

A Weierstrass-type representation for Lagrangian surfaces in complex Euclidean 2-space

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Abstract

We derive a Weierstrass-type formula for conformal Lagrangian immersions in complex Euclidean 2-space, and show that the data satisfies a Dirac-type equation with complex potential. We apply the representation to the Hamiltonian-stationary case to construct all possible tori and obtain a first approach to a moduli space in terms of a simple algebraic-geometric problem on the plane. Results described here are contained in two articles in collaboration with Frédéric HÉLEIN.

1 Introduction

The vector space \mathbb{C}^2 , seen as a *real* vector space (thus identified with \mathbb{R}^4), is endowed with a double structure: on one side the metric (namely the scalar product that we will denote by \langle, \rangle) and on the other the complex structure coming from the multiplication by i , embodied by the matrix J . If we identify $(x^1, x^2, x^3, x^4) \in \mathbb{R}^4$ with $(x^1 + ix^2, x^3 + ix^4)$, the complex structure is

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We derive a symplectic structure with symplectic (i.e., non-degenerate, closed) 2-form ω given by $\omega(\xi, \eta) = \langle J\xi, \eta \rangle$. Of course the hermitian metric contains both structures, the metric and the symplectic form being simply its real and imaginary parts¹.

When interested in surfaces in \mathbb{R}^4 , we may consider the following special cases:

- the complex curves, i.e. (in real notation) the surfaces whose tangent planes are (globally) J -invariant;
- the Lagrangian surfaces, where the tangent bundle is mapped by J to the normal bundle; these surfaces are as far as possible from being complex.

2000 *Mathematics Subject Classification.* Primary 53D12; Secondary 53C42, 49Q10, 53A05.

¹The reason for sticking to the real notation will become apparent later on: basically we will use a complexification of \mathbb{R}^4 which is rather awkward when writing it as \mathbb{C}^2 .

Another (and more usual) way to define Lagrangian surfaces is to require that ω vanishes on the surface. Lagrangian surfaces (and more generally Lagrangian submanifolds) play a key role in physics and symplectic geometry; see the references for some reading directions.

The expression “Weierstrass representation” has been used widely recently. It refers of course to the old and celebrated Weierstrass-Enneper representation which describes completely all minimal surfaces in \mathbb{R}^3 – to be precise all conformal immersions f of such surfaces – by analytic data: a meromorphic function g and a holomorphic 1-form η , plus an algorithm to recover the immersion from the data and vice-versa:

$$f(z) = \operatorname{Re} \int^z (1 - g^2, i(1 + g^2), 2g)\eta$$

where z is some local conformal coordinate on the surface. Moreover the data has geometric significance since g is the (stereographic projection of the) Gauss map and $g\eta$ is the complexified differential of the height function. Later on many generalizations were built, for instance for surfaces with prescribed mean curvature [Ke], for any surface in \mathbb{R}^3 using spinors [Ko1], or \mathbb{R}^4 [Ko2], or for cmc-1 surfaces in \mathbb{H}^3 [B]. In even greater generality and abstraction, Weierstrass-type representations were proved to exist for harmonic maps of simply-connected two dimensional domains into symmetric spaces [DPW]. Many other examples exist (see [Ko2] for some references). The goal of such a representation is to describe completely the local (and sometimes global) theory of the surfaces, to allow the construction of explicit examples, and maybe even a classification. Its use partially explains the success of classical minimal surface theory.

Those Lagrangian surfaces in \mathbb{R}^4 solutions to some variational problem, namely that are Hamiltonian-stationary, are also given by an abstract Weierstrass-type formula as explained in [HR1]. However there is a simpler way to describe *any* conformal Lagrangian immersion into \mathbb{R}^4 :

THEOREM. *Let L be a simply connected domain in \mathbb{C} . Then for any smooth conformal Lagrangian immersion $f : L \rightarrow \mathbb{R}^4$, there exist smooth functions $\beta : L \rightarrow \mathbb{R}/2\pi\mathbb{Z}$, which is the Lagrangian angle, and $s_1, s_2 : L \rightarrow \mathbb{C}$, not simultaneously vanishing, that satisfy the Dirac-type equation*

$$\left(\begin{pmatrix} 0 & \partial/\partial z \\ -\partial/\partial \bar{z} & 0 \end{pmatrix} - \begin{pmatrix} \bar{U} & 0 \\ 0 & U \end{pmatrix} \right) \begin{pmatrix} s_1 \\ \bar{s}_2 \end{pmatrix} = 0 \quad (1)$$

with complex potential $U = \frac{1}{2} \frac{\partial \beta}{\partial z}$. Conversely any given β and any solution (s_1, s_2) to (1) satisfying $|s_1|^2 + |s_2|^2 > 0$ gives rise to a conformal Lagrangian immersion given by

$$f(z) = \operatorname{Re} \int^z e^{\frac{\beta}{2}J} \begin{pmatrix} s_1 \\ s_2 \\ -is_1 \\ is_2 \end{pmatrix}.$$

2 A Dirac-type equation

As many representations quoted above, our formula requires two ingredients : (i) a good choice of the immersion (most often conformal ones are preferred) and (ii) use of the appropriate group acting on \mathbb{R}^4 – here $U(2)$ is the important group. Let $f : L \rightarrow \mathbb{R}^4$ be a Lagrangian immersion from a contractible domain $L \subset \mathbb{C}$; we may assume without loss of generality that f is conformal. In coordinates, $df = e^\rho(e_1 dx + e_2 dy)$ where $e^{\rho(z)}$ is the conformal factor, and (e_1, e_2) is an orthonormal basis of the tangent plane at $f(z)$. Furthermore f is Lagrangian if and only if (e_1, e_2) is a unitary basis (for the implicit Hermitian structure). Since $U(2)$ – written here as a real subgroup of $GL(4, \mathbb{R})$ – acts transitively on unitary frames, we have $e_j = g e_j$ for $j = 1, 2$ and (ϵ_1, ϵ_2) any fixed unitary frame. However $U(2)$ splits into a product $U(1) \times SU(2)$ with factors unique up to sign, and $U(1)$ consisting of complex-diagonal matrices, i.e. $\exp(\mathbb{R}J)$. Hence $g = e^{\frac{\beta}{2}J} k$ where $\beta(z) = \arg \det_{\mathbb{C}} g(z)$ is known as the Lagrangian angle and defined modulo 2π . The sign ambiguity poses no problem on a simply-connected domain. Now set

$$\alpha = e^{-\frac{\beta}{2}J} df = e^\rho k(\epsilon_1 dx + \epsilon_2 dy)$$

and consider its $(1, 0)$ part $\alpha' = u dz = k \epsilon dz$ where $\epsilon = (\epsilon_1 - i\epsilon_2)/2$. A priori u takes values in $(\mathbb{R}^4)^{\mathbb{C}} = \mathbb{C}^4$, but remarkably it stays in a two-dimensional complex vector subspace V :

LEMMA. *The $(1, 0)$ part of $e^{-\frac{\beta}{2}J} df$ takes values in the orbit under $\mathbb{R}_+^* \times SU(2)$ of ϵ which is the vector space $V = \mathbb{C}\epsilon \oplus \mathbb{C}J\bar{\epsilon}$, minus the origin.*

The proof goes by direct computation. Still one may wonder why. One (good) reason is that the space V is not accidental, but rather comes as an eigenspace for some order four automorphism on the Lie algebra of the group of unitary displacements $U(2) \ltimes \mathbb{C}^2$, acting on \mathbb{R}^4 .

As a consequence we may write the integrability condition $d^2 f = 0$ as

$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2} \frac{\partial \beta}{\partial z} J \bar{u}.$$

Setting $u = s_1 \epsilon + s_2 J \bar{\epsilon}$ we derive easily the Dirac equation (1). Since the integrability condition above is the only *local* condition, we have proved the announced theorem. Note that allowing s_1 and s_2 to vanish at points amounts to replacing conformal immersion by weakly-conformal maps.

As a conclusion we may note that this formula is quite similar in spirit to the classical Weierstrass representation, with β playing the role of the Gauss map (indeed β is a part of the Gauss map) and (s_1, s_2) equivalent to η . In a way Lagrangian surfaces in \mathbb{R}^4 behave like surfaces in 3-space due to the additional Lagrangian constraint. However, while the integration procedure is simpler than the classical case, ours relies on solving a non obvious Dirac equation², very close to the equation for the spinor representation of surfaces in \mathbb{R}^3 . Still our solutions (s_1, s_2) are not spinors but 1-forms. .

²For instance it is not clear for which angular functions β there exist solutions to (1).

3 The Hamiltonian-stationary case

Our formula is even more interesting in the case of Hamiltonian-stationary surfaces. These are the Lagrangian surfaces that are critical for the area functional but under a restricted set of possible deformations: instead of all compactly supported vector fields, one restricts to *Hamiltonian* vector fields, i.e. symplectic gradients of functions. Namely $X = J\nabla\phi$ for some compactly supported ϕ . The one-parameter group generated by X will then preserve the symplectic structure as well³. Hamiltonian stationary surface have been introduced and studied (in \mathbb{C}^n) by Y.-G. OH [O1,O2], and later on by R. SCHOEN and J. WOLFSON [ScW] with a different point of view. In particular Y.-G. OH showed that rectangular tori – cartesian products of circles $r_1S^1 \times r_2S^1 \subset \mathbb{C} \times \mathbb{C}$ are local minima of area. However it was probably unclear at that time whether many such tori existed or not. This is where our formalism comes into play.

The condition for being Hamiltonian stationary has a very neat formulation:

$$\Delta\beta = 0 \tag{2}$$

where Δ is the Laplacian for the induced metric. Due to the representation formula, finding Hamiltonian stationary examples reduces to solving (1) with a harmonic potential (we can think of the conformal Laplacian now). Thus we have replaced a third-order non linear PDE with two successive linear PDES, plus an integration procedure.

This point of view applies remarkably to the toric case. Let Γ be a lattice in \mathbb{C} , Γ^* the dual lattice, and $\mathbb{T} = \mathbb{C}/\Gamma$ a torus with universal cover \mathbb{C} . Then $e^{i\beta}$ is a well-defined harmonic Γ -periodic function on \mathbb{C} if and only if $\beta(z) = 2\pi\text{Re}(\beta_0\bar{z})$ for some frequency $\beta_0 \in \Gamma^*$ (up to a rotation in space, or equivalently an affine change of variable). Equation (1) simplifies dramatically using Fourier expansions to yield:

$$s_1 = \sum_{\gamma} a_{\gamma} e^{2i\pi\text{Re}(\gamma\bar{z})}, \quad s_2 = \frac{2}{i\beta_0} \sum_{\gamma} \bar{\gamma} \bar{a}_{\gamma} e^{2i\pi\text{Re}(\gamma\bar{z})}$$

where the sum is taken over all frequencies $\gamma \in \Gamma^*/2$ such that

$$|\gamma| = \left| \frac{\beta_0}{2} \right| \tag{3}$$

Further careful analysis reveals that frequencies $\pm \frac{\beta_0}{2}$ yield non periodic terms (though df will be periodic) and hence should be discarded. We end up with an interesting geometric problem: find couples (Γ, β_0) where Γ is a lattice and $\beta_0 \in \Gamma^*$ is a frequency such that the circle of radius β_0 meets the lattice at points other than the obvious $\pm\beta_0$. Then any such configuration gives rise to an (weakly-immersed) Lagrangian surface. Known

³Minimal Lagrangian surfaces are also worth studying; they coincide with *special Lagrangian* surfaces, i.e. (calibrated) surfaces with constant Lagrangian angle. However special Lagrangian surfaces in \mathbb{R}^4 are none other than complex curves for a different complex structure. Nonetheless our analysis still applies, though we recover the classical Weierstrass representation in \mathbb{R}^4 , with the added Lagrangian condition.

examples are well characterized by that special lattice configuration. It may also be proven that the number of possible frequencies γ is a lower bound for the *type number* when using finite gap methods to construct tori.

Finally we may consider the following questions/problems:

- What are the possible conformal types ? All previously known examples were rectangular but through our formulas, we were able to construct a rhombic torus⁴.
- Can there be arbitrarily many frequencies ? The answer is yes but should be analyzed in view of the next question.
- Choosing a set of frequencies guarantees that the solution f is Γ -periodic; however Γ may not be *the* lattice of periods of f (in other words $f : \mathbb{C}/\Gamma \rightarrow \mathbb{R}^4$ may be a multiple cover of some torus). When is a configuration optimal in that sense ?
- What are the local (global) minima for area ? Y.-G. OH [O2] conjectured that product of circles were minimizers in their Hamiltonian isotopy class. Recently H. ANCIAUX [A] showed the conjecture under the assumption that smooth minimizers exist (in that class); he also pointed out another isotopy class where no minimizer were known (the rhombic example's).

REMARK. Although hidden from the first glimpse, the quaternions play a major role in our representation. Indeed we may identify \mathbb{R}^4 with \mathbb{H} , the group $SU(2)$ with the right multiplication by unit quaternions, and J with left multiplication by i, j or k . A elegant one-line formulation of our representation in terms of quaternions can be found in [HR2].

References

- [A] H. ANCIAUX: *An isoperimetric inequality for Hamiltonian stationary Lagrangian tori in \mathbb{C}^2 related to Oh's conjecture*, CMLA preprint.
- [B] R. BRYANT: *Surfaces of mean curvature one in hyperbolic space*, Théorie des variétés minimales et applications (Palaiseau, 1983–1984). Astérisque No. 154-155, (1987), 12, 321–347, 353 (1988).
- [DPW] J. DORFMEISTER, F. PEDIT, H.-Y. WU: *Weierstrass type representation of harmonic maps into symmetric spaces*, Comm. in Analysis and Geometry Vol 6, Number 4 (1998), 633-668.
- [HR1] Frédéric Hélein, Pascal Romon: *Hamiltonian stationary Lagrangian surfaces in \mathbb{C}^2* , arXiv:math.DG/0009202.

⁴Moreover the rhombic torus has a different Maslov class hence a different isotopy class than the rectangular ones.

- [HR2] Frédéric Hélein, Pascal Romon: *Weierstrass representation of Lagrangian surfaces in four-dimensional space using spinors and quaternions*, to appear in *Commentarii Mathematici Helvetici*.
- [Ke] K. KENMOTSU: *Weierstrass formula for surfaces of prescribed mean curvature*, *Math. Ann.* 254:88-99 (1979).
- [Ko1] B. G. KONOPELCHENKO: *Induced surfaces and their integrable dynamics*, *Stud. Appl. Math.* 96:9-51 (1996).
- [Ko2] B. G. KONOPELCHENKO: *Weierstrass representations for surfaces in 4D spaces and their integrable deformations via DS hierarchy*, *Ann. Global Anal. Geom.* 18 (2000), no. 1, 61–74.
- [O1] Y.G. OH, *Second variation and stabilities of minimal Lagrangian submanifolds in Kähler manifolds*, *Invent. Math.* 101 (1990), 501-519.
- [O2] Y.G. OH, *Volume minimization of Lagrangian submanifolds under Hamiltonian deformations*, *Math. Z.* 212 (1993), 175-192.
- [ScW] R. SCHOEN, J. WOLFSON: *Minimizing volume among Lagrangian submanifolds*, in *Differential equations: La Pietra 1996 (Florence)*, 181-199, *Proc. Sympos. Pure Math.*, 65, Amer. Math. Soc., Providence, RI (1999).
- [SYZ] A. STROMINGER, S.T. YAU, E. ZASLOW, *Mirror symmetry is T-duality*, *Nuclear Phys. B* 479 (1996), no. 1-2, 243–259.

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