$O(n^3 \log n)$ time complexity for the Optimal Consensus Set computation for Andres and 4-connected Flake circles

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Abstract. This paper presents a method for fitting Andres circles as well as 4-connected digital circles to a given set of points in 2D images in the presence of noise by maximizing the number of inliers, namely the optimal consensus set, while fixing the thickness. Our approach based on one or several parameter spaces has a $O(n^3 logn)$ time complexity, $O(n)$ space complexity, n being the number of points, which is lower than previous known methods while still guaranteeing optimal solution(s).

Keywords: keywords Shape fitting; consensus set; outliers; 4-connected digital circle; Andres circle

1 Introduction

In the present paper, we are considering the fitting problem of a set of points in a noisy 2D image by two types of digital circles with fixed thickness, the Andres circle [12] and the 0-flake (4-connected) circle [5]. These type of circles are defined by two morphological based digitization schemes and have an analytically characterization. The thickness of the digital circles is fixed. The set of points (inliers) which fits a model is called a consensus set. The idea of using such consensus sets was proposed for the RANdom Sample Consensus (RANSAC) method [6], which is widely used in the field of computer vision. However RANSAC is inherently probabilistic in its approach and does not guarantee optimality. This paper aims at proposing a new lower time complexity for the computation of the optimal consensus set. This means that we are searching to maximize the number of inliers. Non Probablistic methods that detect annuli have been proposed. Most of these algorithms minimize the thickness of the annuli which is not adequate when considering digital where the thickness is fixed. Only few algorithms deal with outliers [14, 7, 15] but the number of outliers is usually predefined [7, 15] and the problem consists again in minimizing the width. The method proposed by O'Rourke et al. [16, 17] that transforms a circle separation problem into a plane separability problem, is also not well suited because the fixed thickness of the digital circles translates into non fixed vertical thicknesses for the planes. In this case, the approach is complicated (See [4] for some solutions on howto handle this difficulty).

The digital circles are analytically defined as digital points inside an offset region. In [1] and [2], a brute force algorithm was proposed to compute the optimal consensus set. It was shown that if an optimal solution exists then there exists a finite number of equivalent optimal solution (with the same set of inliers) with three points on the boundary (internal and/or external) of the offset region. Testing all the configurations of three points and counting the inliers leads therefore to all the possible optimal solutions within a time complexity of $O(n^4)$ where n is the number of points to fit. A new method is proposed in this paper, that requires just two points to be located on the boundary. The centers of all the circles with two specific points on the boundary corresponds to a straight line (for Andres circles) or a set of straight lines (for 0-Flake circles). By considering this straight line as a parametric axis, we build a dual space where the points of the set enter or exit the circles for some parameter values (Section 3.1). By considering all the sets of two points, we are able to construct the exhaustive set of all optimal consensus sets in $O(n^3 \log n)$.

The paper is organized as follows: in Section 2 we expose some properties and characterizations of the digital circles. Section 3 provides the algorithm for finding the optimal consensus sets for Andres circles and flake circles. Section 4 presents some results. Finally Section 5 proposes a conclusion and some perspectives.

2 Annular Characterizations

In [1] and [2], we proposed a brute force algorithm. We have shown that if an optimal solution exists then there exists an equivalent optimal solution (with the same set of inliers) with three points on the boundary (internal and/or external). In this section we are considering the problem of characterizing the annuli that are equivalent to some optimal solution with only two points of the boundary of the analytical region defining the annuli. Let us first introduce some basic notations as well as the analytical definitions of the Andres and 0-Flake circles. In a second part of this section, we will look at the annuli characterization for Andres and 0-Flake circles.

2.1 Notations and basic definitions

In this section, we present both Andres and 0-Flake digital circles with the associated notations and definitions.

An Andres circle A of width ω and radius R centered at $C(C_x, C_y)$, is defined by the set of points in R^2 satisfying two inequalities:

$$A = \{(P_x, P_y) \in \mathbb{R}^2 : R^2 \leq (P_x - C_x)^2 + (P_y - C_y)^2 \leq (R + \omega)^2\} \tag{1}$$

where $C(C_x, C_y) \in \mathbb{R}^2$ and R, $\omega \in \mathbb{R}_+$. We denote B_i (resp. B_e) the internal (resp. external) boundary of the annulus defined as the set of points located at distance R (resp. $R+\omega$) from C.

The second digitization scheme we are considering is an *Adjacency Flake Digitization* [5]. It is based on a morphological based digitization scheme with a structuring element called an *Adjacency Flake*. In this paper we are limiting ourself to 0-adjacency flake (or simply 0-Flake) circles for the sake of simplicity. This corresponds to 4-connected digital circles when the width is equal to one. However, the proposed fitting method works as well for 2D 1-adjacency flake circles (8-connected circles). The figure 1.a shows the 0-flake.

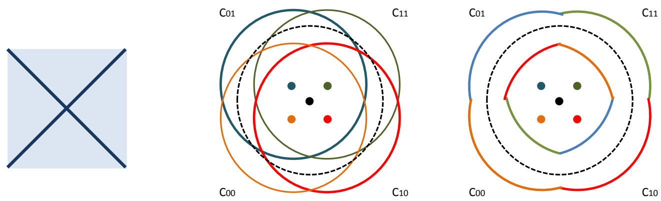

Fig. 1. 0-Flake, 0-Flake boundary circles and 0-Flake offset for the digitization.

The 0-Flake digitization of circle $\mathcal{C}(C,R)$ of center C and radius R is analytically described by:

$$D_{F_0}(\mathcal{C}(C,R)) = \left\{ x \in \mathbb{Z}^2 : -|x-C_x| - |y-C_y| - \tfrac{1}{2} \leq (x-C_x)^2 + (y-C_y)^2 - R^2 \leq |x-C_x| + |y-C_y| + \tfrac{1}{2} \right\}$$

The smallest possible 0-Flake circle is of radius $\sqrt{2}/2$. With a flake structuring element, circles of smaller radii are not correctly defined. This is one of the limitations of the flake model. It is however not a big constraint as it corresponds to a circle that spans only a couple of voxels.

We call **boundary circles** the 4 circles that form the boundary of the 0-Flake offset, i.e. the circles centered on $(C_x \pm \frac{1}{2}, C_y \pm \frac{1}{2})$. On figure 1.b, we can see the four boundary circles C_{00}, C_{01}, C_{10} and C_{11}:

Definition 1. *Let $\mathcal{C}_{ij}$ be a boundary circle of the flake annulus $\mathcal{C}(C_x, C_y)$ of radius R. $\mathcal{C}_{ij}$ is defined as the circle of center $(C_x, C_y) + (1/2, 1/2) - (i, j)$ and radius R.*

The digital 0-Flake circle is the set of digital points in the 0-flake offset (see fig 1.c).

2.2 Optimal two point Andres annuli characterization

In [1] we have shown that for a given optimal solution, there exists an equivalent optimal solution (with the same inliers) with three points on the inner or outer boundary of the annuli defining the Andres circle. It is thus obvious that there exists an equivalent solution with only two points on the boundary. What is however not immediate, from the proof presented in [1], is that there are always equivalent solutions with two points on the **outer** boundary of the annuli. The following theorem states that given a width ω, and given an Andres circle A covering a set of points S, there exists at least one other Andres circle A' of same width, that covers S with at least 2 points of S on the external boundary.

Theorem 1. *Let S be a set of n $(n \geq 2)$ points in $\mathbb{R}^2$. Let $A = (C(C_x, C_y), R, \omega)$ be an Andres circle of center $C(C_x, C_y)$, of internal radius R and of width ω such that $\forall (P_x, P_y) \in S,\ R^2 \leq (P_x - C_x)^2 + (P_y - C_y)^2 \leq (R + \omega)^2$. Then it exists $A' = (C'(C'_x, C'_y), R', \omega)$ such that:*

$$\exists P_0, P_1 \in S,\ \forall i \in [0,1], P_i \in B_e$$

Proof. Let S be a set of n $(n \geq 2)$ points in $\mathbb{R}^2$. Let $A = (C(C_x, C_y), R, \omega)$ be an Andres circle of center $C(C_x, C_y)$ with internal radius R and width ω such that A covers S: i.e. $\forall (P_x, P_y) \in S,\ R^2 < (P_x - C_x)^2 + (P_y - C_y)^2 < (R + \omega)^2$. Our first assumption is that no point of S is on the annulus boundarys.

The theorem proof is given in two steps:

A First step: decreasing radius. This step consists in decreasing the internal and external radius (the thickness is fixed) until reaching a first point P_0 on the external boundary B_e (Figure 2a). However while decreasing the radius we may reach an annulus with internal radius 0 before we have external point P_0 on B_e (Figure 2b); if this is the case, a translation toward the closest inlier to B_e is performed (Figure 2c and Figure 2d).

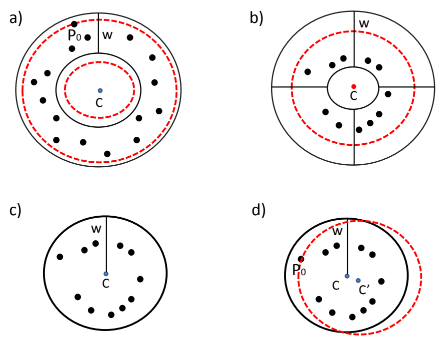

Fig. 2. First step of the proof.

B Second step: decreasing the radius while maintaining P_0 on B_e. In this step the radius is decreased by moving the radius along the axis CP_0; this is

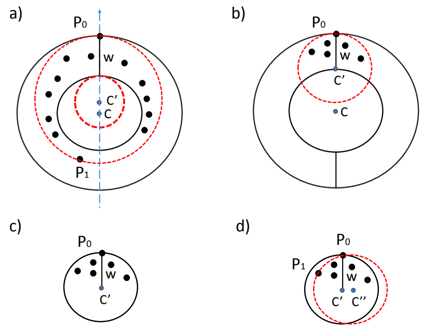

Fig. 3. second step of the proof.

continuously done until reaching a second point P_1 on B_e. However decreasing the radius while keeping P_0 on B_e can be done until the internal radius becomes 0 (Figure 3b). If the value of the internal radius is 0 and no point P_1 is found, the radius cannot be decreased anymore. The points of S are all in disk (Figure 3b and Figure 3c). If this is the case, a rotation centered in P_0 of the disk is done until reaching a second point P_1 on B_e. Figure 3d shows an example of this rotation done for the annulus in Figure 3c in order to reach P_1.

In all cases, if an Andres circle of width ω covers S, then it is possible to build an annulus of same width that has 2 points of S on the external boundary. □

Let us suppose we have an optimal consensus set S and an Andres annulus $A\left(C(C_x, C_y), R, \omega\right)$ with two points P_0 and P_1 on its outer boundary. The set of all the centers of the annuli containing S with P_0 and P_1 on the outer boundary, is a straight line segment, half line or a straight line. It is obvious that the center of an annulus that has two points P_0 and P_1 on its outer boundary belongs to the perpendicular bisector of $[P_0P_1]$. When all the points of the set S belong to the straight line segment $[P_0P_1]$ then all the circles centered on the perpendicular bisector with with P_0 and P_1 on the outer boundary cover the set S. If only some of the points belong to $[P_0P_1]$ and all the other points are on one side of the halfspace delimited by P_0P_1 then the set of all centers wee are looking for is a halfline and in all the other cases it is only a straight line subsegment of the perpendicular bisector of $[P_0P_1]$.

2.3 Optimal two point 0-Flake annuli characterization

In [2], it has been proven that given a Flake circle covering a set of points there exists an equivalent Flake circle which has at least two points of the set on its boundary circles. For Andres circles, there is an inner and outer circle defining the boundary. This is not the directly the case for Flake circles. The boundary circles are only half parts on the actual boundary of the annulus, one quarter on the inner boundary and one quarter on the outer part of the annulus boundary. What has been proven in [2] is that there are two points on the boundary circles which means that they are not necessarily on the actual inner or outer boundary of the annulus. This, however, is not a problem for the fitting method we are going to present.

Proposition 1. *Let us suppose we have an optimal consensus set S and an 0-Flake annulus $F\left(C(C_x, C_y), R,\right)$ with two points P_0 and P_1 on its boundary circles. The set of all the centers of the annuli containing S with P_0 and P_1 on the boundary circles, is a set of a maximum of 16 straight line segments, half lines or straight lines.*

Proof. The proof is simple. First, let us note that if we consider equivalent Flake annuli with two points on boundary circles, we have several possibilities since P_0 and P_1 may belong to the boundary circles C_{00}, C_{01}, C_{10} or C_{11}. There are 16 different configurations. If P_0 and P_1 belong to the same boundary circle C_{ij} of center (C_{ijx}, C_{ijy}) then the center of the 0-Flake annulus $F\left(C(C_x, C_y), R,\right)$ belongs to the parallel of the perpendicular bisector of P_0 and P_1 passing through $C(C_x, C_y) = (C_{ijx}, C_{ijy}) + (1/2, 1/2) - (i, j)$. If P_0 belongs to the boundary circle C_{ij} and P_1 to the boundary circle C_{kl} then $P_1' = P_1 + (i - k, j - l)$ belongs to C_{ij} and the previous reasoning works with P_0 and P_1'. The remaining of the argument is similar to the one of the Andres circles. □

3 Fitting Algorithms

Using the above proposed annuli characterization, our fitting problem can then be described as follows: given a finite set $S = \left\{(P_x, P_y) \in \mathbb{R}^2\right\}$ of n points such that $n \geq 2$, and given a width ω we would like to find an Andres circle A of width ω or a 0-flake annulus such that it contains the maximum number of points of S. Points belonging to the annulus are called inliers; otherwise they are called outliers.

We just showed in the previous section that for each maximal set of inliers S, there exists equivalent optimal solutions (with the same set S of inliers) with at least two points of the set S on the boundary circle for both digital circle models. Furthermore, the centers of the annuli with two points on the boundary belong to a straight line.

The principle of the algorithms work as follow: for each couple of points (p, q) of the set S, we define a dual space where one axis corresponds to the straight line where the centers of the annulus are. In the case of Andres circles, the centers are on the bisector B_{pq} of the segment $[pq]$. In the case of Flake annuli there are 16 different straight lines corresponding to p and q belonging to the different boundary circles. For each other point of S, we compute the intervals on the center axis that describe when the point is inside the annulus (Section 3.1). The intervals are sorted and the subinterval where the most points belong to annulus is determined. The complexity is given by considering all combinations of two points, sorting the n intervals and counting the number of inliers among those sorted intervals: this leads to an $O(n^3 \log n)$ time complexity.

In the following we detail the method for each type of digital circle.

3.1 Digital Andres circle fitting algorithm

Andres circle fitting algorithm that can be seen as an annulus of fixed width works as follows: considering a couple of points (p, q) we consider all the annuli that have those two points on their external boundary B_e. With a distance dual space inspired by [13] we can determine the annuli with the maximum number of inliers among all those having the points p and q on its external boundary. Doing this for all the couple of points among the set of points to fit yields the optimal annuli in terms of number of inliers.

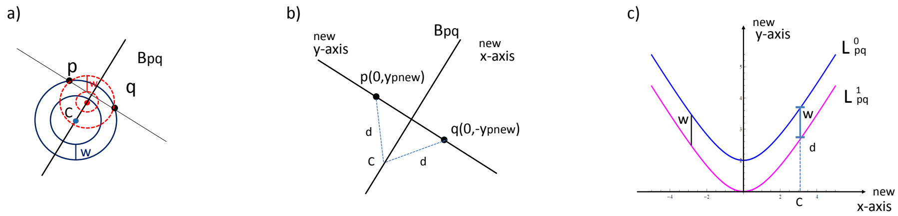

Fig. 4. a) The annulus having p and q as boundary points has its centers on the bisector B_{pq} of p and q, b) An axis transformation is done so that the bisector B_{pq} becomes the new x-axis. c) All annuli for which p and q are on B_e correspond to the set of all the vertical line segments of length w having one of its endpoints on B^0_{pq}.

Such an annulus has its center on the bisector B_{pq} of p and q (Figure 4a). By performing an axis transform and moving the x-axis toward the bisector B_{pq} of p and q; the new coordinates of p and q become respectively $p(0, y_{pnew})$and $q(0, y_{qnew})$ where $y_{qnew} = -y_{pnew}$ (Figure 4b). Any annulus A having a center $C(x, 0)$ on B_{pq} and p and q on B_e has the external radius $R_e = \sqrt{(0-x)^2 + {y_{pnew}}^2}$ and internal radius $R_i = (\sqrt{(0-x)^2 + {y_{pnew}}^2} - \omega)$ since we are dealing with annuli of fixed width ω.

Given a point $t(x_t, y_t)$ of S. The point t is inlier to the annulus A if $R_i \leq \sqrt{(x_t - x)^2 + {y_t}^2} \leq R_e$. We therefore define a dual space that associates to each point t in the image the curve $L_t = \sqrt{(x_t - x)^2 + {y_t}^2}$. L_t represents actually the distance of a point t to every point $(x, 0)$ of the x-axis (B_{pq}). In this dual space, p and q are represented by the same curve $L^0_{pq} = \sqrt{x^2 + {y_{pnew}}^2}$ (Figure 4c). The curve associated to p and q also represents R_e. By translating this curve by $-w$ we obtain the curve that represents R_i. Thus an annulus A of center $C(x, 0)$ and with the external boundary points p and q, corresponds in the dual space to the vertical line segment $[L^0_{pq}(C), L^1_{pq}(C)]$ of length w. For every point L_t in the

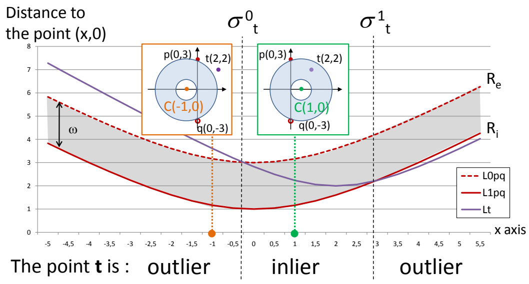

Fig. 5. Point localisation.

image it is possible to see if it is inlier or outlier to A by examining its dual curve L_t. The point t is inlier if L_t intersects the vertical segment $[L^0_{pq}(C), L^1_{pq}(C)]$ (Figure 4c) since in this case it is between $L^0_{pq}(C) = R_e$ and $L^1_{pq}(C) = R_i$ (see Figure 5).

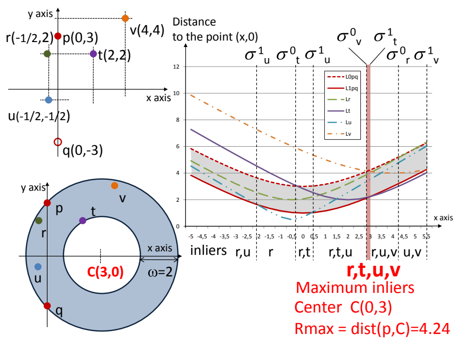

Fig. 6. Six points p, q, r, u, v, and t in the primal space and their corresponding dual curves. The maximum number of inliers for an annulus having p and q on B_e is reached when the center has an x-value around 3.

In order to find the largest consensus set in a strip for a given couple (p,q), we have to know the number of inliers within any annulus with the external boundary points p and q. We therefore check the intersections σ^0_t and σ^1_t of L^0_t for every point t in the image with the strip boundaries, L^0_{pq} and L^1_{pq}. This check is important since any annulus corresponding to a vertical segment between the two intersections σ^0_t and σ^1_t in the strip always contains t as an inlier; outside this interval, t is always an outlier (Figure 5).

We then sort all the intersections for all the points of the image. Each time a point enter the strip we add 1, each exit count for -1. By looking for the maximum, we obtain the center location(s) corresponding to the maximum optimal consensus set(s) having p and q as external boundary points. Figure 6 shows an example of this algorithm; the annulus in the primal having p and q on its external boundary is optimal in terms of inliers at a center around 3 between σ_v^0 and σ_t^1 when all the dual curves are in the strip formed by L_{pq}^0 and L_{pq}^1 (i.e. when all the points are inliers).

This work is repeated for all couple of points in the image until finding the center(s) of the annulus (annuli) having two points on B_e that maximizes the number of inliers. Since a sorting of complexity $n \log n$ of the intersection is needed and since the algorithm is repeated for every couple of two points, the final complexity is $O(n^3 \log n)$.

3.2 Flake annulus fitting algorithm

Let us detail the method for 4-connected circles based on a Flake digitization scheme. The idea behind the fitting is similar to the one for Andres circles. The main difference comes from the fact that for two given points, the center of the 4-connected circle may follow different paths depending on which of the four boundary circles the points are located. The second difference comes from the fact that we do not use a dual space based on the distance to the center since the distance from p and q to the center of the circle is not the identical.

The annulus fitting algorithm works as follows: Considering a couple of points (p, q), we consider all the Flake annuli that have these two points on their boundary circles. Since the boundary of the Flake circle is composed of 4 boundary circles, there are 16 localization configurations for the points p and q: Each point may belong to one of the four circles C_{00}, C_{11}, C_{01} or C_{10}. We have seen that for a given configuration, let's say that p belongs to C_{ij} and q to C_{kl}, the center of the circle follows a straight line. Each position on this line determines one flake annulus. Figure 7 shows an example of the line Δ_{pq} where the center is on.

In order to determine the number of inliers within any flake annulus with boundary points p and q, we must compute when a point t enters and exits the flake annuli that have p and q on two of its boundary circles. Let us suppose that we are considering the case where $p \in C_{ij}$ and $q \in C_{kl}$. It is easy to see that we can right away discard the configurations where $q' = q + (i - k, j - l) = p$. In this case both points are located at the same spot on both boundary circles and do not constrain the centers of the Flake annuli to a straight line.
Let us now note that there exists an unique center I_{uv} such that t belongs to the boundary circle C_{uv}. Indeed, let's say that p belongs to C_{ij}, q belongs to C_{kl} and we look for all the Flake circles such that t belongs to C_{uv}, where $i, j, k, l, u, v \in 0, 1$. This is similar to having $p \in C_{ij}$, $q' = q + (i - k, j - l) \in C_{ij}$ and $t' = t + (i - u, j - v) \in C_{ij}$. If we discard the cases where $p = q', p = t'$ and $q' = t'$ or when p, q' and t' are aligned where there is no solution, we have an unique boundary circle C_{ij} with these three points and thus an unique Flake circle, of center I_{uv} with $t \in C_{uv}$. The intersection centers I_{00}, I_{01}, I_{10} and I_{11} for which t belongs to C_{00}, C_{01}, C_{10} and C_{11} are determined this way. Each of these points I_{uv} corresponds to an half-line on the center axis where on one side I_{uv}, t is outside the boundary circle C_{uv} and inside on the other side. It is easy now

to determine the interval (possibly not finite) on the center axis when t is inside $1, 2$ or 3 boundary circles. This is when t belongs to the Flake annuli. When t is inside 0 or 4 boundary circles then t does not belong to a Flake annuli. Figure 7 gives an example of a Flake annuli with $p \in C_{11}$ and $q \in C_{00}$. Doing this for all the couple of points among the set of points to fit yields the optimal flake annuli in terms of number of inliers. A sorting of the intervals is needed just as for Andres circles and this same work is repeated for all the combination of two points in the image until finding all the center intervals for which an optimal consensus set is reached. Since a sorting of complexity $n \log n$ of the intersection is needed and since the algorithm is repeated for every couple of points, the final complexity is here as well $O(n^3 \log n)$.

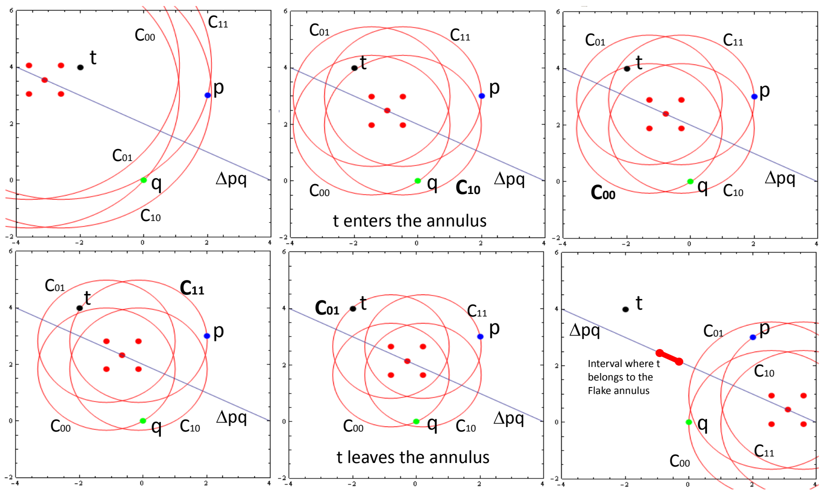

Fig. 7. The Flake annuli with $p \in C_{11}$ and $q \in C_{00}$ and an interval where the point t belongs to the Annulus.

4 Experiments

We used Mathematica for implementing our methods. We applied our method for 2D noisy Andres circles as shown in Fig. 8. For this set of points, an annulus of width $\omega = 1$ is used. Two optimal consensus sets are found, since two annulus having the same number of inliers can be fitted. This proves that our method is capable of detecting all optimal consensus sets. We also applied our method for 2D noisy Flake annulus as shown in Figure 9(a). All the possible (center,radius) solutions corresponding to optimal consensus sets for this image are shown in Figure 9(b).

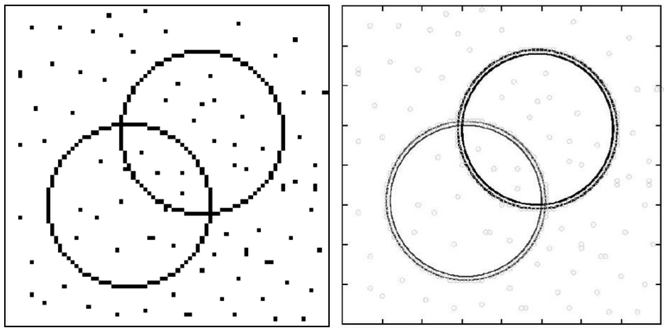

Fig. 8. Annulus fitting for two noisy circles of width 1.

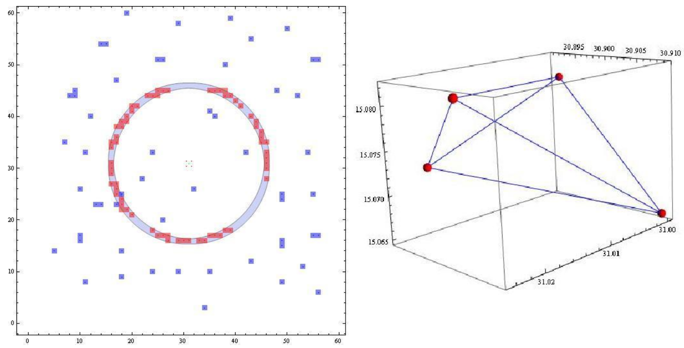

Fig. 9. a) Fitting of 2D noisy 0-flake circles. b) bounding region of all the optimal centers,

5 Conclusion and Perspectives

In this paper we have presented a new method for fitting Andres circles and flake annuli to a set of points while fixing the width. Our approach guarantees optimal results from the point of view of maximal consensus sets: we are guaranteed to fit an annulus with the least amount of outliers. In terms of computation time, these approaches are lower in terms complexity than the one presented in [1] and [2]. One of the future work concerns fitting of 3D annuli. The method seems to extends pretty well to higher dimension. A last perspective is of course fitting of other type of curves such as conics for instance.

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References

1. Zrour, R., Largeteau-Skapin, G., Andres, A.: Optimal consensus set for annulus fitting. In: Proceedings of the 16th International Conference on Discrete Geometry for Computer Imagery, pp. 358–368. LNCS, Springer Verlag, Nancy, France (2011)
2. Andres, E., Largeteau-Skapin, G., Zrour, R., Sugimoto, A., Kenmochi, Y.: Optimal Consensus Set and Preimage of 4-connected circles in a noisy environment. In: 21st International Conference on Pattern Recognition, pp. ???–???. IEEE Xplore, Tsukuba Science City, Japan (2012)
3. Andres, E., Menon, R., Acharya, R., Sibata, C.: Discrete Linear Objects in Dimension n : The Standard Model. Graphical Models. 65, 92–111 (2003)
4. Phan,M.S., Yukiko Kenmochi, Y., Sugimoto, A., Talbot, H., Andres, E., Zrour, R.: Efficient robust digital annulus Fitting with bounded error. In: Submitted to DGCI 2013, Sevilla, Spain (2013)
5. Fiorio, S., Toutant, J.L.: Arithmetic Discrete Hyperspheres and Separatingness. In: 13th international conference on Discrete Geometry for Computer Imagery (DGCI, pp. 425–436. LNCS, Springer Verlag, Berlin, Germany (2006)
6. Fischler, M.A., Bolles, R.C.: Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography. Communications of the ACM. 24, 381–395 (1981)
7. Har-Peled, S.,Wang, Y.: Shape Fitting with Outliers. SIAM Journal on Computing. 33 (2), 269–285 (2004)
8. Isack, H., Boykov, Y.: Energy-Based Geometric Multi-model Fitting. International Journal of Computer Vision. 1–25 (2011)
9. Raguram, R., Frahm, J-M.: RECON: Scale-adaptive robust estimation via Residual Consensus. In: ICCV, pp. 1299–1306. IEEE (2011)
10. Rivlin, T.J.: Approximation by circles. Computing. 21 (2), 93–104 (1979)
11. Toutant, J-L.,Andres, E., Roussillon, T.: Digital Circles, Spheres and Hyperspheres: From Morphological Models to Analytical Characterizations and Topological Properties. Submitted to Discrete Applied Mathematic Journal. (2012)
12. Andres, E., Jacob, M.-A: The Discrete analytical hyperspheres. IEEE Transactions on Visualization and Computer Graphics. 3, 75–86 (1997)
13. Daz-Bez, J.M., Hurtado, F., Meijer, H., Rappaport, D., Sellars, T: The largest empty annulus problem. International Journal of Computational Geometry and Applications. 13 (4), 317–325 (2003)
14. Dunagan, J., Vempala, S.: Optimal outlier removal in high-dimensional spaces. Journal of Computer and System Sciences. 68 (2), 335–373 (2004)
15. Matousek, J.: On enclosing k points by a circle. Information Processing Letters. 53 (4), 217 – 221 (1995)
16. O'Rourke, J., Kosaraju, S.R., Megiddo, N.: Computing circular separability. Discrete and Computational Geometry. 1, 105–113 (1986)
17. Roussillon, T., Sivignon, S., Tougne, L.: On three constrained versions of the digital circular arc recognition problem. In: 15-th International Conference on Discrete Geometry for Computer Imagery (DGCI09), pp. 341–45. LNCS, Springer Verlag (2009)