A human proof of Gessel's lattice path conjecture

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A HUMAN PROOF OF GESSEL'S LATTICE PATH CONJECTURE
(PRELIMINARY VERSION)

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Abstract. Gessel walks are planar walks confined to the positive quarter plane, that move by unit steps in any of the following directions: West, North-East, East and South-West. In 2001, Ira Gessel conjectured a closed-form expression for the number of Gessel walks of a given length starting and ending at the origin. In 2008, Kauers, Koutschan and Zeilberger gave a computer-aided proof of this conjecture. The same year, Bostan and Kauers showed, using again computer algebra tools, that the trivariate generating function of Gessel walks is algebraic. In this article we propose the first "human proofs" of these results. They are derived from a new expression for the generating function of Gessel walks.

1. Introduction

Main results. Gessel walks are planar walks confined to the positive quarter plane $\mathbf{Z}_+^2$, that move by unit steps in any of the following directions: West, North-East, East and South-West, see Figure 1 below. For $(i,j) \in \mathbf{Z}_+^2$ and $n \geqslant 0$, let

$$q(i,j;n) = \#\{\text{Gessel walks of length } n \text{ starting at } (0,0) \text{ and ending at } (i,j)\}.$$

Gessel walks have been puzzling the combinatorics community since 2001, when Ira Gessel conjectured that:

(A) For all $n \geqslant 0$, the following closed-form expression holds for the number of Gessel excursions (Gessel walks starting and ending at the origin)

$$q(0,0;2n) = 16^n \frac{(5/6)_n (1/2)_n}{(2)_n (5/3)_n}, \tag{1}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol.

Notice that obvisouly $q(0,0;2n+1) = 0$ for any $n \geqslant 0$. In 2008, Kauers, Koutschan and Zeilberger [20] gave a computer-aided proof of this conjecture. A second intriguing problem was to decide whether or not:

(B) The (trivariate) generating function (GF) of Gessel walks

$$Q(x,y;z) = \sum_{i,j,n \geqslant 0} q(i,j;n) x^i y^j z^n \tag{2}$$

Date: September 4, 2013.

1991 *Mathematics Subject Classification.* Primary 05A15; Secondary 30F10, 30D05.

Key words and phrases. Walks in the quarter plane; Gessel walks; Counting generating function; Elliptic functions.

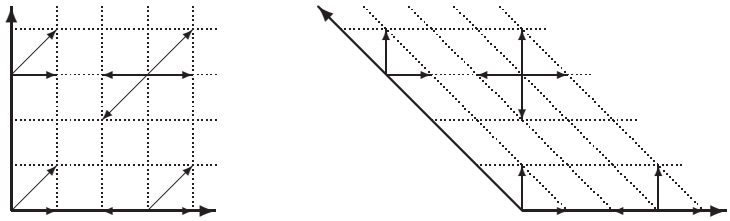

Figure 1. Gessel's model (note that on the boundary, the jumps are the natural ones: those that would take the walk out of $\mathbf{Z}_+^2$ are discarded)

is holonomic,[1] or even algebraic.

The answer to this question (namely, the algebraicity of $Q(x,y;z)$) was finally obtained by Bostan and Kauers [4], using computer algebra techniques.

In summary, the only existing proofs for Problems (A) and (B) used heavy computer calculations in a crucial way. In this article we obtain a new explicit expression for $Q(x,y;z)$, from which we derive the first "human proofs" of (A) and (B).

Context of Gessel's conjecture. In 2001, the motivation of considering Gessel's model was twofold. First, by an obvious linear transformation, Gessel's walk can be viewed as the simple walk (i.e., with jumps to the West, North, East and South) constrained to lie in a cone with angle 135°, see on the right in Figure 1. It turns out that before 2001, the simple walk was well studied in different cones. Pólya [33] first studied the simple walk in the whole plane, and remarked that the probability that a simple random walk ever returns to the origin is one. This is a consequence of the fact that there are exactly $\binom{2n}{n}^2$ unrestricted excursions of length $2n$ in the plane $\mathbf{Z}^2$. There also exist formulæ for excursions of length $2n$ evolving in other regions of $\mathbf{Z}^2$: $\binom{2n+1}{n}C_n$ for the half plane, and C_nC_{n+1} for the quarter plane, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the Catalan number [2]. Gouyou-Beauchamps [17] found a similar formula $C_nC_{n+2} - C_{n+1}^2$ for the cone with angle 45° (the first octant). It was thus natural to consider the cone with angle 135°, and this is what Gessel did.

The second part of the motivation is that Gessel walks are particular instances of walks in the quarter plane. Although it is since 2008 that most of the articles concerning walks in the quarter plane appeared, in 2001 there were already famous examples of such models: Kreweras' walk [22, 14, 15] (with jumps to the West, North-East and South) for which the GF (2) is algebraic; Gouyou-Beauchamps' walk [17]; the simple walk [18]. Further, around 2000, the walks in the quarter plane were brought up to date, notably by Bousquet-Mélou and Petkovšek [8, 9]. Indeed, they were used to illustrate the following phenomenon: although the numbers of walks satisfy a (multivariate) linear recurrence with constant coefficients, their GF (2) might be non-holonomic; see [9] for the example of the knight walk.

[1] The function $Q(x,y;z)$ is holonomic if the vector space over $\mathbf{C}(x,y,z)$—the field of rational functions in the three variables x,y,z—spanned by the set of all derivatives of $Q(x,y;z)$ is finite dimensional, see for instance [13, Appendix B.4].

Existing results in the literature. After 2001, many approaches to treat walks in the quarter plane appeared. Bousquet-Mélou and Mishna initiated a systematic study of the walks with small steps (this means that the step set, i.e., the set of possible steps for the walk, is a subset of the eight nearest neighbors). Mishna [30, 31] first considered the case of step sets of cardinality three. She presented a complete classification of the GF (2) of these walks with respect to the classes algebraic, transcendental holonomic and non-holonomic. Bousquet-Mélou and Mishna [7] then considered all 79 small step sets.[2] They considered a functional equation for the GF that counts walks in such a model leading to a group[3] of birational transformations of $\mathbf{C}^2$. In 23 cases out of 79 this group turns out to be finite, and the corresponding functional equations were solved in 22 out of 23 cases. The remaining case is the one of Gessel walks. In 2008, a method using computer algebra techniques was proposed by Kauers, Koutschan and Zeilberger [21, 20]. Kauers and Zeilberger [21] first obtained a computer-aided proof of the algebraicity of the GF counting Kreweras' walks. A few months later, this approach was enhanced to cover Gessel's case, and the conjecture (Problem (A)) was proved [20]. At the same time, Bostan and Kauers [4] showed, using heavy computer calculations, that the trivariate GF counting Gessel walks is algebraic (Problem (B)). Using the minimal polynomials obtained by Bostan and Kauers, van Hoeij [4, Appendix] managed to obtained an explicit expression for the GF of Gessel walks.

Purely mathematical analysis of the GF of Gessel walks (but without answering Gessel's conjectures) were proposed in [23, 35, 11, 3, 34, 37]. Kurkova and Raschel [23] obtained an explicit integral representation of $Q(x, y; z)$. This was done by solving a boundary value problem, a method inspired by the book [10]. This approach has been generalized for all models of walks with small steps in the quarter plane, see [35]. In [11], Fayolle and Raschel gave another proof of the algebraicity of the GF (Problem (B)), using probabilistic and algebraic methods of the seventies. In [3], Ayyer proposed a combinatorial approach inspired by representation theory. He interpreted Gessel walks as words on certain alphabets. He then reformulated $q(i, j; n)$ as numbers of words, and computed very particular numbers of Gessel walks. Petkovšek and Wilf [34] stated new conjectures, closed to Gessel's. They found an expression for Gessel's numbers in terms of determinants of matrices, by showing that the numbers of walks are solution to an infinite system of equations. Ping [37] introduced a probabilistic model for Gessel walks, and reduced the computation of $q(i, j; n)$ to the computation of a certain probability. Using then probabilistic methods (such as the reflection principle) he proved two conjectures of Petkovšek and Wilf [34].

Presentation of our method and organization of the article. First of all, we fix $z \in]0, 1/4[$. To solve Problems (A) and (B), we start from the GFs $Q(x, 0; z)$ and $Q(0, y; z)$ and from the functional equation (3) of Bousquet-Mélou and Mishna (valid at any $(x, y; z)$ with $|x| < 1$

[2] A priori, there are $2^8 = 256$, but the authors of [7] showed that, after eliminating trivial cases, and also those which can be reduced to the walks in a half plane, there remain 79 inherently different models.

[3] Historically, this group was introduced by Malyshev [26, 27, 28] in the seventies. For details on this group we refer to Section 2, in particular to equation (12).

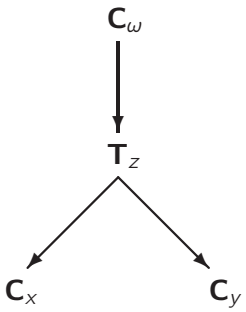

Figure 2. Different levels in our proof

and $|y| < 1$)

$$K(x,y;z)Q(x,y;z) = \\ K(x,0;z)Q(x,0;z) + K(0,y;z)Q(0,y;z) - K(0,0;z)Q(0,0;z) - xy. \quad (3)$$

Above, $K(x,y;z)$ is the kernel of the walk, given by

$$K(x,y;z) = xyz(xy + x + 1/x + 1/(xy) - 1/z) = xyz\left(\sum_{(i,j)\in\mathfrak{G}} x^i y^j - 1/z\right), \quad (4)$$

where $\mathfrak{G} = \{(1,1),(1,0),(-1,0),(-1,-1)\}$ denotes Gessel's step set. Our main idea is to construct all branches of these functions, in other words, to consider the meromorphic continuations of $x \mapsto Q(x,0;z)$ and $y \mapsto Q(0,y;z)$ along any path of the complex plane (and thus not only in their natural domains of definition $\{|x| < 1\}$ and $\{|y| < 1\}$). This idea is motivated by the fact that a function is algebraic if and only if it has a finite number of branches.

To achieve this objective, we need to consider the elliptic curve $\mathbf{T}_z$ defined by the zeros of the kernel $K(x,y;z)$:

$$\mathbf{T}_z = \{(x,y) \in (\mathbf{C} \cup \{\infty\})^2 : K(x,y;z) = 0\}, \quad (5)$$

and to introduce the universal covering of $\mathbf{T}_z$, which is the complex plane $\mathbf{C}_\omega$ with a new variable ω.

The functions $x \mapsto Q(x,0,z)$ and $y \mapsto Q(0,y,z)$ can be lifted on their respective natural domains of definition on $\mathbf{T}_z$ and next on the corresponding domains of the universal covering $\mathbf{C}_\omega$, namely $\{\omega \in \mathbf{C}_\omega : |x(\omega)| < 1\}$ and $\{\omega \in \mathbf{C}_\omega : |y(\omega)| < 1\}$. It turns out that the latter domains are vertical strips. This lifting procedure is illustrated on Figure 2. The first level (at the bottom) represents the complex planes $\mathbf{C}_x$ and $\mathbf{C}_y$, where $Q(x,0;z)$ and $Q(0,y;z)$ are defined in $\{|x| < 1\}$ and $\{|y| < 1\}$. The second level, where the variables x and y are not independent anymore, is given by $\mathbf{T}_z$. The third level is $\mathbf{C}_\omega$, the universal covering of $\mathbf{T}_z$. All this construction has been first elaborated by Malyshev [26] for stationary probability GFs of random walks in $\mathbf{Z}_+^2$ and it has been further developed in [10]. We make it explicit in the context of Gessel walks in Section 2.

The key-point of all our approach is the following: defining the lifted function $r_x(\omega) = K(x(\omega),0;z)Q(x(\omega),0;z)$, we have the identity

$$r_x(\omega - \omega_3) = r_x(\omega) + f_x(\omega), \qquad \forall \omega \in \mathbf{C}_\omega, \quad (6)$$

where the shift positive vector ω_3 and the function f_x are explicit (and relatively simple). A similar equation holds for the lifted function of $Q(0, y; z)$. Equation (6) has many consequences.

Firstly, due to (6), the function r_x can be continued from its initial domain of definition (a vertical strip) to the whole plane $\mathbf{C}_\omega$. By projecting back on $\mathbf{C}_x$, we recover all branches of $Q(x, 0; z)$.

Secondly, we notice that there are only a finite number of different branches, which yields the algebraicity of $Q(x, 0; z)$. Using a similar result on $Q(0, y; z)$ and the functional equation (3), we derive in this way the solution to Problem (B), see Section 5.

Thirdly, from (6), we can deduce the poles of r_x. In general, it is clearly impossible to deduce the expression of a function from the knowledge of its poles. A notable exception is constituted by elliptic functions. In our case, it happens that the poles of r_x form a two-dimensional lattice, and that the residues (all poles are of order 1) are periodic; the function r_x is therefore elliptic. From this fact we deduce an explicit expression of r_x in terms of elliptic ζ-functions. By projection on $\mathbf{C}_x$, this gives a new explicit expression of $Q(x, 0; z)$ for Gessel walks as an infinite series. An analogous result holds for $Q(0, y; z)$ and (3) then leads to a new explicit expression for $Q(x, y; z)$, see Section 3.

Fourthly, evaluating this expression of $Q(x, 0; z)$ at $x = 0$ and performing several simplifications, we obtain the solution of Problem (A), see Section 4.

2. Meromorphic continuation of the GFs

Branch points. For brevity, we drop the variable z (which is fixed in $]0, 1/4[$) from the notations when no ambiguity can arise, writing for instance $Q(x, y)$ instead of $Q(x, y; z)$ and $\mathbf{T}$ instead of $\mathbf{T}_z$. The kernel $K(x, y)$ defined in (4) is a second degree polynomial in both x and y. The algebraic function $X(y)$ defined by $K(X(y), y) = 0$ has thus two branches and four branch points, that we call y_i, $i \in \{1, \dots, 4\}$. They are the roots of the discriminant of the second degree equation $K(x, y) = 0$ in the variable x:

$$\widetilde{d}(y) = (-y)^2 - 4z^2(y^2 + y)(y + 1).$$

We have $y_1 = 0$, $y_4 = \infty$, and

$$y_2 = \frac{1 - 8z^2 - \sqrt{1 - 16z^2}}{8z^2}, \qquad y_3 = \frac{1 - 8z^2 + \sqrt{1 - 16z^2}}{8z^2},$$

in such a way that $y_1 < y_2 < y_3 < y_4$. Since there are four distinct branch points, the Riemann surface of $X(y)$ is a torus $\mathbf{T}^y$ (i.e., a Riemann surface of genus 1). The analogous statement is true for the algebraic function $Y(x)$ defined by $K(x, Y(x)) = 0$. Its four branch points x_i, $i \in \{1, \dots, 4\}$, are the roots of

$$d(x) = (zx^2 - x + z)^2 - 4z^2x^2. \tag{7}$$

They are real and numbered in such a way that $x_1 < x_2 < x_3 < x_4$:

$$x_1 = \frac{1 + 2z - \sqrt{1 + 4z}}{2z}, \qquad x_2 = \frac{1 - 2z - \sqrt{1 - 4z}}{2z}, \qquad x_3 = 1/x_2, \qquad x_4 = 1/x_1.$$

The Riemann surface of $Y(x)$ is also a torus $\mathbf{T}^x$. Since $\mathbf{T}^x$ and $\mathbf{T}^y$ are equivalent, in what follows we shall consider a single Riemann surface $\mathbf{T}$ with two different coverings $x, y : \mathbf{T} \to \mathbf{S}$; see Figure 2.

Universal covering. The torus $\mathbf{T}$ is isomorphic to a quotient space $\mathbf{C}/(\omega_1\mathbf{Z} + \omega_2\mathbf{Z})$, where ω_1, ω_2 are complex numbers linearly independent on $\mathbf{R}$. This set can obviously be thought as the (fundamental) parallelogram $[0, \omega_1] + [0, \omega_2]$, the opposed edges of which are identified. The periods ω_1, ω_2 are unique (up to a unimodular transform) and are found in [10, Lemma 3.3.2]:

$$\omega_1 = i\int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{-d(x)}}, \qquad \omega_2 = \int_{x_2}^{x_3} \frac{\mathrm{d}x}{\sqrt{d(x)}}. \tag{8}$$

The universal covering of $\mathbf{T}$ has the form $(\mathbf{C}, \lambda)$, where $\mathbf{C}$ is the complex plane (the union of infinitely many parallelograms (here, all parallelograms will be rectangles)

$$\Pi_{m,n} = \omega_1[m, m+1[+ \omega_2[n, n+1[, \qquad m, n \in \mathbf{Z},$$

glued together) and $\lambda : \mathbf{C} \to \mathbf{T}$ is a non-branching covering map. For any $\omega \in \mathbf{C}$ such that $\lambda\omega = s \in \mathbf{T}$, we have $x(\lambda\omega) = x(s)$ and $y(\lambda\omega) = y(s)$. The uniformization formulæ [10, Lemma 3.3.1] are

$$\begin{cases} x(\omega) &= x_4 + \dfrac{d'(x_4)}{\wp(\omega) - d''(x_4)/6}, \\ y(\omega) &= \dfrac{1}{2a(x(\omega))}\left(-b(x(\omega)) + \dfrac{d'(x_4)\wp'(\omega)}{2(\wp(\omega) - d''(x_4)/6)^2}\right), \end{cases} \tag{9}$$

where $a(x) = zx^2$, $b(x) = zx^2 - x + z$, d is defined in (7), and $\wp$ is the Weierstrass elliptic function with periods ω_1, ω_2 (its expansion is given in (51)). Throughout, we shall write $x(\lambda\omega) = x(\omega)$ and $y(\lambda\omega) = y(\omega)$. Due to (9), these are elliptic functions on $\mathbf{C}$ with periods ω_1, ω_2. Clearly

$$K(x(\omega), y(\omega)) = 0, \qquad \forall \omega \in \mathbf{C}. \tag{10}$$

Furthermore, since each parallelogram $\Pi_{m,n}$ represents a torus $\mathbf{T}$ composed of two complex spheres, the function $x(\omega)$ (resp. $y(\omega)$) takes each value of $\mathbf{C} \cup \{\infty\}$ twice within this parallelogram, except for the branch points x_i, $i \in \{1, \ldots, 4\}$ (resp. y_i, $i \in \{1, \ldots, 4\}$). The points $\omega_{x_i} \in \Pi_{0,0}$ such that $x(\omega_{x_i}) = x_i$, $i \in \{1, \ldots, 4\}$ are represented on Figure 3. They are equal to

$$\omega_{x_1} = \omega_2/2, \qquad \omega_{x_2} = (\omega_1 + \omega_2)/2, \qquad \omega_{x_3} = \omega_1/2, \qquad \omega_{x_4} = 0.$$

The points ω_{y_i} such that $y(\omega_{y_i}) = y_i$ are just the shifts of ω_{x_i} by a real vector $\omega_3/2$ (to be defined below): $\omega_{y_i} = \omega_{x_i} + \omega_3/2$ for $i \in \{1, \ldots, 4\}$, see also Figure 3. We refer to [10, Chapter 3] and to [23, 35] for proofs of these facts. The vector ω_3 is defined as

$$\omega_3 = \int_{-\infty}^{x_1} \frac{\mathrm{d}x}{\sqrt{d(x)}}.$$

For Gessel's model we have the following important relation [23, Proposition 14], which holds for all $z \in]0, 1/4[$:

$$\omega_3/\omega_2 = 3/4. \tag{11}$$

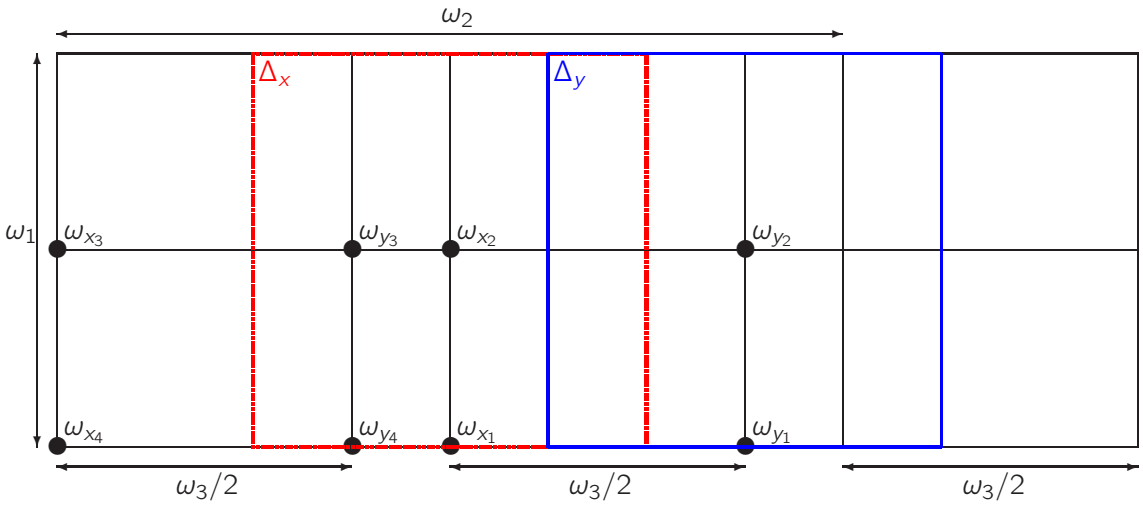

Figure 3. The fundamental parallelograms for the functions r_x and r_y, namely, $\Pi_{0,0} = \omega_1[0,1[+\omega_2[0,1[$ and $\Pi_{0,0} + \omega_3/2 = \omega_1[0,1[+\omega_2[0,1[+\omega_3/2$, and some important points and domains on them

Galois automorphisms. It is easy to see that the functions ξ and η of $\mathbf{C}^2$

$$\xi(x,y) = \left(x, \frac{1}{x^2 y}\right), \qquad \eta(x,y) = \left(\frac{1}{xy}, y\right) \tag{12}$$

leave invariant the quantity $\sum_{(i,j)\in\mathfrak{G}} x^i y^j$ (and therefore also the set $\mathbf{T}$ in (5) for any fixed $z \in]0, 1/4[$). The group $\langle \xi, \eta \rangle$ of birational transformations of $\mathbf{C}^2$ that they span is of order 8, see [7]. Furthermore,

$$\xi^2 = \eta^2 = \mathrm{id}. \tag{13}$$

This group was first defined in a probabilistic context by Malyshev [26, 27, 28]; it was introduced for the combinatorics of walks with small steps in the quarter plane by Bousquet-Mélou and Mishna [7]. In (12), it is defined as a group on $\mathbf{C}^2 = \mathbf{C}_x \times \mathbf{C}_y$, i.e., at the bottom level of Figure 2. We now lift it to the upper levels of Figure 2.

First, we lift it on the intermediate level $\mathbf{T}$ as the restriction of $\langle \xi, \eta \rangle$ on $\mathbf{T}$. Namely, any point $s \in \mathbf{T}$ admits the two "coordinates" $(x(s), y(s))$, which satisfy $K(x(s), y(s)) = 0$ by construction. For any $s \in \mathbf{T}$, there exists a unique s' (resp. s'') such that $x(s') = x(s)$ (resp. $y(s'') = y(s)$). The values $x(s), x(s')$ (resp. $y(s), y(s'')$) are the two roots of the second degree equation $K(x, y(s)) = 0$ (resp. $K(x(s), y) = 0$) in x (resp. y). The automorphism $\xi : \mathbf{T} \to \mathbf{T}$ (resp. $\eta : \mathbf{T} \to \mathbf{T}$) is defined by the identity $\xi s = s'$ (resp. $\eta s = s''$) and is called a Galois automorphism, following the terminology of [26, 27, 28, 10]. Clearly by (12) and (13), we have, for any $s \in \mathbf{T}$,

$$x(\xi s) = x(s), \quad y(\xi s) = \frac{1}{x^2(s) y(s)}, \quad x(\eta s) = \frac{1}{y(s) x(s)}, \quad y(\eta s) = y(s), \quad \xi^2(s) = \eta^2(s) = s.$$

Finally $\xi s = s$ (resp. $\eta s = s$) if and only if $x(s) = x_i$, $i \in \{1, \dots, 4\}$ (resp. $y(s) = y_i$, for some $i \in \{1, \dots, 4\}$).

There are many ways to lift ξ and η from $\mathbf{T}$ to the universal covering $\mathbf{C}$. We follow the way of [10] and [24], lifting them on $\mathbf{C}$ in such a way that ω_{x_2} and ω_{y_2} stay their fixed points, respectively (see Figure 3):

$$\xi\omega = -\omega + \omega_1 + \omega_2, \qquad \eta\omega = -\omega + \omega_1 + \omega_2 + \omega_3, \qquad \forall \omega \in \mathbf{C}.$$

Let us remind that $\omega_1 + \omega_2 = 2\omega_{x_2}$ and that $\omega_1 + \omega_2 + \omega_3 = 2\omega_{y_2}$. One has

$$x(\xi\omega) = x(\omega), \qquad y(\eta\omega) = y(\omega), \qquad \forall \omega \in \mathbf{C}.$$

Lifting of the GFs on the universal covering. The domains

$$\{\omega \in \mathbf{C} : |x(\omega)| < 1\}, \qquad \{\omega \in \mathbf{C} : |y(\omega)| < 1\}$$

consist of infinitely many curvilinear strips, which differ from translations by a multiple of ω_2. We denote by Δ_x (resp. Δ_y) the strip that is within $\cup_{n\in Z}\Pi_{0,n}$ (resp. $\cup_{n\in Z}\Pi_{0,n} + \omega_3/2$). The domain Δ_x (resp. Δ_y) is delimited by vertical lines, see [23, Proposition 26], and is represented on Figure 3. We notice that the function $Q(x(\omega), 0)$ (resp. $Q(0, y(\omega))$) is well defined in Δ_x (resp. Δ_y), by (2). Let us put

$$\begin{aligned} r_x(\omega) &= K(x(\omega), 0)Q(x(\omega), 0), \qquad \forall \omega \in \Delta_x, \\ r_y(\omega) &= K(0, y(\omega))Q(0, y(\omega)), \qquad \forall \omega \in \Delta_y. \end{aligned}$$

The domain $\Delta_x \cap \Delta_y$ is a non-empty open strip, see Figure 3. It follows from (3) and (10) that

$$r_x(\omega) + r_y(\omega) - K(0,0)Q(0,0) - x(\omega)y(\omega) = 0, \qquad \forall \omega \in \Delta_x \cap \Delta_y. \tag{14}$$

Meromorphic continuation of the GFs on the universal covering. Let $\Delta = \Delta_x \cup \Delta_y$. Due to (14), the functions $r_x(\omega)$ and $r_y(\omega)$ can be continued as meromorphic functions on the whole of Δ, by setting

$$\begin{aligned} r_x(\omega) &= -r_y(\omega) + K(0,0)Q(0,0) + x(\omega)y(\omega), \qquad \forall \omega \in \Delta_y, \\ r_y(\omega) &= -r_x(\omega) + K(0,0)Q(0,0) + x(\omega)y(\omega), \qquad \forall \omega \in \Delta_x. \end{aligned}$$

To continue the functions from Δ to the whole of $\mathbf{C}$, we first notice that

$$\bigcup_{n\in Z}(\Delta + n\omega_3) = \mathbf{C},$$

as proved in [10, 24] and illustrated on Figure 3. Let

$$\left\{ \begin{array}{rcl} f_x(\omega) & = & y(\omega)[x(-\omega + 2\omega_{y_2}) - x(\omega)], \\ f_y(\omega) & = & x(\omega)[y(-\omega + 2\omega_{x_2}) - y(\omega)]. \end{array} \right. \tag{15}$$

The following result holds true, see [24].

Lemma 1 ([24]). *The functions $r_x(\omega)$ and $r_y(\omega)$ can be continued meromorphically to the whole of* $\mathbf{C}$. *Further, for any $\omega \in \mathbf{C}$, we have*

$$r_x(\omega - \omega_3) = r_x(\omega) + f_x(\omega),$$
$$r_y(\omega + \omega_3) = r_y(\omega) + f_y(\omega), \tag{16}$$
$$r_x(\omega) + r_y(\omega) - K(0,0)Q(0,0) - x(\omega)y(\omega) = 0, \tag{17}$$
$$\begin{cases} r_x(\xi\omega) = r_x(\omega), \\ r_y(\eta\omega) = r_y(\omega), \end{cases} \tag{18}$$
$$\begin{cases} r_x(\omega + \omega_1) = r_x(\omega), \\ r_y(\omega + \omega_1) = r_y(\omega). \end{cases} \tag{19}$$

3. GFs in terms of Weierstrass Zeta-functions

Statements of the results. The aim of this section is to prove the following results: let ω_1, ω_2 be defined as in (8), and let $\zeta_{1,3}$ be the ζ-Weierstrass function with periods $\omega_1, 3\omega_2$, see Lemma 12 in the appendix for its definition and some of its properties.

Theorem 2. *We have*

$$Q(0,0) = \tag{20}$$
$$\frac{\zeta_{1,3}(\omega_2/4) - 3\zeta_{1,3}(2\omega_2/4) + 2\zeta_{1,3}(3\omega_2/4) + 3\zeta_{1,3}(4\omega_2/4) - 5\zeta_{1,3}(5\omega_2/4) + 2\zeta_{1,3}(6\omega_2/4)}{2z^2}.$$

Theorem 3. *We have, for all $\omega \in \mathbf{C}$,*

$$\begin{aligned} r_y(\omega) = c &+ \frac{1}{2z}\zeta_{1,3}(\omega - (1/8)\omega_2) - \frac{1}{2z}\zeta_{1,3}(\omega - (3/8)\omega_2) \\ &+ \frac{1}{2z}\zeta_{1,3}(\omega - (1+3/8)\omega_2) - \frac{1}{2z}\zeta_{1,3}(\omega - (1+5/8)\omega_2) \\ &- \frac{1}{2z}\zeta_{1,3}(\omega - (1+7/8)\omega_2) + \frac{1}{z}\zeta_{1,3}(\omega - (2+1/8)\omega_2) \\ &- \frac{1}{z}\zeta_{1,3}(\omega - (2+5/8)\omega_2) + \frac{1}{2z}\zeta_{1,3}(\omega - (2+7/8)\omega_2), \end{aligned} \tag{21}$$

where c is a constant.

Note that the constant c above is immediately made explicit from these two theorems. In fact, the point $\omega_0^y = 7\omega_2/8 \in \Delta_y$ is such that $y(\omega_0^y) = 0$ (see Lemma 5 below). Hence the value of $r_y(7\omega_2/8) = K(0, y(7\omega_2/8))Q(0, y(7\omega_2/8)) = K(0,0)Q(0,0) = zQ(0,0)$ is found in Theorem 2. The value of c then is $c = zQ(0,0) - \widehat{\zeta}_{1,3}(7\omega_2/8)$, where $\widehat{\zeta}_{1,3}(\omega)$ is the sum of the eight ζ-functions in (21).

A similar expression to (21) holds for $r_x(\omega)$ (with another constant c). There are two ways to obtain this expression: the first one consists in doing the same analysis as for r_y; the second one is to express r_x from equation (17) in terms of r_y and to apply Theorems 2 and 3. The results of both approaches presented in terms of ζ-functions are rigorously the same.

We shall explain in Section 5 how to deduce from Theorems 2 and 3 explicit expressions for $Q(0, y; z)$ and $Q(x, 0; z)$. Using the functional equation (3), we shall then obtain a new expression for $Q(x, y; z)$.

Preliminary results. The poles of f_y will play a crucial role in our analysis. They are given in the lemma hereafter.

Lemma 4. *In the fundamental parallelogram $\omega_1[0,1[+\omega_2[0,1[$, the function f_y has poles at $\omega_2/8$, $3\omega_2/8$, $5\omega_2/8$ and $7\omega_2/8$. These poles are simple, with residues equal to $-1/(2z)$, $1/(2z)$, $1/(2z)$ and $-1/(2z)$, respectively.*

Before proving Lemma 4, we recall from [23, Lemma 28] the following result, dealing with the zeros and poles of $x(\omega)$ and $y(\omega)$:

Lemma 5 ([23])**.** *In the fundamental parallelogram $\omega_1[0,1[+\omega_2[0,1[$, the only poles of x (of order one) are at $\omega_2/8$, $7\omega_2/8$, and its only zeros (of order one) are at $3\omega_2/8$, $5\omega_2/8$. The only pole of y (of order two) is at $3\omega_2/8$, and its only zero (of order two) is at $7\omega_2/8$.*

Proof of Lemma 4. With the definition (15) of the function $f_y(\omega)$ and (9), we derive that

$$f_y(\omega) = \frac{1}{2z}\frac{x'(\omega)}{x(\omega)}.$$

Accordingly, if $x(\omega)$ has a simple zero (resp. a simple pole) at ω_0, then $f_y(\omega)$ has a simple pole at ω_0, with residue $1/(2z)$ (resp. $-1/(2z)$). Lemma 4 then follows from Lemma 5. □

The following lemma will shorten the proof of Theorem 3.

Lemma 6. *The function r_y is elliptic with periods $\omega_1, 3\omega_2$.*

Proof. The function r_y is meromorphic and ω_1-periodic due to (19). Further, by Lemma 1,

$$r_y(\omega+4\omega_3) - r_y(\omega) = f_y(\omega) + f_y(\omega+\omega_3) + f_y(\omega+2\omega_3) + f_y(\omega+3\omega_3), \qquad \forall \omega \in \mathbf{C}. \quad (22)$$

By the analysis of Lemma 4 and the fact (11) that $\omega_3 = 3\omega_2/4$, the elliptic function

$$\mathcal{O}(\omega) = \sum_{k=0}^{3} f_y(\omega + k\omega_3)$$

has no poles on $\mathbf{C}$. Hence, with property (P2) of Lemma 12, $\mathcal{O}(\omega)$ must be a constant c, so that $r_y(\omega+4\omega_3) = r_y(\omega)+c$ for all $\omega \in \mathbf{C}$. In particular, $r_y(\omega_{y_2}-4\omega_3)+2c = r_y(\omega_{y_2}+4\omega_3)$. But in view of (18), $r_y(\omega_{y_2} - 4\omega_3) = r_y(\omega_{y_2} + 4\omega_3)$, and then $c = 0$. It follows that $r_y(\omega)$ is also $4\omega_3 = 3\omega_2$-periodic, and thus elliptic with periods $\omega_1, 3\omega_2$. □

Note that this lemma also follows from [25, Proposition 10]: by Lemma 4, the assumption of this proposition holds for Gessel's model. Finally, Lemma 6 is proved in [24, Proposition 11] as well, using the representation of $\mathcal{O}(\omega)$ as the so-called orbit-sum:

$$\begin{aligned}\mathcal{O}(\omega) &= \sum_{1\leqslant k\leqslant 4} (xy)(\omega + k\omega_3) - (xy)(\eta(\omega + k\omega_3)) \\ &= \sum_{1\leqslant k\leqslant 4} (xy)((\eta\xi)^k\omega) - (xy)(\xi(\eta\xi)^{k-1}\omega) \\ &= \sum_{\theta\in\langle\xi,\eta\rangle} (-1)^\theta xy(\theta(\omega)),\end{aligned}$$

where $(-1)^\theta$ is the signature of θ, i.e., $(-1)^\theta = (-1)^{\ell(\theta)}$, where $\ell(\theta)$ is the smallest ℓ for which we can write $\theta = \theta_1 \circ \cdots \circ \theta_\ell$, with $\theta_1, \ldots, \theta_\ell$ equal to ξ or η.

Proof of Theorems 2 and 3.

Proof of Theorem 3. Since r_y is elliptic with periods $\omega_1, 3\omega_2$ (Lemma 6), and since any elliptic function is characterized by its poles in a fundamental parallelogram, it suffices to study r_y in $\omega_1[-1/2,1/2[+\omega_2[-5/2,1/2[$. To do so, we shall use [25, Theorem 6], which gives the poles and the principal parts at these poles of r_y in terms of the function f_y, for any model of walks with small steps in the quarter plane (and rational ω_2/ω_3). Specifically, [25, Theorem 6] says that a pole d of r_y must satisfy $\mathcal{N}_d^- \neq \emptyset$, where (recall that $\Re\omega_{x_1} = \omega_2/2$)

$$\mathcal{N}_d^- = \{n \geqslant 0 : d + n\omega_3 \text{ is a pole of } f_y \text{ with } -5\omega_2/2 < \Re d + n\omega_3 < \omega_2/2\}.$$

If a point d is such that $\mathcal{N}_d^- \neq \emptyset$, then function r_y has the following principal part $R_{d,y}$ at d:

$$R_{d,y}(\omega) = \sum_{n\in\mathcal{N}_d^-} -(\lfloor n/4\rfloor + 1)F_{d+n\omega_3,y}(\omega + n\omega_3), \tag{23}$$

where $F_{d+n\omega_3,y}$ is the principal part of f_y at its pole $d + n\omega_3$, and $\lfloor x\rfloor$ is the lower integer part of $x \in \mathbf{R}$.

We thus first need to find the poles d of f_y in $\omega_1[-1/2,1/2[+\omega_2[-5/2,1/2[$. By Lemma 4, these are the points of the set

$$P = \{3\omega_2/8 - 2k\omega_2/8 : k \in \{0,\dots,11\}\}. \tag{24}$$

We thus have $\mathcal{N}_d^- = \{n \geqslant 0 : d + n\omega_3 \in P\}$. It is then obvious that the points d of the parallelogram $\omega_2[-5/2,1/2[+\omega_1[-1/2,1/2[$ such that $\mathcal{N}_d^- \neq \emptyset$ must be among the points of P. We now study each of them.

For $k \in \{0,1,2\}$ in (24), the associated points $d = 3\omega_2/8, \omega_2/8, -\omega_2/8 \in P$ are such that $\mathcal{N}_d^- = \{0\}$. This implies that there is one single term in formula (23), namely $R_{d,y}(\omega) = -F_{d,y}(\omega)$. With Lemma 4 we conclude that r_y has a pole of order 1 at d, with residue $-1/(2z), 1/(2z), 1/(2z)$, respectively.

For $k \in \{3,4,5\}$ in (24), we have the points $d = -3\omega_2/8, -5\omega_2/8, -7\omega_2/8 \in P$, which are such that $\mathcal{N}_d^- = \{0,1\}$. Let us find the principal part at $d = -3\omega_2/8$. With (23) we have

$$\begin{aligned} R_{-3\omega_2/8,y}(\omega) &= -(\lfloor 0/4\rfloor + 1)F_{-3\omega_2/8,y}(\omega) - (\lfloor 1/4\rfloor + 1)F_{3\omega_2/8,y}(\omega + 6\omega_2/8) \\ &= -\frac{1}{2z}\frac{1}{\omega + 3\omega_2/8} - \frac{1}{2z}\frac{1}{\omega + 6\omega_2/8 - 3\omega_2/8} \\ &= -\frac{1}{z}\frac{1}{\omega + 3\omega_2/8}, \end{aligned}$$

where the second line above is a consequence of Lemma 4. Hence $d = -3\omega_2/8$ is a simple pole of r_y, with residue $-1/z$. In the same way $R_{-5\omega_2/8,y}(\omega) = 0$, so that $-5\omega_2/8$ is not a pole of r_y, and $R_{-7\omega_2/8,y}(\omega) = (1/z)/(\omega + 7\omega_2/8)$, so that $-7\omega_2/8$ is a simple pole of r_y, with residue $1/z$.

For $k \in \{6,7,8\}$ in (24), points $-9\omega_2/8, -11\omega_2/8, -13\omega_2/8$ are with $\mathcal{N}_d^- = \{0,1,2\}$. The principal parts (23) of r_y at them are computed as the sums of the principal parts of f_y at three poles. This eventually shows that $-9\omega_2/8, -11\omega_2/8, -13\omega_2/8$ are simple poles of r_y with the residues $-1/(2z), -1/(2z), 1/(2z)$, respectively.

Finally, for $k \in \{9, 10, 11\}$ in (24), points $-15\omega_2/8, -17\omega_2/8, -19\omega_2/8 \in P$ are such that $\mathcal{N}_d^- = \{0, 1, 2, 3\}$. The principal parts of r_y (23) at them are computed as the sums of the principal parts of f_y at four poles, which all turn out to be zero. These points are thus not poles for r_y.

Applying the property (P6) of Lemma 12, we finally reach the conclusion that

$$\begin{aligned} r_y(\omega) = c &- \frac{1}{2z}\zeta_{1,3}(\omega - 3\omega_2/8) + \frac{1}{2z}\zeta_{1,3}(\omega - \omega_2/8) \\ &+ \frac{1}{2z}\zeta_{1,3}(\omega + \omega_2/8) - \frac{1}{z}\zeta_{1,3}(\omega + 3\omega_2/8) \\ &+ \frac{1}{z}\zeta_{1,3}(\omega + 7\omega_2/8) - \frac{1}{2z}\zeta_{1,3}(\omega + 9\omega_2/8) \\ &- \frac{1}{2z}\zeta_{1,3}(\omega + 11\omega_2/8) + \frac{1}{2z}\zeta_{1,3}(\omega + 13\omega_2/8). \end{aligned}$$

Using the fact that $\zeta_{1,3}(\omega + 3\omega_2) = \zeta_{1,3}(\omega) + 2\zeta_{1,3}(3\omega_2/2)$, see property (P8) of the same lemma, we close the proof of Theorem 3, with another constant c. □

Proof of Theorem 2. Equation (17) yields $r_x(\omega) = x(\omega)y(\omega) - r_y(\omega) + K(0,0)Q(0,0)$. We compute the constant $K(0,0)Q(0,0)$ as $r_y(\omega_0^y) = K(0, y(\omega_0^y))Q(0, y(\omega_0^y))$, where $\omega_0^y \in \Delta_y$ is such that $y(\omega_0^y) = 0$. Lemma 5 gives a unique possibility for ω_0^y, namely, $\omega_0^y = 7\omega_2/8$. Hence $r_x(\omega) = x(\omega)y(\omega) - r_y(\omega) + r_y(7\omega_2/8)$. Let us substitute $\omega = 5\omega_2/8$ in this equation. The point $5\omega_2/8$ is a zero of $x(\omega)$ that lies in Δ_x, so that

$$r_x(5\omega_2/8) = K(x(5\omega_2/8), 0)Q(x(5\omega_2/8), 0) = K(0,0)Q(0,0) = zQ(0,0).$$

This point is not a pole of $y(\omega)$, in such a way that $x(5\omega_2/8)y(5\omega_2/8) = 0$. We obtain

$$zQ(0,0) = r_y(7\omega_2/8) - r_y(5\omega_2/8). \tag{25}$$

Note in particular that in order to obtain the expression (25) of $Q(0,0)$, there is no need to know the constant c in Theorem 3.

With Theorem 3 and (25), $Q(0,0)$ can be written as a sum of 16 $\zeta_{1,3}$-Weierstrass functions (each of them being evaluated at a rational multiple of ω_2). Using the fact that $\zeta_{1,3}$ is an odd function and using property (P8), we can perform many easy simplifications in (25), and, this way, we obtain (20). □

4. Proof of Gessel conjecture (Problem (A))

Preliminary results. In this section we shall deal with elliptic functions with different pairs of periods. We shall denote by $\zeta, \wp$ the elliptic functions with periods ω_1, ω_2, and by $\zeta_{1,3}, \wp_{1,3}$ the elliptic functions with periods $\omega_1, 3\omega_2$, see Lemma 12 for their definition. Further, we recall that elliptic functions are alternatively characterized by their periods (see equation (51)) or by their invariants. The invariants of $\wp$ are denoted by g_2, g_3. They are such that

$$\wp'(\omega)^2 = 4\wp(\omega)^3 - g_2\wp(\omega) - g_3, \qquad \forall \omega \in \mathbf{C}. \tag{26}$$

Recall from [23, Lemma 12] the following result that provides explicit expressions for the invariants g_2, g_3.

Lemma 7 ([23]). *We have*

$$g_2 = (4/3)(1 - 16z^2 + 16z^4), \qquad g_3 = -(8/27)(1 - 8z^2)(1 - 16z^2 - 8z^4). \tag{27}$$

Likewise, we define the invariants $g_2^{1,3}, g_3^{1,3}$ of $\wp_{1,3}$. To compute them, we introduce the algebraic function R as the unique positive root[4] of

$$X^4 - 2g_2X^2 + 8g_3X - g_2^2/3 = 0, \tag{28}$$

where g_2 and g_3 are the invariants (27). Using equation (27) in (28), we obtain the local expansion $R(z) = 2 - 16z^2 - 48z^4 + O(z^6)$.

Lemma 8. *One has*

$$\begin{aligned} \wp_{1,3}(\omega_2) &= R/6, \\ g_2^{1,3} &= -g_2/9 + 10R^2/27, \\ g_3^{1,3} &= -35R^3/729 + 7g_2R/243 - g_3/27, \end{aligned}$$

where expressions for g_2 and g_3 are given in (27).

Proof. Using the properties (P4) and (P7) from Lemma 12, one can write, for any $\omega \in \mathbf{C}$,

$$\wp(\omega) = -4\wp_{1,3}(\omega_2) - \wp_{1,3}(\omega) + \frac{\wp'_{1,3}(\omega)^2 + \wp'_{1,3}(\omega_2)^2}{2(\wp_{1,3}(\omega) - \wp_{1,3}(\omega_2))^2}. \tag{29}$$

We then make a local expansion at the origin of the both sides of this equation, using property (P3) from Lemma 12. We obtain

$$\begin{aligned} &\frac{1}{\omega^2} + \frac{g_2}{20}\omega^2 + \frac{g_3}{28}\omega^4 + O(\omega^6) = \\ &\frac{1}{\omega^2} + \left(6\wp_{1,3}(\omega_2)^2 - \frac{9g_2^{1,3}}{20}\right)\omega^2 + \left(10\wp_{1,3}(\omega_2)^3 - \frac{3g_2^{1,3}\wp_{1,3}(\omega_2)}{2} - \frac{27g_3^{1,3}}{28}\right)\omega^4 + O(\omega^6). \end{aligned}$$

Identifying the expansions above, we obtain two equations for the three unknowns $\wp_{1,3}(\omega_2)$, $g_2^{1,3}$ and $g_3^{1,3}$ (remember that g_2 and g_3 are known from Lemma 7). We add a third equation by noticing that $\wp_{1,3}(\omega_2)$ is the only real positive solution to (see, e.g., [23, Proof of Lemma 22])

$$X^4 - \frac{g_2^{1,3}}{2}X^2 - g_3^{1,3}X - \frac{(g_2^{1,3})^2}{48} = 0.$$

We then have a (non-linear) system of three equations with three unknowns. Some easy computations finally lead to the expressions of $\wp_{1,3}(\omega_2)$, $g_2^{1,3}$ and $g_3^{1,3}$ of Lemma 8. □

[4]To prove this, we need to introduce the discriminant of the fourth-degree polynomial $P(X)$ defined by (28). Since $\deg P(X) = 4$ and since the leading coefficient of $P(X)$ is 1, its discriminant equals the resultant of $P(X)$ and $P'(X)$. Some elementary computations give that it equals $cz^{16}(1-16z^2)^2$, where c is a negative constant. The discriminant is thus negative (for any $z \in]0, 1/4[$). On the other hand, the discriminant can be interpreted as $\prod_{1\leqslant i<j\leqslant 4}(R_i - R_j)^2$, where the R_i, $i \in \{1, \dots, 4\}$, are the roots of $P(X)$. The negative sign of the discriminant implies that $P(X)$ has two complex conjugate roots and two real roots. Further, the product of the roots is clearly negative, see (28), so that one of the two real roots is negative while another one is positive.

Proof of Gessel conjecture (Problem (A)). Our starting point for proving Problem (A) is the decomposition[5]

$$Q(0,0) = \frac{1}{2z^2}[V_1 + V_2 + V_3 + V_4], \tag{30}$$

with

$$V_1 = 2(\zeta_{1,3}(6\omega_2/4) - \zeta_{1,3}(4\omega_2/4) - \zeta_{1,3}(2\omega_2/4)), \tag{31}$$
$$V_2 = 5(\zeta_{1,3}(\omega_2/4) + \zeta_{1,3}(4\omega_2/4) - \zeta_{1,3}(5\omega_2/4)), \tag{32}$$
$$V_3 = 2(\zeta_{1,3}(3\omega_2/4) - 3\zeta_{1,3}(\omega_2/4)), \tag{33}$$
$$V_4 = 2\zeta_{1,3}(\omega_2/4) - \zeta_{1,3}(2\omega_2/4). \tag{34}$$

Our first result in this section is to express $Q(0,0)$ in terms of generalized hypergeometric functions. First of all, we recall the definition of hypergeometric functions (see [13, Section B.15]):

$${}_pF_q([a_1,\ldots,a_p],[b_1,\ldots,b_q],z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{35}$$

where $(c)_n = c(c+1)\cdots(c+n-1)$ denotes the Pochhammer symbol. It is natural to search for an expression of $Q(0,0)$ in terms of hypergeometric functions, as Gessel's conjecture is equivalent to (see (1))

$$Q(0,0) = \frac{{}_2F_1\left(\left[-\frac{1}{2},-\frac{1}{6}\right],\left[\frac{2}{3}\right],16z^2\right) - 1}{2z^2}$$

and to

$$Q(0,0) = {}_3F_2\left(\left[\frac{1}{2},\frac{5}{6},1\right],\left[\frac{5}{3},2\right],16z^2\right).$$

Our result is the following:

Theorem 9. *The sum $V_1 + V_2$ admits the expression in terms of hypergeometric functions:*

$$V_1 + V_2 = {}_2F_1\left(\left[-\frac{1}{2},\frac{1}{6}\right],\left[\frac{1}{3}\right],16z^2\right) - 22\,z^6\,{}_3F_2\left(\left[2,\frac{5}{2},\frac{17}{6}\right],\left[\frac{11}{3},4\right],16z^2\right) + 2z^2 + 2. \tag{36}$$

The sum $V_3 + V_4$ admits the expression in terms of hypergeometric functions:

$$V_3 + V_4 = -{}_2F_1\left(\left[-\frac{1}{2},\frac{1}{6}\right],\left[\frac{1}{3}\right],16z^2\right) + 4\,z^4\,{}_3F_2\left(\left[\frac{3}{2},\frac{11}{6},2\right],\left[\frac{8}{3},3\right],16z^2\right) - 2. \tag{37}$$

Before starting the proof, we need to introduce some notations. Let R be defined in (28), and $g_2^{1,3}$ and $g_3^{1,3}$ in Lemma 8. Define T_6 by

$$T_6 = \frac{R + 1 - 8z^2 - \sqrt{3R^2 - 4R(1-8z^2) + 4(1-8z^2)^2 - 6g_2}}{9}. \tag{38}$$

We also introduce T_2 by

$$T_2 = \frac{R + 1 - 8z^2}{6} - \frac{T_6}{2}. \tag{39}$$

[5] In Section 5, we shall use the same decomposition to prove the algebraicity of the trivariate GF.

Finally, define T_1 and T_5 as the unique solutions of the equation

$$X^3 - \left(\frac{R}{3} + \frac{1+4z^2}{3}\right)X^2 + \left(\frac{R(1+4z^2)}{9} + \frac{R^2}{108} + \frac{g_2}{18}\right)X + \left(\frac{23R^3}{2916} - \frac{R^2(1+4z^2)}{108} + \frac{g_3}{27} - \frac{19Rg_2}{972}\right) = 0 \quad (40)$$

such that at 0, $T_1 = 1/3 + 4z^2/3 - 4z^6 - 56z^8 + O(z^{10})$ and $T_5 = 1/3 - 8z^2/3 - 8z^4 - 64z^6 + O(z^8)$.

Proof of Theorem 9. To simplify V_1, we shall use the Frobenius-Stickelberger identity, see (P9). We obtain

$$V_1 = -2\sqrt{\wp_{1,3}(6\omega_2/4) + \wp_{1,3}(4\omega_2/4) + \wp_{1,3}(2\omega_2/4)}.$$

All quantities above are known in terms of the variable z: first, one has $\wp_{1,3}(\omega_2) = R/6$, see Lemma 8; the other two quantities are computed in Lemma 10. We obtain

$$V_1 = -2\sqrt{T_2 + T_6 + R/6}.$$

To simplify the expression of V_2 we also make use of the Frobenius-Stickelberger formula. This way, we obtain

$$V_2 = 5\sqrt{T_1 + T_5 + R/6}.$$

The quantity $V_1 + V_2$ is thus explicit, and we now prove that equation (36) holds.

Let T_3 be the third solution of the equation (40). It is an algebraic function that satisfies

$$T_3^4 - \frac{4}{9}(4z^2+1)T_3^3 + \frac{10}{243}(4z^2+1)^2T_3^2 - \frac{4}{19683}(448z^6 + 21072z^4 - 1212z^2 + 7)T_3 + \frac{1}{59049}(256z^8 + 368896z^6 - 38304z^4 + 976z^2 + 1) = 0 \quad (41)$$

and it admits the expansion $T_3 = 1/3 - 8z^2/3 - 8z^4 - 60z^6 + O(z^8)$ at 0. Equation (40) implies that $T_1 + T_3 + T_5 = R/3 + (1+4z^2)/3$, so that the expression of V_2 simplifies to

$$V_2 = 5\sqrt{R/2 + 1/3 + 4z^2/3 - T_3}.$$

From there, resultant computations show that V_2 satisfies the equation

$$19683V_2^8 - 437400(4z^2+1)V_2^6 + 101250(-208z^4 + 760z^2 - 13)V_2^4 + 250000(-320z^6 - 14820z^4 + 2208z^2 - 5)V_2^2 - 390625(16z^4 - 328z^2 - 1)^2 = 0. \quad (42)$$

Similarly, V_1 satisfies the equation

$$19683V_1^8 + 69984(8z^2-1)V_1^6 + 2592(2048z^4 + 208z^2 - 13)V_1^4 + 1024(8z^2-1)(2048z^4 - 80z^2 + 5)V_1^2 - 256(4z-1)^2(4z+1)^2 = 0. \quad (43)$$

Another resultant computation shows that $V_1 + V_2$ is a root of the polynomial

$$27X^8 - 216(12z^2+1)X^6 + 18(-592z^4 + 488z^2 - 13)X^4 + (-14336z^6 + 1199552z^4 + 162560z^2 - 80)X^2 - (80z^4 + 3592z^2 + 3)^2. \quad (44)$$

From there, it follows that the coefficients $(u_n)_{n\geqslant 0}$ of the power series expansion $V_1+V_2 = 3-2z^2-14z^4-126z^6+O(z^8)$ satisfy the following third order linear recurrence with polynomial coefficients:

$$\begin{aligned}256(n-1)(3n-1)(3n+1)(n+1)(3n^2+36n+70)u_n\\ -32(n+2)(n+1)(27n^4+351n^3+1128n^2+1164n+280)u_{n+2}\\ +(3n+8)(3n+10)(n+4)(n+2)(3n^2+24n+10)u_{n+4}=0. \quad (45)\end{aligned}$$

Now, the coefficients in the expansion at the origin of the sum of hypergeometric functions on the right-hand side of equation (36) satisfy the very same recurrence, with the same initial values. This concludes the proof of equation (36).

Equation (37) is proved in a very similar manner. □

To conclude the proof of Theorem 9, it remains to express these quantities in terms of the announced T_1, T_2, T_6. This is the subject of the next lemma.

Lemma 10. *One has the following formulæ:*

(i) $\wp_{1,3}(\omega_2/4) = T_1$, *$T_1$ being the only solution of (40) such that in the neighborhood of 0,* $T_1 = 1/3+4z^2/3-4z^4-56z^6+O(z^8)$.
(ii) $\wp_{1,3}(2\omega_2/4) = T_2$, *with T_2 defined in (39).*
(iii) $\wp_{1,3}(5\omega_2/4) = T_5$, *$T_5$ being the only solution of (40) such that in the neighborhood of 0,* $T_5 = 1/3-8z^2/3-8z^4-64z^6+O(z^8)$.
(iv) $\wp_{1,3}(6\omega_2/4) = T_6$, *with T_6 defined in (38).*

Proof. Let us first compute $\wp_{1,3}(\omega)$ for some given value of ω as a root of (47) below. By property (P7) we find

$$\wp(\omega) = -4\wp_{1,3}(\omega_2)-\wp_{1,3}(\omega)+\frac{\wp'_{1,3}(\omega)^2+\wp'_{1,3}(\omega_2)^2}{2(\wp_{1,3}(\omega)-\wp_{1,3}(\omega_2))^2}, \quad \forall \omega\in\mathbf{C},$$

where by Lemma 8, one has $\wp_{1,3}(\omega_2) = R/6$. Then $\wp'_{1,3}(\omega_2)^2 = 4(R/6)^3-g_2^{1,3}R/6-g_3^{1,3}$, and this way, we obtain

$$\begin{aligned}\wp_{1,3}(\omega)^3-\left(\frac{R}{3}+\wp(\omega)\right)\wp_{1,3}(\omega)^2+\left(\frac{R\wp(\omega)}{3}+\frac{R^2}{108}+\frac{g_2}{18}\right)\wp_{1,3}(\omega)\\ +\left(\frac{23R^3}{2916}-\frac{\wp(\omega)R^2}{36}+\frac{g_3}{27}-\frac{19Rg_2}{972}\right)=0. \quad (46)\end{aligned}$$

For a given value of ω (and thus for a given value of $\wp(\omega)$), the three solutions of

$$x^3-\left(\frac{R}{3}+\wp(\omega)\right)x^2+\left(\frac{R\wp(\omega)}{3}+\frac{R^2}{108}+\frac{g_2}{18}\right)x+\left(\frac{23R^3}{2916}-\frac{\wp(\omega)R^2}{36}+\frac{g_3}{27}-\frac{19Rg_2}{972}\right)=0 \quad (47)$$

are

$$\{\wp_{1,3}(\omega), \wp_{1,3}(\omega+\omega_2), \wp_{1,3}(\omega+2\omega_2)\}.$$

We start the proof of the lemma by showing (i). Using [23, Lemma 19] one has $\wp(\omega_2/4) = \wp(3\omega_2/4) = (1+4z^2)/3$. Then the equation (40) is exactly (47) with $\omega = \omega_2/4$. The three roots of (40) are $\wp_{1,3}(\omega_2/4)$, $\wp_{1,3}(5\omega_2/4)$ and $\wp_{1,3}(9\omega_2/4) = \wp_{1,3}(3\omega_2/4)$. By

using standard properties of $\wp$-Weierstrass functions, $\wp_{1,3}(\omega_2/4)$ is the biggest of the three quantities (and this for any $z \in]0, 1/4[$). Further, since (46) is a polynomial of degree 3, we can find its roots in terms of the variable z. This way, we find that the three solutions admit the expansions $1/3 + 4z^2/3 - 4z^4 - 56z^6 + O(z^8)$, $1/3 - 8z^2/3 - 8z^4 - 60z^6 + O(z^8)$ and $1/3 - 8z^2/3 - 8z^4 - 64z^6 + O(z^8)$. Accordingly, $T_1 = \wp_{1,3}(\omega_2/4)$ corresponds to the first one and T_5 to the last one.

We now prove (ii) and (iv). Using again [23, Lemma 19], one derives that $\wp(2\omega_2/4) = (1-8z^2)/3$. The three roots of equation (47) with $\omega = 2\omega_2/4$ are $\wp_{1,3}(2\omega_2/4)$, $\wp_{1,3}(6\omega_2/4)$ and $\wp_{1,3}(10\omega_2/4)$. Since $\wp_{1,3}(10\omega_2/4) = \wp_{1,3}(2\omega_2/4)$, (47) with $\omega = 2\omega_2/4$ has a double root (that we call t_1) and a simple root (t_2). It happens to be simpler to deal now with the derivative of the polynomial in the left hand side of (47). It is an easy exercise to show that this derivative polynomial has roots t_1 and $(t_1 + 2t_2)/3$. This way, we obtain expressions for $\wp_{1,3}(2\omega_2/4)$ and $(\wp_{1,3}(2\omega_2/4) + 2\wp_{1,3}(6\omega_2/4))/3$, which are equal to

$$\frac{R + 1 - 8z^2 \pm \sqrt{3R^2 - 4R(1-8z^2) + 4(1-8z^2)^2 - 6g_2}}{9}. \tag{48}$$

Since $\wp_{1,3}(2\omega_2/4) > \wp_{1,3}(6\omega_2/4)$, $\wp_{1,3}(2\omega_2/4)$ corresponds to the sign + in (48). This way we immediately make explicit $\wp_{1,3}(2\omega_2/4)$ and $\wp_{1,3}(6\omega_2/4)$ and finish the proof of the lemma. □

5. Proof of the algebraicity of the GF (problem (B))

Algebraicity of the series of the excursions. In this section, we prove that $Q(0,0)$ is an algebraic function of z.[6] We start with the expression (20) of Theorem 2, where $Q(0,0)$ is written as a weighted sum of six ζ-functions. It turns out that a single term $\zeta_{1,3}(k\omega_2/4)$, $k \in \{1, \dots, 6\}$, in (20) is not algebraic (though it is holonomic); however, one can use the decomposition (30), and we now show that each term above is algebraic in z. For (34), applying the duplication formula (P10), we see that it is sufficient to show the algebraicity of $\wp'_{1,3}(\omega_2/4)$ and $\wp''_{1,3}(\omega_2/4)$ as functions of z, which is proved in Lemma 11 below. Likewise, the proof of the fact that (33) is algebraic uses the triplication formula (P10) and Lemma 11. The proof of the algebraicity of (32) and (31) is similar: we apply (P9) (which is due to Frobenius and Stickelberger, see [38, page 446]) as well as Lemma 11.

Lemma 11. *For any $k \in \mathbf{Z}$ and any $\ell \in \mathbf{Z}_+$, $\wp^{(\ell)}(k\omega_2/8)$ and $\wp^{(\ell)}_{1,3}(k\omega_2/8)$ are (infinite or) algebraic functions of z.*

Proof. First, for any $\ell \in \mathbf{Z}_+$ and $k \in 8\mathbf{Z}$ (resp. $k \in 24\mathbf{Z}$), $\wp^{(\ell)}(k\omega_2/8)$ (resp. $\wp^{(\ell)}_{1,3}(k\omega_2/8)$) is infinite. For other values of k, they are finite. By periodicity and parity, it is enough to prove the algebraicity for $k \in \{1, \dots, 4\}$ (resp. $k \in \{1, \dots, 12\}$).

It is important to notice that it suffices to prove the result for $\ell = 0$. Indeed, all the invariants g_2, g_3, $g_2^{1,3}$ and $g_3^{1,3}$ are algebraic functions of z (see Lemmas 7 and 8), so that using inductively the differential equation (26), we obtain the algebraicity for values of $\ell \geqslant 1$ from the algebraicity for $\ell = 0$.

We first consider $\wp$. It is demonstrated in [23, Lemma 19] that $\wp(k\omega_2/8) = \wp^{(0)}(k\omega_2/8)$ is algebraic for $k = 2$ and $k = 4$. For $k = 1$ this follows from the bisection formula (P11)

[6] Here and throughout, we shall always consider algebraic functions over the field $\mathbf{Q}$ of rational numbers.

and from the case $k = 2$ (note that $\wp(\omega_1/2)$, $\wp(\omega_2/2)$ and $\wp((\omega_1+\omega_2)/2)$ are algebraic in z, see [23, Lemma 12]). For $k = 3$, this is a consequence of the addition formula (P4).

We now deal with $\wp_{1,3}$. Let $k \in \{1, \dots, 12\}$. Using (29), we easily derive that $\wp_{1,3}(\omega_0)$ is algebraic in z as soon as $\wp(\omega_0)$ is algebraic in z: indeed, in (29) $\wp_{1,3}(\omega_2)$ and $\wp'_{1,3}(\omega_2)$ are algebraic in z, as a consequence of Lemma 8. This remark works for all k, except $k = 8$, since then $\wp(k\omega_2/8) = \infty$. In fact, for $k = 8$ the situation is also simple, as $\wp_{1,3}(k\omega_2/8) = R/6$ (see Lemma 8) is algebraic. □

Branches of the GFs and algebraicity of $Q(x, y)$ in the variables x, y. In this section we prove a weakened version of problem (B): we show that $Q(x, y)$ is algebraic in x, y (we do not prove here that the latter function is algebraic in x, y, z, which is much stronger). This is not necessary for our analysis, but this illustrates that our approach easily yields to algebraicity results.

We first propose two proofs of the algebraicity of $Q(0, y)$ as a function in y. The first proof is an immediate application of property (P5). The sum of the residues (i.e., the multiplicative factors in front of the ζ-functions) in the formula (21) of Theorem 3 is clearly 0, so that $r_y(\omega)$ is an algebraic function of $\wp_{1,3}(\omega)$, by (P5). Further, by (P7), $\wp_{1,3}(\omega)$ is an algebraic function of $\wp(\omega)$, and finally by (9), $\wp(\omega)$ is algebraic in $y(\omega)$. This eventually implies that $r_y(\omega)$ is algebraic in $y(\omega)$, or equivalently that $Q(0, y)$ is algebraic in y.

The second proof is based on the meromorphic continuation of the GFs on the universal covering, which is done in section 2. The restrictions of $r_y(\omega)/K(0, y(\omega))$ on

$$\mathcal{D}_{k,\ell} = \omega_3/2 + \omega_1[\ell, \ell+1[+\omega_2]k/2, (k+1)/2]$$

for $k, \ell \in \mathbf{Z}$ provide all branches on $\mathbf{C} \setminus ([y_1, y_2] \cup [y_3, y_4])$ of $Q(0, y)$ as follows:

$$Q(0, y) = \{r_y(\omega)/K(0, y(\omega)) : \omega \text{ is the (unique) element of } \mathcal{D}_{k,\ell} \text{ such that } y(\omega) = y\}.$$

Due to the ω_1-periodicity of $r_y(\omega)$ and $y(\omega)$ (see (19) and (9), respectively), the restrictions of these functions on $\mathcal{D}_{k,\ell}$ do not depend on $\ell \in \mathbf{Z}$, and therefore determine the same branch as on $\mathcal{D}_{k,0}$ for any ℓ. Furthermore, due to equation (18), the restrictions of $r_y(\omega)/K(0, y(\omega))$ on $\mathcal{D}_{-k+1,0}$ and on $\mathcal{D}_{k,0}$ lead to the same branches for any $k \in \mathbf{Z}$. Hence, the restrictions of $r_y(\omega)/K(0, y(\omega))$ on $\mathcal{D}_{k,0}$ with $k \geqslant 1$ provide all different branches of this function. In addition, Lemma 6 says that r_y is $3\omega_2$-periodic. This fact yields that $Q(0, y)$ has (at most) six branches, and is thus algebraic.

An analogous statement holds for (the restrictions of) $r_x(\omega)/K(x(\omega), 0)$ and then for $Q(x, 0)$. Using the functional equation (3), we conclude that $Q(x, y)$ is algebraic in the two variables x, y.

In the section below, we refine the previous statement, by proving that $Q(x, y)$ is algebraic in x, y, z (problem (B)).

Proof of the algebraicity of the trivariate GF. We start by proving the algebraicity of $Q(0, y)$ as a function of y, z. We consider the representation of $r_y(\omega)$ given in Theorem 3 and apply eight times the addition theorem (P4) for ζ-functions, namely (for suitable values of $k \in \mathbf{Z}$ given in (21))

$$\zeta_{1,3}(\omega - k\omega_2/8) = \zeta_{1,3}(\omega) - \zeta_{1,3}(k\omega_2/8) + \frac{1}{2}\frac{\wp'_{1,3}(\omega) + \wp'_{1,3}(k\omega_2/8)}{\wp_{1,3}(\omega) - \wp_{1,3}(k\omega_2/8)}.$$

We then make the weighted sum of the eight identities above (corresponding to the good values of k in (21)); this way, we obtain

$$r_y(\omega) = U_1(\omega) + U_2 + U_3(\omega),$$

where $U_1(\omega)$ is the weighted sum of the eight functions $\zeta_{1,3}(\omega)$, U_2 is the sum of c and of the weighted sum of the eight quantities $\zeta_{1,3}(k\omega_2/8)$, and $U_3(\omega)$ is the weighted sum of the eight quantities

$$\frac{\wp'_{1,3}(\omega) + \wp'_{1,3}(k\omega_2/8)}{\wp_{1,3}(\omega) - \wp_{1,3}(k\omega_2/8)}. \tag{49}$$

Since the sum of the residues in the formula (21) equals 0, the coefficients in front of $\zeta_{1,3}(\omega)$ is 0, so that $U_1(\omega)$ is identically zero. To prove that U_2 is algebraic in z, it suffices to use similar arguments as we did to prove that $Q(0,0)$ is algebraic (first part of section 5); we do not repeat the arguments here. Finally, we show that $U_3(\omega)$ is algebraic in $y(\omega)$ over the field of algebraic functions in z. In other words, we show that there exists a non-zero bivariate polynomial P such that $P(U_3(\omega), y(\omega)) = 0$, where the coefficients of P are algebraic in z. This is enough to conclude to the algebraicity of $Q(0,y)$ as a function of y, z.

To prove the latter fact, we shall prove that each term (49) satisfies the property above (with different polynomials P, of course). First, Lemma 11 implies that $\wp_{1,3}(k\omega_2/8)$ and $\wp'_{1,3}(k\omega_2/8)$ are both algebraic in z. The function $\wp_{1,3}(\omega)$ is algebraic in $y(\omega)$ over the field of algebraic functions in z, this follows from (P7). For $\wp'_{1,3}(\omega)$, this comes from the fact above together with the differential equation (26) satisfied by the Weierstrass elliptic functions.

The proof of the algebraicity of $Q(x,0)$ as a function of x, z is analogous. With equation (3) the algebraicity of $Q(x,y)$ as a function of x, y, z is proved.

6. Conclusion

Application of our results to other end points. In this article we have presented the first human proofs of Gessel conjecture (problem (A)) and of the algebraicity of the trivariate GF counting Gessel walks (problem (B)). We have deduced the closed-form expression (1) of the numbers of walks $q(0,0;n)$ from a new algebraic expression of the GF $\sum_{n\geqslant 0} q(0,0;n)z^n$.

With a very similar analysis, we could obtain an expression for the series $\sum_{n\geqslant 0} q(i,j;n)z^n$, for any fixed couple $(i,j) \in \mathbf{Z}_+^2$. Let us illustrate this fact with the example $(i,j) = (0,j)$. Let

$$g_j(z) = \sum_{n\geqslant 0} q(0,j;2n)z^{2n}$$

be the function counting walks ending at the point $(0,j)$ of the vertical axis. We obviously have

$$Q(0,y) = \sum_{j\geqslant 0} y^j g_j(z).$$

In particular, the functions $g_j(z)$ are exactly the successive derivatives of $Q(0,y)$ with respect to (w.r.t.) the variable y evaluated at $y = 0$ (after dividing by $j!$). First, $g_0(z) = Q(0,0)$.

Further, one has $r_y(\omega) = z(y(\omega)+1)Q(0, y(\omega))$. Differentiating w.r.t. ω and evaluating at ω_0^y (which is such that $y(\omega_0^y) = 0$), we find

$$g_1(z) = \frac{r_y'(\omega_0^y)}{zy'(\omega_0^y)} - Q(0,0).$$

All quantities above can be computed, and a similar analysis as in section 4 could lead to a closed-form expression for the numbers of walks $q(0,1;2n)$. Similarly, one could compute $g_2(z)$, $g_3(z)$, etc.

New Gessel conjectures. For any $j \geqslant 0$, define

$$f_j(z) = (-1)^j(2j+1)z^j + 2z^{j+1}\sum_{n\geqslant 0} q(0,j;2n)z^n.$$

Then the closed-form expression (1) for the $q(0,0;2n)$ is equivalent to [4, 16]

$$f_0\left(z\frac{(1+z)^3}{(1+4z)^3}\right) = \frac{1+8z+4z^2}{(1+4z)^{3/2}}. \tag{50}$$

On Mars 2013, Ira Gessel [16] proposed the following new conjectures: for any $j \geqslant 1$,

$$f_j\left(z\frac{(1+z)^3}{(1+4z)^3}\right) = (-z)^j\frac{p_j(z)}{(1+4z)^{3/2+3j}},$$

where $p_j(z)$ is a polynomial of degree $3j+2$ with positive coefficients (due to (50), these new conjectures generalize the original one).

In this article we shall not prove these conjectures. However, we do think that following the method sketched in the first part of section 6, it could be possible to prove them, at least for small values of $j \geqslant 0$. On the other hand, our approach does not explain why the quantity

$$z\frac{(1+z)^3}{(1+4z)^3}$$

seems to be so important and natural for Gessel walks.

Appendix A. Some properties of elliptic functions

In this appendix, we gather the results we used on $\wp$- and ζ-Weierstrass functions.

Lemma 12. *Let ζ and $\wp$ be the Weierstrass functions with certain periods $\overline{\omega}, \widehat{\omega}$.*

(P1) *We have the expansion*

$$\zeta(\omega) = \frac{1}{\omega} + \sum_{(\overline{n},\widehat{n})\in Z^2\setminus\{(0,0)\}}\left(\frac{1}{\omega-(\overline{n\omega}+\widehat{n\omega})}+\frac{1}{\overline{n\omega}+\widehat{n\omega}}+\frac{\omega}{(\overline{n\omega}+\widehat{n\omega})^2}\right), \qquad \forall \omega \in \mathbf{C}.$$

As for the expansion of $\wp(\omega)$, it is given in

$$\wp(\omega) = \frac{1}{\omega^2} + \sum_{(\overline{n},\widehat{n})\in Z^2\setminus\{(0,0)\}}\left(\frac{1}{(\omega-(\overline{n\omega}+\widehat{n\omega}))^2}-\frac{1}{(\overline{n\omega}+\widehat{n\omega})^2}\right), \qquad \forall \omega \in \mathbf{C}. \tag{51}$$

In particular, in the fundamental parallelogram $[0,\overline{\omega}[+[0,\widehat{\omega}[$, ζ (resp. $\wp$) has a unique pole. It is of order 1 (resp. 2), at 0, and has residue 1 (resp. 0, and principal part $1/\omega^2$).

(P2) *An elliptic function with no poles in the fundamental parallelogram* $[0,\overline{\omega}[+[0,\widehat{\omega}[$ *is constant.*

(P3) *In the neighborhood of* 0*, the function* $\wp(\omega)$ *admits the expansion*

$$\wp(\omega)=\frac{1}{\omega^2}+\frac{g_2}{20}\omega^2+\frac{g_3}{28}\omega^4+O(\omega^6).$$

(P4) *We have the addition theorems*

$$\zeta(\omega+\widetilde{\omega})=\zeta(\omega)+\zeta(\widetilde{\omega})+\frac{1}{2}\frac{\wp'(\omega)-\wp'(\widetilde{\omega})}{\wp(\omega)-\wp(\widetilde{\omega})},\qquad\forall\omega,\widetilde{\omega}\in\mathbf{C}.$$

and

$$\wp(\omega+\widetilde{\omega})=-\wp(\omega)-\wp(\widetilde{\omega})+\frac{1}{4}\left(\frac{\wp'(\omega)-\wp'(\widetilde{\omega})}{\wp(\omega)-\wp(\widetilde{\omega})}\right)^2,\qquad\forall\omega,\widetilde{\omega}\in\mathbf{C}.$$

(P5) *For given* $\widetilde{\omega}_1,\dots,\widetilde{\omega}_p\in\mathbf{C}$*, define*

$$f(\omega)=c+\sum_{1\leqslant\ell\leqslant p}r_\ell\zeta(\omega-\widetilde{\omega}_\ell),\qquad\forall\omega\in\mathbf{C}.\tag{52}$$

The function f *above is elliptic if and only if* $\sum_{1\leqslant\ell\leqslant p}r_\ell=0$*.*

(P6) *Let* f *be an elliptic function with periods* $\overline{\omega},\widehat{\omega}$ *such that in the fundamental parallelogram* $[0,\overline{\omega}[+[0,\widehat{\omega}[$*,* f *has only poles of order* 1*, at* $\widetilde{\omega}_1,\dots,\widetilde{\omega}_p$*, with residues* $r_1,\dots,r_p$*, respectively. Then there exists a constant* c *such that* (52) *holds.*

(P7) *Let* p *be some positive integer. The Weierstrass elliptic function with periods* $\overline{\omega},\widehat{\omega}/p$ *can be written in terms of* $\wp$ *as*

$$\wp(\omega)+\sum_{1\leqslant\ell\leqslant p-1}[\wp(\omega+\ell\widehat{\omega}/p)-\wp(\ell\widehat{\omega}/p)],\qquad\forall\omega\in\mathbf{C}.$$

(P8) *The function* ζ *is quasi-periodic, in the sense that*

$$\zeta(\omega+\overline{\omega})=\zeta(\omega)+2\zeta(\overline{\omega}/2),\qquad\zeta(\omega+\widehat{\omega})=\zeta(\omega)+2\zeta(\widehat{\omega}/2),\qquad\forall\omega\in\mathbf{C}.$$

(P9) *If* $\alpha+\beta+\gamma=0$ *then*

$$(\zeta(\alpha)+\zeta(\beta)+\zeta(\gamma))^2=\wp(\alpha)+\wp(\beta)+\wp(\gamma).$$

(P10) *We have the duplication and triplication formulæ:*

$$\zeta(2\omega)=2\zeta(\omega)+\frac{\wp''(\omega)}{2\wp'(\omega)},\qquad\zeta(3\omega)=3\zeta(\omega)+\frac{4\wp'(\omega)^3}{\wp'(\omega)\wp'''(\omega)-\wp''(\omega)^2},\qquad\forall\omega\in\mathbf{C}.$$

(P11) *We have the bisection formula:*

$$\begin{aligned}\wp(\omega/2)=\wp(\omega)&+\sqrt{(\wp(\omega)-\wp(\omega_1/2))(\wp(\omega)-\wp(\omega_2/2))},\\&+\sqrt{(\wp(\omega)-\wp(\omega_1/2))(\wp(\omega)-\wp((\omega_1+\omega_2)/2))}\\&+\sqrt{(\wp(\omega)-\wp(\omega_2/2))(\wp(\omega)-\wp((\omega_1+\omega_2)/2))},\qquad\forall\omega\in\mathbf{C}.\end{aligned}$$

Proof. All these properties of elliptic functions are classical; they can be found in [1, 19, 38] (more precise references are given in [25]). □

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