



HAL
open science

About Brezis-Merle Problem with holderian condition: the case of three blow-up points.

Samy Skander Bahoura

► **To cite this version:**

Samy Skander Bahoura. About Brezis-Merle Problem with holderian condition: the case of three blow-up points.. 2013. hal-00858892v1

HAL Id: hal-00858892

<https://hal.science/hal-00858892v1>

Preprint submitted on 6 Sep 2013 (v1), last revised 3 Apr 2014 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ABOUT BREZIS-MERLE PROBLEM WITH HOLDERIAN CONDITION: THE CASE OF THREE BLOW-UP POINTS.

SAMY SKANDER BAHOURA

ABSTRACT. We consider the following problem on open set Ω of \mathbb{R}^2 :

$$\begin{cases} -\Delta u_i = V_i e^{u_i} & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega. \end{cases}$$

We assume that :

$$\int_{\Omega} e^{u_i} dy \leq C,$$

and,

$$0 \leq V_i \leq b < +\infty$$

On the other hand, if we assume that V_i s -holderian with $1/2 < s \leq 1$, and,

$$\int_{\Omega} V_i e^{u_i} dy \leq 32\pi - \epsilon, \quad \epsilon > 0$$

then we have a compactness result, namely:

$$\sup_{\Omega} u_i \leq c = c(b, C, A, s, \Omega).$$

where A is the holderian constant of V_i .

1. INTRODUCTION AND MAIN RESULTS

We set $\Delta = \partial_{11} + \partial_{22}$ on open set Ω of \mathbb{R}^2 with a smooth boundary.

We consider the following problem on $\Omega \subset \mathbb{R}^2$:

$$(P) \begin{cases} -\Delta u_i = V_i e^{u_i} & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega. \end{cases}$$

We assume that,

$$\int_{\Omega} e^{u_i} dy \leq C,$$

and,

$$0 \leq V_i \leq b < +\infty$$

The previous equation is called, the Prescribed Scalar Curvature equation, in relation with conformal change of metrics. The function V_i is the prescribed curvature.

Here, we try to find some a priori estimates for sequences of the previous problem.

Equations of this type were studied by many authors, see [7, 8, 10, 12, 13, 17, 18, 21, 22, 25]. We can see in [8], different results for the solutions of those type of equations with or without boundaries conditions and, with minimal conditions on V , for example we suppose $V_i \geq 0$ and $V_i \in L^p(\Omega)$ or $V_i e^{u_i} \in L^p(\Omega)$ with $p \in [1, +\infty]$.

Among other results, we can see in [8], the following important Theorem,

Theorem A (Brezis-Merle [8]). *If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the previous problem (P) with, $0 < a \leq V_i \leq b < +\infty$, then, for all compact set K of Ω ,*

$$\sup_K u_i \leq c = c(a, b, m, K, \Omega) \quad \text{if} \quad \inf_{\Omega} u_i \geq m.$$

A simple consequence of this theorem is that, if we assume $u_i = 0$ on $\partial\Omega$ then, the sequence $(u_i)_i$ is locally uniformly bounded. We can find in [8] an interior estimate if we assume $a = 0$, but we need an assumption on the integral of e^{u_i} .

If, we assume V with more regularity, we can have another type of estimates, $\sup + \inf$. It was proved, by Shafrir, see [22], that, if $(u_i)_i, (V_i)_i$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C \left(\frac{a}{b} \right) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

We can see in [12], an explicit value of $C \left(\frac{a}{b} \right) = \sqrt{\frac{a}{b}}$. In his proof, Shafrir has used the Stokes formula and an isoperimetric inequality, see [6]. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with A the Lipschitz constant, then, $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$, see Brézis-Li-Shafrir [7]. This result was extended for Hölderian sequences $(V_i)_i$ by Chen-Lin, see [12]. Also, we can see in [17], an extension of the Brezis-Li-Shafrir to compact Riemann surface without boundary. We can see in [18] explicit form, $(8\pi m, m \in \mathbb{N}^*$ exactly), for the numbers in front of the Dirac masses, when the solutions blow-up. Here, the notion of isolated blow-up point is used. Also, we can see in [13] and [25] refined estimates near the isolated blow-up points and the bubbling behavior of the blow-up sequences.

In the similar way, we have in dimension $n \geq 3$, with different methods, some a priori estimates of the type $\sup \times \inf$ for equation of the type:

$$-\Delta u + \frac{n-2}{4(n-1)} R_g(x)u = V(x)u^{(n+2)/(n-2)} \quad \text{on } M.$$

where R_g is the scalar curvature of a riemannian manifold M , and V is a function. The operator $\Delta = \nabla^i(\nabla_i)$ is the Laplace-Beltrami operator on M .

When $V \equiv 1$ and M compact, the previous equation is the Yamabe equation. T. Aubin and R. Schoen solved the Yamabe problem, see for example [1]. Also, we can have an idea on the Yamabe Problem in [15]. If V is not a constant function, the previous equation is called a prescribing curvature equation, we have many existence results see also [1].

Now, if we look at the problem of a priori bound for the previous equation, we can see in [2, 3, 4, 5, 11, 16, 20] some results concerning the $\sup \times \inf$ type of inequalities when the manifold M is the sphere or more generality a locally conformally flat manifold. For these results, the moving-plane was used, we refer to [9, 14, 19] to have an idea on this method and some applications of this method.

Also, there are similar problems defined on complex manifolds for the Complex Monge-Ampere equation, see [23, 24]. They consider, on compact Kahler manifold (M, g) , the following equation:

$$\begin{cases} (\omega_g + \partial\bar{\partial}\varphi)^n = e^{f-t\varphi}\omega_g^n, \\ \omega_g + \partial\bar{\partial}\varphi > 0 \text{ on } M \end{cases}$$

And, they prove some estimates of type $\sup_M(\varphi - \psi) + m \inf_M(\varphi - \psi) \leq C(t)$ or $\sup_M(\varphi - \psi) + m \inf_M(\varphi - \psi) \geq C(t)$ under the positivity of the first Chern class of M .

The function ψ is a C^2 function such that:

$$\omega_g + \partial\bar{\partial}\psi \geq 0 \text{ and } \int_M e^{f-t\psi}\omega_g^n = Vol_g(M),$$

Our main result is:

Theorem. Assume that, V_i is uniformly s -holderian with $1/2 < s \leq 1$, and,

$$\int_{B_1(0)} V_i e^{u_i} dy \leq 32\pi - \epsilon, \quad \epsilon > 0,$$

then we have:

$$\sup_{\Omega} u_i \leq c = c(b, C, A, s, \Omega).$$

where A is the holderian constant of V_i .

2. PROOF OF THE RESULT:

Proof of the theorem:

Without loss of generality, we can assume that $\Omega = B_1(0)$ the unit ball centered on the origin.

Here, G is the Green function of the Laplacian with Dirichlet condition on $B_1(0)$. We have (in complex notation):

$$G(x, y) = \frac{1}{2\pi} \log \frac{|1 - \bar{x}y|}{|x - y|},$$

we can write:

$$u_i(x) = \int_{B_1(0)} G(x, y) V_i(y) e^{u_i(y)} dy,$$

We assume that we are in the case of one blow-up point.

We set:

$$r_i = e^{-u_i(x_i)/2},$$

we write, for $|\theta| \leq \frac{\delta_i \epsilon'}{r_i}$, $0 < \epsilon' < \frac{1}{4}$,

$$\begin{aligned} u_i(x_i + r_i \theta) &= \int_{\Omega} G(x_i + r_i \theta, y) V_i(y) e^{u_i(y)} dx = \\ &= \int_{\Omega - B(x_i, 2\delta_i \epsilon')} G(x_i, y) V_i e^{u_i(y)} dy + \int_{B(x_i, 2\delta_i \epsilon')} G(x_i + r_i \theta, y) V_i e^{u_i(y)} dy = \end{aligned}$$

We write, $y = x_i + r_i \tilde{\theta}$, with $|\tilde{\theta}| \leq 2 \frac{\delta_i \epsilon'}{r_i}$,

$$\begin{aligned} u_i(x_i + r_i \theta) &= \int_{B(0, 2 \frac{\delta_i \epsilon'}{r_i})} \frac{1}{2\pi} \log \frac{|1 - (\bar{x}_i + r_i \bar{\theta})(x_i + r_i \tilde{\theta})|}{r_i |\theta - \tilde{\theta}|} V_i e^{u_i(y)} r_i^2 dy + \\ &\quad + \int_{\Omega - B(x_i, 2\delta_i \epsilon')} G(x_i + r_i \theta, y) V_i e^{u_i(y)} dy \end{aligned}$$

$$u_i(x_i) = \int_{\Omega - B(x_i, 2\delta_i \epsilon')} G(x_i, y) V_i e^{u_i(y)} dy + \int_{B(x_i, 2\delta_i \epsilon')} G(x_i, y) V_i e^{u_i(y)} dy$$

Hence,

$$\begin{aligned} u_i(x_i) &= \int_{B(0, 2 \frac{\delta_i \epsilon'}{r_i})} \frac{1}{2\pi} \log \frac{|1 - \bar{x}_i(x_i + r_i \tilde{\theta})|}{r_i |\tilde{\theta}|} V_i e^{u_i(y)} r_i^2 dy + \\ &\quad + \int_{\Omega - B(x_i, 2\delta_i \epsilon')} G(x_i, y) V_i e^{u_i(y)} dy \end{aligned}$$

We look to the difference,

$$v_i(\theta) = u_i(x_i + r_i\theta) - u_i(x_i) = \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon')} \frac{1}{2\pi} \log \frac{|\tilde{\theta}|}{|\theta - \tilde{\theta}|} V_i e^{u_i(y)} r_i^2 dy + h_1 + h_2,$$

where,

$$h_1(\theta) = \int_{\Omega - B(x_i, 2\delta_i\epsilon')} G(x_i + r_i\theta, y) V_i e^{u_i(y)} dy - \int_{\Omega - B(x_i, 2\delta_i\epsilon')} G(x_i, y) V_i e^{u_i(y)} dy,$$

and,

$$h_2(\theta) = \int_{B(0, 2\delta_i\epsilon')} \frac{1}{2\pi} \log \frac{|1 - (\bar{x}_i + r_i\tilde{\theta})y|}{|1 - \bar{x}_i y|} V_i e^{u_i(y)} dy.$$

Remark that, h_1 and h_2 are two harmonic functions, uniformly bounded.

According to the maximum principle, the harmonic function $G(x_i + r_i\theta, \cdot)$ on $\Omega - B(x_i, 2\delta_i\epsilon')$ take its maximum on the boundary of $B(x_i, 2\delta_i\epsilon')$, we can compute this maximum:

$$G(x_i + r_i\theta, y_i) = \frac{1}{2\pi} \log \frac{|1 - (\bar{x}_i + r_i\tilde{\theta})y_i|}{|x_i + r_i\theta - y_i|} \simeq \frac{1}{2\pi} \log \frac{(|1 + |x_i||\delta_i - \delta_i(3\epsilon' + o(1))|}{\delta_i\epsilon'} \leq C_{\epsilon'} < +\infty$$

with $y_i = x_i + 2\delta_i\theta_i\epsilon'$, $|\theta_i| = 1$, and $|r_i\theta| \leq \delta_i\epsilon'$.

We can remark, for $|\theta| \leq \frac{\delta_i\epsilon'}{r_i}$, that v_i is such that:

$$v_i = h_1 + h_2 + \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon')} \frac{1}{2\pi} \log \frac{|\tilde{\theta}|}{|\theta - \tilde{\theta}|} V_i e^{u_i(y)} r_i^2 dy,$$

$$v_i = h_1 + h_2 + \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon')} \frac{1}{2\pi} \log \frac{|\tilde{\theta}|}{|\theta - \tilde{\theta}|} V_i(x_i + r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta},$$

with h_1 and h_2 , the two uniformly bounded harmonic functions.

Remark: In the case of 2 or 3 blow-up points, and if we consider the half ball, we have supplementary terms, around the 2 other blow-up terms. Note that the Green function of the half ball is quasi-similar to the one of the unit ball and our computations are the same if we consider the half ball.

We assume that, the blow-up limit is 0 and we take:

$$G(x, y) = \frac{1}{2\pi} \log \frac{|1 - \bar{x}y|}{|x - y|} - \frac{1}{2\pi} \log \frac{|1 - xy|}{|\bar{x} - y|},$$

Asymptotic estimates and the case of one, two three blow-ups:

By the asymptotic estimates of Cheng-Lin, we can see that, we have the following uniform estimates at infinity:

Lemma 2.1.

$\forall \epsilon, \epsilon' > 0$, $\exists k_{\epsilon, \epsilon'} \in \mathbb{R}_+$, $i_{\epsilon, \epsilon'} \in \mathbb{N}$ and $C_{\epsilon, \epsilon'} > 0$, such that for $i \geq i_{\epsilon, \epsilon'}$ and $k_{\epsilon, \epsilon'} \leq |\theta| \leq \frac{\delta_i\epsilon'}{r_i}$

$$(-4 - \epsilon) \log |\theta| - C_{\epsilon, \epsilon'} \leq v_i(\theta) \leq (-4 + \epsilon) \log |\theta| + C_{\epsilon, \epsilon'},$$

For the proof, we consider the three following sets:

$$A_1 = \{\tilde{\theta}, |\tilde{\theta}| \leq k_\epsilon\}, \quad A_2 = \{\tilde{\theta}, |\theta - \tilde{\theta}| \leq \frac{|\theta|}{2}, |\tilde{\theta}| \geq k_\epsilon\},$$

and,

$$A_3 = \{\tilde{\theta}, |\theta - \tilde{\theta}| \geq \frac{|\theta|}{2}, |\tilde{\theta}| \geq k_\epsilon\}.$$

where k_ϵ is such that;

$$8\pi(1 - \epsilon) \leq \int_{B(0, k_\epsilon)} V_i(x_i + r_i \tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} = \int_{B(x_i, k_\epsilon e^{-u_i(x_i)/2})} V_i e^{u_i(y)} dy \leq 8\pi(1 + \epsilon).$$

In fact, if we assume that we have one blow-up point:

$$\int_{B(0, \frac{\delta_i}{2r_i})} V_i(x_i + r_i \tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} = \int_{B(x_i, \frac{\delta_i}{2})} V_i e^{u_i(y)} dy \rightarrow 8\pi,$$

To have the uniform bounds $C_\epsilon > 0$, we need to bound uniformly the following quantity:

$$A_i = \int_{B(0, \frac{\delta_i}{2r_i})} \frac{1}{2\pi} \log |\tilde{\theta}| V_i e^{u_i(y)} r_i^2 dy = \int_{B(0, \frac{\delta_i}{2r_i})} \frac{1}{2\pi} \log |\tilde{\theta}| V_i e^{v_i(\tilde{\theta})} d\tilde{\theta}.$$

To obtain this uniform bound, we use the CC.Chen and C.S. Lin computations to have the existence of a sequence $l_i \rightarrow +\infty$ such that:

$$\int_{B(0, l_i)} \frac{1}{2\pi} \log |\tilde{\theta}| V_i e^{v_i(\tilde{\theta})} d\tilde{\theta} \leq C,$$

and, on the other hand, the computations of YY.Li and I. Shafrir to have, for $l_i \leq |\tilde{\theta}| \leq \frac{\delta_i}{2r_i}$:

$$e^{v_i(\tilde{\theta})} \leq \frac{C}{|\tilde{\theta}|^{2\beta+2}},$$

for some $0 < \beta < 1$.

Finally,

$$A_i \leq C.$$

Remark that, in the estimate of CC.Chen and C.S Lin, we need the assumption that V_i is s -holderian with $0 < s \leq 1$.

Now, we extend the previous asymptotic estimates to the first derivatives: we have, after derivation under the integral:

$$\partial_j v_i = \partial_j h_1 + \partial_j h_2 + \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon')} \frac{1}{2\pi} \frac{\theta_j - \tilde{\theta}_j}{|\theta - \tilde{\theta}|^2} V_i e^{u_i(y)} r_i^2 dy,$$

In other words, we have:

$$\partial_j v_i = \partial_j h_1 + \partial_j h_2 + \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon')} \frac{1}{2\pi} \frac{\theta_j - \tilde{\theta}_j}{|\theta - \tilde{\theta}|^2} V_i(x_i + r_i \tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta},$$

Here, we use the fact that $\partial_j v_i(0) = 0$, because x_i is a maximum of u_i , and, we take the difference $\partial_j v_i - \partial_j v_i(0)$.

We can say that, we have the following:

$$|\partial_j v_i(\theta) - \partial_j v_i(0)| \leq \frac{\epsilon}{|\theta|} + C \left(\frac{r_i}{\delta_i} \right)^2 |\theta|,$$

But if we consider directly the quantity, $\partial_j v_i$, we have the following:

Lemma 2.2. $\forall \epsilon, \epsilon' > 0 \exists k_{\epsilon, \epsilon'} \in \mathbb{R}_+, i_{\epsilon, \epsilon'} \in \mathbb{N}$, such that, for $i \geq i_{\epsilon, \epsilon'}$ and $k_{\epsilon, \epsilon'} \leq |\theta| \leq \frac{\delta_i \epsilon'}{r_i}$,

$$\partial_j v_i(\theta) \simeq \partial_j u_0(\theta) \pm \frac{\epsilon}{|\theta|} + C \left(\frac{r_i}{\delta_i} \right),$$

where u_0 is the solution to:

$$-\Delta u_0 = V(0)e^{u_0}, \quad \text{in } \mathbb{R}^2.$$

For the proof, we consider the three following sets:

$$A_1 = \{\tilde{\theta}, |\tilde{\theta}| \leq k_\epsilon\}, \quad A_2 = \{\tilde{\theta}, |\theta - \tilde{\theta}| \leq \frac{|\theta|}{2}, |\tilde{\theta}| \geq k_\epsilon\},$$

and,

$$A_3 = \{\tilde{\theta}, |\theta - \tilde{\theta}| \geq \frac{|\theta|}{2}, |\tilde{\theta}| \geq k_\epsilon\}.$$

where k_ϵ is such that;

$$8\pi(1 - \epsilon) \leq \int_{B(0, k_\epsilon)} V_i(x_i + r_i \tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta} = \int_{B(x_i, k_\epsilon e^{-u_i(x_i)/2})} V_i e^{u_i(y)} dy \leq 8\pi(1 + \epsilon).$$

Remark 1: In the case of 2 or 3 blow-up points, and if we consider the half ball, we have supplementary terms, around the 2 other blow-up terms. Note that the Green function of the half ball is quasi-similar to the one of the unit ball. In the case of 3 blow-up points, we have the following supplementary term (x_i is the principal blow-up point and y_i and t_i the 2 other blow-up points):

$$C_1 \left(\frac{r_i}{d(x_i, y_i)} \right) + C_2 \left(\frac{r_i}{d(x_i, t_i)} \right).$$

We assume that, the blow-up limit is 0 and we take:

$$G(x, y) = \frac{1}{2\pi} \log \frac{|1 - \bar{x}y|}{|x - y|} - \frac{1}{2\pi} \log \frac{|1 - xy|}{|\bar{x} - y|},$$

Remark 2: We have two possibilities, first, we estimate directly $\partial_j v_i$ we can write:

$$\partial_j v_i \simeq \partial_j u_0(\theta) \pm \frac{\epsilon}{|\theta|} + C \left(\frac{r_i}{\delta_i} \right)^2 |\theta| + m \times \left(\frac{r_i}{\delta_i} \right)$$

with, $m \neq 0$, "the reste of the Green function and the regular part of the Green function, is such that when we integrate the quantity:

$$\lim_{\epsilon'} \lim_{\epsilon} \lim_i \int_{\partial B(0, \frac{\delta_i \epsilon'}{r_i})} \left(\frac{(\partial_1 v_i)^2}{2} \nu_1 - \frac{(\partial_2 v_i)^2}{2} \nu_1 + (\partial_1 v_i)(\partial_2 v_i) \nu_2 \right) d\sigma_i = m' > 0,$$

Or, we take the difference $\partial_j v_i - \partial_j v_i(0)$, but we need to estimate:

$$\int_{B(0, 2\frac{\delta_i \epsilon'}{r_i})} \frac{1}{2\pi} \frac{\tilde{\theta}_j}{|\tilde{\theta}|^2} V_i(x_i + r_i \tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta},$$

In the previous computations, we have considered the unit ball, but by a conformal transformation, we can have the same estimates on the half ball, with a coefficient of the conformal transformation. We can assume the estimates on the half ball.

Now, we consider the following term of the Pohozaev identity

$$J_i = \int_{B(x_i, \delta_i \epsilon')} \langle x_1^i |\nabla(u_i - u) \rangle (V_i - V_i(x_i)) e^{u_i} dy,$$

We want to show that this term tends to 0 as i tends to infinity. We can reduce the problem, after integration by parts, to the following integral:

$$J'_i = \delta_i \int_{B(x_i, \delta_i \epsilon')} \partial_1 u_i V_i e^{u_i} dy = \delta_i \int_{B(x_i, \delta_i \epsilon')} \partial_1 u_i (-\Delta u_i)$$

But, if we take $y = x_i + r_i \theta$, with, $|\theta| \leq \frac{\delta_i \epsilon'}{r_i}$, we have:

$$\begin{aligned}
J'_i &= \frac{\delta_i}{r_i} \int_{B(0, \frac{\delta_i}{r_i} \epsilon')} \partial_1 v_i (-\Delta v_i) = \\
&= \frac{\delta_i}{r_i} \int_{\partial B(0, \frac{\delta_i}{r_i} \epsilon')} \left(\frac{(\partial_1 v_i)^2}{2} \nu_1 - \frac{(\partial_2 v_i)^2}{2} \nu_1 + (\partial_1 v_i)(\partial_2 v_i) \nu_2 \right) d\sigma_i,
\end{aligned}$$

Thus, if we use the uniform asymptotic estimates, we can see that, we reduce the computation to the Pohozaev identity for the limit blow-up function (which equal to 0), plus terms in $\epsilon|\theta|$ and $|\theta|$. First, we tend i to infinity, after ϵ to 0 and finally, we tend ϵ' to 0.

With this method we can have a compactness result for 3 blow-up points. First, we can see the case of 3 exterior blow-up points, then by the previous formulation we have a compactness result, it is the case for one of the following cases (if we set δ_i, δ'_i and δ''_i for the radii of each exterior blow-up) :

$$\frac{d(x_i, y_i)}{\delta_i} \rightarrow +\infty \quad \text{and} \quad \frac{d(x_i, t_i)}{\delta_i} \rightarrow +\infty,$$

or,

$$\frac{d(y_i, x_i)}{\delta'_i} \rightarrow +\infty \quad \text{and} \quad \frac{d(y_i, t_i)}{\delta'_i} \rightarrow +\infty,$$

or,

$$\frac{d(t_i, x_i)}{\delta''_i} \rightarrow +\infty \quad \text{and} \quad \frac{d(t_i, y_i)}{\delta''_i} \rightarrow +\infty,$$

or,

the case when the distance to two exterior blow-up points is of order the radii. In this last case, we divide the region in 3 parts and use the Pohozaev identity directly. In fact, we are reduced to the case of two blow-up points.

In fact, in the case of 3 exterior blow-up points. By the previous formulation around each exterior blow-up point we look to the one of the 3 first cases. For example, assume the first case. Then we work around the first blow-up. In fact we have, for 3 blow-up points :

Lemma 2.3. $\forall \epsilon > 0, \epsilon' > 0 \exists k_{\epsilon, \epsilon'} \in \mathbb{R}_+, i_{\epsilon, \epsilon'} \in \mathbb{N}$ and $C_{\epsilon, \epsilon'} > 0$, such that, for $i \geq i_{\epsilon, \epsilon'}$ and $k_{\epsilon, \epsilon'} \leq |\theta| \leq \frac{\delta_i \epsilon'}{r_i}$,

$$(-4 - \epsilon) \log |\theta| - C_{\epsilon, \epsilon'} \leq v_i(\theta) \leq (-4 + \epsilon) \log |\theta| + C_{\epsilon, \epsilon'},$$

and,

$$\partial_j v_i \simeq \partial_j u_0(\theta) \pm \frac{\epsilon}{|\theta|} + C \left(\frac{r_i}{\delta_i} \right)^2 |\theta| + m \times \left(\frac{r_i}{\delta_i} \right) + C_1 \left(\frac{r_i}{d(x_i, y_i)} \right) + C_2 \left(\frac{r_i}{d(x_i, t_i)} \right)$$

(to understand this, it is sufficient to do the computations for the half ball directly by using the Green function of the half ball directly).

We have after using the previous term of the Pohozaev identity:

$$o(1) = J'_i = m' + C_1 o(1) + C_2 o(1),$$

$$0 = \lim_{\epsilon'} \lim_{\epsilon} \lim_i J'_i = m',$$

which contradict the fact that $m' > 0$.

REFERENCES

- [1] T. Aubin. Some Nonlinear Problems in Riemannian Geometry. Springer-Verlag 1998
- [2] H. Brezis and F. Merle, Uniform estimates and blow-up behavior for solutions of $-\Delta u = Ve^u$ in two dimensions, *Commun Partial Differential Equations* 16 (1991), 1223-1253.
- [3] L. Caffarelli, B. Gidas, J. Spruck. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. *Comm. Pure Appl. Math.* 37 (1984) 369-402.
- [4] W. Chen, C. Li. A priori Estimates for solutions to Nonlinear Elliptic Equations. *Arch. Rational. Mech. Anal.* 122 (1993) 145-157.
- [5] C-C. Chen, C-S. Lin. A sharp sup+inf inequality for a nonlinear elliptic equation in \mathbb{R}^2 . *Commun. Anal. Geom.* 6, No.1, 1-19 (1998).
- [6] K-S. Cheng, C-S. Lin On the asymptotic behavior of solutions of the conformal Gaussian curvature equations in \mathbb{R}^2 . *Math. Ann.* 308 (1997), no. 1, 119-139
- [7] B. Gidas, W-Y. Ni, L. Nirenberg. Symmetry and related properties via the maximum principle. *Comm. Math. Phys.* 68 (1979), no. 3, 209-243.
- [8] Y.Y. Li. Harnack Type Inequality: the Method of Moving Planes. *Commun. Math. Phys.* 200, 421-444 (1999).
- [9] Y.Y. Li, I. Shafrir. Blow-up Analysis for Solutions of $-\Delta u = Ve^u$ in Dimension Two. *Indiana. Math. J.* Vol 3, no 4. (1994). 1255-1270.
- [10] L. Ma, J-C. Wei. Convergence for a Liouville equation. *Comment. Math. Helv.* 76 (2001) 506-514.
- [11] I. Shafrir. A sup+inf inequality for the equation $-\Delta u = Ve^u$. *C. R. Acad.Sci. Paris Sér. I Math.* 315 (1992), no. 2, 159-164.
- [12] Y-T. Siu. The existence of Kahler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group. *Ann. of Math. (2)* 127 (1988), no. 3, 585-627
- [13] G. Tian. A Harnack type inequality for certain complex Monge-Ampere equations. *J. Differential Geom.* 29 (1989), no. 3, 481-488.
- [14] L. Zhang. Blowup solutions of some nonlinear elliptic equations involving exponential nonlinearities. *Comm. Math. Phys.* 268 (2006), no. 1, 105-133.

DEPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ PIERRE ET MARIE CURIE, 2 PLACE JUSSIEU, 75005, PARIS, FRANCE.

E-mail address: samybahoura@yahoo.fr