Elements related to the largest complete excursion of a reflected BM stopped at a fixed time. Application to local score.

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Abstract

We calculate the density function of $\big(U^*(t),\theta^*(t)\big)$, where $U^*(t)$ is the maximum over $[0,g(t)]$ of a reflected Brownian motion U, where $g(t)$ stands for the last zero of U before t, $\theta^*(t)=f^*(t)-g^*(t)$, $f^*(t)$ is the hitting time of the level $U^*(t)$, and $g^*(t)$ is the left-hand point of the interval straddling $f^*(t)$. We also calculate explicitly the marginal density functions of $(U^*(t)$ and $\theta^*(t)$. Let U^*_n and θ^*_n be the analog of $U^*(t)$ and $\theta^*(t)$ respectively where the underlying process (U_n) is the Lindley process, i.e. the difference between a centered real random walk and its minimum. We prove that $\left(\dfrac{U^*_n}{\sqrt{n}},\dfrac{\theta^*_n}{n}\right)$ converges weakly to $\big(U^*(1),\theta^*(1)\big)$ as $n\to\infty$.

Key words: Lindley process, local score, Donsker invariance Theorem, reflected Brownian motion, inverse of the local time, Brownian excursions.
MSC: 60 F 17, 60 G 17, 60 G 40, 60 G 44, 60 G 50, 60 G 52, 60 J 55, 60 J 65.

1 Introduction

1.1 The local score is a probabilistic tool which is often used by molecular biologists to study sequences of either amino-acids or nucleotides as DNA. In particular its statistical properties allow to determine the most significant

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Preprint submitted to Stochastic Processes and their Applications June 18, 2014

segment in a given sequence, see for instance [11] and [17]. Any position i in the sequence is allocated a random value ϵ_i. For example, ϵ_i can measure either physical or chemical property of the i-th amino acid or nucleotide of the sequence. It can also code the similarity between two components of two sequences. It is assumed that $(\epsilon_i)_{i\geq 1}$ is a sequence of independent and identically distributed random variables. Rather than considering $(\epsilon_i)_{i\geq 1}$, it is more usefull to deal with:

$$S_n = \epsilon_1 + \cdots + \epsilon_n \quad \text{for } n \geq 1 \;; \quad S_0 = 0. \tag{1.1}$$

Obviously, (S_n) is the random walk starting at 0, with independent increments $(\epsilon_i)_{i\geq 1}$. Let us introduce:

$$\underline{S}_n = \min_{0\leq i\leq n} S_i, \quad n \geq 0. \tag{1.2}$$

The two following processes (U_n) and $(\overline{U}_n)$ play an important role in the study of biological sequences. The first one is called the Lindley process and is defined as:

$$U_n = S_n - \underline{S}_n = S_n - \min_{i\leq n} S_i, \quad n \geq 0. \tag{1.3}$$

The process (U_n) is non negative and further properties can be found either in (Chap. III of [1]) or Chap. I [6]). The local score $\overline{U}_n$ is the supremum of the Lindley process up to time n.

Molecular biologists are interested in "unexpected" large values of (U_n), see [17].

The exact distribution of $\overline{U}_n$ has been determined in [12], using the exponentiation of a suitable matrix and classical tools related to Markov chains theory. Although the given formula in [12] is efficient whatever the sign of $\mathbb{E}(\epsilon_i)$, in practice, it can be only applied to short sequences. However, we are sometimes faced with long sequences and in these situations it is often assumed that they have a negative trend, i.e. $\mathbb{E}(\epsilon_i) < 0$. Then, the local score $\overline{U}_n$ grows as $\ln(n)$ (see [18]) and an asymptotic approximation of the distribution of $\overline{U}_n$ as n is large has been given in [11], [9], using the renewal theory. When $\mathbb{E}(\epsilon_n) = 0$, the asymptotic behavior of the tail distribution of $\overline{U}_n$ has been determined in [7] and the rate of convergence is given in [10].

Although the study of biological sequences is the starting point of this paper, the remainder will only consider the probabilistic model.

Here we consider that the $(\epsilon_i)_{i\geqslant 1}$ are centered with unit variance.

It is clear that the trajectory of (U_n) can be composed of a succession of 0 and excursions above 0. However, we only deal with *complete* excursions up to a fixed time. This leads us to introduce the maximum U_n^* of the heights of all the complete excursions up to time n. The second variable which will play an important role is θ_n^*, the time necessary to reach its maximum height U_n^*. See Section 3 for more informations and detailed definitions of the previous RVs.

We believe that the knowledge of the joint distribution of the pair (U_n^*, θ_n^*) should permit the associated bi-dimensional statistical tests to be more powerful than the usual ones based on the first component. This program should be developed in a forthcoming paper.

1.2 Unfortunately, it is difficult to determine explicitly the law of $(U_n^*,\ \theta_n^*)$ for a fixed n. Bearing in mind applications with long biological sequences, it is relevant to study the distribution of $(U_n^*,\ \theta_n^*)$ where n is large. The functional convergence theorem of Donsker tells us that the initial random walk $(S_k,\ 0 \le k \le n)$ normalized by the factor $1/\sqrt{n}$ converges in distribution as $n \to \infty$ to the Brownian motion $(B(s),\ 0 \le s \le 1)$, see Sections 3.1 and 3.2 for a more precise formulation. It is easy to deduce that the normalized Lindley process $\left(\dfrac{U_k}{\sqrt{n}},\ 0 \le k \le n\right)$ can be approximated by $(\widehat{U}_s,\ 0 \le s \le 1)$ where:

$$\widehat{U}(t) := B(t) - \inf_{0 \le s \le t} B(s), \quad t \ge 0. \tag{1.4}$$

Recall that the process $(\widehat{U}(s),\ s \ge 0)$ is distributed as the reflected Brownian motion, since:

$$\Big(|B(t)|,\ t \ge 0\Big) \overset{(d)}{=} \Big(B(t) - \min_{0 \le u \le t} B(u),\ t \ge 0\Big). \tag{1.5}$$

It turns out that the asymptotic behavior of $(U_n^*,\ \theta_n^*)$ for large n should be closely linked the distribution of $\big(U^*(1), \theta^*(1)\big)$ where $U^*(1)$ and $\theta^*(1)$ are the analog in continuous time of U_n^* and θ^*. Consequently, the knowledge of the distribution of $(U_n^*,\ \theta_n^*)$ for large n reduces to $\Big(U^*(1), \theta^*(1)\Big)$. Let us briefly define these RVs. As we proceed with the random walk (S_n), we introduce the following processes (see Section 2 for more explicit definitions):

(1) the local score $\overline{U}(t)$ which is the maximum of the heights of all the excursions of $U(s)$ up to time t, i.e. $U^*(t) := \sup_{0 \le s \le t} U(s)$,
(2) the maximum $U^*(t)$ of the heights of all the *complete* excursions up to time t,
(3) the time $\theta^*(t)$ taken by U to reach $U^*(t)$ starting from the beginning of this highest excursion.

1.3 Let t be a fixed real number. The density function of $\overline{U}(t)$ is known (see either Subsection 2.11 in [4] or Lemma 3.2 in [15]). Although $U^*(t) = \overline{U}\Big(g(t)\Big)$ and $g(t)$ is not a stopping time it is however easy to calculate the density function of $U^*(t)$. Indeed, the process $\Big(g(t)^{-1/2}B(g(t)s),\ 0 \le s \le 1\Big)$ is distributed as $\Big(b(s),\ 0 \le s \le 1\Big)$ and is independent of $g(t)$, where $g(t)$ is the last zero of $U(s)$ before t and b is the Brownian bridge (see e.g. [2]). Therefore:

$$U^*(t) \overset{(d)}{=} \sqrt{g(t)} \sup_{0 \le s \le 1} |b(s)|. \tag{1.6}$$

Finally, we conclude using the fact that $g(t)$ is distributed as the arcsine law (see again [2]) and the distribution of $\sup_{0 \le s \le 1} |b(s)|$ is given by the Kolmogorov-Smirnov formula (see e.g. [13]). The final and explicit result is given in Theorem 2.6.

However, as far as we know, the distribution of $\Big(U^*(t), \theta^*(t)\Big)$ is unknown. Using the theory of excursions related to the one dimensional Brownian motion, we determine in Theorem 2.3 the density function of the couple $\Big(U^*(t), \theta^*(t)\Big)$. Since we are interested in statistical tests based on the joint law of $(U^*(t), \theta^*(t))$, then we have to determine the quantiles of $(U^*(t), \theta^*(t))$. Unfortunately the expression of the density function is complicated and does not allow us to calculate the distribution function of $(U^*(t), \theta^*(t))$. In Theorem 2.4, we express, for any bounded Borel function $f :]0, \infty[\times]0, \infty[\to \mathbb{R}$, the expectation of $f\Big(U^*(t), \theta^*(t)\Big)$ as $\mathbb{E}(f(A_1)A_2)$ where A_1 and A_2 are RVs which can be simulated. Therefore, the quantity $\mathbb{E}\Big(f\Big(U^*(t), \theta^*(t)\Big)$ can be approximated by a Monte-Carlo scheme.

In Section 2.2 we fix notations related to the setting of processes in continuous time, i.e. here the underlying process is the Brownian motion. The main results are Theorems 2.3, 2.4, 2.5 and 2.6 and they are given in Section 2.3. Theorem 2.3 is based on Propositions 2.2 and 2.1. Although the law of $U^*(t)$ is easy to calculate, $\theta^*(t)$ is more difficult, see Theorem 2.5. We recall in Section 3 the functional approximation of the one dimensional Brownian motion by normalized random walks. Then, with additional technical developments, see Proposition 3.1 and Theorem 3.3 we obtain the weak convergence of $\left(\frac{U_n^*}{\sqrt{n}}, \frac{\theta_n^*}{n}\right)$ as $n \to \infty$ towards $\Big(U^*(1), \theta^*(1)\Big)$. All the proofs which are not immediate have been given in Section 4.

Acknowledgements The authors are greatly indebted to the referee for his fruitful comments, references and suggestions.

2 Theoretical results

2.1 Notation

Let $(B(t),\ t \geq 0)$ be a standard Brownian motion started at 0 and $U(t)$ is the reflected Brownian motion at time t:

$$U(t) := |B(t)|, \quad t \geq 0. \tag{2.1}$$

The excursion (above 0) straddling t starts at $g(t)$ and ends at $d(t)$, namely

$$g(t) = \sup\{s \leq t,\ U(s) = 0\}, \quad d(t) = \inf\{s \geq t,\ U(s) = 0\}, \quad t \geq 0. \tag{2.2}$$

Let $\overline{U}(t)$ be the supremum of U over $[0, t]$

$$\overline{U}(t) := \sup_{0 \leq s \leq t} U(s), \quad t \geq 0. \tag{2.3}$$

Then, the highest height $U^*(t)$ of all the *complete* excursions of the process $(U(r)\ ;\ 0 \leq r \leq t)$ equals

$$U^*(t) := \overline{U}(g(t)) = \sup_{0 \leq s \leq g(t)} U(s), \quad t \geq 0. \tag{2.4}$$

Let $f^*(t)$ be the unique time which achieves the maximum of U over $[0,\ g(t)]$:

$$f^*(t) := \sup\{r \leq g(t)\ ;\ U(r) = U^*(t)\}, \quad t \geq 0. \tag{2.5}$$

It is worth introducing the left end-point $g^*(t)$ of the excursion straddling $f^*(t)$:

$$g^*(t) := g(f^*(t)) = \sup\{r \leq f^*(t)\ ;\ U(r) = 0\}, \quad t \geq 0 \tag{2.6}$$

as well as the right end-point $d^*(t)$ of this excursion:

$$d^*(t) := d(f^*(t)) = \inf\{r \geq f^*(t)\ ;\ U(r) = 0\}, \quad t \geq 0. \tag{2.7}$$

It is convenient to visualize the different variables in Figure 1.

We are interested in the joint law of $U^*(t)$ and $\theta^*(t)$ where the second variable is defined as

$$\theta^*(t) := f^*(t) - g^*(t), \quad t \geq 0. \tag{2.8}$$

It is convenient to introduce the following notation which will be used extensively in the sequel.

1) $(\xi_n)_{n\geq 1} \cup \{\xi, \xi'\}$ is a family of i.i.d. r.v.s such that

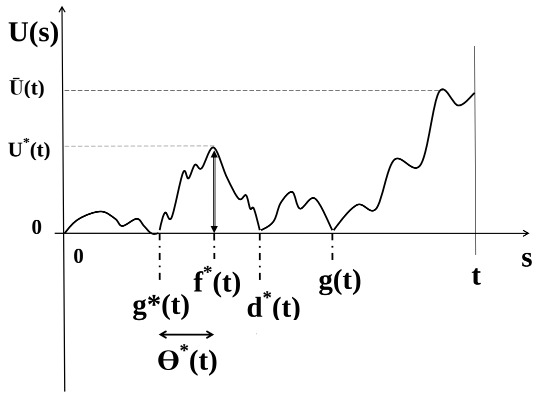

Figure 1. Notation $U(t)$

$$\xi \overset{(d)}{=} \xi' \overset{(d)}{=} \xi_n \overset{(d)}{=} T_1(R) \tag{2.9}$$

with

$$T_x(R) = \inf\{s \geq 0 \ ; \ R(s) = x\}, \quad x > 0 \tag{2.10}$$

and $(R(s), s \geq 0)$ stands for a 3-dimensional Bessel process started at 0. The density p_ξ is explicitly known and is given by

$$p_\xi(u) = \frac{1}{\sqrt{2\pi}u^{3/2}} \sum_{k\in\mathbb{Z}} \left(-1 + \frac{(1+2k)^2}{u}\right) \exp\left(-\frac{(1+2k)^2}{2u}\right) \tag{2.11}$$

$$= \frac{d}{du}\left(\sum_{k\in\mathbb{Z}} (-1)^k \exp\left(-\frac{k^2\pi^2 u}{2}\right)\right) \tag{2.12}$$

(see for instance [3] p 8 and 24). In perspective of simulation, let us mention that an efficient algorithm to simulate very quickly the r.v. ξ is given in [8].

2) e_0', $(e_n)_{n\geq 0}$ is a sequence of i.i.d. exponential r.v.s.

3) $(\lambda(x), x \geq 0)$ is the process defined by

$$\lambda(x) := x^2(\xi_1 + \xi_2) + \sum_{k\geq 1} \frac{\xi_{2k+1} + \xi_{2k+2}}{\left(\frac{1}{x} + e_1 + \cdots + e_k\right)^2}, \quad x \geq 0. \tag{2.13}$$

The sum converges a.s. and in L^1 (see Lemma 4.5). The Laplace transform of $\lambda(x)$ has been calculated in (4.20).

4) α_1 and α_2 are two $[0,1]$ valued r.v.s; α_2 is uniformly distributed and the density function of α_1 is $\frac{2}{\pi}\frac{1}{\sqrt{1-s^2}}\mathbb{1}_{[0,1]}(s)$.

We always assume in the sequel that

$$e'_0,\ (e_n)_{n\geq 0},\ (\xi_n)_{n\geq 1},\ \xi,\ \xi',\ \alpha_1,\ \alpha_2 \text{ and } (U(t))_{t\geq 0} \text{ are independent.} \tag{2.14}$$

2.2 Distribution of the pair $(U^*(t), \theta^*(t))$

The main results are Theorems 2.3, 2.4, 2.5 and 2.6. All the proof of results stated in this section will be developed in Section 4.
In Theorem 2.3, we determine the density function of $(U^*(t), \theta^*(t))$. Its proofs is based on the theory of excursion related to the Brownian motion, see for instance Chap XII in [14]. Let us briefly recall the ingredients which are needed. Let $(L(t),\ t \geq 0)$ be the local time process at 0 related to the Brownian motion $(B(t),\ t \geq 0)$. The random function $t \mapsto L(t)$ is continuous and non-decreasing. Let $(\tau_s,\ s \geq 0)$ be its right inverse. The proof of Theorem 2.3 has two main steps. In Proposition 2.1 below we begin with expressing the distribution of $\Big(U^*(t), \theta^*(t)\Big)$ in terms of the one of $\Big(\overline{U}(\tau_1),\ \tau_1\Big)$.

Proposition 2.1 *Let t be a fixed positive real number. Then, the density function of $(U^*(t), \theta^*(t))$ is given by*

$$p_{(U^*(t),\theta^*(t))} = \sqrt{\frac{2}{\pi}}\rho(x,y)p_\xi\left(\frac{y}{x^2}\right)\frac{1}{x^4}, \quad x > 0, 0 < y < t \tag{2.15}$$

where p_ξ is the density function of ξ (see (2.11)-(2.12)),

$$\rho(x,y) := \int_0^{+\infty} \mathbb{E}\left(\frac{1}{\tau_1}\left\{\sqrt{(\rho_1)_+} - \sqrt{\left(\rho_1 - \frac{x^2\tau_1}{(\overline{U}(\tau_1))^2}\right)_+}\right\}\right) p_\xi(u)du, \tag{2.16}$$

$x_+ := \sup\{x, 0\}$ and $\rho_1 := t - y - x^2 u$.

We are then naturally lead to determine the distribution $\Big(\overline{U}(\tau_1), \tau_1\Big)$.

Proposition 2.2 *(1) For any $x \geq 0$, the sum in (2.13) converges a.s. and in L^1.*
(2) The r.v. $\overline{U}(\tau_1)^{-1}$ is exponentially distributed and conditionally on $\Big\{\overline{U}(\tau_1) = x\Big\}$, $x > 0$,

$$\tau_1 \overset{(d)}{=} \lambda(x). \tag{2.17}$$

Finally, combining Propositions 2.1 and 2.2 provides the density function of $\Big(U^*(t), \theta^*(t)\Big)$.

Theorem 2.3 *For any $t > 0$, the pair $\Big(U^*(t), \theta^*(t)\Big)$ has the density function $p_{(U^*(t),\theta^*(t))}$ given by* (2.15) *where*

$$\rho(x,y) = \int_{\mathbb{R}_+^2} \mathbb{E}\left(\frac{1}{\lambda(v)}\left\{\sqrt{(\rho_1)_+} - \sqrt{\left(\rho_1 - \frac{x^2}{v^2}\lambda(v)\right)_+}\right\}\right) p_\xi(u)\frac{e^{-1/v}}{v^2}\, dudv \tag{2.18}$$

and ρ_1 has been defined in Proposition 2.1.

Formula (2.18) has the disadvantage to be not completely explicit and therefore it does not allow a direct calculation of $\mathbb{E}\left[f\Big(U^*(t), \theta^*(t)\Big)\right]$ for a given bounded Borel function f. For instance, for our biological motivation explained in the Introduction, it would be interesting to calculate $\mathbb{P}\Big(U^*(t) \le a,\ \theta^*(t) \le b\Big)$ for any $a, b > 0$. Rewriting the proof of Theorem 2.3 leads us to an equivalent formulation of Theorem 2.3 which gives rise to a more useful formula.

Theorem 2.4 *Let $f : \mathbb{R}_+^2 \to \mathbb{R}$ be a bounded Borel function. Then*

$$\mathbb{E}\left(f(U^*(t), \theta^*(t))\right) = \sqrt{\frac{\pi}{2}}\mathbb{E}\left[f\left(\frac{\alpha_1\sqrt{t}}{\sqrt{Z}}, \frac{t\alpha_1^2\xi}{Z}\right)\frac{\alpha_2 e_0'^2}{\sqrt{Z}}\right] \tag{2.19}$$

where $Z = \xi + \xi' + e_0'^2\alpha_2^2\lambda(1/e_0')$.

2.3 Distributions of $U^*(t)$ and $\theta^*(t)$

We begin with the distribution of $\theta^*(t)$.

Theorem 2.5 *For any $t > 0$, $\theta^*(t)$ admits the following density function*

$$f_{\theta^*(t)}(x) = \frac{1}{x}\sum_{k\ge 1}(-1)^{k+1}\frac{\sinh\left(\pi k\sqrt{\frac{x}{t-x}}\right)}{\cosh^2\left(\pi k\sqrt{\frac{x}{t-x}}\right)}\mathbb{1}_{[0,t]}(x). \tag{2.20}$$

We now consider the law of $U^*(t)$.

Theorem 2.6 *Let $t > 0$.*

(1) We have the following identity in law

$$U^*(t) \overset{(d)}{=} \sqrt{t}\sqrt{g(1)}b^*, \tag{2.21}$$

where $g(1)$ and b^* are two independent r.v.s such that

$$\mathbb{P}(g(1) \in dx) = \frac{1}{\pi\sqrt{x(1-x)}} 1\!\!1_{[0,1]}(x)\, dx, \tag{2.22}$$

$$\mathbb{P}(b^* > x) = 2\sum_{k\geq 1}(-1)^{k-1}e^{-2k^2x^2}, \quad x > 0. \tag{2.23}$$

(2) $U^*(t)$ admits the following density function

$$f_{U^*(t)}(x) = 4\sqrt{\frac{2}{\pi t}}\left(\sum_{k\geq 1}(-1)^{k-1}ke^{-\frac{2k^2x^2}{t}}\right), \quad x > 0. \tag{2.24}$$

The proof of item 1 in Theorem 2.6 is straightforward and has been developed in the Introduction. Note that this direct approach does not use the knowledge of the density function of $(U^*(t), \theta^*(t))$. However, Lemma 4.1 permits to get another expression of the distribution of $U^*(t)$.

Proposition 2.7 *We define two other stopping times $T_U(a) = \inf\{t \geq 0,\ U(t) = a\}$ and $T_{\widehat{B}}(a) = \inf\{t \geq 0,\ \widehat{B}(t) = a\}$ where $(\widehat{B}(t),\ t \geq 0)$ is a one dimensional Brownian motion independent of U. Then*

$$\mathbb{P}(U^*(t) > a) = \mathbb{P}\left(T_U(a) + T_{\widehat{B}}(a) < t\right), \quad t > 0, a > 0. \tag{2.25}$$

Remark 2.8 *It is clear that* (2.25) *allows to compute the cumulative distribution function of $U^*(t)$ and gives a complement to* (2.24). *The distributions of $T_{\widehat{B}}(a)$ and $T_U(a)$ are explicitly known: the density function of $T_{\widehat{B}}(a)$ is $\frac{a}{\sqrt{2\pi t^3}}e^{-a^2/2t}1\!\!1_{\{t>0\}}$ and (see Lemma 3.2 in [16])*

$$\mathbb{P}\left(T_U(a) > t\right) = \frac{4}{\pi}\sum_{k\geq 0}\frac{(-1)^k}{2k+1}\exp\left\{-\frac{(2k+1)^2\pi^2}{8a^2}t\right\},\ t > 0.$$

3 Application to the discrete case

Recall that the r.v. S_n and the Lindley process U_n are associated with the sequence $(\epsilon_i)_{i\geq 1}$ via (1.1) and (1.3) respectively. The process (U_k) is a non negative Markov chain. In the case where $(\epsilon_i)_{i\geq 1}$ are symmetric Bernoulli r.v.'s (i.e. $P(\epsilon_i = \pm 1) = 1/2$), then (U_k) takes its values in $\mathbb{N}$ and moves as a symmetric random walk in $\{1, 2, \cdots\}$ and being at 0, it either stays at this level with probability 1/2 or jumps to 1 with probability 1/2.

In general, the trajectory of (U_k) can be decomposed in a succession of 0 and excursions above 0. An excursion of (U_k) starting at g and ending at d is a process $(e(k),\ 0 \leq k \leq \zeta)$, where

$$e(0) = U(g) = 0, \quad e(\zeta) = U(d) = 0, \quad \zeta := d - g > 0$$

and $e(k) := U(g+k) > 0$, for any $0 < k < \zeta$. As mentioned in the Introduction, the local score $\overline{U}_n$ is the maximum of (U_k) up to time n and can be interpreted as the maximum of all the heights of the excursions up to time n. Namely

$$\overline{U}_n := \max_{0 \le k \le n} U_k, \quad n \ge 0. \tag{3.1}$$

We are interested in the highest *complete* excursion up to time n. We proceed as in the continuous time setting introducing

$$\begin{aligned}
&g_n := \max\{k \le n\ ;\ U_k = 0\}, && U_n^* := \overline{U}_{g_n} = \max_{0 \le k \le g_n} U_k,\\
&f_n^* := \max\{k \le g_n;\ U_k = U_n^*\}, && g_n^* := g_{f_n^*} = \max\{k \le f_n^*\ ;\ U_k = 0\},\\
&d_n^* := \inf\{k \ge f_n^*\ ;\ U_k = 0\}, && \theta_n^* := f_n^* - g_n^*.
\end{aligned} \tag{3.2}$$

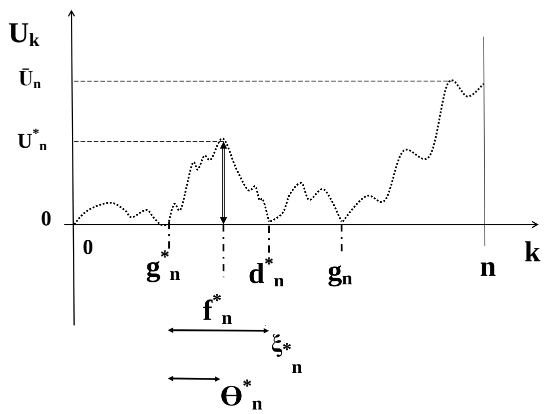

Figure 2. Notation U_n

In Section 3.1, we define a continuous process $(U^M(t), t \ge 0)$ as the classical linear interpolation of $(U_n, n \ge 0)$. We naturally introduce the highest high $U^{M,*}(t)$ and length $\theta^{M,*}(t)$ of the complete excursion until time t of $(U^M(t), t \ge 0)$. We conclude linking $(U^{M,*}(1), \theta^{M,*}(1))$ to (U_M^*, θ_M^*). Then we prove in Section 3.2 the convergence of $(U^{M,*}(t), \theta^{M,*}(t))$ to $(U^*(t), \theta^*(t))$. Since the distribution of $(U^*(t), \theta^*(t))$ has been computed in Section 2, we then get an approximation of the distribution of (U_M^*, θ_M^*).

3.1 The linear interpolation of $(U_k)_k$

We keep notation given above and the one introduced in Section 2. Recall in particular that $(B(t),\ t \geq 0)$ stands for a standard Brownian motion started at 0 and $(U(t),\ t \geq 0)$ is the reflected Brownian motion defined by (2.1). Let $M > 0$ be a scale parameter which allows to obtain the convergence of the normalized random walk to the Brownian motion $(B(t))$ as $M \to \infty$ (see Section 3.2). The classical continuous process $\left(B^M(t),\ t \geq 0\right)$ associated with (S_n) and normalizing factor M is classically defined as $B^M\left(\frac{k}{M}\right) = \frac{1}{\sqrt{M}} S_k$ and for any k such that $\frac{k}{M} \leq t \leq \frac{k+1}{M}$

$$B^M(t) = B^M\left(\frac{k}{M}\right) + M\left(t - \frac{k}{M}\right)\left(B^M\left(\frac{k+1}{M}\right) - B^M\left(\frac{k}{M}\right)\right).$$

We are interested here by the process $(U^M(t),\, t \geq 0)$

$$U^M(t) = B^M(t) - \min_{s \leq t} B^M(s), \quad t \geq 0. \tag{3.3}$$

Note that

$$U^M\left(\frac{k}{M}\right) = \frac{1}{\sqrt{M}} U_k, \quad k \geq 0 \tag{3.4}$$

where (U_k) is the Lindley process associated with (S_k) via (1.3).

We define the analog of r.v.s introduced in the discrete setting of Lindley process, see (3.2) and (3.1) in the continuous time setting of $\left(B^M(t)\right)$

$$\begin{aligned}
\overline{U}^M(t) &:= \sup_{0 \leq s \leq t} U^M(s), \qquad g^M(t) := \sup\left\{s \leq t\ ;\ U^M(s) = 0\right\}, \\
U^{M,*}(t) &:= \overline{U}^M(g^M(t)) = \sup_{0 \leq s \leq g^M(t)} U^M(s), \\
f^{M,*}(t) &:= \sup\left\{r \leq g^M(t);\ U^M(r) = U^{M,*}(t)\right\}, \\
g^{M,*}(t) &:= g^M(f^{M,*}(t)) = \sup\left\{r \leq f^{M,*}(t)\ ;\ U^M(r) = 0\right\}, \\
d^{M,*}(t) &:= \inf\left\{s \geq f^{M,*}(t)\ ;\ U(s) = 0\right\}, \qquad \theta^{M,*}(t) := f^{M,*}(t) - g^{M,*}(t).
\end{aligned} \tag{3.5}$$

Using the definition (3.2) of θ^*_M and U^*_M we deduce easily that these r.v.'s can be expressed in terms of their analog in continuous time.

Proposition 3.1 *We have the following scaling properties*

$$\frac{\theta^*_M}{M} = \theta^{M,*}(1) \quad \textit{and} \quad \frac{U^*_M}{\sqrt{M}} = U^{M,*}(1). \tag{3.6}$$

3.2 Convergence of $\Big(U^{M,*}(t), \theta^{M,*}(t)\Big)$ *to* $\Big(U^*(t), \theta^*(t)\Big)$

The key ingredient of our convergence results is the Donsker Theorem, see Section 2.10 in [4]: the processus $\Big(B^M(t),\ t \geq 0\Big)$ converges weakly to the Brownian motion $\Big(B(t),\ t \geq 0\Big)$ when $M \to +\infty$. Using moreover (1.5) we get the following useful result.

Proposition 3.2 $\Big(U^M(t),\ t \geq 0\Big)$ *converges weakly to* $\Big(U(t),\ t \geq 0\Big)$.

Note that it is unclear that the map $\omega \mapsto \Big(g^{M,*}(t), f^{M,*}(t), d^{M,*}(t), \theta^{M,*}(t)\Big)$ defined from $U^*(t)$ is continuous. Therefore the weak convergence of $\Big(g^{M,*}(t), f^{M,*}(t), d^{M,*}(t), \theta^{M,*}(t), U^{M,*}(t)\Big)$ as $M \to \infty$ is not a straightforward consequence of Proposition 3.2.

Theorem 3.3 *Let* $t > 0$*. The 5-uplet* $\Big(g^{M,*}(t), f^{M,*}(t), d^{M,*}(t), \theta^{M,*}(t), U^{M,*}(t)\Big)$ *converges weakly to* $\Big(g^*(t), f^*(t), d^*(t), \theta^*(t), U^*(t)\Big)$ *as* $M \to \infty$ *where the r.v.s* $g^*(t)$*,* $f^*(t)$*,* $d^*(t)$*,* $\theta^*(t)$*,* $U^*(t))$ *have been defined by relations* (2.4)-(2.8).

4 Proofs

We follow the notation introduced in Sections 2 and 3.

4.1 Proof of Proposition 2.1

We first link the distribution of $(U^*(t), \theta^*(t))$ to that of $(\overline{U}(\tau_s), \theta^*(\tau_s), \tau_s)$.

Lemma 4.1 *Let* $f : \mathbb{R}^2_+ \to \mathbb{R}$ *be a bounded Borel function. Then*

$$\mathbb{E}\Big[f(U^*(t), \theta^*(t))\Big] = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \mathbb{E}\left[f\Big(\overline{U}(\tau_s), \theta^*(\tau_s)\Big) \frac{1}{\sqrt{t - \tau_s}} \mathbb{1}_{\{\tau_s < t\}}\right] ds. \tag{4.1}$$

Proof The real number $s = L(t)$ is the unique s such that $\tau_{s_-} < t < \tau_s$. Thus,

$$f\Big(U^*(t), \theta^*(t)\Big) = \sum_{s \geq 0} 1\!\!1_{\{\tau_{s_-} < t < \tau_s\}} f\Big(\overline{U}(\tau_{s_-}), \theta^*(\tau_{s_-})\Big)$$

since $B(\tau_{s_-}) = 0$ implies that $U^*(t) = \overline{U}(\tau_{s_-})$ and $\theta^*(t) = \theta^*(\tau_{s_-})$.

Denote e_s the Brownian excursion

$$e_s(v) : = \begin{cases} B(\tau_{s_-} + v), \ 0 \leq v \leq \tau_s - \tau_{s_-} \text{ for } \tau_s - \tau_{s_-} > 0 \\ [\delta] \ \text{ otherwise} \end{cases}$$

and $\zeta(e_s) := \tau_s - \tau_{s_-}$ its lifetime. Since $\tau_s = \tau_{s_-} + \zeta(e_s)$,

$$\mathbb{E}\Big[f(U^*(t), \theta^*(t))\Big] = \mathbb{E}\left[\sum_{s \geq 0} f\Big(\overline{U}(\tau_{s-}), \theta^*(\tau_{s-})\Big) 1\!\!1_{\{\tau_{s-} < t < \tau_{s-} + \zeta(e_s)\}}\right].$$

Applying Proposition 2.6 in [14] (consequence of the Master Formula stated in Proposition 1.10, Chapter XII), one gets

$$\mathbb{E}\Big[f(U^*(t), \theta^*(t))\Big] = \mathbb{E}\left[\int_0^{+\infty} \left\{\int f\Big(\overline{U}(\tau_s), \theta^*(\tau_s)\Big) 1\!\!1_{\{\tau_s < t < \tau_s + \zeta(w)\}} n(dw)\right\} ds\right],$$

$n(dw)$ being a σ-finite measure on the set of all positive excursions. According to Proposition 2.8, Chapter XII in [14], $n\Big(\zeta(\omega) > \varepsilon\Big) = \sqrt{\frac{2}{\pi\varepsilon}}$. Identity (4.1) then follows. ■

Since for any $a > 0$, the process $(U(sa)/\sqrt{a}; s \geq 0)$ is distributed as $(U(s); s \geq 0)$, we deduce the following scaling property

$$(\overline{U}(\tau_s), \theta^*(\tau_s), \tau_s) \stackrel{(d)}{=} (s\overline{U}(\tau_1), s^2\theta^*(\tau_1), s^2\tau_1), \quad s > 0. \tag{4.2}$$

Now we express the distribution of $\Big(\overline{U}(\tau_1), \theta^*(\tau_1), \tau_1\Big)$ in terms of that of $\Big(\overline{U}(\tau_1), \tau_1\Big)$.

Lemma 4.2 *Let $h : \mathbb{R}_+^3 \to \mathbb{R}$ be a bounded Borel function. Then*

$$\mathbb{E}\left[h\left(\overline{U}(\tau_1), \theta^*(\tau_1), \tau_1\right)\right] = \int_0^{+\infty} \mathbb{E}\left[h\Big(x, x^2\xi, x^2(\xi + \xi') + \tau_1\Big) 1\!\!1_{\{\overline{U}(\tau_1) < x\}}\right] \frac{dx}{x^2}.$$

Proof It can be deduced from Theorem 1 of [16] that

$$\mathbb{P}\Big(\overline{U}(\tau_1) < x\Big) = e^{-1/x} \quad x > 0. \tag{4.3}$$

Moreover conditionally on $\{\overline{U}(\tau_1) = x\}$,

(1) the r.v. $f^*(\tau_1) - g^*(\tau_1)$, $d^*(\tau_1) - f^*(\tau_1)$ and $\tau_1 - d^*(\tau_1) + g^*(\tau_1)$ are independent;
(2) $f^*(\tau_1) - g^*(\tau_1) \overset{(d)}{=} d^*(\tau_1) - f^*(\tau_1) \overset{(d)}{=} T_x(R)$;
(3) $\tau_1 - d^*(\tau_1) + g^*(\tau_1)$ is distributed as τ_1 conditionally on $\{\overline{U}(\tau_1) < x\}$.

Now assume that $\widetilde{R}$ is distributed as R such that $(R,\widetilde{R})$ is independent of U. By the definition of $\theta^*(\tau_1)$ and a wise decomposition of τ_1, we get

$$\mathbb{E}\Big[h(\overline{U}(\tau_1), \theta^*(\tau_1), \tau_1)\Big] = \int_0^{+\infty} \frac{e^{-1/x}}{x^2} \mathbb{E}\left[h\Big(x, T_x(R), T_x(R) + T_x(\widetilde{R}) + \tau_1\Big) | \overline{U}(\tau_1) < x\right] dx. \tag{4.4}$$

The result is a direct consequence of (2.9) and the scaling property

$$T_x(R) \overset{(d)}{=} x^2 T_1(R). \tag{4.5}$$

■

Proof of Proposition 2.1 Denote $\Delta := \mathbb{E}\left[f(U^*(t), \theta^*(t))\right]$, where $f :\ [0,\infty[\times[0,\infty[\to \mathbb{R}$ is a bounded Borel function. According to Lemma 4.1, we have

$$\Delta = \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}_+^2} f(y, y^2 z) \mathbb{E}\left(\psi(y) 1\!\!1_{\{y^2(\xi'+z)<t\}}\right) \frac{1}{y^2} p_\xi(z)\, dydz$$

where

$$\psi(y) := \int_0^{+\infty} \frac{s\, ds}{\sqrt{t - y^2(z+\xi') - s^2\tau_1}} 1\!\!1_{\{s<s_*\}} \tag{4.6}$$

$$= \frac{1}{\tau_1}\left[\sqrt{t - y^2(z+\xi')} - \sqrt{t - y^2(z+\xi') - s_*^2 \tau_1}\right] \tag{4.7}$$

with $s_* = \frac{y}{\overline{U}(\tau_1)} \wedge \sqrt{\frac{t - y^2(z+\xi')}{\tau_1}}$ and $a \wedge b = \inf\{a, b\}$.

It is easy to prove that on $\{s < s_*\}$,

$$\psi(y) = \frac{1}{\tau_1}\left[\sqrt{(t - y^2(z+\xi'))_+} - \sqrt{\left(t - y^2\left\{z + \xi' - \frac{\tau_1}{\left(\overline{U}(\tau_1)\right)^2}\right\}\right)_+}\right].$$

Then (2.15) follows. ■

Since the density function of $\overline{U}(\tau_1)$ is explicit (see (4.3)), that of $\Big(\tau_1, \overline{U}(\tau_1)\Big)$ will be determined once the conditional distribution of τ_1 given $\overline{U}(\tau_1)$ will be known. Our proof is based on the study of the process $(\widehat{\lambda}(x),\ x > 0)$ such that conditionally on $\{\overline{U}(\tau_1) = x\}$,

$$\tau_1 \overset{(d)}{=} \widehat{\lambda}(x). \tag{4.8}$$

Obviously (4.8) is equivalent to

$$E\Big[f(\tau_1)g\Big(\overline{U}(\tau_1)\Big)\Big] = E\Big[f\Big(\widehat{\lambda}\Big(\overline{U}(\tau_1)\Big)\Big)g\Big(\overline{U}(\tau_1)\Big)\Big] \tag{4.9}$$

for any bounded Borel functions $f, g : [0,\infty[\to \mathbb{R}$. We will show that $(\widehat{\lambda}(x),\ x > 0)$ satisfies an equation which has a unique solution.

Lemma 4.3 *Let $x > 0$ and $n \geq 0$. Then,*

$$\widehat{\lambda}(x) \overset{(d)}{=} \Lambda_n + \widehat{\lambda}\left(\frac{1}{\frac{1}{x} + e_1 + \cdots + e_{n+1}}\right) \tag{4.10}$$

where

$$\Lambda_n := x^2(\xi_1 + \xi_2) + \sum_{k=1}^{n} \frac{\xi_{2k+1} + \xi_{2k+2}}{\Big(\frac{1}{x} + e_1 + \cdots + e_k\Big)^2}, \quad n \geq 0$$

with the classical convention $\sum\limits_{1}^{0} = 0$.

Proof First we prove

$$\widehat{\lambda}(x) \overset{(d)}{=} x^2(\xi_1 + \xi_2) + \widehat{\lambda}\left(\frac{1}{\frac{1}{x} + e_1}\right), \quad x > 0.$$

Let $f_1,\ f_2 : [0,+\infty[\to [0,+\infty[$ be two bounded Borel functions and

$$A := \mathbb{E}\left[f_1(\tau_1)f_2\Big(\overline{U}(\tau_1)\Big)\right] = \int_0^{+\infty} e^{-1/x} f_2(x)\mathbb{E}[f_1(\widehat{\lambda}(x))]\,\frac{dx}{x^2} \tag{4.11}$$

by (4.3) and (4.8). Applying formula (4.4) to $h(x_1, x_2, x_3) = f_1(x_3)f_2(x_1)$ leads to

$$\begin{aligned} A &= \int_0^{+\infty} e^{-1/x}\mathbb{E}\left[f_1\Big(T_x(R) + T_x(\widetilde{R}) + \tau_1\Big)\Big|\overline{U}(\tau_1) < x\right] f_2(x)\,\frac{dx}{x^2} \\ &= \int_0^{+\infty}\left(\int_0^x \frac{e^{-1/y}}{y^2}\mathbb{E}\left[f_1\left(x^2(\xi_1 + \xi_2) + \widehat{\lambda}(y)\right)\right] dy\right) f_2(x)\,\frac{dx}{x^2} \end{aligned}$$

using (4.5), (4.3) (2.9) and (4.9). Identifying with (4.11) implies

$$\mathbb{E}\left[f_1\Big(\widehat{\lambda}(x)\Big)\right] = e^{1/x}\int_0^x e^{-1/y}\mathbb{E}\left[f_1\Big(x^2(\xi_1+\xi_2)+\widehat{\lambda}(y)\Big)\right]\,\frac{dy}{y^2}.$$

Let Y be the r.v. defined by $Y = 1\Big/\left(\frac{1}{x}+e_1\right)$ whose density is obviously given by $e^{1/x-1/y}\,1\!\!1_{[0,x]}(y)/y^2$. Thus $\mathbb{E}\left[f_1\Big(\widehat{\lambda}(x)\Big)\right]$ can be rewritten as $\mathbb{E}\left[f_1\Big(x^2(\xi_1+\xi_2)+\widehat{\lambda}(Y)\Big)\right]$ which means that

$$\widehat{\lambda}(x) \overset{(d)}{=} x^2(\xi_1+\xi_2)+\widehat{\lambda}(Y) = x^2(\xi_1+\xi_2)+\widehat{\lambda}\left(\frac{1}{\frac{1}{x}+e_1}\right), \quad x>0.$$

Iterating this procedure leads to (4.10). ∎

Lemma 4.4 *For any $x>0$, $\mathbb{E}\left(\widehat{\lambda}(x)\right) = \frac{2}{3}(x+x^2)$.*

Proof Using for instance Exercise (4.9) Chap VI in [14] we get that

$$M(t) := \Big\{\cosh\Big(\lambda|B(t)|\Big)+b\sinh\Big(\lambda|B(t)|\Big)\Big\}\exp\left\{-\frac{\lambda^2}{2}t-b\lambda L(t)\right\}$$

is a local martingale for $\lambda>0$. Let $r>0$, $b=-\dfrac{\cosh(\lambda r)}{\sinh(\lambda r)}$ and

$$\sigma_r := \inf\{s\geq 0\ ;\ |B(s)|=r\} = \inf\{s>0\ ;\ U(s)=r\}.$$

The process $(M_{t\wedge\tau_1\wedge\sigma_r};t\geq 0)$ being bounded, we can apply the stopping theorem to obtain $\mathbb{E}(M(\tau_1\wedge\sigma_r)) = \mathbb{E}(M(0))$. It is clear that $|B(\sigma_r)| = U(\sigma_r) = r$, $\quad B(\tau_1)=0$ and $L(\tau_1)=1$. Our choice of b implies that $M(\sigma_r)=0$. Consequently, $M(\tau_1\wedge\sigma_r) = M(\tau_1)\,1\!\!1_{\{\tau_1<\sigma_r\}}$ and $e^{-b\lambda}\mathbb{E}\left[e^{-\lambda^2\tau_1/2}\,1\!\!1_{\{\tau_1<\sigma_r\}}\right]=1$. Since $\{\tau_1<\sigma_r\} = \{\overline{U}(\tau_1)<r\}$, the previous identity can be rewritten as

$$\mathbb{E}\left[e^{-\lambda^2\tau_1/2}\,1\!\!1_{\{\overline{U}(\tau_1)<r\}}\right] = e^{b\lambda} = \exp\left\{-\lambda\frac{\cosh(\lambda r)}{\sinh(\lambda r)}\right\}, \tag{4.12}$$

that leads to

$$\mathbb{E}\left[e^{-\lambda\tau_1}\,1\!\!1_{\{\overline{U}(\tau_1)<r\}}\right] = \exp\left\{-\sqrt{2\lambda}\,\frac{\cosh(r\sqrt{2\lambda})}{\sinh(r\sqrt{2\lambda})}\right\} = e^{-1/r}\left(1-\frac{2\lambda r}{3}+o(\lambda)\right). \tag{4.13}$$

Taking the derivative at 0, we get

$$\mathbb{E}\left(\tau_1 1\!\!1_{\{\overline{U}(\tau_1)<r\}}\right) = \frac{2r}{3}\,e^{-1/r}. \tag{4.14}$$

Let φ be the function defined by $\varphi(x) := \mathbb{E}\left[\widehat{\lambda}(x)\right]$. Therefore, taking the conditional expectation with respect to $\overline{U}(\tau_1)$ in (4.14) and using (4.3), we have $\int_0^r e^{-1/x}\varphi(x)\,dx/x^2 = \frac{2r}{3}\, e^{-1/r} \quad r > 0$, which conduces to
$\varphi(x) = \dfrac{2}{3}\left(x + x^2\right)$. ■

Lemma 4.5 *(1) Λ_n converges a.s. and in L^1 while $n \to +\infty$.*
(2) For any $x > 0$, $\widehat{\lambda}\left(\frac{1}{\frac{1}{x}+e_1+\cdots+e_n}\right)$ converges to 0 in L^1 while $n \to +\infty$.

Proof 1) Since all the r.v.s under concern are positive, Λ_n converges a.s. while $n \to \infty$ to the positive r.v.

$$\Lambda_\infty := x^2(\xi_1 + \xi_2) + \sum_{k\geq 1} \frac{\xi_{2k+1} + \xi_{2k+2}}{\left(\frac{1}{x} + e_1 + \cdots + e_k\right)^2}. \tag{4.15}$$

One way to prove that Λ_∞ is a.s. finite is to show that its expectation is finite.

Note that (see [5] p 463)

$$\mathbb{E}\left[e^{-\lambda\xi_1}\right] = \mathbb{E}\left[e^{-\lambda T_1(R)}\right] = \frac{\sqrt{2\lambda}}{\sinh(\sqrt{2\lambda})} = 1 - \frac{\lambda}{3} + o(\lambda), \quad \lambda > 0.$$

that conduces by derivation to $\mathbb{E}(\xi_1) = \frac{1}{3}$. Now using the fact that $e_1 + \cdots + e_k$ is $\gamma(k)$-distributed, we have successively

$$\begin{aligned}\mathbb{E}(\Lambda_\infty) - \frac{2}{3}x^2 &= \mathbb{E}\left(\sum_{k\geq 1} \frac{\xi_{2k+1} + \xi_{2k+2}}{\left(\frac{1}{x} + e_1 + \cdots + e_k\right)^2}\right)\\ &= \frac{2}{3}\sum_{k\geq 1}\int_0^{+\infty} \frac{1}{\left(\frac{1}{x}+y\right)^2}\frac{y^{k-1}}{(k-1)!}\, e^{-y}\,dy = \frac{2}{3}\int_0^{+\infty}\frac{dy}{\left(\frac{1}{x}+y\right)^2} < +\infty\end{aligned}$$

which proves item 1 of Lemma 4.5.

2) Since $\widehat{\lambda}(y) \geq 0$, it is sufficient to check that $\lim_{n\to+\infty} \mathbb{E}\left[\widehat{\lambda}\left(\frac{1}{\frac{1}{x}+e_1+\cdots+e_n}\right)\right] = 0$.

As $0 < \frac{1}{\frac{1}{x}+e_1+\cdots+e_n} \leq x$ and $\lim_{n\to+\infty} \frac{1}{\frac{1}{x}+e_1+\cdots+e_n} = 0$ a.s. (by the Law of Large Numbers), the Lebesgue's dominated convergence theorem directly implies

$$\lim_{n\to+\infty} \mathbb{E}\left(\frac{1}{\frac{1}{x} + e_1 + \cdots + e_n}\right) = \lim_{n\to+\infty} \mathbb{E}\left(\frac{1}{\left(\frac{1}{x} + e_1 + \cdots + e_n\right)^2}\right) = 0.$$

It remains to use Lemma 4.4 to get

$$\mathbb{E}\left[\widehat{\lambda}\left(\frac{1}{\frac{1}{x}+e_1+\cdots+e_n}\right)\right]=\frac{2}{3}\left\{\mathbb{E}\left(\frac{1}{\frac{1}{x}+e_1+\cdots+e_n}\right)+\mathbb{E}\left(\frac{1}{\left(\frac{1}{x}+e_1+\cdots+e_n\right)^2}\right)\right\}.$$

and conclude the proof. ■

4.3 Proof of Theorem 2.4

We revisit the results of Sections 4.1 and 4.2, keeping the notation introduced there. Interpreting the Lebesgue integral as an expectation in Lemma 4.1 gives:

$$\mathbb{E}\Big[f(U^*(t),\theta^*(t))\Big]=\sqrt{\frac{\pi}{2}}\mathbb{E}\left[f\left(\sqrt{\frac{t}{\tau_1}}\alpha_1\overline{U}(\tau_1),\frac{t}{\tau_1}\alpha_1^2\theta^*(\tau_1)\right)\frac{1}{\sqrt{\tau_1}}\right].$$

By the same reasoning, Lemma 4.2 can be modified as:

$$\mathbb{E}\left[h\Big(\overline{U}(\tau_1),\theta^*(\tau_1),\tau_1\Big)\right]=\mathbb{E}\left[h\left(\frac{\overline{U}(\tau_1)}{\alpha_2},\frac{\overline{U}(\tau_1)^2}{\alpha_2^2}\xi,\frac{\overline{U}(\tau_1)^2}{\alpha_2^2}(\xi+\xi')+\tau_1\right)\frac{1}{\overline{U}(\tau_1)}\right].$$

Then the two previous equations and the following identity in law: $\overline{U}(\tau_1)\overset{(d)}{=}1/e_0'$ imply (2.19). ■

4.4 Proof of Theorem 2.5

For any $a>0$, b and $c\in\mathbb{R}$, we set

$$H(a,b):=\mathbb{E}\left[\frac{1}{\sqrt{b-a\xi}}1\!\!1_{\{b-a\xi>0\}}\right]$$
$$\widehat{H}(a,b,c):=\mathbb{E}\left[\frac{1}{(a\xi-b)^{3/2}}\exp\left(-\frac{c}{a\xi-b}\right)1\!\!1_{\{a\xi-b>0\}}\right].$$

The proof of Theorem 2.5 is based on the following Lemma.

Lemma 4.6 *We have*

$$H(a,b) = \frac{a}{b^{3/2}} \sum_{k\in\mathbb{Z}} |1+2k| \exp\left(-\frac{(1+2k)^2}{2}\frac{a}{b}\right) \tag{4.16}$$

$$\widehat{H}(a,b,c) = \frac{\pi^{5/2}}{2a\sqrt{c}} \sum_{k\in\mathbb{Z}} (-1)^{k+1} k^2 \exp\left(-\frac{k^2\pi^2 b}{2a} - |k|\pi\sqrt{\frac{2c}{a}}\right). \tag{4.17}$$

Proof 1) Since $H(a,b)=0$ for $b\le 0$, we assume from now on $b>0$. By (2.11)

$$H(a,b) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left(-H_{1,k} + (1+2k)^2 H_{2,k}\right)$$

with

$$H_{1,k} := \int_0^{b/a} \frac{1}{u^{3/2}} \frac{1}{\sqrt{b-au}} \exp\left(-\frac{(1+2k)^2}{2u}\right) du,$$

$$H_{2,k} := \int_0^{b/a} \frac{1}{u^{5/2}\sqrt{b-au}} \exp\left(-\frac{(1+2k)^2}{2u}\right) du.$$

The change of variable $z = 1/u - 1/a$ in the above integrals gives

$$H_{1,k} = \sqrt{\frac{2\pi}{b}} \frac{1}{|1+2k|} \exp\left(-\frac{(1+2k)^2}{2}\frac{a}{b}\right),$$

$$H_{2,k} = \left(\frac{a\sqrt{2\pi}}{b^{3/2}} \frac{1}{|1+2k|} + \frac{\sqrt{2\pi}}{\sqrt{b}} \frac{1}{|1+2k|^3}\right) \exp\left(-\frac{(1+2k)^2}{2}\frac{a}{b}\right).$$

From these relations we deduce the identity $-H_{1,k} + (1+2k)^2 H_{2,k} = \frac{a\sqrt{2\pi}}{b^{3/2}} |1+2k| \exp\left(-\frac{(1+2k)^2}{2}\frac{a}{b}\right)$ and finally (4.16).

2) Using (2.12) and an integration by parts lead to $\widehat{H}(a,b,c) = a\sum_{n\in\mathbb{Z}}(-1)^n \widehat{H}_n$ with

$$\widehat{H}_n = \int_{b/a}^{\infty} \left(\frac{3}{2(au-b)^{5/2}} - \frac{c}{(au-b)^{7/2}}\right) \exp\left(-\frac{c}{au-b} - \frac{n^2\pi^2 u}{2}\right) du.$$

With the change of variable $s = c/(au-b)$, we get

$$\widehat{H}_n = \frac{1}{a}\left(\widehat{H}_n^1 + \widehat{H}_n^2\right) \exp\left(-\frac{n^2\pi^2}{2}\frac{b}{a}\right)$$

where

$$\widehat{H}_n^1 = \frac{3}{2c^{3/2}} \int_0^\infty \sqrt{s} \exp\left(-s - \frac{n^2\pi^2 c}{2a}\frac{1}{s}\right) ds,$$
$$\widehat{H}_n^2 = -\frac{1}{c^{3/2}} \int_0^\infty s^{3/2} \exp\left(-s - \frac{n^2\pi^2 c}{2a}\frac{1}{s}\right) ds.$$

The Bessel functions K_ν admits the following integral representation (see formula (15) p 183 in [19]):

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{1}{s^{\nu+1}} \exp\left(-s - \frac{z^2}{4s}\right) ds.$$

Since $K_{-\nu}(z) = K_\nu(z)$ (see formula (8) p 79 in [19], we obtain

$$\widehat{H}_n^1 = \frac{3}{c^{3/2}}\left(\frac{z}{2}\right)^{3/2} K_{3/2}(z) \quad \text{and} \quad \widehat{H}_n^2 = -\frac{2}{c^{3/2}}\left(\frac{z}{2}\right)^{5/2} K_{5/2}(z)$$

where $z = |n|\pi\sqrt{\frac{2c}{a}}$. The functions $K_{3/2}$ and $K_{5/2}$ are explicit (see formula (12) p 80 in [19]):

$$K_{3/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}\left(1 + \frac{1}{z}\right) \quad \text{and} \quad K_{5/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}\left(1 + \frac{3}{z} + \frac{3}{z^2}.\right)$$

Then, we deduce $\widehat{H}_n^1 + \widehat{H}_n^2 = -\frac{\sqrt{\pi}}{4c^{3/2}} z^2 e^{-z}$ and (4.17). ■

Proof of Theorem 2.5 Using Proposition 2.1 and (4.16), for $0 < x < t$, we get

$$\begin{aligned} f_{\theta^*(t)}(x) &= \frac{1}{\sqrt{2\pi t x}} \int_{\mathbb{R}_+^3} \mathbb{E}\left[H\left(\frac{x}{ty}, 1 - \frac{x}{t} - \frac{u^2 x}{ty}\lambda\left(\frac{1}{v}\right)\right) \mathbb{1}_{\left\{1 - \frac{x}{t} - \frac{u^2 x}{ty}\lambda(1/v) > 0\right\}}\right] \\ &\qquad \frac{u}{\sqrt{y}} e^{-v} \mathbb{1}_{\{u<v\}} p_\xi(y)\, du\, dv\, dy \\ &= \frac{1}{\sqrt{2\pi t x}} \int_{\mathbb{R}_+^2} u e^{-v} \mathbb{1}_{\{0<u<v\}} f_1(u,v)\, du\, dv \end{aligned} \tag{4.18}$$

where

$$f_1(u,v) := \int_0^\infty \mathbb{E}\left[H\left(\frac{x}{ty}, 1 - \frac{x}{t} - \frac{u^2 x}{ty}\lambda\left(\frac{1}{v}\right)\right) 1_{\left\{1 - \frac{x}{t} - \frac{u^2 x}{ty}\lambda(\frac{1}{v}) > 0\right\}}\right] \frac{p_\xi(y)}{\sqrt{y}}\, dy.$$

Using (4.16) with $a = \frac{x}{ty}$ and $b = 1 - \frac{x}{t} - \frac{u^2 x}{ty}\lambda\left(\frac{1}{v}\right)$, we get

$$\frac{H()}{\sqrt{y}} = \frac{x\sqrt{t}}{[(t-x)y - u^2 x\lambda(1/v)]^{3/2}} \sum_{k\in\mathbb{Z}} |1+2k| \exp\left\{-\frac{(1+2k)^2}{2}\frac{x}{(t-x)y - u^2 x\lambda(1/v)}\right\}.$$

Using the definition of the function $\widehat{H}$, we have:

$$f_1(u,v) = x\sqrt{t}\sum_{k\in\mathbb{Z}} |1+2k|\mathbb{E}\left[\widehat{H}\left(t-x, u^2x\lambda(1/v), \frac{(1+2k)^2}{2}x\right)\right].$$

Thus the density function of $\theta^*(t)$ can be written as follows, for $0<x<t$,

$$\begin{aligned} f_{\theta^*(t)}(x) = & \frac{\sqrt{x}}{\sqrt{2\pi}}\int_{\mathbb{R}_+^2} ue^{-v}1\!\!1_{\{0<u<v\}} \\ & \left\{\sum_{k\in\mathbb{Z}} |1+2k|\mathbb{E}\left[\widehat{H}\left(t-x, u^2x\lambda(1/v), \frac{(1+2k)^2}{2}x\right)\right]\right\} du\, dv. \end{aligned} \tag{4.19}$$

Set $a=t-x$, $b=u^2x\lambda(1/v)$ and $c=\frac{(1+2k)^2}{2}x$. From (4.17), we get

$$\begin{aligned} & \mathbb{E}\left[\widehat{H}\left(t-x, u^2x\lambda(1/v), \frac{(1+2k)^2}{2}x\right)\right] = \frac{\pi^{5/2}}{\sqrt{2x}(t-x)|1+2k|} \\ & \sum_{n\in\mathbb{Z}}(-1)^{n+1}n^2\exp\left(-\pi|n||1+2k|\sqrt{\frac{x}{t-x}}\right)\mathbb{E}\left[\exp -\left(\frac{n^2\pi^2}{2(t-x)}u^2x\lambda(1/v)\right)\right]. \end{aligned}$$

Identity (4.12) and item 2 of Proposition 2.2 give

$$\int_0^r \mathbb{E}\left(e^{-\mu\lambda(x)}\right)e^{-1/x}\frac{dx}{x^2} = \exp\left\{-\sqrt{2\mu}\coth\left(r\sqrt{2\mu}\right)\right\}$$

and a derivation with respect to r leads to

$$\mathbb{E}[\exp\{-\mu\lambda(r)\}] = r^2e^{1/r}\left(\frac{2\mu}{\sinh^2(r\sqrt{2\mu})}\right)\exp\left\{-\sqrt{2\mu}\coth(r\sqrt{2\mu})\right\}. \tag{4.20}$$

Taking $\mu=\frac{n^2\pi^2}{2(t-x)}u^2x$ we get

$$\begin{aligned} & \mathbb{E}\left[\widehat{H}\left(t-x, u^2x\lambda(1/v), \frac{(1+2k)^2}{2}x\right)\right] = \frac{\pi^{9/2}u^2e^v\sqrt{x}}{\sqrt{2}|1+2k|(t-x)^2v^2} \\ & \sum_{n\in\mathbb{Z}^*}(-1)^{n+1}n^4\frac{\exp\left\{-\frac{\pi|n|\sqrt{x}}{\sqrt{t-x}}\left(|1+2k|+u\coth\left(\frac{\pi u|n|}{v}\sqrt{\frac{x}{t-x}}\right)\right)\right\}}{\sinh^2\left(\frac{\pi u|n|}{v}\sqrt{\frac{x}{t-x}}\right)}. \end{aligned} \tag{4.21}$$

Equations (4.19) and (4.21) imply that $f_{\theta^*(t)}(x)=\sum_{k,n\in\mathbb{Z}^*}f_{k,n}(x)$ where

$$f_{k,n}(x) = \sqrt{\frac{x}{2\pi}}|1+2k|\frac{\pi^{9/2}\sqrt{x}}{\sqrt{2}|1+2k|(t-x)^2}(-1)^{n+1}n^4$$
$$\int_{\mathbb{R}^2}\frac{u^3}{v^2}\frac{\exp\left\{-\frac{\pi|n|\sqrt{x}}{\sqrt{t-x}}\left(|1+2k|+u\coth\left(\frac{\pi u|n|}{v}\sqrt{\frac{x}{t-x}}\right)\right)\right\}}{\sinh^2\left(\frac{\pi u|n|}{v}\sqrt{\frac{x}{t-x}}\right)}1\!\!1_{\{u<v\}}\,du\,dv.$$

Letting $u = vs$ (v being fixed) and integrating with respect to dv, we obtain

$$\begin{aligned}
f_{k,n}(x) &= (-1)^{n+1}\pi|n|\sqrt{\frac{1}{x(t-x)}}\exp\left(-\pi|n||1+2k|\sqrt{\frac{x}{t-x}}\right) \\
&\quad\times\int_0^1\frac{\sinh\left(\pi s|n|\sqrt{\frac{x}{t-x}}\right)}{\cosh^2\left(\pi|n|\sqrt{\frac{x}{t-x}}\right)}\,ds \\
&= (-1)^{n+1}\frac{1}{2x}\exp\left(-\pi|n||1+2k|\sqrt{\frac{x}{t-x}}\right)\tanh^2\left(\pi|n|\sqrt{\frac{x}{t-x}}\right)
\end{aligned}$$

and straightforward computation leads to

$$\sum_{k\in\mathbb{Z}^*} f_{k,n}(x) = (-1)^{n+1}\frac{1}{2x}\tanh^2\left(\pi|n|\sqrt{\frac{x}{t-x}}\right)\left(\sinh\left(\pi|n|\sqrt{\frac{x}{t-x}}\right)\right)^{-1}$$

that finally conduces to (2.20). ■

4.5 Proof of (2.24) *in Theorem 2.6*

It is clear that (2.21)-(2.23) directly imply

$$\mathbb{P}(U^*(t) > x) = \mathbb{E}\left(2\sum_{k\geq 1}(-1)^{k-1}e^{-\frac{2k^2x^2}{tg(1)}}\right) = \frac{2}{\pi}\sum_{k\geq 1}(-1)^{k-1}\int_0^1 e^{-\frac{2k^2x^2}{ty}}\,\frac{dy}{\sqrt{y(1-y)}}.$$

We take the x-derivative and we set $u = 1/y - 1$. (2.24) follows easily. ■

4.6 Proof of Proposition 2.7

Let us introduce $d_a := \inf\{t \geq T_U(a),\ U(t) = 0\}$. It is clear that $\{U^*(t) > a\} = \{d_a < t\}$. The process $(U(s+T_U(a)) - a,\ 0 \leq s \leq d_a - T_U(a))$ is distributed as $(B(s),\ 0 \leq s \leq T_B(-a))$ and is independent of $(U(s),\ s \leq T_U(a))$, then $d_a \overset{(d)}{=} T_U(a) + T_{\widehat{B}}(a)$. This shows (2.25). ■

4.7 Proof of Theorem 3.3

4.7.1 Auxiliary results in the discrete setting

Let us go back to the random walk defined by (1.1) and introduce for any integer $n_1 > 0$,

$$S'_k := S_{n_1+k} - S_{n_1}, \quad k \geq 0.$$

Lemma 4.7 (Key Property) *(1)* $U_{n+1} = \max(U_n + \epsilon_{n+1}, 0)$.
(2) Let k *be an integer such as* $k > 0$. *Then*

$$U_{n_1+i} > 0 \ \ \forall i \in \{0, \cdots, k\} \iff U_{n_1} > 0 \text{ and } U_{n_1} + S'_i > 0 \ \ \forall i \in \{1, \cdots, k\}.$$

In such a case

$$U_{n_1+i} = U_{n_1} + S'_i \text{ for } 1 \leq i \leq k.$$

Now consider

$$N := \{g^*_n < n_1, \ n_2 < f^*_n < n_3, \ n_4 < d^*_n < n_5, \ U^*_n \geq b\} \tag{4.22}$$

where $0 < n_1 < \cdots < n_5 < n$ are integers and see (3.2) and (1.3) for the definition of the r.v.'s $g^*_n, f^*_n, d^*_n, U^*_n$ and (U_k).
Define

$$\overline{U}(m_1, m_2) := \max_{m_1 \leq i \leq m_2} U_i, \quad \underline{U}(m_1, m_2) := \min_{m_1 \leq i \leq m_2} U_i$$

and

$$n'_i := n_i - n_1, \quad 2 \leq i \leq 5, \quad n' := n - n_1.$$

The event D can be decomposed as

$$N = N^1 \cap N^2 \cap N^3 \cap N^4 \tag{4.23}$$

where

$$N^1 := \Big\{U_k > 0, \ n_1 \leq k \leq n_4\Big\} = \Big\{\underline{U}(n_1, n_4) > 0\Big\} \tag{4.24}$$

$$N^2 := \Big\{\overline{U}(n_2, n_3) \geq \overline{U}(0, n_2) \vee b\Big\} \tag{4.25}$$

$$N^3 := \Big\{\overline{U}(n_2, n_3) > \overline{U}(n_3, n)\Big\} \tag{4.26}$$

$$N^4 := \Big\{\exists \, k, \ U_k = 0, \ n_4 \leq k \leq n_5\Big\} = \Big\{\underline{U}(n_4, n_5) \leq 0\Big\} \tag{4.27}$$

Now note that

$$U_{n_1} + S'_i > 0 \iff S_{n_1+k} - \underline{S}(0, n_1) > 0. \tag{4.28}$$

Moreover by the definitions of the n_j's, one has $U_{n_j} > 0$ and $S_{i+n_j} - \underline{S}(0, n_j) > 0 \ \forall i = 1 \ldots n_5$. Consequently, we successively have

$$N^1 = \left\{ \underline{S'}(0, n'_4) > -U_{n_1} \right\} \tag{4.29}$$

$$N^2 = \left\{ \overline{S'}(n'_2, n'_3) \geq -U_{n_1} + \max \left[\overline{U}(0, n_1),\ b,\ \overline{S'}(0, n'_2) + U_{n_1} \right] \right\} \tag{4.30}$$

$$N^3 = \left\{ \overline{S'}(n'_2, n'_3) \geq -U_{n_1} + \overline{U}(n_3, n) \right\} \tag{4.31}$$

$$N^4 = \left\{ \underline{S'}(n'_4, n'_5) \leq -U_{n_1} \right\}. \tag{4.32}$$

The above equalities can be directly read on Figure 4.7.2 (a dash line representing a level that could not be crossed by the process).

We want to express N^3 in terms of $T'_k := S_{t_3+k} - S_{t_3}$. We have

$$\underline{S}(0, n_3 + k) = \min \left\{ \underline{S}(0, n_3),\ S_{n_3} + \underline{T'}(0, k) \right\}$$

and

$$U_{n_3+k} = S_{n_3+k} - \underline{S}(0, n_3 + k) = T'_k + \max \left\{ U_{n_3},\ -\underline{T'}(0, k) \right\}.$$

As a result

$$N^3 = \left\{ \overline{S'}(n'_2, n'_3) > -U_{n_1} + \max_{0 \leq k \leq n - n_3} \left[T'_k + \max \left\{ U_{n_3},\ -\underline{T'}(0, k) \right\} \right] \right\}.$$

4.7.2 Back to the continuous case

1) Let $t_1, t_2, \ldots, t_5$ be positive real numbers such that $0 < t_1 < \cdots < t_5$ and $b > 0$. Let us introduce

$$A^1_M = \{ g^{M,*}(t) < t_1,\ t_2 < f^{M,*}(t) < t_3,\ t_4 < d^{M,*}(t) < t_5,\ U^{M,*}(t) > b \} \tag{4.33}$$

where $g^{M,*}(t)$, $f^{M,*}(t)$, $d^{M,*}(t)$ and $U^{M,*}(t)$) have been defined by (3.5). The goal is to show

$$\lim_{M \to \infty} P(A^1_M) = P(A^1) \tag{4.34}$$

where

$$A^1 := \{ g^*(t) < t_1,\ t_2 < f^*(t) < t_3,\ t_4 < d^*(t) < t_5,\ U^*(t) > b \} \tag{4.35}$$

and the r.v.'s $U^*(t)$, $g^*(t)$, $f^*(t)$ and $d^*(t)$ have been defined by (2.4)-(2.7).

2) In view of the discrete case, let us consider the sets of dyadic points

$$D = \bigcup_{m \in \mathbb{N}} D_m \quad \text{where} \quad D_m = \left\{ \frac{k}{2^m}, \quad k \in \{0, 1, \ldots\} \right\}.$$

Since D is dense in $\mathbb{R}$ and $D_n \subset D_m$ as soon as $n \le m$, we can choose without loss of generality positive integers L_0, l and l_i for $i = 1 \dots 5$ such as

$$t_i = \frac{l_i}{2^{L_0}},\ 1 \le i \le 5, \quad t = \frac{l}{2^{L_0}}.$$

Recall that $\Big(U^M(t),\ t \ge 0\Big)$ is the continuous process defined by (3.3) and the linear interpolation of $\left(\frac{1}{\sqrt{M}} U_k,\ k \ge 0\right)$.

3) For any continuous function $\omega :\ [0, \infty[\to \mathbb{R}$, we denote

$$\overline{\omega}(u, v) := \max_{u \le r \le v} \omega(r), \quad \underline{\omega}(u, v) := \min_{u \le r \le v} \omega(r), \quad 0 \le u \le v. \tag{4.36}$$

Following the procedure presented in the discrete case, the event A^1_M can be decomposed as

$$A^1_M = A^{1,1}_M \cap A^{1,2}_M \cap A^{1,3}_M \cap A^{1,4}_M \tag{4.37}$$

where for $i = 1, \dots, 4$ $A^{1,i}_M$ is the analog of N_i obtained by replacing $\underline{U}$ (resp. $\overline{U}$, n_i, $i = 1, \dots, 5$) by $\underline{U}^M$ (resp. $\overline{U}^M$, t_i, $i = 1, \dots, 5$).

By Corollary 3.2, $\underline{U}^M(t_4, t_5)$ converges weakly to $\underline{U}(t_4, t_5)$, as $M \to \infty$. Thus we want to study the limit when M goes to infinity and apply the following lemma

Lemma 4.8 *Let (ξ^M) be a sequence of r.v.'s valued in $\mathbb{R}^d$ and converging weakly to ξ when $M \to \infty$. Then the Porte-Manteau's lemma (see e.g. [4]) asserts that for any Borel Λ in $\mathbb{R}^d$,*

$$\lim_{M \to \infty} \mathbb{P}(\xi^M \in \Lambda) = P(\xi \in \Lambda) \tag{4.38}$$

if $\mathbb{P}(\xi \in \partial\Lambda) = 0$.

Unfortunately, the distribution of $\underline{U}(t_4, t_5)$ (being bounded below by 0) has an atom at 0; therefore we cannot conclude directly that $\lim_{M \to \infty} \mathbb{P}\Big(\underline{U}^M(t_4, t_5) = 0\Big) = \mathbb{P}\Big(\underline{U}(t_4, t_5) = 0\Big)$. This is the reason why we will introduce the processes W and Z in the sequel.

4) We follow now the procedure developed in section 4.7.1. It is worth introducing $t'_i := t_i - t_1$, $i \in \{2,3,4,5\}$, $t' := t - t_1$ and W^M the process

$$W^M(s) := B^M(t_1 + s) - B^M(t_1), \quad s \geq 0.$$

Note that the process $\Big(W^M(s), s \geq 0\Big)$ is the linear interpolation of $\Big(\frac{1}{\sqrt{M}}(S_{k+n_1} - S_{n_1}, k \in \mathbb{N}\Big)$. We deduce from the previous step that $A^1_M = A^2_M$ where

$$A^2_M := A^{2,1}_M \cap A^{2,2}_M \cap A^{2,3}_M \cap A^{2,4}_M$$

and

$$\begin{aligned}
A^{2,1}_M &:= \Big\{\underline{W}^M(0, t'_4) > -U^M(t_1)\Big\} \\
A^{2,2}_M &:= \Big\{\overline{W}^M(t'_2, t'_3) \geq -U^M(t_1) + \max\Big[\overline{U}^M(0, t_1),\ b,\ \overline{W}^M(0, t'_2) + U^M(t_1)\Big]\ \Big\} \\
A^{2,3}_M &:= \Big\{\overline{W}^M(t'_2, t'_3) > -U^M(t_1) + \max_{0 \leq u \leq t - t_3}\Big[Z^M(u) + \max\Big\{U^M(t_3),\ -\underline{Z}^M(0, u)\Big\}\Big]\Big\} \\
A^{2,4}_M &:= \Big\{\underline{W}^M(t'_4, t'_5) \leq -U^M(t_1)\Big\}.
\end{aligned}$$

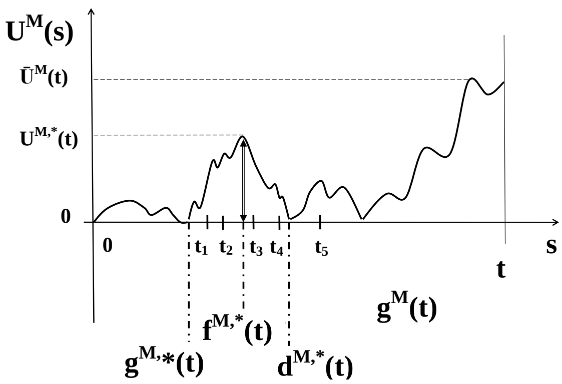

Figure 3. Sequence of t_i

5) To conclude the proof by taking the limit in M, it remains to express the limit subsets in the same way. In that view, let us introduce

$$W(s) := B(t_1 + s) - B(t_1), \quad Z(s) := B(t_3 + s) - B(t_3), \quad s \geq 0$$

and

$$A^2 := A^{2,1} \cap A^{2,2} \cap A^{2,3} \cap A^{2,4}$$

with

$$
\begin{aligned}
A^{2,1} &:= \Big\{ \underline{W}(0, t'_4) > -U(t_1) \Big\} \\
A^{2,2} &:= \Big\{ \overline{W}(t'_2, t'_4) \geq -U(t_1) + \max \Big[\overline{U}(0, t_1),\ b,\ \overline{W}(0, t'_2) + U(t_1) \Big] \ \Big\} \\
A^{2,3} &:= \Big\{ \overline{W}(t'_2, t'_3) > -U(t_1) + \max_{0 \leq u \leq t - t_3} \Big[Z(u) + \max \Big\{ U(t_3),\ -\underline{Z}(0, u) \Big\} \Big] \Big\} \\
A^{2,4} &:= \Big\{ \underline{W}(t'_4, t'_5) \leq -U(t_1) \Big\}.
\end{aligned}
$$

Recall that for any $u > 0$, the r.v.s $\max\limits_{0 \leq r \leq u} B(r)$ and $\min\limits_{0 \leq r \leq u} B(r)$ have a density function. Therefore we can apply (4.38) to get

$$
\lim_{M \to \infty} \mathbb{P}(A^{2,M}) = \mathbb{P}(A^2).
$$

As done in the discrete setting, we deduce that $A^2 = A^1$ where A^1 has been defined by (4.35). It is now clear that (4.34) follows.

■

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