-\min_{\Omega} q^-).$$

By the previous computations we have:

$$\min_{\Omega} q^- = \min_{\partial\Omega} q^- = \min_{\partial\Omega} (v_i - h_1 - h_2)^- \geq (-4 - \epsilon) \log |\theta| - C,$$

and by the previous mean value estimate, we have:

$$q(0) = \int_{\{|\tilde{\theta}-\theta|=r=|\theta|^{-\sigma}\}} (v_i - h_1 - h_2) \leq (-4 + \epsilon) \log |\theta|.$$

Thus,

$$q(\theta) \leq C \log |\theta|.$$

Here C is a constant independent of i and σ .

We have by the same computations as in the proof of the theorem 1.1 of [8] to conclude that:

$$\int_{\{|\tilde{\theta}-\theta|\leq|\theta|^{-\sigma}\}} e^{2v_i} \leq |\theta|^{-2\sigma+2C},$$

and by Cauchy-Schwarz inequality, we have:

$$\left(\frac{1}{2\pi} \int_{\{|\tilde{\theta}-\theta|\leq|\theta|^{-\sigma}\}} \log \frac{1}{|\theta-\tilde{\theta}|} V_i(x_i+r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} \right)^2 \leq C,$$

and that, for θ and σ large enough:

$$v_i - h_1 - h_2 \leq (-4 + \epsilon) \log |\theta| + C.$$

Now, we extend the previous asymptotic estimates to the first derivatives: we have, after derivation under the integral:

$$\partial_j v_i = \partial_j h_1 + \partial_j h_2 + \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon')} \frac{1}{2\pi} \frac{\theta_j - \tilde{\theta}_j}{|\theta - \tilde{\theta}|^2} V_i e^{u_i(y)} r_i^2 dy,$$

In other words, we have:

$$\partial_j v_i = \partial_j h_1 + \partial_j h_2 + \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon')} \frac{1}{2\pi} \frac{\theta_j - \tilde{\theta}_j}{|\theta - \tilde{\theta}|^2} V_i(x_i+r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta},$$

We have the following lemma:

Lemma 2.2. $\forall \epsilon, \epsilon' > 0 \exists k_{\epsilon, \epsilon'} \in \mathbb{R}_+, i_{\epsilon, \epsilon'} \in \mathbb{N}$, such that, for $i \geq i_{\epsilon, \epsilon'}$ and $k_{\epsilon, \epsilon'} \leq |\theta| \leq \frac{\delta_i \epsilon'}{r_i}$,

$$\partial_j v_i(\theta) \simeq \partial_j u_0(\theta) \pm \frac{\epsilon}{|\theta|} + C \left(\frac{r_i}{\delta_i} \right),$$

where u_0 is the solution to:

$$-\Delta u_0 = V(0)e^{u_0}, \quad \text{in } \mathbb{R}^2.$$

For the proof, we consider the three following sets:

$$A_1 = \{\tilde{\theta}, |\tilde{\theta}| \leq k_\epsilon\}, \quad A_2 = \{\tilde{\theta}, |\theta - \tilde{\theta}| \leq \frac{|\theta|}{2}, |\tilde{\theta}| \geq k_\epsilon\},$$

and,

$$A_3 = \{\tilde{\theta}, |\theta - \tilde{\theta}| \geq \frac{|\theta|}{2}, |\tilde{\theta}| \geq k_\epsilon\}.$$

where k_ϵ is such that;

$$8\pi(1 - \epsilon) \leq \int_{B(0, k_\epsilon)} V_i(x_i+r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} = \int_{B(x_i, k_\epsilon e^{-u_i(x_i)/2})} V_i e^{u_i(y)} dy \leq 8\pi(1 + \epsilon).$$

Remark 1: In the case of 2 or 3 blow-up points, and if we consider the half ball, we have supplementary terms, around the 2 other blow-up terms. Note that the Green function of the half ball is quasi-similar to the one of the unit ball. In the case of 3 blow-up points, we have the following supplementary term (x_i is the principal blow-up point and y_i and t_i the 2 other blow-up points):

$$C_1 \left(\frac{r_i}{d(x_i, y_i)} \right) + C_2 \left(\frac{r_i}{d(x_i, t_i)} \right).$$

We assume that, the blow-up limit is 0 and we take:

$$G(x, y) = \frac{1}{2\pi} \log \frac{|1 - \bar{x}y|}{|x - y|} - \frac{1}{2\pi} \log \frac{|1 - xy|}{|\bar{x} - y|},$$

In the previous computations, we have considered the unit ball, but by a conformal transformation, we can have the same estimates on the half ball, with a coefficient of the conformal transformation. We can assume the estimates on the half ball.

Now, we consider the following term of the Pohozaev identity

$$J_i = \int_{B(x_i, \delta_i \epsilon')} < x_i^i |\nabla(u_i - u)| > (V_i - V_i(x_i)) e^{u_i} dy,$$

We want to show that this term tends to 0 as i tends to infinity. We can reduce the problem, after integration by parts, to the following integral:

$$J'_i = \delta_i \int_{B(x_i, \delta_i \epsilon')} \partial_1 u_i V_i e^{u_i} dy = \delta_i \int_{B(x_i, \delta_i \epsilon')} \partial_1 u_i (-\Delta u_i)$$

But, if we take $y = x_i + r_i \theta$, with, $|\theta| \leq \frac{\delta_i \epsilon'}{r_i}$, we have:

$$\begin{aligned} J'_i &= \frac{\delta_i}{r_i} \int_{B(0, \frac{\delta_i \epsilon'}{r_i})} \partial_1 v_i (-\Delta v_i) = \\ &= \frac{\delta_i}{r_i} \int_{\partial B(0, \frac{\delta_i \epsilon'}{r_i})} \left(\frac{(\partial_1 v_i)^2}{2} \nu_1 - \frac{(\partial_2 v_i)^2}{2} \nu_1 + (\partial_1 v_i)(\partial_2 v_i) \nu_2 \right) d\sigma_i, \end{aligned}$$

Thus, if we use the uniform asymptotic estimates, we can see that, we reduce the computation to the Pohozaev identity for the limit blow-up function (which equal to 0), plus terms in $\epsilon|\theta|$ and $|\theta|$. First, we tend i to infinity, after ϵ to 0 and finally, we tend ϵ' to 0.

With this method we can have a compactness result for 3 blow-ups points. First, we can see the case of 3 exterior blow-up points, then by the previous formulation we have a compactness result, it is the case for one of the following cases (if we set δ_i, δ'_i and δ''_i for the radii of each exterior blow-up) :

$$\frac{d(x_i, y_i)}{\delta_i} \rightarrow +\infty \text{ and } \frac{d(x_i, t_i)}{\delta_i} \rightarrow +\infty,$$

or,

$$\frac{d(y_i, x_i)}{\delta'_i} \rightarrow +\infty \text{ and } \frac{d(y_i, t_i)}{\delta'_i} \rightarrow +\infty,$$

or,

$$\frac{d(t_i, x_i)}{\delta''_i} \rightarrow +\infty \text{ and } \frac{d(t_i, y_i)}{\delta''_i} \rightarrow +\infty,$$

or,

the case when the distance to two exterior blow-up points is of order the radii. In this last case, we divide the region in 3 parts and use the Pohozaev identity directly. In fact, we are reduced to the case of two blow-up points.

In fact, in the case of 3 exterior blow-up points. By the previous formulation around each exterior blow-up point we look to the one of the 3 first cases. For example, assume the first case. Then we work around the first blow-up. In fact we have, for 3 blow-up points :

Lemma 2.3. $\forall \epsilon > 0, \epsilon' > 0 \exists k_{\epsilon, \epsilon'} \in \mathbb{R}_+, i_{\epsilon, \epsilon'} \in \mathbb{N}$ and $C_{\epsilon, \epsilon'} > 0$, such that, for $i \geq i_{\epsilon, \epsilon'}$ and $k_{\epsilon, \epsilon'} \leq |\theta| \leq \frac{\delta_i \epsilon'}{r_i}$,

$$(-4 - \epsilon) \log |\theta| - C_{\epsilon, \epsilon'} \leq v_i(\theta) \leq (-4 + \epsilon) \log |\theta| + C_{\epsilon, \epsilon'},$$

and,

$$\partial_j v_i \simeq \partial_j u_0(\theta) \pm \frac{\epsilon}{|\theta|} + C \left(\frac{r_i}{\delta_i} \right)^2 |\theta| + m \times \left(\frac{r_i}{\delta_i} \right) + C_1 \left(\frac{r_i}{d(x_i, y_i)} \right) + C_2 \left(\frac{r_i}{d(x_i, t_i)} \right)$$

Proof of the compactness :

By using the lemma we can see that we have the compactness, because:

(to understand this, it is sufficient to do the computations for the half ball directly by using the Green function of the half ball directly).

We have after using the previous term of the Pohozaev identity:

$$o(1) = J'_i = m' + C_1 o(1) + C_2 o(1),$$

$$0 = \lim_{\epsilon'} \lim_{\epsilon} \lim_i J'_i = m',$$

which contradict the fact that $m' > 0$.

Proof of the second estimate of the lemma:

For the proof, we consider the three following sets:

$$A_1 = \{\tilde{\theta}, |\tilde{\theta}| \leq k_\epsilon\}, \quad A_2 = \{\tilde{\theta}, |\theta - \tilde{\theta}| \leq \frac{|\theta|}{2}, |\tilde{\theta}| \geq k_\epsilon\},$$

and,

$$A_3 = \{\tilde{\theta}, |\theta - \tilde{\theta}| \geq \frac{|\theta|}{2}, |\tilde{\theta}| \geq k_\epsilon\}.$$

where k_ϵ (large enough), is such that;

$$8\pi(1 - \epsilon) \leq \int_{B(0, k_\epsilon)} V_i(x_i + r_i \tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} = \int_{B(x_i, k_\epsilon e^{-u_i(x_i)/2})} V_i e^{u_i(y)} dy \leq 8\pi(1 + \epsilon).$$

We write:

$$\partial_j v_i - \partial_j h_1 - \partial_j h_2 = \int_{B(0, 2\frac{\delta_i}{r_i} \epsilon')} \frac{1}{2\pi} \frac{\theta_j - \tilde{\theta}_j}{|\theta - \tilde{\theta}|^2} V_i(x_i + r_i \tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta},$$

$$\partial_j v_i - \partial_j h_1 - \partial_j h_2 = \int_{A_1} \frac{1}{2\pi} \frac{\theta_j - \tilde{\theta}_j}{|\theta - \tilde{\theta}|^2} V_i(x_i + r_i \tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} + \int_{A_2 \cup A_3} \frac{1}{2\pi} \frac{\theta_j - \tilde{\theta}_j}{|\theta - \tilde{\theta}|^2} V_i(x_i + r_i \tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta}$$

Using the estimates of v_i , we obtain:

$$\partial_j v_i - \partial_j h_1 - \partial_j h_2 = \frac{o(1)}{|\theta|} + \int_{A_1} \frac{1}{2\pi} \frac{\theta_j - \tilde{\theta}_j}{|\theta - \tilde{\theta}|^2} V_0 e^{u_0(\tilde{\theta})} d\tilde{\theta} + \int_{A_2 \cup A_3} \frac{1}{2\pi} \frac{\theta_j - \tilde{\theta}_j}{|\theta - \tilde{\theta}|^2} V_i(x_i + r_i \tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta}$$

Thus,

$$\partial_j v_i - \partial_j h_1 - \partial_j h_2 = \partial_j u_0 + \frac{o(1)}{|\theta|} + \int_{A_2 \cup A_3} \frac{1}{2\pi} \frac{\theta_j - \tilde{\theta}_j}{|\theta - \tilde{\theta}|^2} V_i(x_i + r_i \tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta}$$

Finally,

$$\partial_j v_i - \partial_j h_1 - \partial_j h_2 - \partial_j u_0 = \frac{o(1)}{|\theta|} + \int_{A_2 \cup A_3} \frac{1}{2\pi} \frac{\theta_j - \tilde{\theta}_j}{|\theta - \tilde{\theta}|^2} V_i(x_i + r_i \tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta}.$$

For A_2 and A_3 , we have:

$$\left| \int_{A_3} \frac{1}{2\pi} \frac{\theta_j - \tilde{\theta}_j}{|\theta - \tilde{\theta}|^2} V_i(x_i + r_i \tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} \right| \leq \frac{1}{2\pi} \int_{\{|\tilde{\theta}| \geq k_\epsilon\}} \frac{1}{|\theta|} V_i(x_i + r_i \tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} \leq \frac{\epsilon}{|\theta|},$$

because,

$$\int_{\{|\tilde{\theta}| \geq k_\epsilon\}} V_i(x_i + r_i \tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} \rightarrow 0,$$

for k_ϵ large enough.

For $\theta \in A_2$, $|\tilde{\theta}| \geq \frac{|\theta|}{2}$ and, $|\tilde{\theta}| \geq \frac{|\theta - \tilde{\theta}|}{2}$, and we use the estimate of v_i to have:

$$\left| \int_{A_2} \frac{1}{2\pi} \frac{\theta_j - \tilde{\theta}_j}{|\theta - \tilde{\theta}|^2} V_i(x_i + r_i \tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} \right| \leq \frac{C|\theta|}{|\theta|^{4-\epsilon_0}} \leq \frac{\epsilon}{|\theta|},$$

for θ large enough and ϵ_0 small enough.

Finally, we have:

$$|\partial_j v_i - \partial_j h_1 - \partial_j h_2 - \partial_j u_0| \leq \frac{\epsilon}{|\theta|},$$

for θ large enough.

Now, it is easy to see from the definition of h_1 and h_2 that:

$$|\partial_j h_1 - \partial_j h_2 - m \frac{r_i}{\delta_i}| \leq C_1 \left(\frac{r_i}{d(x_i, y_i)} \right) + C_2 \left(\frac{r_i}{d(x_i, t_i)} \right)$$

Thus,

$$|\partial_j v_i - \partial_j u_0 - m \frac{r_i}{\delta_i}| \leq C_1 \left(\frac{r_i}{d(x_i, y_i)} \right) + C_2 \left(\frac{r_i}{d(x_i, t_i)} \right)$$

for θ large enough.

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