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Digital Circles, Spheres and Hyperspheres: From Morphological Models to Analytical Characterizations and Topological Properties

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Abstract

In this paper we provide an analytical description of various classes of digital circles, spheres and in some cases hyperspheres, defined in a morphological framework. The topological properties of these objects, especially the separation of the digital space, are discussed according to the shape of the structuring element. The proposed framework is generic enough so that it encompasses most of the digital circle definitions that appear in the literature and extends them to dimension 3 and sometimes dimension n .

Keywords: digital geometry, digital topology, mathematical morphology, digital circle and sphere, analytical characterization

1. Introduction

Digital circle generation, characterization and recognition have been important topics for many years in the digital geometry and pattern recognition communities. It is well known now that all digital straight lines are some sort of Reveillès digital straight line [1]. This arithmetical framework provides a way of defining digital hyperplanes too [1, 2]. What is less well-known is that there is not only one but many different types of digital circles in the literature. This is a problem when dealing with circle recognition algorithms. Most recognition algorithms provide parameters of a Euclidean circle while the corresponding type of digital circle is implicit [3, 4, 5, 6, 7, 8, 9]. This makes comparison between different algorithms dubious because different sets may or may not be recognized as a digital circle by different algorithms. A second problem arises from the way a digital circle is defined. Digital circles are defined as the result of an algorithm or implicitly by a set of (topological) properties. A typical example is the Bresenham's circle [10] which is either defined by its generation algorithm or topologically characterized as a 0-connected (8-connected in classical notation) digital approximation of a Euclidean circle of integer radius and integer coordinate center. This does not lead to a global mathematical definition of the object. Extensions to higher dimensions are thus complicated: a revealing fact is that there are almost no definition of digital spheres or hyperspheres in the literature [11].

In this paper, we propose a unified framework allowing to analytically characterize most of, if not all, known digital circles appearing in the literature [10, 12, 13, 14, 15, 16, 17, 18]. Each of these digital circles is defined as the set of integer solutions of a system of analytical inequalities. Such a global mathematical definition provides natural extensions to the different types of digital circles, in particular, extension of the parameter domains and extension in dimensions. For instance, the Bresenham's circle [10] can be easily extended to a digital circle that is not limited to integer radii or integer coordinate centers. It can also be extended to digital spheres or hyperspheres. This is a step forward compared to the results previously presented in [19].

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In an n -dimensional Euclidean space, a sequence of morphological operations (dilations by a structuring element) and set-theoretic operations (intersection, union) is applied to a hypersurface \mathcal{S} in order to define an offset region. The digitization of \mathcal{S} is then the digitization of this offset region, i.e, the set of the integer coordinate points lying in it.

According to the type of structuring elements, two families of morphological digitization models are proposed. For both of them, the offset regions of a circle, a sphere and, in some cases, a hypersphere, are analytically described and the topological properties of their digitizations are studied.

For the first family of digitization models, the structuring elements correspond to norm based balls. The norms we are considering are the Euclidean norm and the *adjacency norms* that encompass the ℓ^∞ - and the ℓ^1 -norms. The adjacency norms allow us to define k -separating digital hyperspheres. Analytical characterizations are provided for circles and spheres whatever the norm used while hyperspheres admit only a simple analytical characterization for the Euclidean norm.

The second family of digitization models is based on structuring elements, called *adjacency flakes*, that are subsets of the adjacency norm based balls. The resulting digital hyperspheres are still k -separating, and even strictly k -separating (without any k -simple point) for one model. Besides they have much simpler analytical characterizations.

In Section 2, we introduce families (closed or semi-open and Gaussian or centered) of digitization models that are morphological in nature. Each model is parametrized by a structuring element. This allows to define different types of digital hyperspheres according to the shape of the structuring element.

In Section 3 and Section 4, we propose digital hyperspheres based on balls of different norms. We recall some results for the digital hyperspheres based on the Euclidean norm [20] before introducing the *adjacency norms*. The adjacency balls enable us to define thin digital hyperspheres that separate \mathbb{Z}^n . We provide analytical characterizations only for circles and spheres. According to the adjacency norm considered, we define the Chebyshev and the Manhattan families. Supercover circles and spheres [21] are then closed centered Chebyshev circles and spheres. Bresenham's circles [10] are centered Manhattan circles.

Analytical characterizations for n -dimensional hyperspheres are proposed in Section 5, with a family of even thinner digital hyperspheres based on another kind of structuring elements. These structuring elements, called *adjacency flakes*, are specific subsets of the adjacency balls. The digital hyperspheres thus defined are compared with existing definitions in the literature, their topological properties are discussed and we provide their analytical characterization in any dimension.

1.1. Recalls and notations

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the canonical basis of the n -dimensional Euclidean vector space. We denote by x_i the i -th coordinate of a point or a vector \mathbf{x} , that is its coordinate associated to \mathbf{e}_i . A *digital object* is a set of integer points. A *digital inequality* is an inequality with coefficients in \mathbb{R} from which we retain only the integer coordinate solutions. A *digital analytical object* is a digital object defined by a finite set of digital inequalities.

For all $k \in \{0, \dots, n-1\}$, two integer points \mathbf{v} and \mathbf{w} are said to be *k-adjacent* or *k-neighbors*, if for all $i \in \{1, \dots, n\}$, $|v_i - w_i| \leq 1$ and $\sum_{j=1}^n |v_j - w_j| \leq n - k$. In the 2-dimensional plane, the 0- and 1-neighborhood notations correspond respectively to the classical 8- and 4-neighborhood notations. In the 3-dimensional space, the 0-, 1- and 2-neighborhood notations correspond respectively to the classical 26-, 18- and 6-neighborhood notations.

A *k-path* is a sequence of integer points such that every two consecutive points in the sequence are k -adjacent. A digital object E is *k-connected* if there exists a k -path in E between any two points of E . A maximum k -connected subset of E is called a *k-connected component*. Let us suppose that the complement of a digital object E , $\mathbb{Z}^n \setminus E$, admits exactly two k -connected components F_1 and F_2 , or in other words that there exists no k -path joining integer points of F_1 and F_2 , then E is said to be *k-separating* in \mathbb{Z}^n . If there is no path from F_1 to F_2 then E is said to be 0-separating or simply *separating*. A point \mathbf{v} of a k -separating object E is said to be a *k-simple point* if $E \setminus \{\mathbf{v}\}$ is still k -separating. A k -separating object that has no k -simple points is said to be *strictly k-separating*.

The logical *and* and *or* operators are denoted \wedge and \vee respectively.

Let \oplus be the Minkowski addition, known as dilation, such that $\mathcal{A} \oplus \mathcal{B} = \cup_{\mathbf{b} \in \mathcal{B}} \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in \mathcal{A}\}$.

In the present paper, the focus is only on the n -dimensional hypersphere $\mathcal{S}_{\mathbf{c},r}$ of center $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{R}^n$ and radius $r \in \mathbb{R}^+$ which is analytically defined by:

$$\mathcal{S}_{\mathbf{c},r} = \{\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n : s_{\mathbf{c},r}(\mathbf{x}) = 0\}, \quad \text{with } s_{\mathbf{c},r}(\mathbf{x}) = \left(\sum_{i=1}^n (x_i - c_i)^2 \right) - r^2.$$

We also introduce notations for the inside and the outside (strict or large) of such an hypersphere:

$$\mathcal{S}_{\mathbf{c},r}^{-*} = \{\mathbf{x} \in \mathbb{R}^n : s_{\mathbf{c},r}(\mathbf{x}) < 0\}, \quad \mathcal{S}_{\mathbf{c},r}^{+*} = \{\mathbf{x} \in \mathbb{R}^n : s_{\mathbf{c},r}(\mathbf{x}) > 0\}, \quad \mathcal{S}_{\mathbf{c},r}^- = \mathcal{S}_{\mathbf{c},r}^{-*} \cup \mathcal{S}_{\mathbf{c},r} \quad \text{and} \quad \mathcal{S}_{\mathbf{c},r}^+ = \mathcal{S}_{\mathbf{c},r}^{+*} \cup \mathcal{S}_{\mathbf{c},r}.$$

2. Digitization Models

Since the direct digitization of a hypersphere $\mathcal{S}_{\mathbf{c},r}$ has obviously not enough integer points to ensure good topological properties such as separation of the space, one first applies a sequence of morphological operations (dilations) to \mathcal{S} in order to define a region \mathcal{O} located around or close to $\mathcal{S}_{\mathbf{c},r}$ called *offset region*. The digitization of $\mathcal{S}_{\mathbf{c},r}$ is then the set of integer coordinate points lying in this offset region.

In what follows we consider various digitization models. They are morphological in nature. Whatever the dimension, the shape of the hypersphere and the shape of the structuring element using for dilation, we define first digitization models centered on the hypersphere, either closed or semi-open (the inner or the outer boundary of the offset region is not taken into account). Then, we define digitization models non centered on the hypersphere, such that the resulting digital set lies only on one side of the hypersphere.

We assume that the structuring element has a central symmetry (ie. $\mathbf{x} \in \mathcal{A} \Rightarrow -\mathbf{x} \in \mathcal{A}$).

2.1. The closed model

Let us assume that the structuring element \mathcal{A} is closed (it is equal to its closure).

The digitization $D_{\mathcal{A}}(\mathcal{S}_{\mathbf{c},r})$ according to the structuring element \mathcal{A} and centered on the hypersphere $\mathcal{S}_{\mathbf{c},r}$ is defined from the offset region $\mathcal{O}_{\mathcal{A}}(\mathcal{S}_{\mathbf{c},r})$:

$$D_{\mathcal{A}}(\mathcal{S}_{\mathbf{c},r}) = \mathcal{O}_{\mathcal{A}}(\mathcal{S}_{\mathbf{c},r}) \cap \mathbb{Z}^n = (\mathcal{S}_{\mathbf{c},r} \oplus \mathcal{A}) \cap \mathbb{Z}^n.$$

The offset region is closed since \mathcal{A} is closed. Unit balls for a given norm are good candidates for such structuring elements as we will see in Section 3 and Section 4. Among those models we can mention the *Pythagorean* model, which is based on the Euclidean norm, the *supercover* model based on the ℓ^∞ -norm and the *closed naive* model based on the ℓ^1 -norm [22].

2.2. Avoiding simple points : the semi-open models

Closed models are known to contain simple points for lines, planes or hyperplanes and, as a direct consequence, for more general thin objects such as hyperspheres. The supercover model is a good example.

A supercover line contains many simple points. Removing one boundary of the offset region (for example the upper one) allows to exclude these simple points while preserving the separation of the space by the digital line. This defines the *Standard digitization model* [23], which has been defined, however, only for linear objects (flats and simplices).

The idea here, with the semi-open models, is to proceed similarly for circular objects: removing one of the boundaries of the offset region in order to remove (at least some) simple points in the digitization.

Such a model can be described with two structuring elements: a closed element \mathcal{A} and a second element \mathcal{A}^* which is defined as \mathcal{A} deprived of the boundary of its convex hull.

We consider the two semi-open digitizations $D_{\mathcal{A}}^+(\mathcal{S}_{\mathbf{c},r})$ and $D_{\mathcal{A}}^-(\mathcal{S}_{\mathbf{c},r})$ centered on $\mathcal{S}_{\mathbf{c},r}$ defined from the following offset region:

$$\begin{aligned} \mathcal{O}_{\mathcal{A}}^+(\mathcal{S}_{\mathbf{c},r}) &= (\mathcal{S}_{\mathbf{c},r}^+ \oplus \mathcal{A}) \cap (\mathcal{S}_{\mathbf{c},r}^- \oplus \mathcal{A}^*), \\ \mathcal{O}_{\mathcal{A}}^-(\mathcal{S}_{\mathbf{c},r}) &= (\mathcal{S}_{\mathbf{c},r}^+ \oplus \mathcal{A}^*) \cap (\mathcal{S}_{\mathbf{c},r}^- \oplus \mathcal{A}). \end{aligned}$$

The offset region used to define $D_{\mathcal{A}}^+(\mathcal{S}_{c,r})$ is open on the $\mathcal{S}_{c,r}^+$ side, whereas the one used to define $D_{\mathcal{A}}^-(\mathcal{S}_{c,r})$ is open on the $\mathcal{S}_{c,r}^-$ side.

Note that we will not discuss models based only on open structuring elements in this paper. Such models do not have particular properties that seem relevant. If need be, it is rather trivial to write the corresponding equations to them.

2.3. The inner and outer Gaussian digitization models

As reported in [24], Gauss used a method to measure an approximation of the area of a planar set by counting the number of integer points inside the set. This can also be seen as a digitization model of a planar set. In the present paper, we are interested in the digitization of circles and hyperspheres and not disks and balls. The 0-connected or 1-connected boundary of a Gaussian disk, or of its complement, can however be a way to define a digital circle. We define such digitization schemes we call *inner and outer Gaussian digitization models*.

More precisely, the inner semi-open $D_{i\mathcal{A}}^-(\mathcal{S}_{c,r})$ and the outer semi-open Gaussian digitizations $D_{o\mathcal{A}}^+(\mathcal{S}_{c,r})$ of a hypersphere $\mathcal{S}_{c,r}$ are defined as the digitizations of the following offset regions:

$$\begin{aligned} \mathcal{O}_{i\mathcal{A}}^-(\mathcal{S}_{c,r}) &= (\mathcal{S}_{c,r}^+ \oplus 2\mathcal{A}^*) \cap \mathcal{S}_{c,r}^-, \\ \mathcal{O}_{o\mathcal{A}}^+(\mathcal{S}_{c,r}) &= (\mathcal{S}_{c,r}^- \oplus 2\mathcal{A}^*) \cap \mathcal{S}_{c,r}^+. \end{aligned}$$

Note that we dilate $\mathcal{S}_{c,r}$ with a structuring element twice as big ($2\mathcal{A}$) in order to have an offset region as thick as the ones of the previous digitization models.

It is of course also possible to define closed Gaussian models $D_{i\mathcal{A}}(\mathcal{S}_{c,r})$, $D_{o\mathcal{A}}(\mathcal{S}_{c,r})$ by considering the closed structuring element $2\mathcal{A}$. The reader should have no problem in handling these cases if need be.

In this section, we have selected morphological models to digitize hyperspheres. They can be used in the more general case of oriented hypersurfaces. In the sequel of the paper, we will consider various structuring elements in order to obtain digital hyperspheres with good topological properties. In particular, in the two following sections we consider digitization models based on structuring elements that are balls for given norms. These balls are illustrated in Fig. 1 and the structuring elements of the general model, \mathcal{A} and \mathcal{A}^* , will be respectively a closed ball $\mathcal{B}_{\|\cdot\|}(\rho)$ and an open ball $\mathcal{B}_{\|\cdot\|}^*(\rho)$ with same norm and radius.



Figure 1: Structuring elements in 2D for the norms $[\cdot]_1 = \ell^1$, ℓ^2 , $[\cdot]_0 = \ell^\infty$ and in 3D for the norms $[\cdot]_2 = \ell^1$ (octahedron), ℓ^2 , $[\cdot]_1$ (cuboctahedron), $[\cdot]_0 = \ell^\infty$.

3. Digital circles and spheres based on the Euclidean norm

We first study the *Euclidean norm* (classical ℓ^2 notation) which allows a very simple analytical characterization.

3.1. The Euclidean norm

The first norm we investigate is the Euclidean norm, ℓ^2 , defined by:

$$\forall \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^n (x_i)^2}.$$

In the case of hypersphere digitization under the introduced models, this norm presents an important advantage: the two boundaries of the offset region are concentric hyperspheres. The offset region of an Euclidean digital hypersphere is thus an annulus and analytical characterizations can be directly deduced.

Proposition 1. *Let $\mathcal{B}_2(\rho)$ be the ball of radius $\rho \in \mathbb{R}^{+*}$ under the Euclidean norm. The analytical characterizations of the Euclidean digitizations (or ℓ^2 -digitizations) of a hypersphere $\mathcal{S}_{\mathbf{c},r}$ are:*

$$\begin{aligned} D_{\mathcal{B}_2(\rho)}(\mathcal{S}_{\mathbf{c},r}) &= \left\{ \mathbf{v} \in \mathbb{Z}^n : (r - \min\{r, \rho\})^2 - r^2 \leq s_{\mathbf{c},r}(\mathbf{v}) \leq 2\rho r + \rho^2 \right\}, \\ D_{\mathcal{B}_2(\rho)}^+(\mathcal{S}_{\mathbf{c},r}) &= \left\{ \mathbf{v} \in \mathbb{Z}^n : (r - \min\{r, \rho\})^2 - r^2 \leq s_{\mathbf{c},r}(\mathbf{v}) < 2\rho r + \rho^2 \right\}, \\ D_{\mathcal{B}_2(\rho)}^-(\mathcal{S}_{\mathbf{c},r}) &= \left\{ \mathbf{v} \in \mathbb{Z}^n : (r - \min\{r, \rho\})^2 - r^2 < s_{\mathbf{c},r}(\mathbf{v}) \leq 2\rho r + \rho^2 \right\}, \\ D_{i\mathcal{B}_2(\rho)}^-(\mathcal{S}_{\mathbf{c},r}) &= \left\{ \mathbf{v} \in \mathbb{Z}^n : (r - \min\{r, 2\rho\})^2 - r^2 < s_{\mathbf{c},r}(\mathbf{v}) \leq 0 \right\}, \\ D_{o\mathcal{B}_2(\rho)}^+(\mathcal{S}_{\mathbf{c},r}) &= \left\{ \mathbf{v} \in \mathbb{Z}^n : 0 \leq s_{\mathbf{c},r}(\mathbf{v}) < 4\rho r + 4\rho^2 \right\}. \end{aligned}$$

The proof of this proposition is immediate.

Note that, if the radius of the hypersphere is too small compared to the radius of the ball used as structuring element, then the offset region is a filled hypersphere, except for the outer Gaussian digitization. Note also that the Gaussian models and the centered models have similar analytical characterizations with different radii. Indeed, we have $D_{o\mathcal{B}_2(\rho)}^+(\mathcal{S}_{\mathbf{c},r}) = D_{\mathcal{B}_2(\rho)}^+(\mathcal{S}_{\mathbf{c},r+\rho})$ and $D_{i\mathcal{B}_2(\rho)}^-(\mathcal{S}_{\mathbf{c},r}) = D_{\mathcal{B}_2(\rho)}^-(\mathcal{S}_{\mathbf{c},r-\rho})$.

The family of hyperspheres, $D_{\mathcal{B}_2(\rho)}^+(\mathcal{S}_{\mathbf{c},r})$, has already been proposed [20] and is known as the Andres hypersphere. It comes with the important property of tiling space (see Fig. 2(b)). Let $(a_i)_{i \in \mathbb{N}}$ be a strictly increasing infinite sequence of positive real values with $a_0 = 0$. The set of intervals $\{[a_i, a_{i+1}[: i \in \mathbb{N}\}$ tiles \mathbb{R}^+ . Let us now consider the sequences $(\rho_i)_{i \in \mathbb{N}^*}$ defined by $\rho_i = (a_i - a_{i-1})/2$ and $(r_i)_{i \in \mathbb{N}^*}$ defined by $r_i = (a_{i-1} + a_i)/2$. Then, the set of digital hyperspheres $\{D_{\mathcal{B}_2(\rho_i)}^+(\mathcal{S}_{\mathbf{c},r_i}) : i \in \mathbb{N}^*\}$ tiles \mathbb{Z}^n . There is a similar result for $D_{\mathcal{B}_2(\rho_i)}^-(\mathcal{S}_{\mathbf{c},r_i})$ except that if \mathbf{c} is an integer point, the set of digital hyperspheres only tiles $\mathbb{Z}^n \setminus \{\mathbf{c}\}$.

An interesting result can be given about the separation of ℓ^2 -digitized hyperspheres, already stated for Andres hyperspheres [20]. Let us consider a ℓ^2 -digitization of an hypersphere $\mathcal{S}_{\mathbf{c},r}$ such that there exists at least one integer point \mathbf{v} of $\mathcal{S}_{\mathbf{c},r}^{-*}$ not in it. The distance from \mathbf{v} to any point \mathbf{x} of $\mathcal{S}_{\mathbf{c},r}^{+*}$ not in the offset region of the hypersphere is at least of 2ρ . We call this bound the *Euclidean thickness* of the digital hypersphere. for any $k \in \{0, \dots, n-1\}$, if the Euclidean thickness is greater than $\sqrt{n-k}$, i.e., $\rho \geq \sqrt{n-k}/2$, it is easy to see that the ℓ^2 -digitized hypersphere is k -separating in \mathbb{Z}^n . The value $\sqrt{n-k}$ corresponds indeed to the maximal distance between two k -adjacent integer points. Once the Euclidean thickness is greater or equal to such a distance, two k -adjacent integer points cannot be on two different sides of such a digital hypersphere anymore. It is however important to notice that the condition of k -separation is sufficient but not necessary.

4. Digital circles and spheres based on the adjacency norms

As seen in the previous section, there is not a strong relationship between the thickness of a Euclidean digital hypersphere and its topological properties. We will now introduce digital circles, spheres and hyperspheres that are k -separating with fewer k -simple points than for the Euclidean digital hyperspheres.

4.1. The adjacency norms

Every digital adjacency relationship can be associated to a norm. This fact is well-known for 0-adjacency and $(n-1)$ -adjacency which are respectively linked to the ℓ^∞ -norm:

$$\mathbf{v} = (v_1, \dots, v_n), \mathbf{w} = (w_1, \dots, w_n) \in \mathbb{Z}^n \text{ are 0-adjacent iff } \|\mathbf{w} - \mathbf{v}\|_\infty = \max_{i \in \{1, \dots, n\}} \{|w_i - v_i|\} = 1,$$

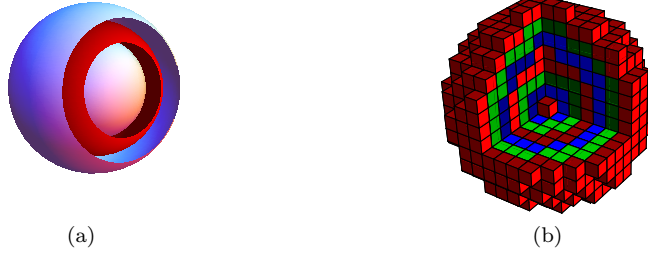


Figure 2: (a) Offset region of a Euclidean digitization of a sphere, (b) an illustration of the space filling property for Andres spheres of center $(0.1, 0.2, 0.4)$, radii $(r + 0.3)_{r \in \mathbb{N}}$ and the ball of radius $\rho = 1/2$ as structuring element.

and to the ℓ^1 -norm:

$$\mathbf{v} = (v_1, \dots, v_n), \mathbf{w} = (w_1, \dots, w_n) \in \mathbb{Z}^n \text{ are } (n-1)\text{-adjacent iff } \|\mathbf{w} - \mathbf{v}\|_1 = \sum_{i=1}^n |w_i - v_i| = 1.$$

We introduce the adjacency norms to extend these results to any digital adjacency.

Definition 1 (Adjacency norms). Let n be the dimension of the space. Let also k be a positive integer lower than n . Then the k -adjacency norm $[\cdot]_k$ is defined as follows:

$$\forall \mathbf{x} \in \mathbb{R}^n, [\mathbf{x}]_k = \max \left\{ \|\mathbf{x}\|_\infty, \frac{\|\mathbf{x}\|_1}{n-k} \right\}.$$

They are norms since they are defined as the maximum of two norms.

Let $\mathcal{B}_{[\cdot]_k}(\rho)$ be the ball of radius ρ under the norm $[\cdot]_k$. The associated distance is denoted by d_k .

It is easy to see that the 0-adjacency norm correspond to the norm ℓ^∞ and the $(n-1)$ -adjacency norm to the norm ℓ^1 . one must be careful here not to confuse the classical ℓ^1 -distance with the 1-adjacency distance d_1 . The classical ℓ^1 -distance corresponds to the adjacency distance d_{n-1} and the ℓ^∞ -distance to the adjacency distance d_0 .

The name *adjacency norms* is justified by the following lemma.

Lemma 1 (digital adjacency and adjacency norms). Let \mathbf{v} and $\mathbf{w} \in \mathbb{Z}^n$. Then, \mathbf{v} and \mathbf{w} are k -adjacent iff $[\mathbf{v} - \mathbf{w}]_k \leq 1$.

PROOF. if \mathbf{v} and \mathbf{w} are k -adjacent, it implies that they are 0-adjacent or, expressed with the norm ℓ^∞ , that $\|\mathbf{v} - \mathbf{w}\|_\infty = 1$. Moreover, to be k -adjacent the two integer points should share at least k identical coordinates, or expressed with the norm ℓ^1 , that $\|\mathbf{v} - \mathbf{w}\|_1 \leq n-k$, which is equivalent to $\|\mathbf{v} - \mathbf{w}\|_1 / (n-k) \leq 1$. Thus, the two k -adjacent integer points satisfy the condition $[\mathbf{v} - \mathbf{w}]_k = 1$.

Now, consider \mathbf{v} and \mathbf{w} such that $[\mathbf{v} - \mathbf{w}]_k = 1$. \mathbf{v} and \mathbf{w} are 0-adjacent. Moreover, $\|\mathbf{v} - \mathbf{w}\|_1 / (n-k) \leq 1$, and the two integer points share at least k equal coordinates. \square

4.2. Topological properties

Since those norms characterize adjacency relationships between integer points, they are also strongly related to the separation of the digital space.

For what follows, we are interested in the minimal (with respect to set inclusion) digital hyperspheres that are k -separating. Intuitively, they should be the ones with a k -adjacency thickness (i.e. the minimal k -adjacency distance between two points not in the offset region of the hypersphere and respectively in $\mathcal{S}_{\mathbf{c},r}^{-,*}$ and in $\mathcal{S}_{\mathbf{c},r}^{+,*}$) equal to 1. As a consequence, we consider only structuring elements based on adjacency norms with radius of 1/2.

For the sake of simplicity, the radius of the adjacency ball is omitted in the model notations so that $D_{\mathcal{B}_{[\cdot]_k}}(\mathcal{S}_{\mathbf{c},r})$ is the closed $[\cdot]_k$ -digitization based on the ball $\mathcal{B}_{[\cdot]_k}(1/2)$ of the hypersphere $\mathcal{S}_{\mathbf{c},r}$. We denote the other $[\cdot]_k$ -digitized hyperspheres with the same convention.

Proposition 2. *The following semi-open $[\cdot]_k$ -digitized hyperspheres are k -separating in \mathbb{Z}^n :*

- $D_{\mathcal{B}_{[\cdot]_k}}^+(\mathcal{S}_{\mathbf{c},r})$ with $r > \frac{\sqrt{n} + \sqrt{n-k}}{2}$,
- $D_{\mathcal{B}_{[\cdot]_k}}^-(\mathcal{S}_{\mathbf{c},r})$ with $r \geq \frac{\sqrt{n} + \sqrt{n-k}}{2}$,
- $D_{i\mathcal{B}_{[\cdot]_k}}^-(\mathcal{S}_{\mathbf{c},r})$ with $r > \frac{\sqrt{n}}{2} + \sqrt{n-k}$,
- $D_{o\mathcal{B}_{[\cdot]_k}}^+(\mathcal{S}_{\mathbf{c},r})$ with $r > \frac{\sqrt{n}}{2}$.

PROOF. The sketch of the proof is the same for all types of $[\cdot]_k$ -digitized hyperspheres : we have to demonstrate that the complement of its offset region, $\bar{\mathcal{O}}$, intersects $\mathcal{S}_{\mathbf{c},r}^{+*}$ and $\mathcal{S}_{\mathbf{c},r}^{-*}$ and that two integer points, one in each of these subsets of the complement, are not k -adjacent. For the sake of clarity, we focus here on only one type of $[\cdot]_k$ -digitized hyperspheres, $D_{\mathcal{B}_{[\cdot]_k}}^+(\mathcal{S}_{\mathbf{c},r})$.

To consider $D_{\mathcal{B}_{[\cdot]_k}}^+(\mathcal{S}_{\mathbf{c},r})$ as a k -separating set, it is necessary that its complement admits two different k -connected components. $\mathcal{S}_{\mathbf{c},r}^{-*} \cap \mathbb{Z}^n$ is a finite set and ensuring that at least one of its elements is not in the digital hypersphere is possible only for sufficiently large radii : the part of $\mathcal{S}_{\mathbf{c},r}^{-*}$ not in the offset region have to include a whole unit hypercube, the minimal subspace containing at least one integer point whatever its position. A unit hypercube, $\mathcal{B}_{[1]_0}(1)$, is included in an Euclidean ball of radius $\sqrt{n}/2$. Moreover, by definition, any point of $\mathcal{S}_{\mathbf{c},r}^{-*}$ in $\mathcal{O}_{\mathcal{B}_{[\cdot]_k}}^+(\mathcal{S}_{\mathbf{c},r})$ is at a k -adjacency distance from $\mathcal{S}_{\mathbf{c},r}$ not greater than $1/2$. In term of Euclidean distance, it corresponds to a distance not greater than $\sqrt{n-k}/2$. As a result, a radius $r > (\sqrt{n-k} + \sqrt{n})/2$ is sufficient to ensure that the complement of $D_{\mathcal{B}_{[\cdot]_k}}^+(\mathcal{S}_{\mathbf{c},r})$ admits integer points in $\mathcal{S}_{\mathbf{c},r}^{+*}$ and also in $\mathcal{S}_{\mathbf{c},r}^{+*}$.

Let us now consider two points $\mathbf{x} \in (\mathcal{S}_{\mathbf{c},r}^+ \setminus \mathcal{O}_{\mathcal{B}_{[\cdot]_k}}^+(\mathcal{S}_{\mathbf{c},r}))$ and $\mathbf{y} \in (\mathcal{S}_{\mathbf{c},r}^- \setminus \mathcal{O}_{\mathcal{B}_{[\cdot]_k}}^+(\mathcal{S}_{\mathbf{c},r}))$. By definition of this offset region and the use of a ball of radius $1/2$ as structuring element, we have $d_k(\mathbf{x}, \mathcal{S}_{\mathbf{c},r}) \geq 1/2$ and $d_k(\mathbf{y}, \mathcal{S}_{\mathbf{c},r}) > 1/2$. Since \mathbf{x} and \mathbf{y} are not on the same side of $\mathcal{S}_{\mathbf{c},r}$, it is easy to see that $[\mathbf{x} - \mathbf{y}]_k > 1$. In the case where \mathbf{x} and \mathbf{y} are integer points, they cannot be k -adjacent. \square

However a $[\cdot]_k$ -digitized hypersphere is not necessarily a strong k -separating set because some k -simple points may still remain. This is actually also the case for classical digital circles such as the Bresenham's circle [25]. The Bresenham's circle of radius 4 is a good example for that.

To conclude on k -separation, notice that $[\cdot]_k$ -digitized hyperspheres are thinner (with fewer simple points) than Andres hyperspheres when they are both k -separating. Indeed an Andres hypersphere is k -separating in \mathbb{Z}^n if $2\rho \geq \sqrt{n-k}$ and a ball of radius greater than $\sqrt{n-k}/2$ under the Euclidean norm contains the ball of radius $1/2$ under the k -adjacency norm.

Another interesting topological result concerns the inner semi-open Gaussian $[\cdot]_k$ -digitized hyperspheres.

Proposition 3. *The digital hypersphere $D_{i\mathcal{B}_{[\cdot]_k}}^-(\mathcal{S}_{\mathbf{c},r})$, is the set of integer points in $\mathcal{S}_{\mathbf{c},r}^-$ k -adjacent to at least one integer point of $\mathcal{S}_{\mathbf{c},r}^{+*}$.*

PROOF. In the case where $\mathcal{S}_{\mathbf{c},r}^- \cap \mathbb{Z}^n = \emptyset$, the proposition is true. We now consider that $\mathcal{S}_{\mathbf{c},r}^-$, and therefore $D_{i\mathcal{B}_{[\cdot]_k}}^-(\mathcal{S}_{\mathbf{c},r})$, contains at least one integer point.

We first show that $s_{\mathbf{c},r}$ assumes a maximum in any k -adjacency ball at one of its vertices. Then, we will show that such a maximum is positive and reached in $\mathcal{S}_{\mathbf{c},r}^{+*} \cap \mathbb{Z}^n$ when considering a k -adjacency ball of radius 1 located at a integer point of $D_{i\mathcal{B}_{[\cdot]_k}}^-(\mathcal{S}_{\mathbf{c},r})$.

For all $\mathbf{x} \in \mathbb{R}^n$ and $\boldsymbol{\varepsilon} \in \mathbb{R}^n$ such that $(\mathbf{x} + \boldsymbol{\varepsilon}) \in (\mathcal{B}_{[\cdot]_k}(\rho) \oplus \{\mathbf{x}\})$, one have:

$$s_{\mathbf{c},r}(\mathbf{x} + \boldsymbol{\varepsilon}) - s_{\mathbf{c},r}(\mathbf{x}) = \sum_{i=1}^n (\varepsilon_i^2 + 2\varepsilon_i(x_i - c_i)).$$

The sign of one component of $\boldsymbol{\varepsilon}$ is not related to the sign of the other ones. We can choose these signs independently. Since we are looking for the maximum value of $s_{\mathbf{c},r}(\mathbf{x} + \boldsymbol{\varepsilon}) - s_{\mathbf{c},r}(\mathbf{x})$, each component of $\boldsymbol{\varepsilon}$ would have the sign of the associated component in $\mathbf{x} - \mathbf{c}$. We can thus, without any loss of generality, rather study the maximum of:

$$\sum_{i=1}^n (\varepsilon_i^2 + 2|\varepsilon_i|(x_i - c_i)).$$

Moreover, $(\mathbf{x} + \boldsymbol{\varepsilon}) \in (\mathcal{B}_{[\cdot]_k}(\rho) \oplus \{\mathbf{x}\})$ induces that $[\boldsymbol{\varepsilon}]_k \leq \rho$. In other words, the sum of the absolute values of the components of $\boldsymbol{\varepsilon}$ is no more than $\rho(n - k)$ and each of these absolute values, taken separately, is no more than ρ . Under such conditions, $s_{\mathbf{c},r}(\mathbf{x} + \boldsymbol{\varepsilon}) - s_{\mathbf{c},r}(\mathbf{x})$ is maximum for a vector $\boldsymbol{\varepsilon}$ having null components except for those associated to the $n - k$ largest, in absolute value, components of $\mathbf{x} - \mathbf{c}$ which are equal to ρ in absolute value. $s_{\mathbf{c},r}$ is then maximum in $\mathcal{B}_{[\cdot]_k}(\rho) \oplus \{\mathbf{x}\}$ at one of its vertices.

By definition, for all $\mathbf{v} \in D_{i\mathcal{B}_{[\cdot]_k}}^-(\mathcal{S}_{\mathbf{c},r})$, there exists $\mathbf{s} \in \mathcal{S}_{\mathbf{c},r}$ such that $[\mathbf{v} - \mathbf{s}]_k < 1$. Thus there exists a Euclidean point $\mathbf{x} \in \mathcal{S}_{\mathbf{c},r}^{+*}$ on the straight line $(\mathbf{v}\mathbf{s})$ such that $[\mathbf{v} - \mathbf{x}]_k = 1$. The maximum of $s_{\mathbf{c},r}$ in $\mathcal{B}_{[\cdot]_k}(1) \oplus \{\mathbf{v}\}$ is then a positive value, necessarily reached at a point in $\mathcal{S}_{\mathbf{c},r}^{+*}$. The vertices of $\mathcal{B}_{[\cdot]_k}(1) \oplus \{\mathbf{v}\}$ being integer points, the maximum is more precisely reached in $\mathcal{S}_{\mathbf{c},r}^{+*} \cap \mathbb{Z}^n$.

Finally, any integer point of $D_{i\mathcal{B}_{[\cdot]_k}}^-(\mathcal{S}_{\mathbf{c},r})$ is k -adjacent to at least one integer point of $\mathcal{S}_{\mathbf{c},r}^{+*} \cap \mathbb{Z}^n$. \square

This result does not apply to the outer digitization $D_{o\mathcal{B}_{[\cdot]_k}}^+(\mathcal{S}_{\mathbf{c},r})$. In general, some integer points of this set are not k -adjacent to any integer point of $\mathcal{S}_{\mathbf{c},r}^{-*}$.

In a 2-dimensional space, $D_{i\mathcal{B}_{[\cdot]_k}}^-(\mathcal{S}_{\mathbf{c},r})$ is known as the circle digitized under Kim scheme [16, 18, 26] and appears in many different recognition algorithms [4, 6, 9, 14, 15, 26].

4.3. Some clues about analytical description of the offset region

Before giving analytical characterizations of the digital circles and spheres based on adjacency norms, let us explain in somewhat informal way how an offset region can be described by inequalities in the case of a convex polytope as ball.

Offset region of semi-open and Gaussian models are defined by the intersection of two dilations. It is also the case for the offset region of the closed model:

$$\mathcal{O}_{\mathcal{B}_{[\cdot]_k}}(\mathcal{S}_{\mathbf{c},r}) = (\mathcal{S}_{\mathbf{c},r}^- \oplus \mathcal{B}_{[\cdot]_k}) \cap (\mathcal{S}_{\mathbf{c},r}^+ \oplus \mathcal{B}_{[\cdot]_k}).$$

Such a decomposition is interesting since each set in the intersection has only one boundary : the boundary of the first set is the outer boundary of the offset region and the boundary of the second set is the inner boundary of the offset region. Moreover, both parts can be related to already studied objects. For the first dilation, we have:

$$\mathcal{S}_{\mathbf{c},r}^- \oplus \mathcal{B}_{[\cdot]_k} = \mathcal{B}_2(r) \oplus (\{\mathbf{c}\} \oplus \mathcal{B}_{[\cdot]_k})$$

Such objects are known as *offset of a polygon (or polyhedron) by a radius* [27, 28]. One can sum up their properties in 2-dimensional and 3-dimensional spaces by the following two lemma:

Proposition 4 (Offsetting of a polygon). *The offsetting by a radius r of a polygon \mathcal{P} with set of edges $E(\mathcal{P})$ and vertices $V(\mathcal{P})$ is the union of:*

- the polygon \mathcal{P} ,
- for each edge $e \in E(\mathcal{P})$, the extrusion of \mathcal{P} between \mathbf{o} and $r\mathbf{n}(e)$, where $\mathbf{n}(e)$ is the outward-pointing unit normal vector to e ,

- for each vertex $\mathbf{v} \in V(\mathcal{P})$, the Euclidean ball of radius r , $\mathcal{B}_2(r)$, centered at \mathbf{v} .

Proposition 5 (Offsetting of a polyhedron). *The offsetting by a radius r of a polyhedron \mathcal{P} with set of faces $F(\mathcal{P})$, of edges $E(\mathcal{P})$ and vertices $V(\mathcal{P})$ is the union of:*

- the polyhedron \mathcal{P} ,
- for each face $f \in F(\mathcal{P})$, the extrusion of \mathcal{P} between \mathbf{o} and $r\mathbf{n}(f)$, where $\mathbf{n}(f)$ is the outward-pointing unit normal vector to f ,
- for each edge $e \in E(\mathcal{P})$, the filled right circular cylinder of radius r based on the segment $[\mathbf{v}_1\mathbf{v}_2]$ where \mathbf{v}_1 and \mathbf{v}_2 are the extremities of e ,
- for each vertex $\mathbf{v} \in V(\mathcal{P})$, the Euclidean ball of radius r , $\mathcal{B}_2(r)$, centered at \mathbf{v} .

Note that in the case of adjacency balls, each extrusion of \mathcal{P} associated to an edge e of the 2-dimensional ball (respectively face f of the 3-dimensional ball), can be replaced only by a translated copy of the adjacency ball by $r\mathbf{n}(e)$ (respectively $r\mathbf{n}(f)$). Such a translated copy is indeed sufficient to cover the interior of the offset not already covered by disks in the 2-dimensional case (respectively cylinders in the 3-dimensional case).

For the second dilation, we have:

$$\mathcal{S}_{\mathbf{c},r}^+ \oplus \mathcal{B}_{[\cdot]_k} = \left\{ \mathbf{x} \in \mathbb{R}^n : \max_{\mathbf{y} \in (\mathcal{B}_{[\cdot]_k} \oplus \{\mathbf{c}\})} \{\|\mathbf{x} - \mathbf{y}\|_2\} \geq r \right\}$$

The maximum distance from a point \mathbf{x} to a convex polytope \mathcal{P} (in any dimension) is the maximum distance from \mathbf{x} to the set of vertices of \mathcal{P} [29]. Thus, $\mathcal{S}_{\mathbf{c},r}^+ \oplus \mathcal{B}_{[\cdot]_k}$ can be seen as the union of the sets of points at a Euclidean distance greater or equal to r of one of the vertices of the adjacency ball centered at \mathbf{c} .

$$\mathcal{S}_{\mathbf{c},r}^+ \oplus \mathcal{B}_{[\cdot]_k} = \bigcup_{\mathbf{v} \in V(\mathcal{B}_{[\cdot]_k})} (\overline{\mathcal{B}_2^*(r)} \oplus \mathbf{v}) \oplus \mathbf{c}.$$

Fig. 3(a) shows the offset region construction in 2D on one quadrant and Fig. 3(b) shows the complete offset region of a closed \mathcal{L}_∞ -digitized circle. The structuring element is an axis-oriented square. In Fig. 6 we can see the offset zones for the three closed adjacency norm based digital spheres.

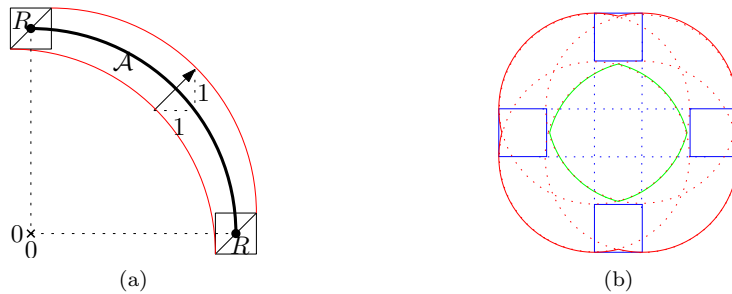


Figure 3: Construction of the offset region of a quadrant (a) and the whole closed \mathcal{L}_∞ -circle (b).

4.4. Digital circles and spheres based on the 0-adjacency norm

The 0-adjacency norm corresponds to the usual \mathcal{L}_∞ -norm. The 0-adjacency ball is an axis aligned hypercube of side 1. Geometrically, in the 2-dimensional space, it is composed of 4 edges and 4 points, and in the 3-dimensional space, of 6 faces, 12 edges and 8 vertices.

The closed digitization model based on this norm is known as the *supercover* digitization model. The supercover model has been extensively studied [21]. Linear objects can be described analytically in this model [30]. We will show that circles and spheres can also be analytically described in this model.

The analytical characterizations of digital circles based on the 0-adjacency norm are given by the following proposition.

Proposition 6 (Analytical characterization of the $[\cdot]_0$ -digitized circles). *The analytical characterizations of the $[\cdot]_0$ -digitizations of a circle $S_{\mathbf{c},r}$ are given by:*

$$\begin{aligned} D_{\mathcal{B}_{[\cdot]_0}}(S_{\mathbf{c},r}) &= \left\{ \mathbf{v} \in \mathbb{Z}^2 : \left(\bigvee_{i=1}^2 \left(\bigvee_{j=1}^2 [(\mathbf{v} - \mathbf{c}) \pm r\mathbf{e}_i]_0 \leq 1/2 \right) \right) \wedge (s_{\mathbf{c},r}(\mathbf{v}) \geq -[\mathbf{v} - \mathbf{c}]_1 - 1/2) \right\}, \\ D_{\mathcal{B}_{[\cdot]_0}^+}(S_{\mathbf{c},r}) &= \left\{ \mathbf{v} \in \mathbb{Z}^2 : \left(\bigvee_{i=1}^2 \left(\bigvee_{j=1}^2 [(\mathbf{v} - \mathbf{c}) \pm r\mathbf{e}_i]_0 < 1/2 \right) \right) \wedge (s_{\mathbf{c},r}(\mathbf{v}) \geq -[\mathbf{v} - \mathbf{c}]_1 - 1/2) \right\}, \\ D_{\mathcal{B}_{[\cdot]_0}^-}(S_{\mathbf{c},r}) &= \left\{ \mathbf{v} \in \mathbb{Z}^2 : \left(\bigvee_{i=1}^2 \left(\bigvee_{j=1}^2 [(\mathbf{v} - \mathbf{c}) \pm r\mathbf{e}_i]_0 \leq 1/2 \right) \right) \wedge (s_{\mathbf{c},r}(\mathbf{v}) > -[\mathbf{v} - \mathbf{c}]_1 - 1/2) \right\}, \\ D_{o\mathcal{B}_{[\cdot]_0}}(S_{\mathbf{c},r}) &= \left\{ \mathbf{v} \in \mathbb{Z}^2 : \left(\bigvee_{i=1}^2 \left(\bigvee_{j=1}^2 [(\mathbf{v} - \mathbf{c}) \pm r\mathbf{e}_i]_0 \leq 1 \right) \right) \wedge (s_{\mathbf{c},r}(\mathbf{v}) \geq 0) \right\}, \\ D_{i\mathcal{B}_{[\cdot]_0}}(S_{\mathbf{c},r}) &= \{ \mathbf{v} \in \mathbb{Z}^2 : (0 \geq s_{\mathbf{c},r}(\mathbf{v}) \geq -2[\mathbf{v} - \mathbf{c}]_1 - 2) \}. \end{aligned}$$

Fig. 3(b) shows the offset region of a closed $[\cdot]_0$ -digitized circle (or supercover circle) and Fig. 6(a) shows the offset region for a closed $[\cdot]_0$ -digitized sphere (or supercover sphere).

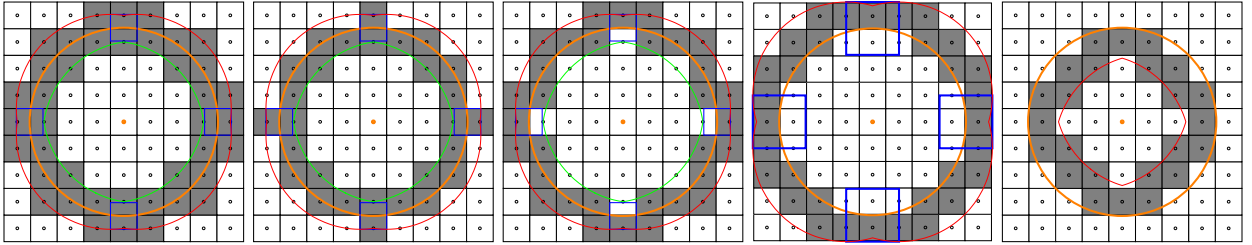


Figure 4: The digitizations $D_{\mathcal{B}_{[\cdot]_0}}(S_{\mathbf{c},r})$, $D_{\mathcal{B}_{[\cdot]_0}^+}(S_{\mathbf{c},r})$, $D_{\mathcal{B}_{[\cdot]_0}^-}(S_{\mathbf{c},r})$, $D_{o\mathcal{B}_{[\cdot]_0}}(S_{\mathbf{c},r})$, $D_{i\mathcal{B}_{[\cdot]_0}}(S_{\mathbf{c},r})$ of a circle $S_{\mathbf{c},r}$ of center $\mathbf{c} = (0, 0)$ and radius $r = \sqrt{10}$.

Let us just recall that, with the adjacency norm notations, we have $[\mathbf{v} - \mathbf{c}]_1 = |v_1 - c_1| + |v_2 - c_2|$ and, for instance, $[\mathbf{v} - (\mathbf{c} + r\mathbf{e}_1)]_0 = \max(|v_1 - c_1 - r|, |v_2 - c_2|)$.

The analytical description of a supercover circle is composed of 4 spheres of radius r (corresponding to, and centered at, each of the 4 vertices of the 0-adjacency ball $\mathcal{B}_{[\cdot]_0}$ centered at c) and 4 copies of $\mathcal{B}_{[\cdot]_0}$ centered at each of the cardinal points of the circle $S_{\mathbf{c},r}$ (corresponding to the 4 edges of $\mathcal{B}_{[\cdot]_0}$). To check if an integer point belongs to such a digital circle requires for the worst case 6 tests and only 2 for the best one.

PROOF. We just consider the case of $D_{\mathcal{B}_{[\cdot]_0}}(S_{\mathbf{c},r})$. The offset region of $D_{\mathcal{B}_{[\cdot]_0}}(S_{\mathbf{c},r})$ can be regarded as the intersection between the offsetting of the convex polygonal ball $\mathcal{B}_{[\cdot]_0}$ by the radius r (which define the outer

boundary of the offset region) and the set of points at a minimum distance of r from $\mathcal{B}_{[\cdot]_0}$ (which define the inner boundary).

The offsetting of a polygon by a radius can be decomposed in the contribution of its vertices (disks) and the contribution of its edge (translated copy of itself). The contribution of one vertex $\mathbf{v} \in V$ is the disk of center $\mathbf{c} + \mathbf{v}$ and radius r . We can describe it as the set $\{\mathbf{x} \in \mathbb{R}^2 : s_{\mathbf{c}+\mathbf{v},r}(\mathbf{x}) \leq 0\}$, or, expressed with the map $s_{\mathbf{c},r}$, as the set $\{\mathbf{x} \in \mathbb{R}^2 : s_{\mathbf{c},r}(\mathbf{x}) \leq \sum_{i=1}^2 (2(x_i - c_i)v_i - v_i^2)\}$. The maximum of $(2x_i v_i - v_i^2)$ is reached when x_i and v_i have the same sign. Moreover, for all $i \in \{1, 2\}$, we have $v_i \in \{-1/2, 1/2\}$. Thus, applying appropriate symmetries, the contribution of V to the offset region is the set $\{\mathbf{x} \in \mathbb{R}^2 : s_{\mathbf{c},r}(\mathbf{x}) \leq \sum_{i=1}^2 (|x_i - c_i| - 1/4)\}$. The contribution of one edge to the offset region is $\{\mathbf{x} \in \mathbb{R}^2 : [\mathbf{x} - (\mathbf{c} + r\mathbf{n}(e))]_k \leq 1/2\}$. The edges are axis-aligned, each admits as outward-pointing unit normal vector $\mathbf{n}(e)$, one of the vectors $\pm \mathbf{e}_1$ or $\pm \mathbf{e}_2$. Consequently the contribution of E to the offset region is $\{\mathbf{x} \in \mathbb{R}^2 : \bigvee_{i=1}^2 [(\mathbf{x} - \mathbf{c}) \pm r\mathbf{e}_i]_0 \leq 1/2\}$.

The same reasoning as the one for the contribution of vertices to the offsetting of $\mathcal{B}_{[\cdot]_0}$ by the radius r can be applied to obtain the analytical characterization of the set of points at a minimum distance of r from $\mathcal{B}_{[\cdot]_0}$. \square

Note that Lincke proposed another interpretation of this result based on mathematical morphology operations [31]. Note in addition that Nakamura and Aizawa, based on a cellular scheme, defined a digital disk [16] that is actually a supercover disk. The outer border of their digital disk is thus also the outer border of a supercover circle.

Let us now consider the dimension three. The analytical characterizations of a digital sphere based on the 0-adjacency norm is given by the following proposition:

Proposition 7 (Analytical characterization of a supercover sphere). *The analytical description of a closed centered $[\cdot]_0$ -digitization of a sphere $\mathcal{S}_{\mathbf{c},r}$, $D_{\mathcal{B}_{[\cdot]_0}}(\mathcal{S}_{\mathbf{c},r})$, is:*

$$\left\{ \mathbf{v} \in \mathbb{Z}^3 : \left(s_{\mathbf{c},r}(\mathbf{v}) \leq [\boldsymbol{\alpha}]_2 - \frac{3}{4} \quad \vee \quad \left(\bigvee_{j=1}^3 \left([\boldsymbol{\alpha} \pm r\mathbf{e}_j]_0 \leq \frac{1}{2} \right) \right) \right) \wedge \left(s_{\mathbf{c},r}(\mathbf{v}) \geq -[\boldsymbol{\alpha}]_2 - \frac{3}{4} \right) \right\},$$

$$\vee \left(\bigvee_{j=1}^3 \left(\sum_{\substack{i=1, \\ i \neq j}}^3 \alpha_i^2 - r^2 \leq \sum_{\substack{i=1, \\ i \neq j}}^3 |\alpha_i| - \frac{1}{2} \right) \wedge \left(|\alpha_j| \leq \frac{1}{2} \right) \right)$$

with $\boldsymbol{\alpha} = \mathbf{v} - \mathbf{c}$.

The analytical description of a supercover sphere is composed of 8 spheres of radius r (corresponding to, and centered at, each of the 8 vertices of the 0-adjacency ball $\mathcal{B}_{[\cdot]_0}$ centered at c), 12 cylinders of radius r and width 1 (corresponding to, and having as axis, each of the 12 edges of $\mathcal{B}_{[\cdot]_0}$ centered at c) and 6 copies of $\mathcal{B}_{[\cdot]_0}$ centered at each of the cardinal points of the sphere $\mathcal{S}_{\mathbf{c},r}$ (corresponding to the 6 faces of $\mathcal{B}_{[\cdot]_0}$). To check if an integer point belongs to the digital sphere requires at worst 14 tests.

PROOF. The inner boundary ($s_{\mathbf{c},r}(\mathbf{v}) \geq -[\boldsymbol{\alpha}]_2 - 3/4$) and the contribution of vertices ($s_{\mathbf{c},r}(\mathbf{v}) \leq [\boldsymbol{\alpha}]_2 - 3/4$) and faces ($\bigvee_{j=1}^3 ([\boldsymbol{\alpha} \pm r\mathbf{e}_j]_0 \leq 1/2)$) to the outer boundary can be easily deduced from the 2-dimensional case introduced above.

The edges of the 0-adjacency ball, $\mathcal{B}_{[\cdot]_0}$, are directed either by \mathbf{e}_1 or \mathbf{e}_2 or \mathbf{e}_3 . Without any loss of generality, let us focus only on the four edges directed by \mathbf{e}_3 . All have one of their extremities in the plane $\{\mathbf{x} \in \mathbb{R}^3 : x_3 = -1/2\}$ and the other in the plane $\{\mathbf{x} \in \mathbb{R}^3 : x_3 = 1/2\}$. Moreover, each of these edges contain a point of $P = \{-1/2, 1/2\} \times \{-1/2, 1/2\} \times \{0\}$. Their contribution is thus the union of the cylinders of radius r , directed by \mathbf{e}_3 , restricted to the thick plane $\{\mathbf{x} \in \mathbb{R}^3 : |x_3| \leq 1/2\}$ and translated to a point of P . The equation of an infinite filled right circular cylinder of radius r and directed by \mathbf{e}_3 at \mathbf{o} is $\{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 - r^2 \leq 0\}$. With the same argument as the one used for the contribution of vertices in the 2-dimensional case, we obtained the following contribution for edges directed by

\mathbf{e}_3 : $\{\mathbf{x} \in \mathbb{R}^3 : ((x_1 - c_1)^2 + (x_2 - c_2)^2 - r^2 \leq |x_1 - c_1| + |x_2 - c_2| - 1/2) \wedge (|x_3 - c_3| \leq 1/2)\}$ and then the general analytical characterization of the contribution of all edges of $\mathcal{B}_{[\cdot]_0}$. \square

In order to save space, we do not present here all the formulas for the semi-open and Gaussian $[\cdot]_0$ -digitized spheres. With the help of the proof of proposition 6, the reader should not have any difficulties to get the corresponding analytical characterizations.

4.5. Digital circles and spheres based on the $(n-1)$ -adjacency norm

The $(n-1)$ -adjacency norm $[\cdot]_{(n-1)}$ corresponds to the usual norm \mathcal{L}_1 . The $(n-1)$ -adjacency ball is the dual polytope of the unit hypercube : the cross-polytope. The closed digitization model based on this norm is known as the closed naive digitization model [22].

The analytical characterizations of digital circles based on the 1-adjacency norm are given by the following proposition:

Proposition 8 (Analytical characterization of the closed centered $[\cdot]_1$ -digitized circle). *The analytical characterization of the closed centered $[\cdot]_1$ -digitization of the circle $\mathcal{S}_{\mathbf{c},r}$ is defined by:*

$$D_{\mathcal{B}_{[\cdot]_1}}(\mathcal{S}_{\mathbf{c},r}) = \left\{ \mathbf{v} \in \mathbb{Z}^2 : \left(\bigvee_{\mathbf{t} \in \{-1,1\}^2} \left[(\mathbf{v} - \mathbf{c}) + \frac{\sqrt{2}}{2} r \mathbf{t} \right]_1 \leq \frac{1}{2} \right) \wedge \left(s_{\mathbf{c},r}(\mathbf{v}) \geq -[\mathbf{v} - \mathbf{c}]_0 - \frac{1}{4} \right) \right\}.$$

The analytical description of a closed naive circle is composed of 4 spheres of radius r (corresponding to, and centered at, each of the 4 vertices of the 1-adjacency ball $\mathcal{B}_{[\cdot]_1}$ centered at \mathbf{c}) and 4 copies of $\mathcal{B}_{[\cdot]_1}$ centered at the intersection of the circle $\mathcal{S}_{\mathbf{c},r}$ and the lines through \mathbf{c} directed by a vector in $\{-1,1\} \times \{1\}$ (corresponding to the 4 edges of $\mathcal{B}_{[\cdot]_1}$). To check if an integer point belongs to such a digital circle requires at worst 6 tests.

PROOF. The set of vertices of the 1-adjacency ball, $\mathcal{B}_{[\cdot]_{B_k}}$, is the set $V = \{(0, 1/2); (1/2, 0); (-1/2, 0), (0, -1/2)\}$. Their contribution to inner and outer boundaries of $D_{\mathcal{B}_{[\cdot]_1}}(\mathcal{S}_{\mathbf{c},r})$ is deduced with the same argument as in the case of $D_{\mathcal{B}_{[\cdot]_0}}(\mathcal{S}_{\mathbf{c},r})$. For all edge e , $\mathbf{n}(e) \in \{-\sqrt{2}/2, \sqrt{2}/2\}^2$. Thus each edge induces a copy of $\mathcal{B}_{[\cdot]_{B_k}}$, translated by a vector $r\mathbf{n}(e)$ from the center \mathbf{c} of the circle $\mathcal{S}_{\mathbf{c},r}$, $\{\mathbf{x} \in \mathbb{R}^2 : [(\mathbf{x} - \mathbf{c}) + r\mathbf{n}(e)]_1 \leq 1/2\}$. \square

And let us now examine the analytical characterization of the closed centered $[\cdot]_2$ -digitized sphere:

Proposition 9 (Analytical characterization of the closed centered $[\cdot]_2$ -digitized sphere). *The analytical characterization of the closed centered $[\cdot]_2$ -digitization of the sphere $\mathcal{S}_{\mathbf{c},r}$ is defined by:*

$$D_{\mathcal{B}_{[\cdot]_2}}(\mathcal{S}_{\mathbf{c},r}) = \left\{ \mathbf{v} \in \mathbb{Z}^3 : \left(\bigvee_{\substack{\mathbf{v}(i,j,k) \in \Pi \\ \mathbf{t} \in \{-1,1\}}} \left((v_i - c_i)^2 + \frac{1}{2} \left(|(v_j - c_j) + t(v_k - c_k)| - \frac{1}{2} \right)^2 \leq r^2 \right) \right) \wedge \left(s_{\mathbf{c},r}(\mathbf{v}) \geq -[\mathbf{v} - \mathbf{c}]_0 - \frac{1}{4} \right) \right\}.$$

where Π is the set of circular shifts of $(1, 2, 3)$.

The analytical description of a closed naive sphere is composed of 6 spheres of radius r (corresponding to, and centered at, each of the 6 vertices of the 2-adjacency ball $\mathcal{B}_{[\cdot]_2}$ centered at c), 12 cylinders of radius r and width $\sqrt{2}/2$ (corresponding to, and having as axis, each of the 12 edges of $\mathcal{B}_{[\cdot]_2}$ centered at c) and 8 copies of $\mathcal{B}_{[\cdot]_2}$ centered at the intersection between the sphere $\mathcal{S}_{c,r}$ and the lines through c and directed by a vector in $\{-1, 1\}^3$ (corresponding to the 8 faces of $\mathcal{B}_{[\cdot]_2}$). To check if an integer point belongs to the digital sphere requires at worst 22 tests.

PROOF. The structuring element for the adjacency norm $[\cdot]_2$ is an octahedron whose vertices correspond to the center of the faces of a unit cube. The analytical description of a closed centered $[\cdot]_2$ -sphere is composed of 6 spheres of radius r (corresponding to, and centered at, each of the 6 vertices of an octahedron centered at c), 12 cylinders of radius r and width 1 (corresponding to, and having as axis, each of the 12 edges of an octahedron) and 8 structuring elements positioned at a distance r from c orthogonally to the faces of an octahedron (corresponding to the 8 faces of an octahedron). The last line of the analytical characterization of the closed centered $[\cdot]_2$ -sphere corresponds to the inner boundary of the offset region while the other lines correspond to the outer boundary. The first and last equation lines are obtained in the same way as for the supercover sphere and $[\cdot]_1$ -circle. The last line of the outer boundary description corresponds to the faces of the structuring element, which is an octahedron, translated by a vector $(r \pm \frac{\sqrt{3}}{3}, r \pm \frac{\sqrt{3}}{3}, r \pm \frac{\sqrt{3}}{3})$. This corresponds to the faces of the structuring element translated orthogonally at a Euclidean distance of r . In order to obtain this face at the good spot, we simply describe the equations of a complete structuring element at these spots.

The cylinders are obtained by developing the formulas describing a cylinder of radius r for each edge of the structuring element centered at c . For instance, the cylinder defined by $\frac{1}{2}((v_2 - c_2) + (v_3 - c_3) - \frac{1}{2})^2 + (v_1 - c_1)^2 \leq r^2$ $\wedge |(v_2 - c_2) - (v_3 - c_3)| \leq \frac{1}{2}$ corresponds to a cylinder of radius r and of axis the edge $(c_1, c_2 + \frac{1}{2}, c_3) - (c_1, c_2, c_3 + \frac{1}{2})$. The planes perpendicular to the edge correspond to $|(v_2 - c_2) - (v_3 - c_3)| \leq \frac{1}{2}$. The equation of the cylinder is simply obtained as the points that are at a maximal Euclidean distance of r from the edge. By doing this for all the edges we obtain the given equations. \square

Based on what has been presented in the proof of proposition 6, the reader should not have any difficulties to get the analytical characterizations of the other types of $[\cdot]_1$ -circles and $[\cdot]_2$ -spheres.

Note that the Bresenham's circle is by construction a 0-connected and 1-separating circle with integer radii and integer coordinate centers. It is actually a particular case of the circles introduced in this section:

Proposition 10 (Bresenham's circle). *Let $\mathcal{S}_{c,r}$ be a circle (2-dimensional hypersphere) with center $c \in \mathbb{Z}^2$ and radius $r \in \mathbb{N}^*$. Then, the Bresenham's circle of center c and radius r is the same set as $D_{\mathcal{B}_{[\cdot]_1}}(\mathcal{S}_{c,r})$, $D_{\mathcal{B}_{[\cdot]_1}}^+(\mathcal{S}_{c,r})$ or $D_{\mathcal{B}_{[\cdot]_1}}^-(\mathcal{S}_{c,r})$.*

PROOF. A Bresenham's circle is, as we mentioned, due to its algorithmic construction, a 0-connected and 1-separating digital circle. Its points are the closest ones to $\mathcal{S}_{c,r}$ [32]. As such it corresponds to one of the closed or semi-open $[\cdot]_1$ -digitizations. Moreover, no point $(x \pm \frac{1}{2}, y)$ or $(x, y \pm \frac{1}{2})$, with $(x, y) \in \mathbb{Z}^2$ belong to a Euclidean circle that has an integer coordinate center and an integer radius. This means that no integer coordinate point lies on the inner or outer boundary of the offset region $\mathcal{S}_{c,r} \oplus \mathcal{B}_{[\cdot]_1}(\frac{1}{2})$, which is removed by choosing either the inner or outer semi-open digitization model. \square

Nevertheless, the extension of Bresenham's circles to non-integer parameters, namely Pham's circle does not fit into the digital circles based on adjacency norms. In fact, Pham's circle corresponds to a flake based circle as we will see in section 4.

4.6. Digital spheres based on the 1-adjacency norm

In a 3-dimensional space, we have not examined the 1-adjacency norm yet. There is no corresponding digital circle. This leads to 1-separating digital spheres.

Proposition 11 (Analytical characterization of the closed centered $[\cdot]_1$ -digitized sphere). *The analytical characterization of the closed centered digitization based on the 1-adjacency norm of the sphere $\mathcal{S}_{\mathbf{c},r}$ is defined by:*

$$D_{\mathcal{B}_{[\cdot]_2}}(\mathcal{S}_{\mathbf{c},r}) = \left\{ \mathbf{v} \in \mathbb{Z}^3 : \left(\begin{array}{l} s_{\mathbf{c},r}(\mathbf{v}) \leq \sum_{i=1}^3 |v_i - c_i| - \min_{1 \leq i \leq 3} \{ |v_i - c_i| \} - \frac{1}{2} \\ \vee \left(\bigvee_{\substack{\forall (i,j,k) \in \Pi \\ t \in \{-1,1\}}} \left(\left(|v_i - c_i| - \frac{1}{2} \right)^2 + \frac{1}{2} \left(|(v_j - c_j) + t(v_k - c_k)| - \frac{1}{2} \right)^2 \leq r^2 \right) \right) \\ \wedge |v_j - c_j| - t(v_k - c_k) \leq \frac{1}{2} \\ \vee \left(\bigvee_{\mathbf{t} \in \{-1,1\}^3} \left(\left[\mathbf{v} - \left(\mathbf{c} + \frac{r}{\sqrt{3}} \mathbf{t} \right) \right]_1 \leq \frac{1}{2} \right) \right) \vee \left(\bigvee_{i=1}^3 \left([\mathbf{v} - (\mathbf{c} \pm r \mathbf{e}_i)]_1 \leq \frac{1}{2} \right) \right) \right) \\ \wedge \left(s_{\mathbf{c},r}(\mathbf{v}) \geq - \sum_{i=1}^3 |v_i - c_i| + \min_{1 \leq i \leq 3} \{ |v_i - c_i| \} - \frac{1}{2} \right) \end{array} \right\}.$$

where Π is the set of circular shifts of $(1, 2, 3)$.

The analytical description of a closed centered $[\cdot]_1$ -digitized sphere is composed of 12 spheres of radius r (corresponding to, and centered at, each of the 12 vertices of the 1-adjacency ball $\mathcal{B}_{[\cdot]_1}$ centered at \mathbf{c}), 24 cylinders of radius r and width $\sqrt{2}/2$ (corresponding to, and having as axis, each of the 24 edges of $\mathcal{B}_{[\cdot]_1}$ centered at \mathbf{c}) and 10 copies of $\mathcal{B}_{[\cdot]_1}$ centered at the intersections between the sphere $\mathcal{S}_{\mathbf{c},r}$ and the lines through \mathbf{c} directed by a vector $\mathbf{v} = \pm \mathbf{e}_i$ (for all $i \in \{1, 2, 3\}$) or $\mathbf{v} \in \{-1, 1\}^2 \times \{1\}$ (corresponding to the 14 faces of $\mathcal{B}_{[\cdot]_1}$). To check if an integer point belongs to the digital sphere requires 28 tests for the worst case and only 2 for the best one.

PROOF. The proof for the 1-adjacency norm sphere is similar to the proofs for the closed centered $[\cdot]_0$ -digitized spheres and $[\cdot]_2$ -digitized spheres. The last line in the equations corresponds to the inner boundary while the other lines describe the outer boundary. The first line of the outer boundary corresponds to the spheres of radius r centered at each of the vertices of the structuring element translated to \mathbf{c} .

The spherical parts in the first line come from developing equations like $(v_1 - c_1 + \frac{1}{2})^2 + (v_2 - c_2 + \frac{1}{2})^2 + (v_3 - c_3)^2 \leq r^2$ for all the 14 vertices of the structuring element centered at \mathbf{c} and applying appropriate symmetries. The final formula, complicated as it seems, is actually very similar to the one corresponding to the 1-separating hyperplanes [2]. It is a consequence of lemma 1. The cylinders are obtained by developing the formulas describing a cylinder of radius r for each edge of the structuring element centered at \mathbf{c} . For example, the edge $(c_1 + 1/2, c_2, c_3 + 1/2) - (c_1, c_2 + 1/2, c_3 + 1/2)$ corresponds to the cylinder $\left\{ \mathbf{x} \in \mathbb{R}^3 : (x_3 - 1/2)^2 + 1/2(x_1 - c_1 + x_2 - c_2 - 1/2)^2 \leq r^2 \right\} \cap \left\{ \mathbf{x} \in \mathbb{R}^3 : |x_1 - c_1 - x_2 + c_2 - 1/2| \leq 1/2 \right\}$. Appropriate symmetries allows to simplify the final expression to the one in the proposition. Finally, there are 14 faces. From those faces, 6 have as normal vector $\mathbf{n}(f) = \pm \mathbf{e}_i$ (for all $i \in \{1, 2, 3\}$) and the 8 other have normal vector $\mathbf{n}(f) \in \{-\sqrt{2}/2, \sqrt{2}/2\}^3$ which explains the analytical expression in the last line of the outer boundary description. \square

We have provided an analytical characterization of 0-separating spheres in section 4.4 (Fig. 5(a)), an analytical characterization of 2-separating spheres in section 4.5 (Fig. 5(c)) and an analytical characterization of a 1-separating sphere in this section (Fig. 5(b)).

5. Digital hyperspheres based on adjacency flakes

In the previous section, the offset region was based on structuring elements that correspond to balls based on norms. We showed that we could define k -separating digital hyperspheres this way. Those hyperspheres have, however, simple points. In the present section, we propose a new type of structuring elements that

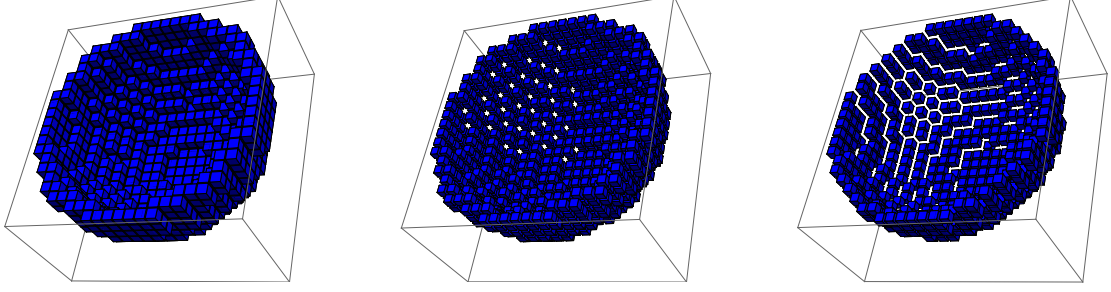


Figure 5: $\mathcal{B}_{[\cdot]_0^-}$, $\mathcal{B}_{[\cdot]_1^-}$ and $\mathcal{B}_{[\cdot]_2^-}$ -digitized spheres of radius $r = 10$ and center $\mathbf{c} = (0, 0, 0)$. They are respectively 0-, 1- and 2-separating in \mathbb{Z}^n .

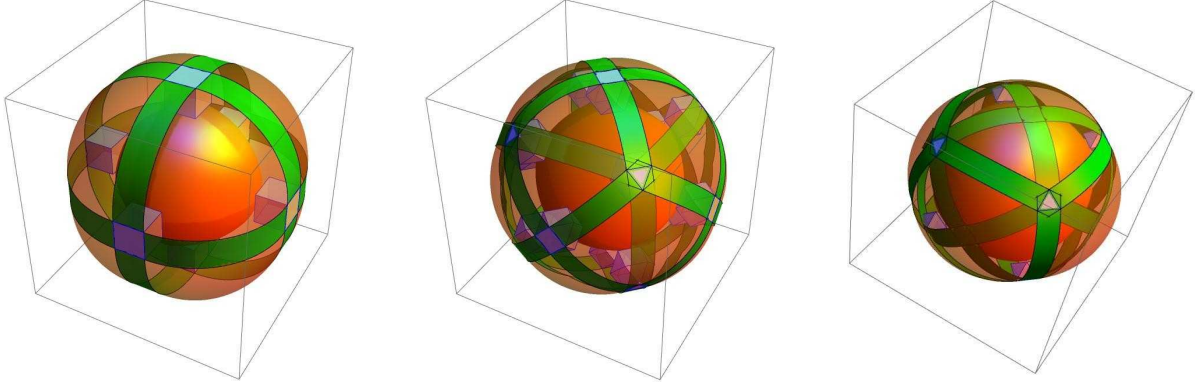


Figure 6: Offset regions for $\mathcal{B}_{[\cdot]_0^-}$, $\mathcal{B}_{[\cdot]_1^-}$ and $\mathcal{B}_{[\cdot]_2^-}$ -digitized spheres of radius 3. They are respectively 0-, 1- and 2-separating in \mathbb{Z}^n .

preserves the k -separation property with fewer simple points. In some cases, strict k -separation can even be achieved for digital hyperspheres.

The new structuring elements we introduce are derived from the k -adjacency balls and we call them *k-adjacency flakes* or simply *adjacency flakes*. Such a set is the intersection of a ball based on an adjacency norm and a finite number of straight lines through the origin.

Definition 2. The *closed k-adjacency flake*, $F_k(\rho)$, based on the k -adjacency norm, $[\cdot]_k$, and with radius $\rho \in \mathbb{R}^+$ is defined by:

$$F_k(\rho) = \mathcal{B}_{[\cdot]_k}(\rho) \cap \left\{ \mathbf{x} \in \{-\alpha, 0, \alpha\}^n : \alpha \in \mathbb{R}^+, \sum_{i=1}^n |x_i| \leq (n-k)\alpha \right\}.$$

The *open k-adjacency flake*, $F_k^*(\rho)$, follows the same definition with an open ball $\mathcal{B}_{[\cdot]_k}^*(\rho)$ instead of the closed one $\mathcal{B}_{[\cdot]_k}(\rho)$.

Fig. 7 shows the different adjacency flakes in 2- and 3-dimensional spaces. In what follows, the structuring elements of the general model \mathcal{A} and \mathcal{A}^* will be respectively a closed flake $F_k(\rho)$ and an open flake $F_k^*(\rho)$ with same associated k -adjacency norm and radius. Note that, since for all $\mathbf{x} \in F_k(\rho)$, $-\mathbf{x}$ is also a point of $F_k(\rho)$, we have the property that for a given $\mathbf{x} \in \mathbb{R}^n$, for all $\mathbf{y} \in (\{\mathbf{x}\} \oplus F_k(\rho))$, $\mathbf{x} \in (\{\mathbf{y}\} \oplus F_k(\rho))$.

For the rest of this section, we assume that $\rho = 1/2$. For the sake of simplicity, the radius is omitted in model notations so that $D_{F_k}^+(\mathcal{S}_{\mathbf{c},r})$, $D_{F_k}^-(\mathcal{S}_{\mathbf{c},r})$, $D_{i_{F_k}}^-(\mathcal{S}_{\mathbf{c},r})$ and $D_{o_{F_k}}^+(\mathcal{S}_{\mathbf{c},r})$, all refer to digitizations of a hypersphere $\mathcal{S}_{\mathbf{c},r}$ based on a k -adjacency flake of radius $1/2$. A F_k -digitized hypersphere under a given model

is by definition included in the $\mathcal{B}_{[\cdot]_k}$ -digitized hypersphere under the same model. Nevertheless, a structuring element F_k is enough to ensure the k -separation property for the four semi-open F_k -digitized hyperspheres.

Proposition 12. *The following digital hyperspheres are k -separating in \mathbb{Z}^n :*

- $D_{F_k}^+(\mathcal{S}_{\mathbf{c},r})$ with $r > (\sqrt{n} + \sqrt{n-k})/2$,
- $D_{F_k}^-(\mathcal{S}_{\mathbf{c},r})$ with $r \geq (\sqrt{n} + \sqrt{n-k})/2$,
- $D_{iF_k}^-(\mathcal{S}_{\mathbf{c},r})$ with $r > \sqrt{n}/2 + \sqrt{n-k}$,
- $D_{oF_k}^+(\mathcal{S}_{\mathbf{c},r})$ with $r > \sqrt{n}/2$.

PROOF. Since $F_k(\rho) \subseteq \mathcal{B}_{[\cdot]_k}(\rho)$, the conditions to ensure that the (digital) complement of a $[\cdot]_k$ -digitized hypersphere admits two distinct k -connected components remain valid for a F_k -digitized hypersphere as depicted in Fig. 8(a) and 8(b).

Let us consider two k -adjacent integer coordinate points $\mathbf{v} \in \mathcal{S}_{\mathbf{c},r}^{+*}$ and $\mathbf{w} \in \mathcal{S}_{\mathbf{c},r}^-$ (or equivalently $\mathbf{v} \in \mathcal{S}_{\mathbf{c},r}^+$ and $\mathbf{w} \in \mathcal{S}_{\mathbf{c},r}^{-*}$), i.e. $[\mathbf{v} - \mathbf{w}]_k = 1$. By definition, we have $(\mathbf{v} - \mathbf{w}) \in \{\mathbf{x} : \mathbf{x} \in \{-1, 0, 1\}^n \wedge \sum_{i=1}^n |x_i| \leq n - k\}$. The first condition is induced by $\|\mathbf{v} - \mathbf{w}\|_\infty = 1$ and the second one by $\|\mathbf{v} - \mathbf{w}\|_1 / (n - k) \leq 1$. Let us now consider $\mathbf{s} = \mathcal{S}_{\mathbf{c},r} \cap [\mathbf{vw}]$ where $[\mathbf{vw}]$ is the straight segment linking \mathbf{v} and \mathbf{w} (\mathbf{s} exists since \mathbf{v} and \mathbf{w} are on each side of the hypersphere $\mathcal{S}_{\mathbf{c},r}$). Since the direction of $[\mathbf{vw}]$ is very constrained according to the previous statement, we have $\{\mathbf{v}, \mathbf{w}\} \subset (\{\mathbf{s}\} \oplus \{\mathbf{x} : \mathbf{x} \in \{-\alpha, 0, \alpha\}^n, \alpha \in \mathbb{R}^+ \wedge \sum_{i=1}^n |x_i| \leq n - k\})$. Let us now show that $\{\mathbf{v}, \mathbf{w}\} \cap (\{\mathbf{s}\} \oplus \mathcal{B}_{[\cdot]_k}(\rho)) \neq \emptyset$ and thus that \mathbf{v} or \mathbf{w} belongs to the F_k -digitized hypersphere. Remember that \mathbf{s} is necessarily between \mathbf{v} and \mathbf{w} because on $[\mathbf{vw}]$.

In the case of a centered type model, $\rho = 1/2$ and three cases can occur: either $[\mathbf{v} - \mathbf{s}]_k = 1/2$ and $[\mathbf{w} - \mathbf{s}]_k = 1/2$, or $[\mathbf{v} - \mathbf{s}]_k < 1/2$, or $[\mathbf{w} - \mathbf{s}]_k < 1/2$. In all three cases, \mathbf{v} or \mathbf{w} belongs to $(\{\mathbf{s}\} \oplus \mathcal{B}_{[\cdot]_k}(1/2))$.

In the case of a Gaussian type model, $\rho = 1$ and three other cases have to be considered: either $[\mathbf{v} - \mathbf{s}]_k < 1$ and $[\mathbf{w} - \mathbf{s}]_k < 1$, or $\mathbf{v} = \mathbf{s}$, or $\mathbf{w} = \mathbf{s}$. In all three cases, \mathbf{v} or \mathbf{w} belongs to $(\{\mathbf{s}\} \oplus \mathcal{B}_{[\cdot]_k}(1))$.

As a consequence, for each couple of k -adjacent integer points $(\mathbf{v}, \mathbf{w}) \in \mathcal{S}_{\mathbf{c},r}^+ \times \mathcal{S}_{\mathbf{c},r}^-$, at least one of them is in the digital hypersphere. \square

The F_k -digitizations of $\mathcal{S}_{\mathbf{c},r}$ come with simple analytical characterizations as soon as the closest and the farthest points to \mathbf{c} , in an adjacency flake translated to any point of $\mathcal{S}_{\mathbf{c},r}$, are vertices of this adjacency flake. Such a condition is fulfilled for reasonably large radii as depicted in Fig. 8(c) and 8(d).

Proposition 13. *For a given $\mathbf{x} \in \mathbb{R}^n$, let $\sigma(\mathbf{x})$ be a permutation of the components of \mathbf{x} such that the terms of the sequence $(\sigma_i(\mathbf{x}))_{1 \leq i \leq n}$ are decreasing in absolute value. Then, we have the following analytic*

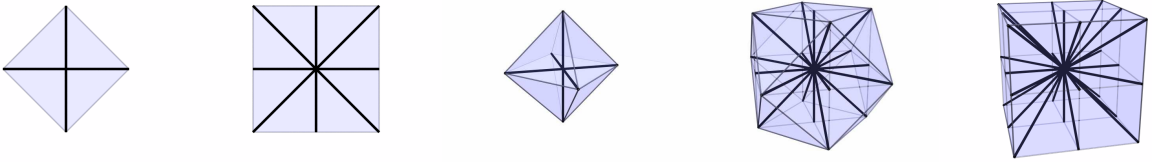


Figure 7: Adjacency flakes $F_1(\rho)$, $F_0(\rho)$ in the 2-dimensional space and $F_2(\rho)$, $F_1(\rho)$, $F_0(\rho)$ in the 3-dimensional space. Adjacency flakes are depicted in black and balls of k -adjacency norms in light blue.

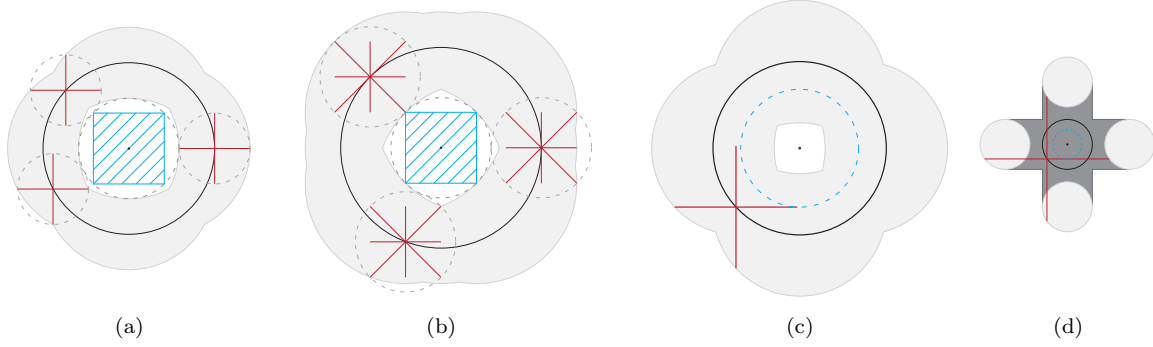


Figure 8: (a),(b)- Sufficient condition to ensure that a hypersphere separates the space: the bounded component of the complement of the offset region contains a unit hypercube. (c),(d)- In light grey, offset regions obtained by considering that the maximum and the minimum distance to the center of the circle are reached in vertices of the flake. In dark grey, the real offset region. For reasonable radii, both are equivalent (c), This is no more the case for small radii (d).

characterizations:

$$\begin{aligned}
D_{F_k}^+(\mathcal{S}_{c,r}) &= \left\{ \mathbf{v} \in \mathbb{Z}^n : -\sum_{i=1}^{n-k} \left(|\sigma_i(\mathbf{v} - \mathbf{c})| + \frac{1}{4} \right) \leq s_{c,r}(\mathbf{v}) < \sum_{i=1}^{n-k} \max \left\{ |\sigma_i(\mathbf{v} - \mathbf{c})| - \frac{1}{4}, 0 \right\} \right\} \quad (\text{if } r > \sqrt{n}/2), \\
D_{F_k}^-(\mathcal{S}_{c,r}) &= \left\{ \mathbf{v} \in \mathbb{Z}^n : -\sum_{i=1}^{n-k} \left(|\sigma_i(\mathbf{v} - \mathbf{c})| + \frac{1}{4} \right) < s_{c,r}(\mathbf{v}) \leq \sum_{i=1}^{n-k} \max \left\{ |\sigma_i(\mathbf{v} - \mathbf{c})| - \frac{1}{4}, 0 \right\} \right\} \quad (\text{if } r \geq \sqrt{n}/2), \\
D_{oF_k}^+(\mathcal{S}_{c,r}) &= \left\{ \mathbf{v} \in \mathbb{Z}^n : 0 \leq s_{c,r}(\mathbf{v}) < \sum_{i=1}^{n-k} \max \{ 2|\sigma_i(\mathbf{v} - \mathbf{c})| - 1, 0 \} \right\} \quad (\text{if } r > \sqrt{n}), \\
D_{iF_k}^-(\mathcal{S}_{c,r}) &= \left\{ \mathbf{v} \in \mathbb{Z}^n : -\sum_{i=1}^{n-k} (2|\sigma_i(\mathbf{v} - \mathbf{c})| + 1) < s_{c,r}(\mathbf{v}) \leq 0 \right\}.
\end{aligned}$$

PROOF. Let us prove the analytic expression for $D_{oF_k}^+(\mathcal{S}_{c,r})$. Remember that for Gauss type models, we use a structuring element twice bigger as usual (ball of radius 1 instead of $1/2$).

A integer point $\mathbf{v} \in \mathcal{S}_{c,r}^+$ belongs to $D_{oF_k}^+(\mathcal{S}_{c,r})$ if $(\{\mathbf{v}\} \oplus F_k^*(1)) \cap \mathcal{S}_{c,r} \neq \emptyset$. In other words, the map $s_{c,r}$ should vanish in a neighborhood of each integer point of the digital hypersphere:

$$\begin{aligned}
D_{oF_k}^+(\mathcal{S}_{c,r}) &= \left\{ \mathbf{v} \in \mathbb{Z}^n : \min_{\boldsymbol{\varepsilon} \in F_k(1)} \{s_{c,r}(\mathbf{v} + \boldsymbol{\varepsilon})\} < 0 \leq s_{c,r}(\mathbf{v}) \right\} \\
&= \left\{ \mathbf{v} \in \mathbb{Z}^n : \sum_{i=1}^n (v_i - o_i)^2 - \min_{\boldsymbol{\varepsilon} \in F_k(1)} \left\{ \sum_{i=1}^n ((v_i - o_i) + \varepsilon_i)^2 \right\} > s_{c,r}(\mathbf{v}) \geq 0 \right\}
\end{aligned}$$

For all $\mathbf{x} = (x_1, \dots, x_n) \in F_k(1)$, and for all $j \in \{1, \dots, n\}$, we have $(\mathbf{x} - 2x_j \mathbf{e}_j) \in F_k(1)$ with $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ the vectors of the canonical basis of \mathbb{R}^n . In other words, even if one changes the sign of some components of \mathbf{x} , \mathbf{x} remains in the adjacency flake. Thus for all $\mathbf{v} \in \mathbb{Z}^n$, there exists $\boldsymbol{\varepsilon}' \in F_k(1)$ such that for all $i \in \{1, \dots, n\}$, $\varepsilon_i' |v_i - o_i| = \varepsilon_i (v_i - o_i)$. Without loss of generality, we consider:

$$D_{oF_k}^+(\mathcal{S}_{c,r}) = \left\{ \mathbf{v} \in \mathbb{Z}^n : \sum_{i=1}^n |v_i - o_i|^2 - \min_{\boldsymbol{\varepsilon} \in F_k(1)} \left\{ \sum_{i=1}^n (|v_i - o_i| + \varepsilon_i)^2 \right\} > s_{c,r}(\mathbf{v}) \geq 0 \right\}.$$

Since $\boldsymbol{\varepsilon} \in F_k(1)$, we have, with $0 \leq \alpha \leq 1$, $\boldsymbol{\varepsilon} \in \{-\alpha, 0, \alpha\}^n$ and $\sum_{i=1}^n |\varepsilon_i| \leq (n-k)\alpha$. $\boldsymbol{\varepsilon}$ admits at least k zero components. According to the condition $r > 2\sqrt{n}$, $\boldsymbol{\varepsilon}$ should belong to $\{-1, 0\}^n$ to minimize the lower bound in the analytic expression. More precisely, for all $i \in \{1, \dots, n\}$, if $|v_i - o_i| < 1/2$, $(|v_i - o_i| + \varepsilon_i)^2$ is

minimal for $\varepsilon_i = 0$ else, $(|v_i - o_i| + \varepsilon_i)^2$ is minimal for $\varepsilon_i = -1$. Then the global minimum is reached for a vector ε :

- with zero at each index of the k small components of $\mathbf{v} - \mathbf{c}$ in absolute value,
- with zero at each index of other components of $\mathbf{v} - \mathbf{c}$ with absolute value lower than $1/2$,
- with the value -1 for each component associated with the remaining indexes.

With σ , it leads to:

$$D_{o_{F_k}}^+(\mathcal{S}_{\mathbf{c},r}) = \left\{ \mathbf{v} \in \mathbb{Z}^n : \sum_{i=1}^{n-k} |\sigma_i(\mathbf{v} - \mathbf{c})|^2 - \sum_{i=1}^{n-k} \min \{ (|\sigma_i(\mathbf{v} - \mathbf{c})| - 1)^2, (|\sigma_i(\mathbf{v} - \mathbf{c})|)^2 \} > s_{\mathbf{c},r}(\mathbf{v}) \geq 0 \right\},$$

and finally to the expression of $D_{o_{F_k}}^+(\mathcal{S}_{\mathbf{c},r})$ given in the proposition.

The proof is similar for other models. \square

Those analytic characterizations allow to easily prove topological properties, in particular for Gaussian type digital hyperspheres.

Proposition 14. *The inner (respectively outer) Gaussian digitization of a hypersphere $\mathcal{S}_{\mathbf{c},r}$, $D_{i_{F_k}}^-(\mathcal{S}_{\mathbf{c},r})$ (respectively $D_{o_{F_k}}^+(\mathcal{S}_{\mathbf{c},r})$), is the set of integer points in $\mathcal{S}_{\mathbf{c},r}^-$ (respectively $\mathcal{S}_{\mathbf{c},r}^+$) k -adjacent to at least one integer point of $\mathcal{S}_{\mathbf{c},r}^{+*}$ (respectively $\mathcal{S}_{\mathbf{c},r}^{-*}$).*

PROOF. The offset region used in $D_{i_{F_k}}^-(\mathcal{S}_{\mathbf{c},r})$ lies entirely in $\mathcal{S}_{\mathbf{c},r}^-$. Proposition 12 induces that the set of integer points in $\mathcal{S}_{\mathbf{c},r}^-$ that are k -adjacent to at least one integer point of $\mathcal{S}_{\mathbf{c},r}^{+*}$ is included in $D_{i_{F_k}}^-(\mathcal{S}_{\mathbf{c},r})$.

Let us consider an integer point $\mathbf{v} \in D_{i_{F_k}}^-(\mathcal{S}_{\mathbf{c},r})$. \mathbf{v} satisfied $\sum_{i=1}^{n-k} (2|\sigma_i(\mathbf{v} - \mathbf{c})| - 1) < s_{\mathbf{c},r}(\mathbf{v}) \leq 0$. It exists $\mathbf{w} \in \mathbb{Z}^n$ k -adjacent to \mathbf{v} such that for all $i \in \{1, \dots, n-k\}$, $\sigma_i(\mathbf{w} - \mathbf{c}) = \sigma_i(\mathbf{v} - \mathbf{c}) + 1$ and for all $i \in \{n-k+1, \dots, n\}$, $\sigma_i(\mathbf{w} - \mathbf{c}) = \sigma_i(\mathbf{v} - \mathbf{c})$. Then, we have $s_{\mathbf{c},r}(\mathbf{w}) \geq 0$ and $D_{i_{F_k}}^-(\mathcal{S}_{\mathbf{c},r})$ is the set of integer points in $\mathcal{S}_{\mathbf{c},r}^-$ k -adjacent to at least one integer point of $\mathcal{S}_{\mathbf{c},r}^{+*}$.

The proposition can be proved for $D_{o_{F_k}}^+(\mathcal{S}_{\mathbf{c},r})$ with the same argument. \square

A direct consequence of this last proposition and proposition 3 is that the digital hyperspheres $D_{i_{B_{| \cdot |}^k}}^-(\mathcal{S}_{\mathbf{c},r})$ and $D_{i_{F_k}}^-(\mathcal{S}_{\mathbf{c},r})$ define the same set of integer points.

$D_{o_{F_k}}^+(\mathcal{S}_{\mathbf{c},r})$ comes with a stronger topological property.

Proposition 15. *The outer Gaussian digitization of a hypersphere $\mathcal{S}_{\mathbf{c},r}$, $D_{o_{F_k}}^+(\mathcal{S}_{\mathbf{c},r})$ is a strict k -separating set in \mathbb{Z}^n .*

PROOF. Let us consider $\mathbf{v} \in D_{o_{F_k}}^+(\mathcal{S}_{\mathbf{c},r})$. It exists $\mathbf{w} \in \mathbb{Z}^n$ k -adjacent to \mathbf{v} such that for all $i \in \{1, \dots, m\}$, $\sigma_i(\mathbf{w} - \mathbf{c}) = \sigma_i(\mathbf{v} - \mathbf{c}) - 1$ and for all $i \in \{m+1, \dots, n\}$, $\sigma_i(\mathbf{w} - \mathbf{c}) = \sigma_i(\mathbf{v} - \mathbf{c})$, with:

$$m = \arg \max_i \{i \in \{1, \dots, n-k\} : \sigma_i(\mathbf{v} - \mathbf{c}) \geq 1\}.$$

Since \mathbf{v} satisfies $0 \leq s_{\mathbf{c},r}(\mathbf{v}) < \sum_{i=1}^{n-k} \max \{2|\sigma_i(\mathbf{v} - \mathbf{c})| - 1, 0\}$, we have $s_{\mathbf{c},r}(\mathbf{w}) < 0$ and $\mathbf{w} \in \mathcal{S}_{\mathbf{c},r}^{-*}$. Thus, $D_{o_{F_k}}^+(\mathcal{S}_{\mathbf{c},r}) \setminus \{\mathbf{v}\}$ is not k -separating in \mathbb{Z}^n . It implies that $D_{o_{F_k}}^+(\mathcal{S}_{\mathbf{c},r})$ is a strict k -separating set. \square

In addition to come with the highlighted topological properties, adjacency flake based models also characterize the Pham's circles [18], that is, the main extension of Bresenham's circles to non integer parameters.

Proposition 16 (Pham's circle). *In a 2-dimensional space, we have, with $\mathbf{c} = (x_0, y_0)$:*

$$D_{F_1}^+(\mathcal{S}_{\mathbf{c},r}) = \left\{ (x, y) \in \mathbb{Z}^2 : -\max \{|x - x_0|, |y - y_0|\} - \frac{1}{4} \leq s_{\mathbf{c},r}(x, y) < \max \{|x - x_0|, |y - y_0|\} - \frac{1}{4} \right\}.$$

This digital circle describes the same set of integer points as the Pham's circle of center \mathbf{c} and radius r .

PROOF. Without loss of generality, we restrict the study to the first octant thanks to symmetries. At each step, the Pham's algorithm consists in computing $\Delta = r^2 - (x - \frac{1}{2} - x_0)^2 - ((y + 1) - y_0)^2$ from the current pixel (x, y) , which belongs to the digital circle. If $\Delta > 0$, then the pixel $(x, y + 1)$ is selected because it belongs to the digital circle. Otherwise, if $\Delta \geq 0$, the pixel $(x - 1, y + 1)$ is selected because it belongs to the digital circle. Then, the algorithm proceeds to the next step with the new current integer point.

Let us now consider a pixel (x, y) in the first octant of $D_{F_1}^+(\mathcal{S}_{c,r})$. In order to belong to $D_{F_1}^+(\mathcal{S}_{c,r})$, the pixels $(x, y + 1)$ and $(x - 1, y + 1)$ should respectively satisfy:

$$\begin{cases} -(x - 1 - x_0) - \frac{1}{4} \leq s_{c,r}(x - 1, y + 1) < (x - 1 - x_0) - \frac{1}{4}, \\ -(x - x_0) - \frac{1}{4} \leq s_{c,r}(x, y + 1) < (x - x_0) - \frac{1}{4}. \end{cases}$$

If we express those conditions according to $s_{c,r}(x - \frac{1}{2}, y + 1)$, we obtain:

$$\begin{cases} 0 \leq s_{c,r}(x - \frac{1}{2}, y + 1), \\ s_{c,r}(x - \frac{1}{2}, y + 1) < 0. \end{cases}$$

Only one of the two pixels belongs to $D_{F_1}^+(\mathcal{S}_{c,r})$ according to the sign of $s_{c,r}(x - \frac{1}{2}, y + 1)$. The fact that $\Delta = s_{c,r}(x - \frac{1}{2}, y + 1)$ proves the equivalence between Pham's circle and $D_{F_1}^+(\mathcal{S}_{c,r})$. \square

Note that the initialization of the Pham's algorithm is questionable. The execution of the algorithm suggests that the first pixel has to be selected according to the 1-adjacency flake. This is not the case in the original algorithm. In Proposition 16, we consider this problem fixed. As a consequence, Bresenham's circle is also a $D_{F_1}^+(\mathcal{S}_{c,r})$. The analytic characterization introduced in [33] is equivalent to the one in Proposition 16 for integer parameters (under such conditions, one can remove the term $-1/4$ in both bounds).

6. Conclusion

In this paper, we introduced a family of morphological digitization models. The digitization of an object is the set of integer coordinate points lying in a so-called *offset region*. The offset regions are analytically characterized in order to mathematically describe the digital object independently of a generation algorithm or of the set of integer points composing it. We focused on orientable hypersurfaces and proposed several digitization models, either closed or semi-open and either on only one side of the hypersurface (inner and outer Gaussian models) or centered on it.

According to the shape of the structuring element, we have introduced digital circles, spheres and in some cases hyperspheres having different topological properties.

First, we focused on balls based on the Euclidean norm or the adjacency norms. From these balls, we analytically characterized several digital circles, spheres and hyperspheres. When they are based on the k -adjacency norm, these digital sets are k -separating.

We then introduced a new type of structuring elements that is still based on the k -adjacency norms but smaller than the balls of the norms. These structuring elements have been called *adjacency flakes*. They lead to thinner k -separating digital hyperspheres, which have been analytically characterized in any dimension. Moreover, we show that the semi-open outer Gaussian model leads to strict (i.e. without simple points) k -separating digital hyperspheres.

The proposed definitions are generic and extend previous definitions (like Bresenham's circle, Kim's circle or Pham's circle) to arbitrary centers and radii, thickness or dimension. the Kovalevsky's circle [17] is the only digital circle not covered in the present paper. Nevertheless, it can be analytically characterized with our models and with a different flake we do not introduce here.

One of the main perspectives of this paper is of course the extensions that analytical descriptions allow: extension to thick digital circles, spheres and hyperspheres, recognition and generation algorithms for these different objects. For the adjacency norms, the analytical description of hyperspheres is a difficult problem that remains largely open. Another perspective is the extension to more complex algebraic curves [34, 35, 36].

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