Foundational aspects of multiscale digitization

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Abstract

In this article, we describe the theoretical foundations of the Ω-arithmetization. This method provides a multi-scale discretization of a continuous function that is a solution of a differential equation. This discretization process is based on the Harthong-Reeb line $\mathcal{HR}_\omega$. The Harthong- Reeb line is a linear space that is both discrete and continuous. This strange line $\mathcal{HR}_\omega$ stems from a nonstandard point of view on arithmetic based, in this paper, on the concept of Ω-numbers introduced by Laugwitz and Schmieden. After a full description of this nonstandard background and of the first properties of $\mathcal{HR}\omega$, we introduce the Ω-arithmetization and we apply it to some significant examples. An important point is that the constructive properties of our approach leads to algorithms which can be exactly translated into functional computer programs without uncontrolled numerical error. Afterwards, we investigate to what extent $\mathcal{HR}_\omega$ fits Bridges's axioms of the constructive continuum. Finally, we give an overview of a formalization of the Harthong-Reeb line with the Coq proof assistant.

Keywords: discrete geometry, nonstandard analysis, multi-resolution analysis, constructive mathematics.

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1. Introduction

Multi-resolution object representation and numerical precision problems are important subjects in computer imagery. In the continuation of previous works of the authors [11, 5, 6], we detail in this article a new method called the Ω-arithmetization. It is a process of multiscale discretization of a continuous function that is a solution of a differential equation. The Ω-arithmetization is an extension of the arithmetization method based on Ω-numbers introduced by Laugwitz and Schmieden [16, 14, 15]

Initially, the arithmetization method was introduced by Georges Reeb and Jacques Harthong. The principle of the arithmetization method has led Reveillès to the definition of the discrete analytical line [26, 27, 28]. Jacques Harthong and Jean-Pierre Reveillès were part of a group of young mathematicians that gathered around Georges Reeb to work on nonstandard analysis. Georges Reeb is well known in mathematics for his work on the geometric topological theory of foliations and in computer science for what is now known as the Reeb graphs. What is less known is that Georges Reeb was also interested in intuitionism with a keen interest in the relations between computers and nonstandard analysis. Jacques Harthong and Georges Reeb proposed a model of the discrete line $\mathbb{Z}$, called the Harthong-Reeb line, that they showed to be equivalent to the real line $\mathbb{R}$. The arithmetization is obtained by transforming, for instance, the classical integration Euler scheme used to compute the curves, solution to differential equations, into an equivalent integer scheme. The transformation from $\mathbb{Z}$ to $\mathbb{R}$ corresponds to a rescaling which induces a strong deformation of space. A rigorous implementation of this approach requires a model of the set $\mathbb{Z}$ of integer numbers together with a notion of infinitely large number (i.e. a scale on $\mathbb{Z}$). In previous works such a model was introduced with the help of an axiomatic version of nonstandard analysis. The major drawback of such an approach is that the infinitely large integers which arise in the corresponding method have only an axiomatic status, i.e. no indication is available to know how to compute these integers.

In the present paper, we propose the arithmetization method based on the notion of Ω-numbers introduced by Laugwitz and Schmieden [16, 14, 15]. Roughly speaking, an Ω-number (natural, integer or rational) is a sequence of numbers of the same nature together with an adapted equality relation.

The sets of Ω-numbers are extending the corresponding sets of usual numbers with the added advantage of providing a natural concept of infinitely large integer numbers: for instance, an Ω-integer α represented by a sequence (α_n) of integers is such that $\alpha \simeq +\infty$ if $\lim_{n\to+\infty} \alpha_n = +\infty$ in the usual meaning. Clearly, these infinite numerical entities are effectively constructive. This is one of the main interest points of the authors in this work. Indeed, from a computer science point of view, it is interesting as practical real numbers are only constructive ones that cannot be other that denumerable.

After choosing an Ω-integer ω such that $\omega \simeq +\infty$, we can define the Harthong-Reeb line $\mathcal{HR}_\omega$ [9] which is a numerical system consisting of Ω-integers with the additional property of being "roughly" equivalent to the real line system. Not only the elements of $\mathcal{HR}_\omega$ have a constructive flavor, but we show that the structure of this system partially fits with the constructive axiomatic developed by Bridges [2]. The constructive properties of the underlying theory that is presented in this paper leads, as we will show, to algorithms which can be exactly translated into functional computer programs without uncontrolled numerical errors. An important part of this work is devoted to the definition and the study of the theoretical framework of the method. It corresponds also to an intuition of Georges Reeb that speculated that the Harthong-Reeb line is constructive in nature.

The Ω-arithmetization is an extension of the arithmetization method based on Ω-numbers. The principle of this method is unchanged and the resulting algorithm is formally the same. The new and crucial facts are the following:

- The algorithm operates on Ω-numbers in a completely constructive way and consequently, in the applications, we can represent adequately all the entities present in the theory.

- The result of the algorithm appears to be an exact discrete multi-resolution representation of the continuous function on which the method is applied. See figure 1.

From the first point, we deduce that the implementation of the method does not lead to uncontrolled approximation errors. The second point is particularly interesting from a computer imagery point of view. This multi-resolution aspect is a direct consequence of the Ω-arithmetization: this is in relation with the nature of the scaling parameter β used in the method (see

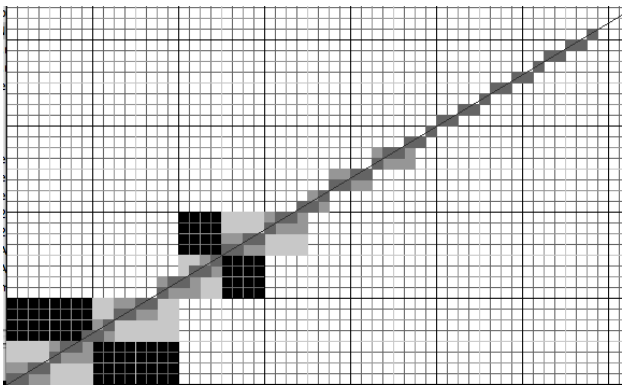


Figure 1: Plot of the multi-resolution aspects of the Ω-arithmetization of the real function $X(T) = 3T/5$. (Full explanation in section (4)).

section 4). This parameter, as an infinitely large Ω-integer, encodes an infinity of increasing scales. The arithmetization process gives simultaneously a discretization of the initial continuous function at each of these scales.

Since nowadays many developments in image analysis, geometric modeling, etc. use multi-resolution approaches and must deal with numerical precision problems, the Ω-arithmetization is a new tool which has the interesting property of taking into account these two aspects. Let us note that our goal is not to define a discretization method that produce "better" images and "faster" algorithms. On the bases of a significant theoretical analysis, our originality is to propose a constructive and exact discrete representation of continuous functions. Moreover, this framework naturally leads to a multi-resolution representation.

However, even if practical handling of Ω-numbers is just manipulations of integer sequences, the underlying concepts are not so easy to apprehend as they are rather far from the usual mathematical practice. In order to help people (and at first the authors) to obtain a good grasp of this unusual mathematical framework, a formalization into the Coq proof assistant [1] is proposed by Magaud and some of the authors [17]. This formalization had helped to verify the proofs that we made, and a quick overview of it is given in this paper.

Let us now clarify the links of the present study with the existing literature. The preceding description has shown what are the main preexisting ideas and works which are directly and explicitly at the root of the present study:

firstly the topics of the Harthong-Reeb line and the arithmetization method introduced in the French school of nonstandard analysis, secondly the theory Ω-numbers of Laugwitz and Schmieden and thirdly, in a lower proportion, Bridges's axiomatic approach of the constructive real line.

On the other hand, since our contribution used some tools of nonstandard analysis and provides a constructive form of the continuum, it is natural to draw an analogy with the realm of constructive nonstandard analysis. The origin of this last theoretical field is the pioneer work of P. Martin-Löf [20] which is also one of our sources of reflections. Following this first headway, various versions of constructive nonstandard analysis were developed in particularly in the works of E. Palmgren [22, 23, 24, 25, 30, 31]. Starting with the use of the rational Ω-numbers of Schmieden and Laugwitz [14], these works have introduced some formal systems and models, thus offering a rich and effective theoretical framework for a constructive practice of nonstandard analysis.

The analogy between these works and the present article comes from the fact that we use Ω-numbers - Ω-integers to be more precise - and that we end up with a nonstandard and constructive representation of certain continuous functions. Nevertheless, we argue that there is no fundamental interference, no theoretical dependence between our work and the quoted works. Indeed, it is not our purpose to define or to use a general version of the constructive nonstandard analysis. Following the reflections of Harthong, Reeb [12, 13] and of our previous own works [11, 5, 6], we essentially want to clarify and to study a new digital system, the Ω-Harthong-Reeb line, as base of a discrete multi-scale representation of certain continuous functions. For that purpose, we absolutely have to stay at the semantic level of the Ω-integers because it is the functional structure of these last ones that is at the heart of our multi-scale discrete representation. Finally, the functional programming allows an exact and effective implementation of this representation on computers, proving, practically, the constructive character of our approach.

The specificity of our work with related developments in the field of constructive nonstandard analysis will be discussed later in more detail about the following points: the treatment of the Euler scheme in the end of the introduction of part 3, the study of the the regular elements of the Harthong-Reeb line in the beginning of part 5.2.2 just before theorem 2.

Finally, let us note that it would certainly be interesting to define and to study an Harthong-Reeb line in one or other of the various approaches of constructive nonstandard analysis mentioned previously. This would be,

however, a rather different subject than the one developed in this article.

The paper is organized as follow: in part 2, we introduce the Ω-numbers and study their general mathematical and logical properties, in part 3, we use the Ω-numbers to define an Harthong-Reeb line $\mathcal{HR}_\omega$. In part 4, we present the Ω-arithmetization and finally in part 5 we present a theoretical analysis on the constructive content of $\mathcal{HR}_\omega$.

2. The Ω-numbers of Laugwitz and Schmieden

In this section we will present the notion of Ω-numbers introduced by Laugwitz and Schmieden [16, 14, 15]. For the most part, we follow the presentations of these authors, but on some points, we have introduced new developments and, from our point of view, important distinctions. The Ω-numbers are nonstandard numbers but the encompassing theory has two complementary characteristics: it seems theoretically weaker than the usual versions of nonstandard analysis [29, 21, 8] but it has an undeniable flavor of constructivity suggesting the possibility of explicit and exact computations. The principal goal of Laugwitz and Schmieden was to build a new approach to real analysis based only on the introduction of a set of Ω-rational numbers which is an extension of the usual set $\mathbb{Q}$. In our case and in view of the arithmetization process, we are mainly interested in Ω-integers but we will occasionally consider Ω-rationals.

2.1. Extension by an infinitely large number Ω

The first step is to extend a given formal theory T (unspecified but including an elementary theory of integer and rational numbers) by introducing a new number constant Ω and a new rule (BD) described thereafter. This leads to a new theory $T\langle\Omega\rangle$ which is an extension of T. Thus, any formula of T is also a formula of $T\langle\Omega\rangle$. In this connection, we will need the following definition.

Definition 1. A formula of the theory $T\langle\Omega\rangle$ is said internal if it can be formulated in the initial theory T

In addition to internal formulas, we can form in $T\langle\Omega\rangle$ new formulas depending on Ω for which the truth is given by the following axiom called the Basic Definition (BD):

Let $S(n)$ an internal formula depending on $n \in \mathbb{N}$. If $S(n)$ is true for almost $n \in N$, then $S(\Omega)$ is true.

We specify that here and in all what follows, the expression "*almost* $n \in \mathbb{N}$" means "for all $n \in \mathbb{N}$ from some level", i.e. "$(\exists N \in \mathbb{N})$ such that $(\forall n \in \mathbb{N})$ with $n > N$". Deriving from (BD), we can verify that Ω is an infinitely large integer, i.e. greater than every element of $\mathbb{N}$. Indeed, for $p \in \mathbb{N}$, we apply (BD) to the statement $p < n$ which is true for almost $n \in \mathbb{N}$; thus $p < \Omega$ for each $p \in \mathbb{N}$.

2.2. The sets of Ω*-numbers*

The second step is to describe a world of mathematical objects which is a realization of the extended theory $T\langle\Omega\rangle$. For this purpose, we consider the set of sequences of integer or rational numbers. On this set, we introduce the equivalence relation $\mathcal{R}$ such that, for $a = (a_n)$ and $b = (b_n)$[1], we have $a\mathcal{R}b$ if and only if $a_n = b_n$ for *almost* $n \in \mathbb{N}$.

Definition 2. *Each equivalence class for the relation* $\mathcal{R}$ *is called an* Ω*-number.*

In the general case, an Ω-number is also called an Ω-rational number. We agree to identify each sequence of numbers $a = (a_n)$ with the Ω-number equal to the equivalence class of a. Given a sequence $a = (a_n)$ such that $a_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$, we can say that $a = (a_n)$ is an Ω-integer. Finally, we decide that the symbol Ω is the name of the particular Ω-number $(n)_{n\in\mathbb{N}}$. The following development will show that these choices are coherent.

Let $\mathbb{Z}_\Omega$ be the set of Ω-integers, $\mathbb{N}_\Omega$ be the set of Ω-integers $c = (c_n)$ such that $c_n \geqslant 0$ for almost $n \in \mathbb{N}$ and $\mathbb{Q}_\Omega$ be the set of Ω-rational numbers. We consider the embedding $i : \mathbb{Z} \to \mathbb{Z}_\Omega$ which associates to each $p \in \mathbb{Z}$ the constant sequence of value p. We distinguish two classes of elements in $\mathbb{Z}_\Omega$: the first deriving from the map i and the second characterizing the infinitely large.

Definition 3. An Ω-integer $a = (a_n)$ is said to be *standard* if a belongs to the image of the preceding embedding, i.e. if there exists $p \in \mathbb{Z}$ such that $a_n = p$ for almost $n \in \mathbb{N}$.

Definition 4. An Ω-integer $a = (a_n)$ is *infinitely large* when $(\forall p \in \mathbb{N})(\exists n \in \mathbb{N})(\forall n > N)(p < |a_n|)$.

[1] Although this is not always indicated, in our sequences, the index n takes all the values $0, 1, \ldots$ in $\mathbb{N}$.

This is equivalent to say that $a = (a_n)$ is infinitely large when $\lim_{n\to+\infty} |a_n| = +\infty$.

Any sequence of integers $f = (f(n))$ is a map $f : \mathbb{N} \to \mathbb{Z}$ which has a natural extension $f : \mathbb{N}_\Omega \to \mathbb{Z}_\Omega$ defined by $f(a) =_{\text{def}} (f(a_n))_{n\in\mathbb{N}}$ for $a = (a_n)$. For each Ω-integer $b = (b_n)$, we can extend the underlying sequence to $\mathbb{N}_\Omega$ and we obtain in particular $b_\Omega = (b_n) = b$. Applying this property to $(n)_{n\in\mathbb{N}}$, we find again $\Omega = (n)_{n\in\mathbb{N}}$, which partly shows the consistency of our previous choice. We do the same for the Ω-rational numbers.

Any operation or relation defined on $\mathbb{Z}$ (or $\mathbb{Q}$) naturally extends to $\mathbb{Z}_\Omega$ (or $\mathbb{Q}_\Omega$). For instance, the following definition give the definitions of operations, relations and the absolute value the for Ω-numbers.

Definition 5. For every $a = (a_n)$ and $b = (b_n) \in \mathbb{Z}_\Omega$ let us set :

- $a + b =_{\text{def}} (a_n + b_n)$ and $-a =_{\text{def}} (-a_n)$ and $a \times b =_{\text{def}} (a_n \times b_n)$;
- $a > b =_{\text{def}} [(\exists N \forall n > N)\ a_n > b_n]$ and $a \geqslant b =_{\text{def}} [(\exists N \forall n > N)\ a_n \geqslant b_n]$;
- $|a| =_{\text{def}} (|a_n|)$.

It is easy to check that $(\mathbb{Z}_\Omega, +, \times)$ is a commutative ring with the constant sequence of value 0 as zero and the constant sequence of value 1 as unit. The previous map $i : \mathbb{Z} \to \mathbb{Z}_\Omega$ is an injective ring homomorphism which allows to identify $\mathbb{Z}$ with the subring of standard elements of $\mathbb{Z}_\Omega$. From now on, we identify any integer $p \in \mathbb{Z}$ with the Ω-integer $i(p)$ equal to the sequence of constant value p.

For the implementation of an arithmetization process based on Ω-integers, we need an extension of the Euclidean division to the Ω-integers and of the floor and the fractional part functions to the Ω-rational numbers.

- Given two Ω-integers $a = (a_n)$ and $b = (b_n)$ verifying $b > 0$, there is an unique $(q, r) \in \mathbb{Z}_\Omega^2$ such that $a = bq + r$ and $0 \leqslant r < b$. Indeed, since $b_n > 0$ from some level $N \in \mathbb{N}$, we can set $q = (q_n)$ and $r = (r_n)$ where, for $n \geqslant N$, q_n is the quotient of a_n by b_n and r_n is the remainder of this Euclidean division. For $n < N$ the values of q_n and r_n are arbitrary (for instance 0). We will use the usual notations $a \div b$ for the quotient q and $a \bmod b$ for the remainder r.
- Given an Ω-rational number $r = (r_n)$, there is a unique $\lfloor r \rfloor \in \mathbb{Z}_\Omega$ and a unique $\{r\} \in \mathbb{Q}_\Omega$ such that $(0 \leqslant \{r\} < 1) \wedge (r = \lfloor r \rfloor + \{r\})$. Indeed, we can choose $\lfloor r \rfloor = (\lfloor r_n \rfloor)$ and similarly $\{r\} = (\{r_n\})$.

Regarding the order relation, the usual properties that are true on $\mathbb{Z}$ are not always verified on $\mathbb{Z}_\Omega$. For instance

$$(\forall a, b \in \mathbb{Z}_\Omega) \quad (a \geqslant b) \vee (b \geqslant a) \tag{1}$$

is not valid as we can see for the particular Ω-integers $a = ((-1)^n)_{n\in\mathbb{N}}$ and $b = ((-1)^{n+1})_{n\in\mathbb{N}}$. Nevertheless, given two arbitrary Ω-integers $a = (a_n)$ and $b = (b_n)$, we have

$$(\forall n \in \mathbb{N}) \quad (a_n \geqslant b_n) \vee (b_n \geqslant a_n). \tag{2}$$

Using (BD), we obtain $(a_\Omega \geqslant b_\Omega) \vee (b_\Omega \geqslant a_\Omega)$ and thus (1) since $a_\Omega = a$ and $b_\Omega = b$. There is a contradiction. To avoid it, we may admit that the application of (BD) leads to a notion of truth weaker than the usual notion. Hence, we introduce an important logical distinction :

Definition 6. Let $P(x_1, x_2, ..., x_n)$ an internal formula with $x_1, x_2, ..., x_n$ free variables in $\mathbb{Z}$. For $a_i = (a_{i,m})_{m\in\mathbb{N}} \in \mathbb{Z}_\Omega$ $i = 1, ..., n$ if

$$(\exists M \in \mathbb{N})(\forall m > M)(P(a_{1,m}, a_{2,m}, ..., a_{n,m})) \text{ is true,}$$

then the formula $P(a_1, a_2, ..., a_n)$ is said *weakly true*.

In contrast to the weak truth, we may use the terms of strong truth for the usual truth. For instance, (1) is weakly true but not strongly true, and the weak truth of (1) means exactly that (2) is (strongly) true. In the sequel, we will use the following properties.

Proposition 1. The following formulas are weakly true on $\mathbb{Z}_\Omega$:
(1) $\forall (x, y) \in \mathbb{Z}_\Omega^2 \quad (x < y) \vee (x \geq y)$;
(2) $\forall (x, y, z) \in \mathbb{Z}_\Omega^3 \quad (x + y \geqslant z) \Rightarrow (2x \geqslant z) \vee (2y \geqslant z)$.

Proof. Let $x = (x_n)$, $y = (y_n)$ and $z = (z_n)$. For each $n \in \mathbb{N}$, we have

$$(x_n < y_n) \vee (x_n \geq y_n) \text{ and } (x_n + y_n > z_n) \Rightarrow (2x_n > z_n) \vee (2y_n > z_n)$$

Thus, we can apply (BD) and we get the two formulas.

Let us remark that the first formula says that the order relation on $\mathbb{Z}_\Omega$ is (weakly) decidable.

2.3. Some details about Ω-rational numbers

Returning to the Ω-rational numbers, we can check that $(\mathbb{Q}_\Omega, +, \times, \geqslant)$ is a commutative ordered field for the weak truth. Given two Ω-integers $a = (a_n)$ and $b = (b_n)$, if $b \neq 0$ in the weak meaning, then b has an inverse b^{-1} in $\mathbb{Q}_\Omega$ and $a/b =_{\text{def}} a \times b^{-1}$ is an Ω-rational number. Conversely, if $r \in \mathbb{Q}_\Omega$ is weakly different from 0, then there is a unique pair $(a, b) \in \mathbb{Z}_\Omega^2$ with $b > 0$ such that $r = a/b$; then, it is easy to check that we have the usual relations $\lfloor r \rfloor = a \div b$ and $\{r\} = (a \bmod b)$.

An Ω-rational number $a = (a_n)$ is said to be *limited* in case there is a standard $p \in \mathbb{N}$ such that $|a| \leqslant p$ where $|a| = (|a_n|)$; this means that $|a_n| \leqslant p$ for almost $n \in \mathbb{N}$. Let $\mathbb{Q}_\Omega^{lim}$ be the set of limited Ω-rational numbers. In the same way, we say that a is *infinitely small* and we write $a \simeq 0$ in case $p|a| \leqslant 1$ for every $p \in \mathbb{N}$. For $a, b \in \mathbb{Q}_\Omega$, we write $a \simeq b$ when $a - b \simeq 0$ and $a \lesssim b$ when $p(a - b) \leq 1$ for every $p \in \mathbb{N}$. It is easy to check that $\simeq$ is an equivalence relation and that $\lesssim$ is an order relation on $\mathbb{Q}_\Omega$. This leads to the numerical system $(\mathbb{Q}_\Omega^{lim}, \simeq, \lesssim, +, \times)$ which is, for Laugwitz and Schmieden [14], an equivalent of the classical system of the real numbers $(\mathbb{R}, =, \leqslant, +, \times)$.

3. Arithmetization with Ω-integers

In the preceding section, we have seen that the concept of Ω-integer numbers provides a relatively constructive version of nonstandard arithmetic. Now, we are going to use this framework to define an arithmetization method in the spirit of what has been done in a less constructive context [5].

Let us recall what is the general principle of the arithmetization method. It is a process which provides discrete equivalents to continuous functions or curves. This method is based on a new perspective on the continuous line. For this purpose, we use an infinitely large integer number ω which is interpreted as the new unit of the integer numerical system; in other words, the distance between two successive integers is assumed to be equal to $1/\omega$. With this interpretation, we get a discrete system $\mathcal{R}$ which looks like the continuous one. Then, given a function h with real variables and real values, we call arithmetization of h a process which transfers in $\mathcal{R}$ a numerical characterization of h. In our context, this process involves two steps: firstly the arithmetization at a global scale, secondly the translation at an intermediate scale.

Our arithmetization method is based on the Euler scheme usually used for the numeric integration of a common differential equation. Before describing in detail our approach, it is interesting to compare it to other methods which also use this integration scheme in a nonstandard context. Taking up with the intuition of the creators of the infinitesimal calculation, nonstandard analysis uses the Euler scheme based on an infinitesimal step to prove the existence of a solution for a Cauchy problem [8]. The same is true for certain constructive approaches of nonstandard analysis as E. Palmgren showed it [24]. What we do with the Euler scheme is of different nature which, actually, requires new developments: we do not want to integrate a differential equation but to give a discrete multiscale representation of a given real function which is already solution of some Cauchy problem.

Following up on the ideas of G. Reeb and JP. Reveillès [28] and our own previous work, our method starts by introducing a kind of translation of the Euler scheme into the discrete world of Ω-integers. The solution of this new scheme appears as a discrete representation of the initial continuous function. Because this translation carries errors, in particular rounding errors, we then have to show that the obtained discrete solution is, in a certain sense, an exact representative of the initial real function.

3.1. Ω-arithmetization at the global scale

Our method applies to functions which are solutions of some differential equations[2]. More precisely, we consider a real function $X : T \mapsto X(T)$ defined on an interval of $\mathbb{R}$ with values in $\mathbb{R}$. We suppose that X is the solution of the Cauchy problem

$$X' = F(T, X) \quad X(A) = B \tag{3}$$

where F is a continuously differentiable function ; for simplicity, we assume that F is defined on the whole plane $\mathbb{R}^2$. To this differential problem is associated the Euler scheme with integration step $\frac{1}{S}$ and real variables T_k and X_k:

$$\begin{cases} T_0 = A \ ; \ X_0 = B \\ T_{k+1} = T_k + \frac{1}{S} \times 1 \\ X_{k+1} = X_k + \frac{1}{S} \times F(T_k, X_k) \end{cases} \tag{4}$$

[2]The set $\mathbb{R}$ of real numbers and the differential equation that follows are supposed defined in the framework of classical analysis.

The scheme (4) provides a sequence of points $(T_k, X_k)_{k\geq 0}$ such that each X_k is an approximation of $X(T_k)$. It is well know that the error of the approximation $|X(T_k) - X_k|$ converges to 0 when $S \to +\infty$.

Now, we want to transfer this scheme in the arithmetical system $\mathbb{Z}_\Omega$ of Ω-relative numbers. To this end, we choose a scale, that is to say a number

$$\omega = (\omega_m)_{m\in\mathbb{N}} \in \mathbb{N}_\Omega \text{ such that } \omega \simeq +\infty. \tag{5}$$

We want to consider that $\mathbb{Z}_\Omega$ is a kind of continuous line in which ω is the unit. In this perspective, we introduce a function $\Psi_\omega : \mathbb{R} \to \mathbb{Z}_\Omega$: for each $U \in \mathbb{R}$, we define $\Psi_\omega(U) = (\lfloor \omega_m U \rfloor)_{m\in\mathbb{N}} \in \mathbb{Z}_\Omega$. The element $\Psi_\omega(U)$ is the representative of U at the scale ω in $\mathbb{Z}_\Omega$. We note that $\Psi_\omega(1) = \omega$ so that ω is the representative of the real number 1. Although the map Ψ_ω is far from being bijective, we can consider it as a kind of change of variables. For instance, we consider that the representative of a function $\Phi : \mathbb{R} \to \mathbb{R}$ is the function $\phi : \mathbb{Z}_\Omega \to \mathbb{Z}_\Omega$ given by

$$\forall t = (t_m)_{m\in\mathbb{N}} \in \mathbb{Z}_\Omega \qquad \phi(t) = (\lfloor \omega_m \Phi(t_m/\omega_m) \rfloor)_{m\in\mathbb{N}} \tag{6}$$

Similarly, the representative of the function $F : \mathbb{R}^2 \to \mathbb{R}$ in the differential equation (3) is the function $f : \mathbb{Z}_\Omega^2 \mapsto \mathbb{Z}_\Omega$ given by

$$\forall (t, x) \in \mathbb{Z}_\Omega^2 \qquad f(t, x) = (\lfloor \omega_m F(t_m/\omega_m, x_m/\omega_m) \rfloor)_{m\in\mathbb{N}} \tag{7}$$

where $t = (t_m)_{m\in\mathbb{N}}$ and $x = (x_m)_{m\in\mathbb{N}}$. In the following, we will focus on the case where this function f has a purely arithmetic expression[3].

Now, we consider the step $\frac{1}{S}$ and its involvement in (4). Through the change of variables (6), the real multiplication $U \mapsto \frac{1}{S} \times U$ is represented in $\mathbb{Z}_\Omega$ by the map $u = (u_m)_{m\in\mathbb{N}} \mapsto (\lfloor \frac{1}{S} u_m \rfloor)_{m\in\mathbb{N}}$. Since we also want to represent the condition $S \to +\infty$, we can replace S by $1/\beta_k$ for each $k \in \mathbb{N}$ where $\beta = (\beta_k)_{k\in\mathbb{N}}$ is a given Ω-natural number such that $\beta \simeq +\infty$. Finally, we can represent both the parameter S and the condition $S \to +\infty$ by deciding that the real multiplication $U \mapsto \frac{1}{S} \times U$ is replaced in $\mathbb{Z}_\Omega$ by the map $u = (u_m)_{m\in\mathbb{N}} \mapsto (\lfloor u_m/\beta_m \rfloor)_{m\in\mathbb{N}} = u \div \beta$ (where $\div$ is the Euclidean division). For instance, the term $\frac{1}{S} \times 1$ is represented by $\omega \div \beta$; in order to get a non trivial term, it is better to suppose that ω converges to $+\infty$

[3]We intend to define algorithms using only arithmetic operations on integer numbers.

more quickly than β. Finally, to avoid the multiplicity of parameters and conditions, we assume the simplifying hypothesis

$$\omega = \beta^2 \text{ where } \beta \in \mathbb{N}_\Omega \text{ such that } \beta \simeq +\infty. \tag{8}$$

so that $\omega_m = \beta_m^2$ for all $m \in \mathbb{N}$ and $\lim_{m\to+\infty} \beta_m = +\infty$. From now on, we suppose that ω and β are fixed Ω-integers subject to the condition (8). Then, working term by term according to the above principle, we are able to define an arithmetic analogue of the Euler Scheme (4).

Definition 7. The *Ω-arithmetization of the Euler scheme (4) at the global scale ω* is the following scheme with variables $x_k, t_k \in \mathbb{Z}_\Omega$

$$\begin{cases} t_0 = a \ ; \ x_0 = b \\ t_{k+1} = t_k + \beta \\ x_{k+1} = x_k + f(t_k, x_k) \div \beta \end{cases} \tag{9}$$

where $a = \lfloor \omega A \rfloor$, $b = \lfloor \omega B \rfloor$ and f is defined by (6).

Note that there is some arbitrariness in this definition since we had to make some choices for going from (4) to the new scheme. Nevertheless, (9) has a strong analogy with (4) and moreover, we will show that this scheme leads to interesting results. Using (9) iteratively, we obtain a sequence of points (t_k, x_k) which is the graph of a discrete function $t \mapsto x(t)$ which is called *the Ω-arithmetization at the scale ω of the initial real function $T \mapsto X(T)$*.

3.2. The convergence of the arithmetized scheme

Since the last definition results of reasonable but somewhat arbitrary choices, it is natural to wonder about the existence of a precise link between (9) and the original Euler scheme (4). We will show that this new scheme contains enough information to reconstruct asymptotically the original Euler scheme. For this purpose, we remark that the variables t_k and x_k represent sequences $(t_{k,m})_{m\in\mathbb{N}}$ and $(x_{k,m})_{m\in\mathbb{N}}$ of integers subject to the following scheme depending on $m \in \mathbb{N}$ with variables $t_{k,m}, x_{k,m} \in \mathbb{Z}$

$$\begin{cases} t_{0,m} = a_m \ ; \ x_{0,m} = b_m \\ t_{k+1,m} = t_{k,m} + \beta_m \\ x_{k+1,m} = x_{k,m} + f_m(t_{k,m}, x_{k,m}) \div \beta_m \end{cases} \tag{10}$$

where
$a_m = \lfloor \omega_m A \rfloor$, $b_m = \lfloor \omega_m B \rfloor$ and $f_m(t_{k,m}, x_{k,m}) = \lfloor \omega_m F(t_{k,m}/\omega_m, x_{k,m}/\omega_m) \rfloor$.

If each line of (10) is divided by ω_m, then we obtain a scheme depending on $m \in \mathbb{N}$ with real variables $T'_k = t_{k,m}/\omega_m$ and $X'_k = x_{k,m}/\omega_m$

$$\left\{ \begin{array}{l} T'_0 = A_m \ ; \ X'_0 = B_m \\ T'_{k+1} = T'_k + 1/\beta_m \\ X'_{k+1} = X'_k + (1/\beta_m) F_m(T'_k, X'_k) \end{array} \right. \tag{11}$$

We remark that (11) is the Euler scheme with integration step $1/\beta_m$ for the new Cauchy problem $X' = F_m(T, X)$ with $X(A_m) = B_m$. When $m \to +\infty$, the step $1/\beta_m$ tends to 0 and the next result shows that, is some sense, the scheme (11) converges to (4).

Proposition 2. The sequence of functions $(F_m)_{m\in\mathbb{N}}$ converges uniformly to F and $\lim_{m\to+\infty} A_m = A$, $\lim_{m\to+\infty} B_m = B$. More precisely

$$|A - A_m| < \frac{1}{\omega_m} \ , \ |B - B_m| < \frac{1}{\omega_m} \ , \ |F(T,X) - F_m(T,X)| < \frac{1}{\omega_m} + \frac{1}{\beta_m} \tag{12}$$

for all $m \in \mathbb{N}$ and all $(T, X) \in \mathbb{R}^2$.

Proof. All the proof is based on the decomposition $U = \lfloor U \rfloor + \{U\}$ of any real number U where $\lfloor U \rfloor \in \mathbb{Z}$ and $0 \leq \{U\} < 1$. For instance

$$A_m = \frac{1}{\omega_m} \lfloor \omega_m A \rfloor = \frac{1}{\omega_m} (\omega_m A - \{\omega_m A\}) = A - \frac{1}{\omega_m} \{\omega_m A\}$$

so that $|A - A_m| < \dfrac{1}{\omega_m}$ and ditto for B_m and B. From the definition of (11), we see that F_m is such that

$$\frac{1}{\beta_m} F_m(T,X) = \frac{1}{\omega_m} (\lfloor \omega_m F(T,X) \rfloor \div \beta_m) = \frac{1}{\omega_m} \lfloor \lfloor \omega_m F(T,X) \rfloor / \beta_m \rfloor$$

which gives

$$F_m(T,X) = F(T,X) - \frac{\{\omega_m F(T,X)\}}{\omega_m} - \frac{1}{\beta_m} \left\{ \frac{\omega_m F(T,X) - \{\omega_m F(T,X)\}}{\beta_m} \right\}$$

Hence $|F(T,X) - F_m(T,X)| < \dfrac{1}{\omega_m} + \dfrac{1}{\beta_m}$.

3.3. Ω-arithmetization at the intermediate scale β

Now, we return to the the Ω-arithmetization at the scale ω of the initial real function $T \mapsto X(T)$, that is to say, the discrete function $t \mapsto x(t)$ given by the scheme (9). This function suffers from a major imperfection: its domain is far from being connected, since $t_{k+1} - t_k = \beta \gg 1$. In order to correct this defect, we perform the following arithmetic scaling:

$$\begin{array}{ccc} \mathbb{Z}_\Omega & \longrightarrow & \mathbb{Z}_\Omega \\ x & \longmapsto & \lfloor x\beta/\omega \rfloor = x \div \beta. \end{array}$$

which sends β to 1. With this transformation, we move from $\mathbb{Z}_\Omega$ equipped with the unit ω to $\mathbb{Z}_\Omega$ equipped with the unit β. This will allow us to observe the arithmetized solution at the intermediate scale β.

In order to compute the effect of this scaling, it is convenient to introduce the following decomposition: for every $x \in \mathbb{Z}_\Omega$, we write $x = \widetilde{x}\beta + \widehat{x}$, where $\widetilde{x} =_{def} x \div \beta$ and $\widehat{x} =_{def} x \bmod \beta$. Thus, $\widetilde{x}$ is the result of the scaling on $x \in \mathbb{Z}_\Omega$. Using this decomposition in (9), we obtain the following scheme.

Definition 8. The *Ω-arithmetization of the Euler Scheme (4) at the intermediary scale* β is the following scheme with variables $\widetilde{t}_k, \widetilde{x}_k, \widehat{x}_k \in \mathbb{Z}_\Omega$

$$\left\{ \begin{array}{l} \widetilde{t}_0 = a \div \beta,\ \widetilde{x}_0 = b \div \beta \text{ and } \widehat{x}_0 = b \bmod \beta \\ \widetilde{t}_{k+1} = \widetilde{t}_k + 1 \\ \widetilde{x}_{k+1} = \widetilde{x}_k + (\widehat{x}_k + \widetilde{f}_k) \div \beta \\ \widehat{x}_{k+1} = (\widehat{x}_k + \widetilde{f}_k) \bmod \beta \end{array} \right. \tag{13}$$

where $\widetilde{f}_k = f(\widetilde{t}_k\beta + a \bmod \beta, \widetilde{x}_k\beta + \widehat{x}_k) \div \beta$, f is defined by (7) and $a = \lfloor \omega A \rfloor$, $b = \lfloor \omega B \rfloor$.

Now, the relevant variables are $\widetilde{t}_k$ and $\widetilde{x}_k$ while the $\widehat{x}_k$ are auxiliary variables that manage the remainder coming from the Euclidean division.

The important outcome of this scaling is that the discrete function $\widetilde{t} \mapsto \widetilde{x}(\widetilde{t})$ whose graph is the set of points $(\widetilde{t}_k, \widetilde{x}_k)$ is now defined over a connected domain, because $\widetilde{t}_{k+1} - \widetilde{t}_k = 1$ for each k. This discrete function is the *arithmetization of the initial real function* $X : T \mapsto X(T)$ *at the intermediate scale* β.

3.4. Ω-iteration

Let us now examine how we can compute the values of the discrete function $\widetilde{t} \mapsto \widetilde{x}\left(\widetilde{t}\right)$ that we have just obtained. Since (13) is an iterative scheme, the usual procedure is to choose a number of iteration $N \in \mathbb{N}$ and to use the scheme to compute the successive values $(\widetilde{t}_k, \widetilde{x}_k)$ for $k = 0,\ ...,\ N$. Hence, we obtain a description of the function on the discrete interval $\{\widetilde{t}_0, \ldots, \widetilde{t}_N\}$. The problem is that this interval is very small at the scale β since N is standard and $\beta \simeq +\infty$. We would like to generalized this procedure in order to obtain a function defined on an interval whose length is of the same order as the unit β. This will be done by giving the definition of a number of iterations which is an arbitrary Ω-natural number.

To this end, we consider the very general framework of a function $\Phi : \mathbb{Z}_\Omega^p \to \mathbb{Z}_\Omega^p$ with $p \in \mathbb{N}$. Every element $u \in \mathbb{Z}_\Omega$ may be put into the form of a sequence $u = (u_m)_{m\in\mathbb{N}}$ where $u_m \in \mathbb{Z}^p$ for all $m \in \mathbb{N}$. We suppose that there is a sequence $(\Phi_m)_{m\in\mathbb{N}}$ of function $\Phi_m : \mathbb{Z}^p \to \mathbb{Z}^p$ such that, for all $u = (u_m)_{m\in\mathbb{N}} \in \mathbb{Z}_\Omega^p$, we have $\Phi(u) = (\Phi_m(u_m))_{m\in\mathbb{N}}$. Given $\alpha \in \mathbb{Z}_\Omega^p$, we know what is $\Phi^N(\alpha)$ for $N \in \mathbb{N}$; we want to generalize to an exponent which is an Ω-natural number.

Definition 9. For $\alpha = (\alpha_m)_{m\in\mathbb{N}} \in \mathbb{Z}_\Omega^p$ and $\nu = (\nu_m)_{m\in\mathbb{N}} \in \mathbb{N}_\Omega$ we define

$$\Phi^\nu(\alpha) =_{\text{def}} (\Phi_m^{\nu_m}(\alpha_m))_{m\in\mathbb{N}}$$

This definition may be applied to any scheme of the form $z_0 = \alpha\ ;\ z_{k+1} = \Phi(z_k)$ provided Φ is a function as above. This is the case of the scheme (9) since this scheme is decomposed into the sequence of schemes (10). For similar reasons, this is also the case for the scheme (13).

With the help of this definition, we are now able to compute the value $x(t)$ for each $t \in \mathbb{Z}_\Omega$ of the form $t_0 + \nu\beta$ where $\nu \in \mathbb{N}_\Omega$. Indeed, this value is obtained through ν iterations of the scheme (9). Similarly, we can compute the value $\widetilde{x}\left(\widetilde{t}\right)$ for each $\widetilde{t} \in \mathbb{Z}_\Omega$ of the form $\widetilde{t}_0 + \nu$ where $\nu \in \mathbb{N}_\Omega$ is again the number of iterations applied now to the scheme (13).

3.5. Ω-arithmetization is an exact representation

Now, we claim that the arithmetization of the real function $T \mapsto X(T)$ is an *exact representation* of this function. This statement means that the information contained in the function $t \mapsto x(t)$ (or in the function $\tilde{t} \mapsto \widetilde{x}(\tilde{t})$) is sufficient to reconstruct $T \mapsto X(T)$. At the first glance, this claim

is surprising since the functions x and $\widetilde{x}$ are obtained from X via the use of a numerical approximation method with additional truncation rounding. Nevertheless, we will show that this property of exact representability follows ultimately the very nature of nonstandard entities that encapsulate in a single object an entire process of convergence.

In order to simplify the proof of the next result, we suppose that the function F satisfies the following global Lipschitz condition[4]: there is a real constant $C > 0$ such that for all $(T_1, X_1), (T_2, X_2) \in \mathbb{R}^2$

$$|F(T_1, X_1) - F(T_2, X_2)| \leq C(|T_1 - T_2| + |X_1 - X_2|) \tag{14}$$

Theorem 1. *We suppose that the solution $T \mapsto X(T)$ of the initial Cauchy problem (3) is defined on the real interval $[A, D]$ with $A < D$. Given a real number T such $T \in [A, D]$, we consider an Ω-natural number $\nu = (\nu_m)_{m\in\mathbb{N}}$ such that*

$$\forall m \in \mathbb{N} \quad \frac{\nu_m}{\beta_m} \leq D - A \quad \textit{and} \quad \lim_{m\to+\infty} \frac{\nu_m}{\beta_m} = T - A \tag{15}$$

Then

$$X(T) = \lim_{m\to+\infty} \frac{(x(t_0 + \nu\beta))_m}{\omega_m} = \lim_{m\to+\infty} \frac{(\widetilde{x}(\tilde{t}_0 + \nu))_m}{\beta_m} \tag{16}$$

where $((x(t_0+\nu\beta))_m)_{m\in\mathbb{N}}$ and $((\widetilde{x}(\tilde{t}_0+\nu))_m)_{m\in\mathbb{N}}$ are the sequences representing respectively the Ω-numbers $x(t_0 + \nu\beta)$ and $\widetilde{x}(\tilde{t}_0 + \nu)$.

Before giving the proof, let us examine the meaning of condition (15). For a fixed $m \in \mathbb{N}$, we consider the following subdivision of the interval $[A, D]$

$$A < A + \frac{1}{\beta_m} < A + \frac{2}{\beta_m} < \cdots < A + \frac{\lfloor \beta_m(D - A)\rfloor}{\beta_m} \leq D \tag{17}$$

that defines a family of intervals of length $1/\beta_m$ except for the length of the last interval that may be less. Then, $A + \nu_m/\beta_m$ is a point of (17) and, when $m \to +\infty$, the sequence $(A + \nu_m/\beta_m)$ converges to T while the step $1/\beta_m$ converges to 0. An example of such an Ω-number $\nu = (\nu_m)$ is given by $\nu_m =_{\text{def}} \lfloor \omega_m(T - A)\rfloor \div \beta_m$ for all $m \in \mathbb{N}$; indeed

$$\frac{\lfloor \omega_m(T - A)\rfloor}{\omega_m} = \frac{\nu_m}{\beta_m} + \rho_m \quad \text{with} \quad 0 \leq \rho_m < \frac{1}{\beta_m}$$

[4]This simplifying assumption could be avoided by the means of some technical complications in the proof.

from which follows condition (15).

PROOF (OF THEOREM 1). The main idea of the proof is to give an estimate of the difference between the solutions of the schemes (4) and (11) both provided with the same step $1/\beta_m$.

According to proposition 2, the difference between the initial data is such that $|T_0 - T'_0| = |A - A_m| < 1/\omega_m$ and $|X_0 - X'_0| = |B - B_m| < 1/\omega_m$.

Regarding the temporal variables, we have $|T_{k+1} - T'_{k+1}| = |T_k - T'_k| = |T_0 - T'_0| < 1/\omega_m$.

For the difference between the spatial variables, we have

$$X_{k+1} - X'_{k+1} = (X_k - X'_k) + \frac{1}{\beta_m}\left(F(T_k, X_k) - F_m(T'_k, X'_k)\right) \tag{18}$$

Since we may write

$$F(T_k, X_k) - F_m(T'_k, X'_k) = (F(T_k, X_k) - F(T'_k, X'_k)) + (F(T'_k, X'_k) - F_m(T'_k, X'_k))$$

we obtain from proposition 2 and condition (14)

$$|F(T_k, X_k) - F_m(T'_k, X'_k)| \leq \frac{1}{\beta_m} + \frac{C+1}{\omega_m} + C|X_k - X'_k|$$

and thus (18) gives

$$|X_{k+1} - X'_{k+1}| \leq \left(1 + \frac{C}{\beta_m}\right)|X_k - X'_k| + \frac{1}{\omega_m} + \frac{C+1}{\beta_m \omega_m}$$

It is easy to check that, given a sequence of inequalities $e_{k+1} \leq ae_k + b$ with $a \neq 1$ and $b \neq 0$, we have $e_k \leq a^k e_0 + b(a^k - 1)/(a - 1)$ for all $k \in \mathbb{N}$. Hence

$$|X_k - X'_k| \leq \left(1 + \frac{C}{\beta_m}\right)^k |X_0 - X'_0| + \frac{\beta_m + C + 1}{C\omega_m}\left(\left(1 + \frac{C}{\beta_m}\right)^k - 1\right)$$

Since $|X_0 - X'_0| < 1/\omega_m$ and $(1 + u)^k \leq e^{ku}$ for all $u \geq 0$ and $k \in \mathbb{N}$, we get

$$|X_k - X'_k| \leq \frac{e^{kC/\beta_m}}{\omega_m} + \frac{\beta_m + C + 1}{C\omega_m}\left(e^{kC/\beta_m} - 1\right)$$

Now, if we choose ν_m for value of k, then we obtain the fundamental estimation

$$|X_{\nu_m} - X'_{\nu_m}| \leq \frac{e^{C(D-A)}}{\omega_m} + \frac{\beta_m + C + 1}{C\omega_m}\left(e^{C(D-A)} - 1\right) \tag{19}$$

The first part of the conclusion follows since:
a) the right member of inequality (19) converges to 0 when $m \to +\infty$,
b) by the definition, $X'_{\nu_m} = (x(t_0 + \nu\beta))_m/\omega_m$,
c) X_{ν_m} is the value in the point ν_m/β_m of the Euler approximation given by the initial scheme (4) with the step $1/\beta_m$,
d) it is well known that this Euler approximation converges to $X(T)$ when $m \to +\infty$.

For the second part of the conclusion, we note that

$$(x(t_0 + \nu\beta))_m = (\widetilde{x}(\tilde{t}_0 + \nu))_m \, \beta_m + r_m \quad \text{with} \quad 0 \leq r_m < \beta_m$$

since $(\widetilde{x}(\tilde{t}_0 + \nu))_m$ is the euclidean quotient of $(x(t_0 + \nu\beta))_m$ by β_m. Dividing both side of the equation by ω_m, we see at once that $(x(t_0 + \nu\beta))_m/\omega_m$ and $(\widetilde{x}(\tilde{t}_0 + \nu))_m/\beta_m$ have the same limit when $m \to +\infty$.

4. Illustration of Ω-arithmetization

In order to plot the results of our computations, we have to give a meaningful geometric representation of the Ω-objects given by the final algorithm (13). Since the output of this algorithm is the sequence $(\widetilde{t}_k, \widetilde{x}_k)_{0 \leq k}$ where $\widetilde{t}_k, \widetilde{x}_k \in \mathcal{HR}_\beta$, it is sufficient to explain how we can represent an element $(a, b) \in \mathcal{HR}_\beta^2$ by taking account two aspects : first the multiplicity of scale in the algorithm (ω and β) and secondly the nature of Ω-numbers which are sequences.

Let us consider such a pair (a, b) where $a = (a_m)_{m \in \mathbb{N}}$ and $b = (b_m)_{m \in \mathbb{N}}$ are two elements of $\mathcal{HR}_\beta$ and that our graphic plane is $\mathbb{R}^2$. If we were only interested by the level m, we plunge (a_m, b_m) in $\mathbb{R}^2$ via the function $\varphi_m : (x, y) \mapsto \left(\dfrac{x}{\beta_m}, \dfrac{y}{\beta_m}\right)$ and we chose to represent (a_m, b_m) by the unit square (at the scale β_m) $C_m(a, b) =_{\text{def}} [a_m/\beta_m, (a_m + 1)/\beta_m] \times [b_m/\beta_m, (b_m + 1)/\beta_m]$. With this choice, each square C_m is a pixel of size $1/\beta_m$ Taking into account all the levels $m \in \mathbb{N}$, the Ω-point (a, b) is represented by the infinite sequence of pixels $(C_m)_{m \geq 0}$ of globally decreasing sizes. We can summarize the representation of an Ω-point in the definition of an Ω-pixel.

Definition 10. An Ω-*pixel* C=$(C_m)_{m>0}$ representing an Ω-point $(a, b) \in \mathcal{HR}_\beta^2$ is defined by the sequence of squares

$$C_m(a, b) =_{\text{def}} [a_m/\beta_m, (a_m + 1)/\beta_m] \times [b_m/\beta_m, (b_m + 1)/\beta_m].$$

Each square C_m is associated to a grid which divided the plan into squares of length $1/\beta_m$. In general, when m varies, these grids are randomly overlapping. But, if we choose β such that for all m, β_m is a factor of β_{m+1}, each grid is obtained by subdivision of the previous. In the following plots, our choice is $\beta = (2^m)_{m\geq 0}$, consequently each square at a level m is regularly decomposed in four squares at the level $m+1$ which leads possible, with the choice of different colors assigned for each level, an unified graphical representation of Ω-numbers. Hence, for instance, the Ω-point $(1,1)$ in $\mathcal{HR}^2_\beta$ is represented in the figure (2) where the origin $(0,0)$ is for each level the square at the bottom right.

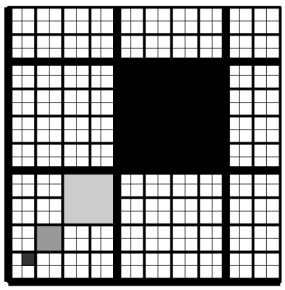
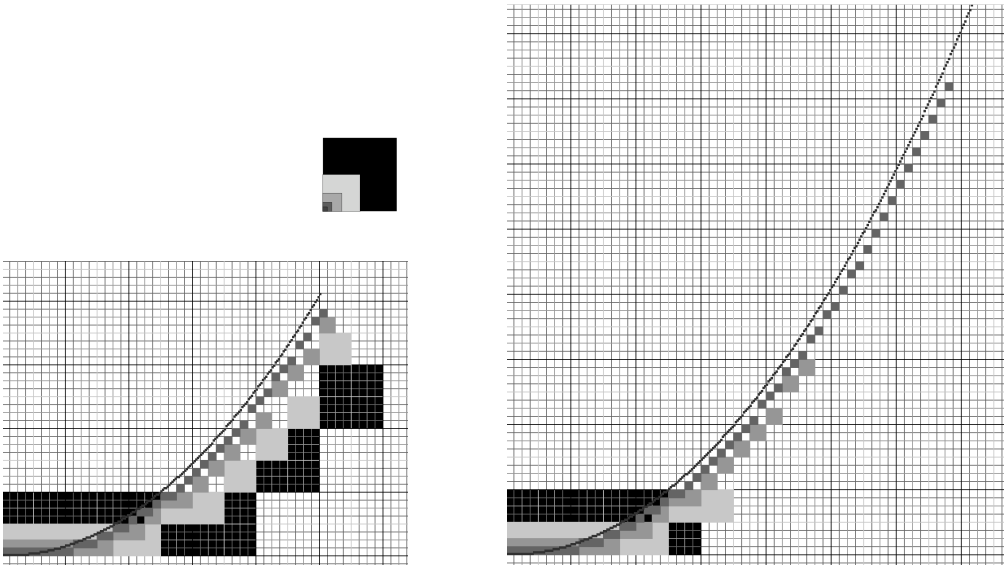


Figure 2: Some scale of the point (1,1).

Let us now show two examples of Ω-arithmetization : a polynomial function and a line. Let the function $x_1 : t \to t^2/6$. Hence, the function f_n in the scheme (13) is $t_n/3$ and the initial conditions are $\widetilde{t}_0 = 0$, $\widetilde{x}_0 = 0$ and $\widehat{x}_0 = 0$. The figures (3(a)) and (3(b)) give two representations of the Ω-arithmetization of the function x_1 at the scale $\beta = (2^m)_{m\geq 0}$ and for two different numbers of iterations. When the number of iteration N is an integer factor of β like in (3(a)) where $N = (5.2^m)_{m\in\mathbb{N}}$, the domain of definition of the function has the same length in each scale. When $N \gg \beta$, like in (3(b)) where $N = (2.3^m)_{m\in\mathbb{N}}$ more the level is high, more the domain of definition is long.

The line on the figure (1) in the introduction is $x_2 : t \to 3t/5$, hence, the function f_n in the corresponding algorithm is the constant $3/5$ and the initial conditions are $(0,0,0)$. $N = (2.3^m)_{m\in\mathbb{N}}$ iterations are made.

The aspect multi-scale of the representation is intrinsic to the structure of the Ω-numbers. In fact, work with sequences allows to consider that each level represents one scale of lecture. Each scale represents one approximation of the real initial function and more the scale is large, better the discrete



(a) (b)

Figure 3: Arithmetization of $x_1 : t \to t^2/6$ with two differents Ω-iterations.

approximation of the initial function is. With the notion of Ω-iteration, we control the link between these approximations and their length of representation.

5. Foundational aspects

In this last section, we first present the Harthong-Reeb line noted $\mathcal{HR}_\omega$, then, we prove that, in the intuitionnistic logic, the $\mathcal{HR}_\omega$ line partly verifies the Bridges' axioms [2]. And, in a third part, we explain the work [17] of Magaud and some of the authors about a formalization with the Coq proof assistant [1, 7] of the Harthong-Reeb line $\mathcal{HR}_\omega$ and the Ω-number system. Some hints about the payoff of such a formalization are also given.

5.1. The Harthong-Reeb line

The Harthong-Reeb line is a numerical line which is, in some meaning, both discrete and continuous. To obtain such a paradoxical space, the basic idea is to make a strong contraction on the set $\mathbb{Z}$ such that the prescribed infinitely large $\omega \in \mathbb{N}$ becomes the new unit; the result of this scaling is a

line which looks like the real line (See figure 4). Historically, this system

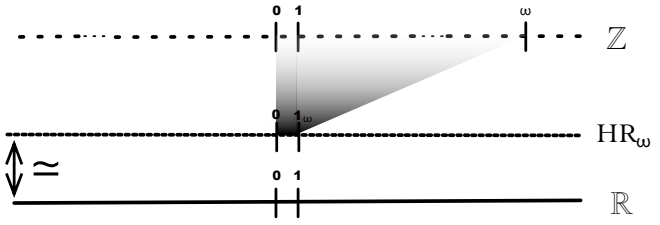


Figure 4: An intuitive representation of the Harthong-Reeb line.

is at the origin of the definition of the analytic discrete line proposed by J.P. Reveillès [26, 27] in discrete geometry. For a rigorous implementation of this idea, we must have a mathematical concept of infinitely large numbers. In previous works [9, 11, 5] on this subject, this was done with the help of an axiomatic version of nonstandard analysis in the spirit of the Internal Set Theory [21]. Our purpose in the present section is to define a Harthong-Reeb line based on the notion of Ω-integers introduced in the previous section. Our main motivation is to obtain a more constructive version of the Harthong-Reeb line allowing an exact translation of the arithmetization process into computer programs.

5.1.1. Formalization of the Harthong-Reeb line

Although the definition has already been stated in the section (2.2), we recall that an Ω-number a is infinitely large if, for all $p \in \mathbb{N}$, we have $p \leq |a|$. If a is infinitely large and $a > 0$ we note $a \simeq +\infty$. We already know that $\Omega = (n)_{n\in\mathbb{N}} \simeq +\infty$. More generally, for $a = (a_n)$, it is easy to check that $a \simeq +\infty$ if and only if $\lim_{n\to+\infty} a_n = +\infty$.

Notation 1. The symbol ω denotes a fixed Ω-integer such that $\omega \simeq +\infty$.

Let us remark that ω may be different from Ω. We only know that there is a sequence (ω_n) of natural numbers such that $\omega = (\omega_n)$ and $\lim_{n\to+\infty} \omega_n = +\infty$. Now, we are going to give the definition of the Harthong-Reeb line which results in the scaling on $\mathbb{Z}_\Omega$ such that ω becomes the new unit.

Definition 11. We consider the following set

$$\mathcal{HR}_\omega = \{x \in \mathbb{Z}_\Omega,\ \exists p \in \mathbb{N},\ |x| \leq p\,\omega\}$$

and the relations, operations and constants on $\mathcal{HR}_\omega$ described by the following definitional equalities: for all $(x, y) \in \mathcal{HR}_\omega^2$, we set

- $(x =_\omega y) =_{\text{def}} (\forall p \in \mathbb{N})\,(p|x - y| \leq \omega)$;
- $(x >_\omega y) =_{\text{def}} (\exists p \in \mathbb{N})\,(p(x - y) \geq \omega)$;
- $(x \neq_\omega y) =_{\text{def}} (x >_\omega y) \vee (x <_\omega y)$;
- $(x \leq_\omega y) =_{\text{def}} (\forall z \in \mathcal{HR}_\omega)\,(z <_\omega x \Rightarrow z <_\omega y)$;
- $(x +_\omega y) =_{\text{def}} (x + y)$ and $0_\omega =_{\text{def}} 0$ and $-_\omega x =_{\text{def}} -x$;
- $(x \times_\omega y) =_{\text{def}} ((x \times y) \div \omega)$ and $1_\omega =_{\text{def}} \omega$ and $x^{(-1)_\omega} =_{\text{def}} (\omega^2 \div x)$ for $x \neq_\omega 0$.

Then, the Harthong-Reeb line is the numerical system $(\mathcal{HR}_\omega, =_\omega, \leq_\omega, +_\omega, \times_\omega)$.

We can say that $\mathcal{HR}_\omega$ is the set of Ω-integers which are limited at the scale ω. Note that the way of introducing separately the two order relations and the non-equality relation is quite traditional from a constructive point of view.

Proposition 3. For every $x = (x_n)$ and $y = (y_n)$ in $\mathcal{HR}_\omega$, we have the following equivalences:

(1) $x =_\omega y \iff \forall p \in \mathbb{N}\ \exists M_p \in \mathbb{N}\ \forall n \geq M_p \quad p|x_n - y_n| \leq \omega_n$

(2) $x >_\omega y \iff \exists p \in \mathbb{N}\ \exists M_p \in \mathbb{N}\ \forall n \geq M_p \quad p(x_n - y_n) \geq \omega_n$

(3) $x \leq_\omega y \iff \forall p \in \mathbb{N},\ p(x - y) \leq \omega$

Proof. The points (1) and (2) result of the definition of the order relation $\leq$ on $\mathbb{Z}_\Omega$. We will only give the outline of a proof for (3).
Let us suppose that $x \leq_\omega y$. For every $p \in \mathbb{N} \setminus \{0\}$, we consider $z_p =_{\text{def}} x - \lfloor \omega/p \rfloor$. Since $z_p <_\omega x$, we obtain $z_p <_\omega y$. Thus, there is $k \in \mathbb{N}$ such that $k\,(y - x + \lfloor \omega/p \rfloor) \geq \omega$. Hence, for every $p \in \mathbb{N}$

$$p(x - y) \leq p\lfloor \omega/p \rfloor - p\omega/k = p\,(\omega/p - \{\omega/p\}) - p\omega/k \leq \omega$$

Let us suppose now that $p(x - y) \leq \omega$ for each $p \in \mathbb{N}$. We consider an arbitrary $z \in \mathcal{HR}_\omega$ such that $z <_\omega x$. Thus, there is $k \in \mathbb{N}$ such that $k(x - z) \geq \omega$. We obtain $k(y - z) \geq k(y - x) + \omega$ and since $2k(y - x) \geq -\omega$ we get $2k(y - z) \geq \omega$ and thus $z <_\omega y$.

5.1.2. Isomorphism with a model of the real line

Now, we want to show that the Harthong-Reeb line is equivalent to the system of real numbers. In this context, the appropriate model for the real line is the system $(\mathbb{Q}_\Omega^{lim}, \simeq, \lesssim, +, \times)$ of limited Ω-rational numbers of Laugwitz and Schmieden described in the section (2.3). To this end, we introduce the two following maps:

$$\left\{ \begin{array}{ccccc} \varphi_\omega & : & \mathcal{HR}_\omega & \to & \mathbb{Q}_\Omega^{lim} \\ & & x & \mapsto & x/\omega \end{array} \right\} \text{ and } \left\{ \begin{array}{ccccc} \psi_\omega & : & \mathbb{Q}_\Omega^{lim} & \to & \mathcal{HR}_\omega \\ & & u & \mapsto & (\lfloor \omega u \rfloor) \end{array} \right\}$$

The proof of the following properties is straightforward.

Proposition 4. For every $x, y \in \mathcal{HR}_\omega$ and $u \in \mathbb{Q}_\Omega^{lim}$, we have :

- $x \leq_\omega y \quad \Rightarrow \quad \varphi_\omega(x) \lesssim \varphi_\omega(y)$;
- $\varphi_\omega(x +_\omega y) \simeq \varphi_\omega(x) + \varphi_\omega(y)$;
- $\varphi_\omega(x \times_\omega y) \simeq \varphi_\omega(x) \times \varphi_\omega(y)$;
- $\varphi_\omega(0_\omega) \simeq 0$ and $\varphi_\omega(1_\omega) \simeq 1$;
- $x =_\omega y \Leftrightarrow \varphi_\omega(x) \simeq \varphi_\omega(y)$;
- $\psi_\omega \circ \varphi_\omega(x) =_\omega x$ and $\varphi_\omega \circ \psi_\omega(u) \simeq u$.

We can summarize these properties by saying that φ_ω *is an isomorphism from* $(\mathcal{HR}_\omega, =_\omega, \leq_\omega, +_\omega, \times_\omega)$ *to* $(\mathbb{Q}_\Omega^{lim}, \simeq, \lesssim, +, \times)$ and that ψ_ω *is the inverse isomorphism.*

5.2. Theoretical analysis of the conctructive content of $\mathcal{HR}_\omega$

We know that the Harthong-Reeb line $\mathcal{HR}_\omega$ is a model of the real line and it was also argued that the presented construction fits into a constructive framework.

We now analyze how the construction of the Harthong-Reeb line over the Ω-number system is consistent with the specific constraints of constructive mathematics. With regards to the constructivism, we only recall that

these mathematics are characterized by the BHK-interpretation of the logical constants[5]. For more precisions the reader should refer to the excellent description given in [2].

Our method is to analyse to what extent the system $\mathcal{HR}_\omega$ fits with the constructive axiomatic of $\mathbb{R}$ proposed by Bridges [2, 3]. We call this axiomatic structure a *Bridges-Heyting ordered field* (abbreviated as BH-ordered field).

5.2.1. Presentation of a Bridges-Heyting ordered field

Definition 12. A *BH-ordered field* is a system of the form

$$(R, >, +, \times, 0, 1, \mathrm{Op}, \mathrm{Inv})$$

where R is a set, $>$ is a binary relation on R, $+$ and $\times$ are two operations $(x, y) \mapsto x + y$ and $(x, y) \mapsto xy$ on R, 0 and 1 are two distinguished elements of R, Op is a function $x \mapsto -x$ from R to R, Inv is a map $x \mapsto x^{-1}$ from a subset R^* of R to R. Then, we define a non-equality relation $\neq$ on R by $x \neq y$ if and only if ($x > y$ or $y > x$), a binary relation $\geq$ by $x \geq y$ if and only if $\forall z(y > z \Rightarrow x > z)$, an equivalence relation $=$ by $x = y$ if and only if ($x \geq y$ and $y \geq x$), the subset $R^* =_{\mathrm{def}} \{x \in R\ ;\ x \neq 0\}$. We suppose that all the preceding relations, operations and functions are extensional relatively to $=$ and that the three following groups of axioms are satisfied:

BH1. *R is a Heyting field:*
$\forall x, y, z \in R$,
(1) $x + y = y + x$,
(2) $(x + y) + z = x + (y + z)$,
(3) $0 + x = x$,
(4) $x + (-x) = 0$,
(5) $x \times y = y \times x$,
(6) $(x \times y) \times z = x \times (y \times z)$,
(7) $1 \times x = x$,
(8) $x \times x^{-1} = 1$ for $x \in R^*$,
(9) $x \times (y + z) = x \times y + x \times z$.

BH2. *Basic properties of $>$:*
$\forall x, y \in R$,
(1) $\neg\,(x > y\ \wedge\ y > x)$,
(2) $(x > y) \Rightarrow \forall z\ (x > z\ \vee\ z > y)$,
(3) $\neg(x \neq y) \Rightarrow x = y$,
(4) $(x > y) \Rightarrow \forall z\ (x + z > y + z)$,
(5) $(x > 0\ \wedge\ y > 0) \Rightarrow x \times y > 0$.

BH3. *Special properties of $>$:*
(1) *Axiom of Archimedes:* $\forall x \in R\ \exists p \in \mathbb{N}\quad p > x$ (we identify every $p \in \mathbb{N}$

[5]The interpretation of Brouwer, Heyting and Kolmogorov which defines the intuitionnistic logic.

with $p1 \in R$ where $p1 =_{\text{def}} 1 + \cdots + 1$ the sum of p terms $1 \in R$).
(2) *The constructive least-upper-bound principle:* Let S be a nonempty subset of R that is bounded above relative to the relation $\geq$, such that the following property $\mathcal{P}(S, R, >)$:

for all $a, b \in R$ with $b > a$, either b is an upper bound of S or else there exists $s \in S$ with $s > a$

is true. Then S has a least upper bound.

Let us recall the classical definition of a *least upper bound* of S: it is an element $\tau \in R$ such that
(I) $(\forall \mu < \tau,\ \exists s \in S$ such that $\mu < s)$ and (II) $(\forall s \in S,\ s \leqslant \tau)$.

5.2.2. Is $\mathcal{HR}_\omega$ a Bridges-Heyting ordered field ?

In order to state our main result about the constructive content of $\mathcal{HR}_\omega$, let us introduce the functions $\text{Op}_\omega : x \mapsto -_\omega x$ and $\text{Inv}_\omega : x \mapsto x^{(-1)_\omega}$. Moreover, we have to introduce a new binary relation: two elements $x = (x_n)$ and $y = (y_n)$ of $\mathcal{HR}_\omega$ are *congruent* and we note $x \triangle y$ if

$$(\forall r \in \mathbb{N})(\exists K \in \mathbb{N})(\forall k \geq K)(\forall l \geq K) \quad \left| \frac{x_k - y_k}{\omega_k} - \frac{x_l - y_l}{\omega_l} \right| \leq \frac{1}{r}$$

In other words, $x \triangle y$ when the sequence of rational numbers $((x_n - y_n)/\omega_n)$ is a Cauchy sequence[6]. This means that the difference $x - y$ is a relatively regular element of $\mathcal{HR}_\omega$. These regular elements correspond to the Cauchy numbers mentioned in the article of P. Schuster[31]. This last work aims to study the fine properties of the rational Ω-number system as model of a constructive and nonstandard real line. Schuster shows in particular that a rational Ω-number is a Cauchy number if and only if, according to his terminology, it is located and bounded. Our work is appreciably different because our objective in this part is, according to the strategy of our previous researches, to investigate to what extent the Harthong-Reeb line is a Bridges-Heyting ordered field.

The next result indicates to what extent this system satisfies the axioms of a BH-ordered field.

[6]It is easy to verify that the relation $\triangle$ is extensional in $\mathcal{HR}_\omega$.

Theorem 2. *The system* $\mathcal{HR}_\omega$[7] *has the following properties:*
(a) *All the axioms of a BH-ordered field except* BH2.(2), BH2.(3) *and* BH3.(2).
(b) BH2.(2)' *If* $x, y \in \mathcal{HR}_\omega$ *are such that* $x <_\omega y$, *then for each* $z \in \mathcal{HR}_\omega$, *there is a* $q \in \mathbb{N}$ *such that* $(q(y-z) \geq \omega) \vee (q(z-x) \geq \omega)$ *is weakly true.*
(c) BH2.(3)' *If* $x, y \in \mathcal{HR}_\omega$ *are such that* $x \triangle y$ *and* $\neg(x \neq_\omega y)$, *then* $x =_\omega y$.
(d) BH3.(2)' *If* S *is a nonempty subset of* $\mathcal{HR}_\omega$ *that is bounded above relative to the relation* $\geq_\omega$ *and such that for all* $(\alpha, \beta) \in \mathcal{HR}_\omega^2$ *where* β *is an upper bound of* S *and* $\alpha \in S$ *and for* $(a, b) \in \mathcal{HR}_\omega^2$ *such that* $\alpha \leq_\omega a \leq_\omega b \leq_\omega \beta$, *either* b *is an upper bound of* S *or else there exists* $s \in S$ *with* $s >_\omega a$.

Then there exists an element $\tau \in \mathcal{HR}_\omega$ *which is a least upper bound of* S *in the following weak meaning:*
(I') $\forall \mu <_\omega \tau,\ \exists s \in S$ *such that* $\mu <_\omega s$ *(identical to* (I)*)*
(II') *(*$\forall \nu \in \mathcal{HR}_\omega$ *such that* $\tau <_\omega \nu)(\exists b$ *upper bound of* $S)$ $\tau \leq_\omega b <_\omega \nu$

Why does $\mathcal{HR}_\omega$ not exactly satisfy all the axioms of a BH-ordered field? The main reason is that there is in this system a lot of very irregular elements, contrary to the usual system of constructive numbers. Thus, it is easy to find, for instance, a counter-example to BH2.(3): let us consider $x = (x_n) \in \mathcal{HR}_\omega$ defined by $x_n = 1$ if n is even and $x_n = \omega_n$ if n is odd; then, it is clear that $\neg(x \neq_\omega 0)$ and $\neg(x =_\omega 0)$. It is likely that the presence of these irregular elements is the price to pay for having a constructive and nonstandard arithmetic. Let us precise without any proof that, in a BH-ordered field, the notion of weak least upper bound defined by (I') and (II') in BH3.(2)' is equivalent to the classical one. In the case of the system $\mathcal{HR}_\omega$ which is not exactly a BH-ordered field, it is not clear that this equivalence is true.

Proof (of the theorem). (a) The verification of the axioms of the group (BH1) is almost straightforward. As an example, we give the proof of (8): $x \times_\omega x^{(-1)_\omega} =_\omega 1_\omega$ for $x \neq_\omega 0_\omega$. Thus, we consider $x \in \mathcal{HR}_\omega$ such that $x \neq_\omega 0_\omega$. Then, $x \times_\omega x^{(-1)_\omega} = (x \times (\omega^2 \div x)) \div \omega$. Using twice the identity $a = (a \div b) \times b + a \bmod b$, we find:

$$x \times_\omega x^{(-1)_\omega} = \omega - (\omega^2 \bmod x)/\omega - ((\omega^2 - \omega^2 \bmod x) \bmod \omega)/\omega.$$

Since $x \in \mathcal{HR}_\omega$, we know there is $k \in \mathbb{N}$ such that $|x| \leq k\omega$. Hence, $|(\omega^2 \bmod x)/\omega| \leq |x|/\omega \leq k$ and $|((\omega^2 - \omega^2 \bmod x) \bmod \omega)|/\omega \leq 1$, we find

[7] Let us note $\mathcal{HR}_\omega$ the complete numerical system $(\mathcal{HR}_\omega, >_\omega, +_\omega, \times_\omega, 0_\omega, 1_\omega, \mathrm{Op}_\omega, \mathrm{Inv}_\omega)$

that $|x \times_\omega x^{(-1)_\omega} - \omega| \leq k+1$. The result follows since $\omega = 1_\omega$, $k+1$ is standard and thus $k+1 =_\omega 0$. In the group (BH2), only the axioms (2) and (3) present a real difficulty; these two properties will be discussed later. Finally, the axiom (1) of the group (BH3) is trivial since, for every $x \in \mathcal{HR}_\omega$, there is $p \in \mathbb{N}$ such that $x \leq |x| \leq p\omega = p1_\omega$.

(b) BH2.(2)' We consider $x, y \in \mathcal{HR}_\omega$ such that $x <_\omega y$. Hence, there is $p \in \mathbb{N}$ such that $p(y-x) \geq \omega$. Thus, for all $z \in \mathcal{HR}_\omega$, we have $p(y-z)+p(z-x) \geq \omega$. From the proposition 1, we get that $(2p(y-z) \geq \omega) \vee (2p(z-x) \geq \omega)$ is weakly true.

(c) BH2.(3)'. We consider $x, y \in \mathcal{HR}_\omega$ such that $x \triangle y$ and $\neg(x \neq_\omega y)$. For each $n \in \mathbb{N}$, let $u_n =_{\text{def}} (x_n - y_n)/\omega_n$. Then the property $x =_\omega y$ results from the following lemma[8].

Lemma 1. Let (u_n) be a sequence of rational numbers such that

(1) $\neg[(\exists q)(\exists N)((\forall n \geq N)\ u_n \geq 1/q) \vee ((\forall n \geq N)\ u_n \leq -1/q)]$,

(2) $(\forall p)(\exists K)(\forall k \geq K)(\forall l \geq K)\ |u_k - u_l| \leq 1/p$,

Then, $(\forall r)(\exists M)(\forall m \geq M)\ |u_m| \leq 1/r$.
(All the quantified variables are assumed to take their values in $\mathbb{N}$.)

Proof (of the lemma). Let us consider an arbitrary $r \in \mathbb{N}$ and let $p =_{\text{def}} 3r$. From the assumption (2), we deduce that there is a number $K \in \mathbb{N}$ such that

$$(\forall k \geq K)(\forall l \geq K)\ |u_k - u_l| \leq 1/(2p).$$

In particular

$$(\forall l \geq K)\ u_K - 1/(2p) \leq u_l \leq u_K + 1/(2p) \tag{20}$$

Otherwise, since u_K is a rational number, the property $(u_K \geq 1/p) \vee (u_K < 1/p)$ is decidable. Let us suppose that $u_K \geq 1/p$. From (20) we obtain that $1/(2p) \leq u_l$ for all $l \geq K$. The assumption (1) of the lemma asserts that this is impossible. Consequently, we are sure that $u_K < 1/p$. Again, from (20) we obtain that $u_l \leq 1/p + 1/(2p) = 3/(2p)$ for every $l \geq K$. Since $3/(2p) = 1/(2r) \leq 1/r$, we get that $(\forall l \geq K)\ u_l \leq 1/r$. Proceeding similarly, we can show that $(\forall l \geq K)\ u_l \geq -1/r$, which gives the expected result.

[8]Actually, this lemma shows that the system of constructive real numbers satisfies the axiom BH.2(3).

(d) BH3.(2)' From now, we identify every rational number $r = p/q$ where $(p, q) \in \mathbb{Z} \times \mathbb{N}^*$ with the element $p\omega \times_\omega (q\omega)^{(-1)_\omega} \in \mathcal{HR}_\omega$; moreover, we omit to mention the symbol $\times_\omega$ of the multiplication in $\mathcal{HR}_\omega$. The proof (BH3.(2)') is rather long and is presented in three steps.

Step 1: construction of two auxiliary sequences (s^n) *and* (b^n). By induction, we are going to construct a monotone nondecreasing sequence (s^n) of elements of S and a sequence (b^n) of strict upper bounds of S such that $b^n - s^n =_\omega (n+1)\left(\frac{2}{3}\right)^n (b^0 - s^0)$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we have $s^n, b^n \in \mathbb{Z}_\Omega$; thus, we introduce the notation $s^n = (s^n_m)_{m\in\mathbb{N}}$ and $b^n = (b^n_m)_{m\in\mathbb{N}}$ where $s^n_m, b^n_m \in \mathbb{Z}$ for every $m \in \mathbb{Z}$. Firstly, we choose a strict upper bound b^0 of S and $s^0 \in S$ such that $b^0 >_\omega s^0 \in \mathcal{HR}_\omega$. Now, we consider the general case: we suppose that we have already defined two finite sequences $(s^k)_{0\leq k\leq n}$ of elements of S and $(b^k)_{0\leq k\leq n}$ of strict upper bounds of S such that $s^0 \leqslant_\omega \cdots \leqslant_\omega s^n$ and for each $k = 0, \ldots, n$ $b^k - s^k =_\omega (k+1)\left(\frac{2}{3}\right)^k (b^0 - s^0)$. Then, we introduce $\alpha^n =_{\text{def}} \frac{2}{3}s^n + \frac{1}{3}b^n$ and $\beta^n =_{\text{def}} \frac{1}{3}s^n + \frac{2}{3}b^n$. Since $\alpha^n <_\omega \beta^n$, the property of S leads to the two following cases where ε^n denotes the decreasing to 0 term $\left(\frac{2}{3}\right)^{n+1}(b^0 - s^0)$:

1. β^n is an upper bound of S in which case we define $b^{n+1} =_{\text{def}} \beta^n + \varepsilon^n$ and $s^{n+1} =_{\text{def}} s^n$;
2. $\exists s \in S$ such that $s >_\omega \alpha^n$; hence, $s >_\omega s^n$ in $\mathbb{Z}_\omega$ and we can suppose that $s = (s_m)_{m\in\mathbb{N}}$ with $s^n_m \leqslant s_m$ for every $m \in \mathbb{N}$; then, we define $b^{n+1} =_{\text{def}} b^n + s - \alpha^n + \varepsilon^n$ and $s^{n+1} =_{\text{def}} s$.

In each case, b^{n+1} is a strict upper bound of S , $s^{n+1} \in S$, $s^n \leq_\omega s^{n+1}$ and it is easy to check that $b^{n+1} - s^{n+1} =_\omega (n+2)\left(\frac{2}{3}\right)^{n+1}(b^0 - s^0)$.

Step 2: construction of the candidate τ. For every $n \in \mathbb{N}$ and for each $k = 0, \ldots, n$, we have $s^n <_\omega b^k$ and thus $s^n < b^k$ in $\mathbb{Z}_\omega$. Hence, there is $M^k_n \in \mathbb{N}$ such that for every $m \geqslant M^k_n$, $s^n_m < b^k_m$ in $\mathbb{Z}$. Now, we consider the sequence $(\mathcal{M}_n)$ of natural numbers such that, $\mathcal{M}_0 = M^0_0$ and $\mathcal{M}_{n+1} = \max(M^0_{n+1}, \ldots, M^{n+1}_{n+1}, \mathcal{M}_n + 1)$ for each $n \in \mathbb{N}$. Finally, we define the sequence $\tau = (t_m)$ such that

$$\begin{array}{|c|}\hline \forall m < \mathcal{M}_0 \quad t_m =_{\text{def}} 0, \\ \forall n \in \mathbb{N}\ \forall m \in \mathbb{N} \text{ such that } \mathcal{M}_n \leqslant m < \mathcal{M}_{n+1} \quad t_m =_{\text{def}} s^n_m \\ \hline \end{array}$$

Step 3. Thus, we have construct an element τ of $\mathcal{HR}_\omega$, a monotone nondecreasing sequence (s^n) of elements of S and a sequence (b^n) of strict upper bounds of S such that, for every $n \in \mathbb{N}$, $s^n \leqslant_\omega \tau \leqslant_\omega b^n$ and $b^n - s^n =_\omega (n+1)\left(\frac{2}{3}\right)^n (b^0 - s^0)$. Then, (I') and (II') can be deduced.

5.3. Formalisation in Coq

The development by P. Martin-Löf of a nonstandard type theory [20] suggests that constructive nonstandard analysis can be effectively represented and treated within the framework of the COQ language. For what concerns us, we are convinced that a treatment in this language of the Harthong-Reeb line is possible and interesting; the following development outlines the first elements of it.

Moreover as it can be seen in the previous section, the verification of the Bridges' axioms is not so easy. The difficulty is mainly induced by the unconventional mathematical framework. The handled arithmetic is in a nonstandard framework and the axioms are in a constructive framework. So, it was not clear that handwritten proofs did not contain subtle mistakes or imprecisions.

In order to settle down this confidence problem into handwritten proofs, we have used the Coq proof assistant [1, 7] to formalize our proofs. The Coq proof assistant implements a higher constructive logic and is also a programming language equipped with inductive definitions and recursive functions. Therefore, it is a very interesting tool to carry out a constructive formalization. Moreover, as a byproduct, the formalization has entailed a better understanding of how concepts and proofs are related to each other.

The formalization we have developed is realized in a wider perspective than Ω-numbers. In [17] N. Magaud and some of the authors have formalized the axiomatic construction of the Harthong-Reeb line presented in [5] and all the handwritten proofs showing that this construction satisfies the 17 Bridges' axioms have been rewritten using the Coq proof assistant. To do so, a parametrized module has been defined in order to be useful to explore different possible constructions of the Harthong-Reeb line.

The parameter module of this parametrized module is a formalization of a nonstandard arithmetic based over five axioms in the spirit of Internal Set Theory [21]. Then using different constructions that satisfy these axioms, different constructions of the Harthong-Reeb line are obtained simply by a module instantiation mechanism. A first attempt to instantiate the parametrized module has been proposed based on the Ω-numbers theory.

A short overview of the formalization is presented here. A complete development and all the proofs can be found at `http://galapagos.gforge.inria.fr/developments.html`.

Following the developments presented in section 2, Ω-numbers are defined

in Coq as sequences indexed by natural numbers (`nat`) whose values are integers (`Z`) by the following Coq instruction:

```
Definition A := nat->Z.
```

This means that `A` is a function type from natural numbers to integer numbers. These functions can be interpreted as sequences. With this definition, the numbers 0 (denoted by `a0`) and 1 (denoted by `a1`) are defined by:

```
Definition a0 : A := fun (n:nat) => 0%Z.
Definition a1 : A := fun (n:nat) => 1%Z.
```

This means that `a0` is of type `A` (i.e. of type `nat -> Z`) and is a constant function that associate to each natural number `n` the integer constant zero, `0%Z`. The constant function `a1` that associates the constant `1%Z` to each natural number is defined in a similar manner.

Hence, the Ω-number $\Omega = (n)_{n\in\mathbb{N}}$ (denoted by `w`) is defined by

```
Definition w :A := fun (n:nat) => (Z_of_nat n).
```

Where the function `Z_of_nat` is the injection of natural numbers to integer numbers.

The equivalence relation $\mathcal{R}$ detailed in section 2 is captured by the definition `ext_almost_everywhere`.

```
Definition ext_almost_everywhere (u v:A) :=
 exists N:nat, forall n:nat, n>N -> u n=v n.
```

This expresses that for two elements in `A`, it exists a natural number `N` such that for all natural numbers `n` greater than `N` the two sequences are equal.

Actually, in our proofs it is usually sufficient to use the axiom `ext` which expresses the extensionally principle for functions:

```
Axiom ext : forall u v:A, (forall n:nat, (u n)=(v n)) -> u = v.
```

And the easily proved lemma `equal_implies_ext`:

```
Lemma equal_implies_ext :
  forall u v , u = v -> ext_almost_everywhere u v.
```

To directly prove that two Ω-numbers are equal.

Arithmetic operations are defined as expected, for example here is the definition of the addition:

```
Definition plusA (u v:A) := fun (n:nat) => Zplus (u n) (v n).
```

Where `Zplus` denotes the addition of two integer numbers. Then we obtain the definition of a type `A` that represents the Ω-numbers.

Now the construction of $\mathcal{HR}_\omega$ over the type `A` of Ω-numbers is as follows. First, the property `P` that characterizes the elements belonging to $\mathcal{HR}_\omega$ is defined[9]:

```
Definition P :=
  fun (x:A) => exists n:A, (lim n /\ 0 ?< n /\ (|x| ?<= n*w)).
```

Where `lim` represents the predicate *limited* and the relations `?<` and `?<=` are comparison relations in `A`. After what $\mathcal{HR}_\omega$ (denoted by `HRw`) is defined as the elements of `A` that satisfy `P`:

```
Definition HRw := {x:A | P x}.
```

Hence, the elements of $\mathcal{HR}_\omega$ are those of the type `A` for which we can have a proof that they satisfy the property `P`. They are represented in Coq by a pair of an element `a` of type `A` and a proof `h` that `a` satisfies `P`.

The arithmetic operations for $\mathcal{HR}_\omega$ are defined on the Ω-numbers. Therefore it is necessary to verify that, for example, given two elements of $\mathcal{HR}_\omega$ the sum and the product is also in $\mathcal{HR}_\omega$. This is done with the following two lemmas[10] using the property `P`:

```
Lemma Pplus : forall x y, P x -> P y -> P (x + y).

Lemma Pmult : forall x y, P x -> P y -> P (( x * y) / w).
```

The proofs of these lemmas explicitly use nonstandard concepts.

Now, arithmetic operations are defined using a matching that decomposes an element of $\mathcal{HR}_\omega$ as an element of the type `A` and a proof that this element belongs to $\mathcal{HR}_\omega$ (i.e. satisfies the property `P`). Here is the definition of the addition in $\mathcal{HR}_\omega$:

[9] The wedge sign /\ stands for the logical *and*.

[10] The lemmas are presented in a "Curryfied" form where tuples are viewed as successive implications, here the couple `(P x, P y)` is replaced by `P x -> P y`.

```
Definition HRwplus (x y: HRw) : HRw :=
 match x with exist xx Hxx => match y with exist yy Hyy =>
  exist P (xx + yy) (Pplus xx yy Hxx Hyy)
end end.
```

The given elements `x` and `y` of $\mathcal{HR}_\omega$ are decomposed into two elements `xx` and `yy` of `A` and proofs `Hxx` and `Hyy` that respectively show that `xx` and `yy` belong to $\mathcal{HR}_\omega$. Then, the addition of `xx` and `yy` is computed and we know by lemma `Pplus` that the result satisfies the property `P` and hence belongs to $\mathcal{HR}_\omega$. The multiplication is defined in the same way using the lemma `Pmult`:

```
Definition HRwmult (x y: HRw) : HRw :=
 match x with exist xx Hxx => match y with exist yy Hyy =>
  exist P ((xx * yy) / w) (Pmult xx yy Hxx Hyy)
end end.
```

Comparison relations ($\geq_\omega$, $_\omega$, $\neq_\omega$) are directly translated in Coq sentences from their definitions given in section 2 using the same decomposition process as for arithmetic operations[11]:

```
Definition HRwgt (y x : HRw) : Prop :=
 match y with exist yy Hyy => match x with exist xx Hxx =>
  (exists n, lim n /\ 0 ?< n /\ (w ?<= (n*(yy+ (-xx)))))
end end.

Definition HRwge (a b : HRw) : Prop :=
 (proj1_sig b) ?<= (proj1_sig a) \/ HRwequal a b.

Definition HRwdiff (x y : HRw) : Prop := HRwgt x y \/ HRwgt y x.
```

Where `(proj1_sig a)` denotes the element `aa` of the type `A` for which we have of proof that it satisfies `P`.

A satisfying result is that we did not find any mistakes in the proofs showing that the axiomatic construction of the Harthong-Reeb line presented in [5] satisfies the 17 Bridges' axioms.

An other formalization of the Harthong-Reeb line in an axiomatic non standard way was done on the assistant proof Isabelle by J. Fleuriot [10].

[11] The wee sign \/ stands for the logical *or*.

6. Conclusion

The work presented in this paper focused on the Ω-arithmetization. In our context, an arithmetization process is a method which gives an exact and discrete representation of real functions which are solutions of some differential equations. This method requires a theoretical background which provides a notion of infinitely large integer number. In the previous works on this subject, this notion resulted of the introduction of an axiomatic version of nonstandard analysis. The main drawback of this approach was a lack of constructivity. As a consequence, it was impossible to have an exact computer implementation of the arithmetization process. This failing is clearly related to the axiomatic status of the infinitely large parameters which occur in the algorithms.

In the present paper, we have introduced an arithmetization method based on the notion of Ω-numbers introduced by Laugwitz and Schmieden. This theoretical framework seems weaker than the usual approach of nonstandard analysis but it has the great advantage of being more constructive. Actually, we have shown that the resulting Ω-arithmetization leads to constructive algorithms which can be exactly translated into functional computer programs. An important point resulting of the constructivity of this approach is that these programs do not generate any numerical error. Moreover, the result of the application of the Ω-arithmetization on a real function is a discrete multi-resolution representation of this function. This very interesting multi-resolution aspect comes from the intrinsic nature of the infinitely large Ω-numbers which is a function that encodes an infinity of increasing scales.

In future works on this subject, we plan to study systematically this form of multi-resolution analysis and its applications to discrete geometry. This work has already been done in the case of an arc of ellipse[4]. More generally, it would be interesting to develop, on the basis of the present approach, a theoretical and systematic bridge between the continuous and discrete worlds for classical concepts of differential geometry.

Nevertheless, it appears that the logical foundations of the theory of Ω-numbers are not entirely satisfactory and natural. On the one hand, the semantic and syntactic levels are not clearly distinguished (as it is often the case in usual mathematics). On the other hand, the use by Laugwitz and Schmieden of classical logic conflicts with the constructive content of the concept of Ω-number and adds some theoretical confusions. Finally,

the notion of weak truth is artificially imposed by the rules without any explanation about the reason and the meaning of this strange constraint.

For these last reasons, in a work in progress on the Harthong-Reeb line, we are changing our general theoretical framework by moving to the formalism of constructive type theory of P. Martin-Löf [18, 19] also called intuitionnistic type theory which will be denoted T in what follows.

The first reason for this change is that this stark approach of mathematics and computer science is well suited for both developing constructive mathematics and writing programs. At the theoretical level, T appears as a constructive foundation of mathematics which is an alternative to usual axiomatic set theories like ZFC (Zermelo-Fraenkel with the axiom of Choice). In T, there is no preexisting universe; the sets are introduced in a controlled way using rules which are deeply inductive. Moreover, T does not depend on a preexisting logic; actually, the logical propositions are also objects of T through the Curry-Howard isomorphism which identifies a proposition with the set of its proofs. Finally, each formal rule is preceded by a semantic explanation which gives its meaning. Accordingly, T is a semantic and syntactic theory. Of course, we have in T a set $\mathbb{N}$ of natural numbers and it is easy to introduce a set $\mathbb{Z}$ of integer numbers.

The second reason for adopting T is that Martin-Löf has defined[20] a nonstandard extension T_∞ of T. This extension provides an infinitely large natural number ∞. The semantic of this nonstandard number is given by a choice sequence and Martin-Löf has shown that the formal theory T_∞ have a natural model $\mathcal{M}_\infty$ which is built inside the standard model $\mathcal{M}$ of T. Since $\mathcal{M}$ is a domain of constructive mathematic entities, we can use the model $\mathcal{M}_\infty$ to develop a deeply constructive approach of the Harthong-Reeb line and of the arithmetization method. Furthermore, this line appears as a new version of the constructive continuum probably related to the intuitionistic continuum of Brouwer.

References

[1] Y. Bertot and P. Castéran. *Interactive Theorem Proving and Program Development, Coq'Art: The Calculus of Inductive Constructions*. Springer, 2004.

[2] D. Bridges. Constructive mathematics: a foundation for computable analysis. *Theoretical Computer Science*, 219:95–109, 1999.

[3] D. Bridges and S. Reeves. Constructive mathematics, in theory and programming practice. Technical Report CDMTCS-068, Centre for Discrete Mathematics and Theorical Computer Science, November 1997.

[4] A. Chollet, G. Wallet, L. Fuchs, E Andres, G. Largeteau-Skapin, and A. Richard. Ω-arithmetization of ellipses. *CompIMAGE'10, LNCS 6026*, pages 24–36, 2010.

[5] A. Chollet, G. Wallet, L. Fuchs, G. Largeteau-Skapin, and E. Andres. Insight in discrete geometry and computational content of a discrete model of the continuum. *Pattern recognition*, 42:2220–2228, 2009.

[6] A. Chollet, G. Wallet, L. Fuchs, G. Largeteau-Skapin, and E. Andres. Ω-arithmetization: a discrete multi-resolution representation of real functions. *IWCIA'09 Springers's LNCS Volume 5852*, 2009.

[7] Coq development team. *The Coq Proof Assistant Reference Manual, Version 8.2*. LogiCal Project, 2008.

[8] F. Diener and G. Reeb. *Analyse Non Standard*. Hermann, Paris, 1989.

[9] M. Diener. Application du calcul de Harthong-Reeb aux routines graphiques. In J.-M. Salanskis and H. Sinaceurs, editors, *Le Labyrinthe du Continu*, pages 424–435. Springer, 1992.

[10] Jacques Fleuriot. Exploring the foundations of discrete analytical geometry in Isabelle/HOL. In Pascal Schreck and Jergen Richter-Gebert, editors, *Proceedings of Automated Deduction in Geometry 2010*, volume 6877 of *LNAI*. Springer, 2011.

[11] L. Fuchs, G. Largeteau-Skapin, G. Wallet, E. Andres, and A. Chollet. A first look into a formal and constructive approach for discrete geometry using nonstandard analysis. *DGCI 2008*, pages 21–32, 2008.

[12] J. Harthong. Éléments pour une théorie du continu. *Astérisque*, 109/110:235–244, 1983.

[13] J. Harthong. Une théorie du continu. In H. Barreau and J. Harthong, editors, *La mathématiques non standard*, pages 307–329. Éditions du CNRS, 1989.

[14] D. Laugwitz. Ω-calculus as a generalization of field extension. In A. Hurd, editor, *Nonstandard Analysis - Recent developments*, Lecture Notes in Mathematics, pages 144–155. Springer, 1983.

[15] D. Laugwitz. Leibniz' principle and omega calculus. In J.M. Salanskis and H. Sinacoeur, editors, *Le Labyrinthe du Continu*, pages 144–155. Springer France, 1992.

[16] D. Laugwitz and C. Schmieden. Eine erweiterung der infinitesimalrechnung. *Mathematische Zeitschrift*, 89:1–39, 1958.

[17] N. Magaud, A. Chollet, and L. Fuchs. Formalizing a discrete model of the continuum in Coq from a discrete geometry perspective. In *Automated Deduction in Geometry, ADG'10*, Munich, Germany, July 22–24 2010.

[18] P. Martin-Löf. Constructive mathematics and computer programming. *In Logic, Methodology and Philosophy of Science VI*, pages 153–175, 1980.

[19] P. Martin-Löf. *Intuitionnistic Type Theory*. Bibliopolis, Napoli, 1984.

[20] P. Martin-Löf. Mathematics of infinity. In *COLOG-88 Computer Logic*, Lecture Notes in Computer Science, pages 146–197. Springer-Verlag Berlin, 1990.

[21] E. Nelson. Internal set theory: A new approach to nonstandard analysis. *Bulletin of the American Mathematical Society*, 83(6):1165–1198, November 1977.

[22] E. Palmgren. A note on mathematics of infinity. *Journal of Symbolic Logic*, 58:1195–1200, 1993.

[23] E. Palmgren. Constructive nonstandard analysis. *A. Petry (ed.) Méthodes et analyse non standard, Cahiers du Centre de Logique*, 9:69–97, 1996.

[24] E. Palmgren. Developments in constructive nonstandard analysis. *The Bulletin of Symbolic Logic*, 4(3):233–272, 1998.

[25] E. Palmgren. Ickestandardanalys och historiska infinitesimaler. *Normat*, 55:166 – 176, 2007.

[26] J-P. Reveillès. *Géométrie discrète, Calcul en nombres entiers et algorithmique*. PhD thesis, Université Louis Pasteur, Strasbourg, France, 1991.

[27] J.-P. Reveillès. Mathématiques discrètes et analyse non standard. In J.-M. Salanskis and H. Sinaceurs, editors, *Le Labyrinthe du Continu*, pages 382–390. Springer, 1992.

[28] J-P. Reveillès and D. Richard. Back and forth between continuous and discrete for the working computer scientist. *Annals of Mathematics and Artificial Intelligence, Mathematics and Informatic*, 16(1-4):89–152, 1996.

[29] A. Robinson. *Non-standard analysis*. American Elsevier, New York, 1974. 2nd ed.

[30] J. Ruokolainen. *Constructive nonstandard analysis without actual infinity*. PhD thesis, University of Helsinki, Helsinki, 2004.

[31] P. Schuster. A constructive look at generalised cauchy reals. *Mathematical Logic Quarterly*, 46:125–134, 2000.