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Moisés Bonilla, Guy Lebret, Jean-Jacques Loiseau, Michel Malabre

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M. Bonilla[a,b], G. Lebret[c,d], J.J. Loiseau[c,f], M. Malabre[c,e]

[a] *CINVESTAV-IPN. Control Automático. AP 14-740 México 07000, MEXICO.*
[b] *UMI 3175, CINVESTAV–CNRS*
[c] *LUNAM Université, IRCCyN, UMR 6597, 1 rue de la Noë, F 44321 Nantes, FRANCE.*
[d] *Ecole Centrale de Nantes,* Guy.Lebret@irccyn.ec-nantes.fr
[e] *CNRS,* Michel.Malabre@irccyn.ec-nantes.fr
[f] *CNRS,* Jean-Jacques.Loiseau@irccyn.ec-nantes.fr

Abstract

In this paper, we give an explicit solution to the behavioral reachability problem for linear time invariant systems, which amounts to finding an explicit control law that reaches a given final input-state pair (u_1, x_1) in a given finite time t_1. We first tackle the case of state space realizations, and we then extend the obtained results to the case of implicit realizations. For this, we use the geometric approach and some results of the viability theory. Some complements are given about the existing relationships between reachability and pole placement, as well as some notions of unicity and existence of solution.

Keywords:
Linear systems, implicit systems, reachability, geometric approach.

Notation. Script capitals $\mathscr{V}$, $\mathscr{W}$, …, denote finite dimensional linear spaces with elements v, w, …; the dimension of a space $\mathscr{V}$ is denoted $\dim(\mathscr{V})$; $\mathscr{V} \approx \mathscr{W}$ stands for $\dim(\mathscr{V}) = \dim(\mathscr{W})$; when $\mathscr{V} \subset \mathscr{W}$, $\frac{\mathscr{W}}{\mathscr{V}}$ or $\mathscr{W}/\mathscr{V}$ stand for the quotient space $\mathscr{W}$ modulo $\mathscr{V}$; the direct sum of independent spaces is written as $\oplus$. $X^{-1}\mathscr{V}$, stands for the inverse image of the subspace $\mathscr{V}$ by the linear transformation X. Given a linear transformation $X : \mathscr{V} \to \mathscr{W}$, $\operatorname{Im} X = X\mathscr{V}$ denotes its image, and $\operatorname{Ker} X$ denotes its kernel; when $\mathscr{V} \approx \mathscr{W}$, we write $X : \mathscr{V} \leftrightarrow \mathscr{W}$; when $\mathscr{U} \subset \mathscr{V}$, $X|_{\mathscr{U}}$ denotes the restriction of X to $\mathscr{U}$. Given the space $\mathscr{X} = \mathscr{S} \oplus \mathscr{T}$, the natural projection, $P : \mathscr{X} \to \mathscr{S}$, on $\mathscr{S}$ along $\mathscr{T}$, is also written as $P : \mathscr{X} \to \mathscr{S}//\mathscr{T}$. The special subspaces $\operatorname{Im} B$, $\operatorname{Ker} E$ and $\operatorname{Ker} C$, are denoted by $\mathscr{B}$, $\mathscr{K}_E$ and $\mathscr{K}_C$, respectively. The zero dimension subspace is denoted $\{0\}$, and the identity operator is denoted I, namely $I\,x = x$. Given the linear transformations $X : \mathscr{V} \to \mathscr{V}$ and $Y : \mathscr{W} \to \mathscr{V}$, $\langle X \,|\operatorname{Im} Y\rangle$ stands for the subspace of $\mathscr{V}$: $\operatorname{Im} Y + X\operatorname{Im} Y + \cdots + X^{\dim(\mathscr{V})-1}\operatorname{Im} Y$. The notations A_{F_p} and E_{F_d} stand for $(A + BF_p)$ and $(E - BF_d)$, respectively.

$\mathrm{BDM}\{X_1, ..., X_k\}$ denotes a block diagonal matrix whose diagonal blocks are the matrices $X_1, \ldots, X_k$, and $\mathrm{DM}\{x_1, ..., x_k\}$ denotes a diagonal matrix whose diagonal elements are $x_1, \ldots, x_k$. The notation $\mathbb{R}^k$ stands for the Euclidean space of dimension k. $\underline{e}_k^i \in \mathbb{R}^k$ stands for the vector whose i-th entry is equal to 1 and the other ones are equal to 0. $\mathrm{T}_u\left\{v^T\right\}$ stands for the upper triangular Toeplitz matrix, whose first row is v^T. $*$ stands for some matrix which exact value has no importance.

$\mathbb{R}^+$, $\mathbb{R}^{+*}$ and $\mathbb{Z}^+$, stand for the sets of non negative real numbers, positive real numbers and non negative integers, respectively. $\mathcal{C}^\infty(\mathbb{R}^+, \mathscr{V})$ and $\mathcal{L}^\infty(\mathbb{R}^+, \mathscr{V})$ are the space of infinitely differentiable functions and the space of bounded functions, $v : \mathbb{R}^+ \to \mathscr{V}$, respectively. $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}^+, \mathscr{V})$ stands for the locally integrable functions.

Geometric Algorithms. Given the linear transformations $X : \mathscr{V} \to \mathscr{W}$, $Y : \mathscr{T} \to \mathscr{W}$, and $Z : \mathscr{V} \to \mathscr{W}$, and the subspace $\mathscr{K} \subset \mathscr{V}$, we have the two following popular geometric algorithms (see mainly Verghese, 1981, Özçaldiran, 1986, Malabre, 1987, 1989, Lewis, 1992):

$$\mathscr{V}^{0}_{[\mathscr{K}:X,Z,Y]} = \mathscr{V}\ , \quad \mathscr{V}^{\mu+1}_{[\mathscr{K}:X,Z,Y]} = \mathscr{K} \cap X^{-1}\left(Z\mathscr{V}^{\mu}_{[\mathscr{K}:X,Z,Y]} + \operatorname{Im} Y \right) \qquad \text{(ALG–V)}$$

$$\mathscr{S}^{0}_{[Z,X,Y]} = \{0\}, \quad \mathscr{S}^{\mu+1}_{[Z,X,Y]} = Z^{-1}\left(X\mathscr{S}^{\mu}_{[Z,X,Y]} + \operatorname{Im} Y \right) \qquad \text{(ALG–S)}$$

where $\mu \in \mathbb{Z}^{+}$. The limit of (ALG–V) is the *supremal* (X, Z, Y) *invariant subspace* contained in $\mathscr{K}$, $\mathscr{V}^{*}_{[\mathscr{K}:X,Z,Y]} := \sup\{\mathscr{S} \subset \mathscr{K} \mid X\mathscr{S} \subset Z\mathscr{S} + \operatorname{Im} Y\}$, and the limit of (ALG–S) is the *infimal* (Z, X, Y) *invariant subspace related to* $\operatorname{Im} Y$, $\mathscr{S}^{*}_{[Z,X,Y]} := \inf\{\mathscr{S} \subset \mathscr{V} \mid \mathscr{S} = Z^{-1}(X\mathscr{S} + \operatorname{Im} Y)\}$.

We distinguish two cases.

- For the square brackets $[\mathscr{V} : X, Z, 0]$ and $[Z, X, 0]$, we write: $\mathscr{V}^{*}_{[X,Z]}$, $\mathscr{V}^{\mu}_{[X,Z]}$, $\mathscr{S}^{*}_{[Z,X]}$ and $\mathscr{S}^{\mu}_{[Z,X]}$, instead of: $\mathscr{V}^{*}_{[\mathscr{V}:X,Z,0]}$, $\mathscr{V}^{\mu}_{[\mathscr{V}:X,Z,0]}$, $\mathscr{S}^{*}_{[Z,X,0]}$ and $\mathscr{S}^{\mu}_{[Z,X,0]}$, respectively, where $\mu \in \mathbb{Z}^{+}$.

- For the square bracket $[\mathscr{K}_{\overline{C}} : \overline{A}, I, Y]$, we write: $\overline{\mathscr{V}}^{*}_{Y}$ and $\overline{\mathscr{V}}^{\mu}_{Y}$, instead of: $\mathscr{V}^{*}_{[\mathscr{K}_{\overline{C}}:\overline{A},I,Y]}$ and $\mathscr{V}^{\mu}_{[\mathscr{K}_{\overline{C}}:\overline{A},I,Y]}$, where $\mu \in \mathbb{Z}^{+}$.

Subspaces. Note that in the particular case $X = A : \mathscr{X} \to \mathscr{X}$, $Y = B : \mathscr{U} \to \mathscr{X}$, and $Z = I$, the equalities $\mathscr{V}^{*}_{[\mathscr{X}:A,I,B]} = \mathscr{X}$ and $\mathscr{S}^{*}_{[I,A,B]} = \langle A \mid \mathscr{B} \rangle$ hold true.

Given the linear transformations $X = A : \mathscr{X}_d \to \mathscr{X}_{eq}$, $Y = B : \mathscr{U} \to \mathscr{X}_{eq}$, and $Z = E : \mathscr{X}_d \to \mathscr{X}_{eq}$, it is observed that

- *the supremal* (A, E, B) *invariant subspace* contained in $\mathscr{X}_d$ and *the infimal* (E, A, B) *invariant subspace related to* $\mathscr{B}$, $\mathscr{V}^{*}_{[\mathscr{X}_d:A,E,B]}$ and $\mathscr{S}^{*}_{[E,A,B]}$, are identified by $\mathscr{V}^*_{\mathscr{X}_d}$ and $\mathscr{S}^*_{\mathscr{X}_d}$, respectively, and the respective subspaces of their algorithms (ALG–V) and (ALG–S) are identified by $\mathscr{V}^{\mu}_{\mathscr{X}_d}$ and $\mathscr{S}^{\mu}_{\mathscr{X}_d}$ $(\mu \in \mathbb{Z}^+)$, respectively;

- *the supremal* (A, E, B) *invariant subspace* contained in $\mathscr{K}_C$, $\mathscr{V}^*_{[\mathscr{K}_C:\, A,E,B]}$, is identified by $\mathscr{V}^*$, and the respective subspaces of its algorithm (ALG–V) are identified by $\mathscr{V}^{\mu}$ $(\mu \in \mathbb{Z}^+)$;

- *the unobservable space* $\mathscr{V}^*_{[\mathscr{K}_C:\, A,E,0]}$ is identified by $\mathscr{N}$; and *the closed loop unobservable space* $\mathscr{V}^*_{[\mathscr{K}_C:\, A_{F_p}, E_{F_d}, 0]}$ is identified by $\mathscr{N}_{(F_p,F_d)}$.

Let us note that:

(i) $\mathscr{V}^*_{[\mathscr{K}:\, A,E,B]} = \mathscr{V}^*_{[\mathscr{K}:\, A_{F_p}, E_{F_d}, B]}$,

(ii) $\mathscr{S}^*_{[\mathscr{K}:\, E,A,B]} = \mathscr{S}^*_{[\mathscr{K}:\, E_{F_d}, A_{F_p}, B]}$, and

(iii) for any F_d, there exists F_p such that: $A_{F_p} \mathscr{V}^*_{[\mathscr{K}:\, A_{F_p}, E_{F_d}, B]} \subset E_{F_d}\, \mathscr{V}^*_{[\mathscr{K}:\, A_{F_p}, E_{F_d}, B]}$. The set of such pairs (F_p, F_d) is identified by $\mathbf{F}(\mathscr{V}^*_{[\mathscr{K}:\, A,E,B]})$.

1. INTRODUCTION

One of the most studied concepts in System Theory is the one of *reachability*. This concept is normally associated with the set of vectors that can be reached from the origin in a finite time, following trajectories solutions of the system, generated by the input system. Here, the term *input system* refers to an exogenous signal which is available for controlling the output system.

1.1. State Space Representations

For the case of state space representations $\mathfrak{R}^{ss}(A,\, B)$,

$$\mathrm{d}x/\mathrm{d}t = Ax + Bu, \tag{1.1}$$

where $u \in \mathscr{U} \approx \mathbb{R}^m$ is the input variable, $x \in \mathscr{X} \approx \mathbb{R}^n$ is the state variable, and with the usual assumption $\operatorname{Ker} B = \{0\}$, Kalman (1960, 1963) introduced his famous reachability matrix: $\mathcal{R}_{[A,\,B]} = \left[\begin{array}{cccc} B & AB & \cdots & A^{n-1}B \end{array}\right]$. He showed that *given any $x_0, x_1 \in \mathscr{X}$, there exists a control law*[1] $u(\cdot) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathscr{U})$*, generating a trajectory $x(\cdot) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathscr{X})$ solution of (1.1), starting from the given initial condition $x(0) = x_0 \in \mathscr{X}$, and reaching the desired final state $x(t_1) = x_1 \in \mathscr{X}$, in a finite time $t_1 \in \mathbb{R}^+$, iff,* $\operatorname{rank}\left(\mathcal{R}_{[A,\,B]}\right) = n$; in this case, the representation (1.1) is called *reachable*. This concept is known as *state reachability*[2], and when the pair $(A,\, B)$ satisfies such a rank condition, we identify it as a *state reachable pair.*

A. State reachability. Brunovsky (1970) showed that for a given *reachable* state space representation (1.1), there exist a linear map $F_{\mathcal{B}} : \mathbb{R}^n \to \mathbb{R}^m$ and isomorphisms $T_{\mathcal{B}} : \mathbb{R}^n \to \mathbb{R}^n$ and $G_{\mathcal{B}} : \mathbb{R}^m \to \mathbb{R}^m$, such that the pair $(A_{\mathcal{B}},\, B_{\mathcal{B}})$, where $A_{\mathcal{B}} = T_{\mathcal{B}}^{-1}(A + BF_{\mathcal{B}})T_{\mathcal{B}}$ and $B_{\mathcal{B}} = T_{\mathcal{B}}^{-1}BG_{\mathcal{B}}$, is expressed in the Brunovsky

[1] In this paper, we restrain our discussion to infinitely differentiable functions. This is not restrictive since $\mathcal{C}^\infty(\mathbb{R}^+, \mathscr{V})$ is dense in $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}^+, \mathscr{V})$ (see Polderman & Willems, 1998, Corollary 2.4.12).

[2] In many text books, this property is called *state controllability*, or simply *controllability*. Let us note that *controllability* only characterizes the system's property of reaching the origin $x_1 = 0$, from any state $x_0 \neq 0$, in a finite time t_1. Since in the continuous time-invariant linear systems case both properties, *reachability* and *controllability*, are mutually implied, they are often treated indistinguishably, but in the general case of the implicit representations, this is no longer the case; for example the implicit representation, $\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] \mathrm{d}x/\mathrm{d}t = \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right] x + \left[\begin{array}{c} 1 \\ 0 \end{array}\right] u$, is trivially controllable but not reachable.

canonical form, namely:

$$\begin{gathered} A_{\mathcal{B}} = \mathrm{BDM}\left\{A_{\mathcal{B}_1}, \ldots, A_{\mathcal{B}_m}\right\},\ B_{\mathcal{B}} = \mathrm{BDM}\left\{b_{\mathcal{B}_1}, \ldots, b_{\mathcal{B}_m}\right\}, \\ [A_{\mathcal{B}_i} \,|\, b_{\mathcal{B}_i}] = \left[\mathrm{T}_u\left\{(\underline{e}^2_{\kappa_i})^T\right\} \,\middle|\, \underline{e}^{\kappa_i}_{\kappa_i}\right],\ i \in \{1, \ldots, m\}, \end{gathered} \tag{1.2}$$

where the set

$$\mathrm{S}_\kappa = \left\{\{\kappa_1, \kappa_2, \ldots, \kappa_m\} \subset \mathbb{Z} \,\middle|\, \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_m \geq 1 \ \&\ \textstyle\sum_{i=1}^m \kappa_i = n\right\} \tag{1.3}$$

is known as the set of *reachability indices*. They are also geometrically characterized as follows

$$\mathrm{card}\,\{\kappa_i \geq 1\} = \dim(\mathscr{B}) \quad \text{and} \quad \mathrm{card}\,\{\kappa_i \geq \mu\} = \dim\left(\frac{\sum_{i=0}^{\mu-1} A^i\mathscr{B}}{\sum_{i=0}^{\mu-2} A^i\mathscr{B}}\right), \quad \forall\, \mu \geq 2\,.$$

Another important success was the introduction of the reachable space $\langle A \mid \mathscr{B}\rangle$. In (Wonham, 1985) is showed that a pair (A, B) is reachable iff:

$$\langle A \mid \mathscr{B}\rangle = \mathscr{X}\ . \tag{1.4}$$

Note that $\langle A \mid \mathscr{B}\rangle = \mathscr{X}$ iff $\mathrm{rank}\left(\mathcal{R}_{[A,\,B]}\right) = n$. Wonham (1985) showed that the reachability Gramian $W_{t_1} = \int_0^{t_1} \exp(\tau A)\, BB^T \exp\left(\tau A^T\right) \mathrm{d}\tau$, with $t \in [0, t_1]$, is non-singular iff (1.4) is satisfied. Thus, with the control law

$$u(t) \quad = B^T \exp\left((t_1 - t)A^T\right) W^{-1}_{t_1}(x_1 - \exp(t_1 A)\, x_0)\ , \tag{1.5}$$

we get a trajectory $x(\cdot) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathscr{X})$ solution of (1.1), such that $x(0) = x_0 \in \mathscr{X}$ and $x(t_1) = x_1 \in \mathscr{X}$.

Another well known result concerning state reachability is the one related with pole assignment. Indeed, *the pair $(A,\, B)$ is reachable iff for every symmetric (with respect to the real line) set of complex numbers Λ, of cardinality n, there exists a proportional state feedback $u = Fx$ such that the spectrum of $(\lambda I - A_F)$ is Λ* (see for example Theorems 2.1 and 9.3.1 of Wonham (1985) and Polderman & Willems (1998), respectively).

B. Behavioral reachability. Willems (1983, 1991) defined an *input/state system* as the triple $\Sigma_{i/s} = \left(\mathbb{R}^+, \mathscr{U} \times \mathscr{X},\, \mathfrak{B}_{[A,B]}\right)$, with behavior[3]

$$\mathfrak{B}_{[A,B]} = \left\{ (u,x) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathscr{U} \times \mathscr{X}) \middle| \begin{bmatrix} (I\frac{\mathrm{d}}{\mathrm{d}t} - A) & -B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0 \right\}. \tag{1.6}$$

In the behavioral framework of Willems (1983, 1991), the system $\Sigma_{i/s} = \left(\mathbb{R}^+, \mathscr{U} \times \mathscr{X},\, \mathfrak{B}_{[A,B]}\right)$ is called[4] *reachable* if *for any given* (u_0, x_0), $(u_1, x_1) \in \mathscr{U} \times \mathscr{X}$ *and* $t_1 > 0$, *it is possible to find a trajectory* $(u, x) \in \mathfrak{B}_{[A,B]}$, *such that* $(u(0), x(0)) = (u_0, x_0)$ *and* $(u(t_1), x(t_1)) = (u_1, x_1)$ (c.f. Polderman & Willems, 1998, Definition 5.2.2). In the following, this *reachability* concept is called *behavioral reachability*.

In (Polderman & Willems, 1998, Theorem 5.2.27) is proved that for the case of state space representations $\mathfrak{R}^{ss}(A, B)$, *state rechability* is equivalent to *behavioral reachability*. Although the *behavioral reachability* is well characterized, it could be interesting to find an explicit control law $u(\cdot) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathscr{U})$, similar to (1.5), which ensures $x(t_1) = x_1$ and $u(t_1) = u_1$. This will be done in Section 2.

1.2. Implicit Representations

As a generalization of proper linear systems, Rosenbrock (1970) introduced the *implicit representations* $\mathfrak{R}^{imp}(E, A, B)$, which are a set of differential

[3] The original definition given by Willems (1983, 1991) is $\mathfrak{B}_{[A,B]} = \Big\{(u,x) \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}^+, \mathscr{U} \times \mathscr{X}) \mid \exists\, x_0 \in \mathscr{X} \text{ s.t. } x(t) = \exp(At)\, x_0 + \int_0^t \exp(A(t-\tau))\, Bu(\tau)\mathrm{d}\tau\Big\}$. But since we restrict our attention to infinitely differentiable functions (see footnote [1]), weak and strong solutions coincide (see Polderman & Willems, 1998, Theorem 2.3.11).

[4] For consistency of this paper, we say *reachable* instead of *controllable*, as is stated in (Polderman & Willems, 1998) (see also footnote [2]) .

and algebraic equations (Brenan *et al*, 1996) of the following form (see also Lewis, 1992)

$$E\mathrm{d}x/\mathrm{d}t = Ax + Bu, \tag{1.7}$$

where $E : \mathscr{X}_d \to \mathscr{X}_{eq}$, $A : \mathscr{X}_d \to \mathscr{X}_{eq}$ and $B : \mathscr{U} \to \mathscr{X}_{eq}$ are linear maps. The linear spaces $\mathscr{X}_d \approx \mathbb{R}^{n_d}$, $\mathscr{X}_{eq} \approx \mathbb{R}^{\underline{n}_{eq}}$, and $\mathscr{U} \approx \mathbb{R}^m$ are called the descriptor, the equation, and the input spaces, respectively. In order to avoid redundant components in the input variable u, and linear dependence on the descriptor equations (1.7), as usually, we assume troughout the paper that the following hypotheses are verified:

[H1] $\mathrm{Ker}\, B = 0$, and

[H2] $\mathrm{Im}\, E + \mathrm{Im}\, A + \mathscr{B} = \mathscr{X}_{eq}$.

For the case of regular implicit representations, *i.e.* representations where the linear transformations E and A are square and the pencil $[\lambda E - A]$ is invertible (Gantmacher, 1977), the *reachability* was studied by Verghese, Lévy and Kailath (1981) from a transfer function point of view, Yip and Sincovec (1981) in the time domain, Cobb (1984) from a distributional point of view, and by Özçaldiran (1985) from a geometric point of view.

In the case of implicit representations, where the linear transformations E and A are square and the pencil $[\lambda E - A]$ is not necessarily invertible, Özçaldiran (1986) extended his *reachability* geometric characterization for the case of regular implicit representations (Özçaldiran, 1985), by means of the supremal (A, E, B) reachability subspace contained in $\mathscr{X}_d$, defined as

$$\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{V}^*_{\mathscr{X}_d} \cap \mathscr{S}^*_{\mathscr{X}_d} \,. \tag{1.8}$$

This is a nice generalization of the classical case, $\mathfrak{R}^{ss}(A,\, B) = \mathfrak{R}^{imp}(I,\, A,\, B)$, where the reachable space $\mathscr{R}^*_{\mathscr{X}_d}$ is equal to $\langle A \mid \mathscr{B} \rangle$, namely equal to $\mathscr{V}^*_{[\mathscr{X} :\, A, I, B]} \cap$

$\mathscr{S}^*_{[I,A,B]}$. Thus, for representations $\mathfrak{R}^{imp}(E, A, B)$, with E and A not necessarily square, it was natural to associate its reachability with $\mathscr{R}^*_{\mathscr{X}_d}$.

Frankowska (1990) firmly established the pertinence of this reachability concept, using differential inclusions to relate it with behavioral properties.

One major difficulty when studying reachability for implicit systems (1.7) is that their solution set does not only depend on the initial conditions $x(0)$ and on the external control input u, but also depends on a possible internal free variable (degree of freedom), which is completely unknown.

1.3. Outline

In this paper, we study the reachability notion in the sense of Frankowska (1990), showing some connections with the important works of Willems (1991) and Geerts (1993), and we consider the relationships between the reachability property and the complete pole assignment ability.

The paper is organized as follows: In Section 2, we consider the behavioral reachability problem for state space representations, namely the ability of reaching the input-state pair $(u(\cdot), x(\cdot))$. In Section 3, we formalize the notion of *implicit systems*, following the behavioral point of view, and we also study the equivalences between the notions of existence of solution and impulse controllability. In Section 4, we study the reachability notion of Frankowska (1990) for implicit systems. In Section 5, we consider the existent relationships between the reachability property and the complete pole assignment ability, and in Section 6, we conclude the paper.

2. BEHAVIORAL REACHABILITY PROBLEM

We consider the following problem.

Problem 1. Let us consider an *input/state system* $\Sigma_{i/s} = (\mathbb{R}^+, \mathscr{U} \times \mathscr{X}, \mathfrak{B}_{[A,B]})$ represented by (1.1), and with the behavior (1.6). Given (u_0, x_0), $(u_1, x_1) \in B^{-1}\langle A \mid \mathscr{B}\rangle \times \langle A \mid \mathscr{B}\rangle$ and $t_1 > 0$, find a trajectory $(u, x) \in \mathfrak{B}_{[A,B]}$, such that $(u(0), x(0)) = (u_0, x_0)$ and $(u(t_1), x(t_1)) = (u_1, x_1)$.

This is the *behavioral reachability* problem, and in (Polderman & Willems, 1998, Theorem 5.2.27) is proved that for the case of *state space representations*, *state reachability* is equivalent to *behavioral reachability*. So, condition (1.4) guarantees the existence of a solution for Problem 1.

One could think that the control law (1.5), proposed by Wonham (1985), solves Problem 1, but this proposition only guarantees the *reachability* of the state variable, $x(0) = x_0$ and $x(t_1) = x_1$, and nothing about the input variable u, which is let completely free at the end points $u(0)$ and $u(t_1)$. An intermediary step towards the solution of Problem 1 is given by the next result proved in Appendix A.

Lemma 1. *Let the state space representation (1.1) be* reachable*, with the* reachability indices *set (1.3). Let the linear map $F_{\mathcal{B}} : \mathbb{R}^n \to \mathbb{R}^m$ and the isomorphisms $T_{\mathcal{B}} : \mathbb{R}^n \to \mathbb{R}^n$ and $G_{\mathcal{B}} : \mathbb{R}^m \to \mathbb{R}^m$ be such that the pair $(A_{\mathcal{B}},\ B_{\mathcal{B}})$, where $A_{\mathcal{B}} = T_{\mathcal{B}}^{-1}(A + BF_{\mathcal{B}})T_{\mathcal{B}}$ and $B_{\mathcal{B}} = T_{\mathcal{B}}^{-1}BG_{\mathcal{B}}$, is expressed in the Brunovsky canonical form (1.2). Let the reachability matrices, $\mathcal{R}_{[A_{\mathcal{B}},\, B_{\mathcal{B}}]}$ and $\mathcal{R}_{[A_{\mathcal{B}_i},\, b_{\mathcal{B}_i}]}$, $i \in \{1, \dots, m\}$, of the pair $(A_{\mathcal{B}}, B_{\mathcal{B}})$ and the pairs $(A_{\mathcal{B}_i}, b_{\mathcal{B}_i})$, $i \in \{1, \dots, m\}$, respectively, be defined as follows:*

$$\begin{aligned} \mathcal{R}_{[A_{\mathcal{B}},\, B_{\mathcal{B}}]} &= \mathrm{BDM}\left\{\mathcal{R}_{[A_{\mathcal{B}_1},\, b_{\mathcal{B}_1}]}, \dots, \mathcal{R}_{[A_{\mathcal{B}_m},\, b_{\mathcal{B}_m}]}\right\}, \\ \mathcal{R}_{[A_{\mathcal{B}_i},\, b_{\mathcal{B}_i}]} &= \begin{bmatrix} b_{\mathcal{B}_i} & A_{\mathcal{B}_i} b_{\mathcal{B}_i} & \cdots & A_{\mathcal{B}_i}^{\kappa_i - 1} b_{\mathcal{B}_i} \end{bmatrix}. \end{aligned} \tag{2.1}$$

Let us assume that we have found trajectories $f_i \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^1)$, satisfying:

(i) for $j = 0, 1$

$$\mathfrak{D}(\tfrac{\mathrm{d}}{\mathrm{d}t}) f(t_j) = G_{\mathcal{B}}^{-1}(u_j - F_{\mathcal{B}} x_j), \tag{2.2}$$

where $\mathfrak{D}(\mathrm{d}/\mathrm{d}t) = \mathrm{DM}\left\{\mathrm{d}^{\kappa_1}/\mathrm{d}t^{\kappa_1},\ \ldots,\ \mathrm{d}^{\kappa_m}/\mathrm{d}t^{\kappa_m}\right\}$, $f(t) = \begin{bmatrix} f_1(t) & \cdots & f_m(t) \end{bmatrix}^T$ *and* $t_0 = 0$ *.*

(ii) If $\overline{w}_i(t) = \begin{bmatrix} \frac{\mathrm{d}^{\kappa_i - 1} f_i(t)}{\mathrm{d}t^{\kappa_i - 1}} & \cdots & \frac{\mathrm{d} f_i(t)}{\mathrm{d}t} & f_i(t) \end{bmatrix}^T$, $1 \le i \le m$, *and* $\overline{w}(t) = \begin{bmatrix} \overline{w}_1^T(t) & \cdots & \overline{w}_m^T(t) \end{bmatrix}^T$ *then, for* $j = 0, 1$,

$$\overline{w}(t_j) = \mathcal{R}_{[A_{\mathcal{B}},\, B_{\mathcal{B}}]}^{-1} T_{\mathcal{B}}^{-1} x_j \ . \tag{2.3}$$

Then, applying the control law,

$$u(t) = F_{\mathcal{B}} x(t) + G_{\mathcal{B}} \mathfrak{D}(\mathrm{d}/\mathrm{d}t) f(t), \tag{2.4}$$

to the system represented by (1.1), we get:

$$x(t) = T_{\mathcal{B}} \mathcal{R}_{[A_{\mathcal{B}},\, B_{\mathcal{B}}]} \overline{w}(t) \ , \tag{2.5}$$

with

$$(u(0), x(0)) = (u_0, x_0) \quad \textit{and} \quad (u(t_1), x(t_1)) = (u_1, x_1) \ . \tag{2.6}$$

Let us now propose the trajectories:[5]

$$\begin{gathered} f_i(t) = \begin{bmatrix} t^{2\kappa_i+1} & \cdots & t^{\kappa_i+1} \end{bmatrix} \mathfrak{a}_{i,1} + \begin{bmatrix} t^{\kappa_i} & \cdots & 1 \end{bmatrix} \mathfrak{a}_{i,0}, \\ \mathfrak{a}_{i,1} = \begin{bmatrix} a_{i,2\kappa_i+1} & \cdots & a_{i,\kappa_i+1} \end{bmatrix}^T \in \mathbb{R}^{\kappa_i+1}, \text{ and } \mathfrak{a}_{i,0} = \begin{bmatrix} a_{i,\kappa_i} & \cdots & a_{i,0} \end{bmatrix}^T \in \mathbb{R}^{\kappa_i+1}, \end{gathered} \tag{2.7}$$

with $i \in \{1, \ldots, m\}$, and let us define the following auxiliary matrices:

[5] Lewis (1986) did a similar proposition when he introduced a "fast" input in his reachability consideration.

$$X_{(i,0)}(t) = \begin{bmatrix} \kappa_i!/0! & 0 & \cdots & 0 & 0 \\ (\kappa_i!/1!)t & (\kappa_i-1)!/0! & 0\cdots & 0 & 0 \\ \vdots & \vdots & \cdots & . & . \\ (\kappa_i!/\kappa_i!)t^{\kappa_i} & ((\kappa_i-1)!/(\kappa_i-1)!)t^{\kappa_i-1} & \cdots & (1!/1!)t & 0!/0! \end{bmatrix}, \quad (2.8)$$

$$X_{(i,1)}(t) = \begin{bmatrix} ((2\kappa_i+1)!/(\kappa_i+1)!)t^{\kappa_i+1} & \cdots & ((\kappa_i+1)!/1!)t \\ \vdots & \cdots & \vdots \\ ((2\kappa_i+1)!/(2\kappa_i+1)!)t^{2\kappa_i+1} & \cdots & ((\kappa_i+1)!/(\kappa_i+1)!)t^{\kappa_i+1} \end{bmatrix}. \quad (2.9)$$

The following Lemma gives a selection of the coefficient vectors $\mathfrak{a}_{i,0}$ and $\mathfrak{a}_{i,1}$, $i \in \{1, \ldots, m\}$, for f in (2.7) to satisfy assumptions (2.2) and (2.3) of Lemma 1 (see Appendix B for the proof).

Lemma 2. *For $i \in \{1, \ldots, m\}$, the determinants of the auxiliary matrices (2.8) and (2.9) satisfy*

$$\det\left(X_{(i,0)}(t)\right) = \prod_{\ell=0}^{\kappa_i} \ell! \quad \textit{and} \quad \det\left(X_{(i,1)}(t)\right) = t^{(\kappa_i+1)^2} \prod_{\ell=0}^{\kappa_i} \ell! \quad (2.10)$$

Moreover, if we select the coefficient vectors $\mathfrak{a}_{i,0}$ and $\mathfrak{a}_{i,1}$, as follows:

$$\begin{gathered} \mathfrak{a}_{i,0} = X_{(i,0)}^{-1}(0)v_0, \quad \mathfrak{a}_{i,1} = X_{(i,1)}^{-1}(t_1)\left(v_1 - X_{(i,0)}(t_1)\mathfrak{a}_{i,0}\right), \\ v_j = \begin{bmatrix} \left((\underline{e}_m^i)^T G_{\mathcal{B}}^{-1}(u_j - F_{\mathcal{B}}x_j)\right)^T & \left(\mathcal{R}^{-1}_{[A_{\mathcal{B}_i},\, B_{\mathcal{B}_i}]} P_i T_{\mathcal{B}}^{-1} x_j\right)^T \end{bmatrix}^T, \ j \in \{0, 1\}, \end{gathered} \quad (2.11)$$

where:

$$P_i = \begin{bmatrix} \underline{e}_n^{\hat{n}_i+1} & \cdots & \underline{e}_n^{\hat{n}_i+\kappa_i} \end{bmatrix}^T, \quad \hat{n}_1 = 0 \quad \textit{and} \quad \hat{n}_{i\geq 2} = \textstyle\sum_{j=1}^{i-1} \kappa_j, \quad (2.12)$$

then the function f defined by (2.7) fulfills assumptions (2.2) and (2.3) of Lemma 1.

Let us note from Lemma 2 that the proposed solutions only depend on the set of reachability indices S_κ, and on the fixed final time t_1. Hence, once S_κ and t_1 are given, the matrices $X_{i,0}(0)$, $X_{i,0}(t_1)$ and $X_{i,1}(t_1)$ are uniquely determined. And thus, the values of $\mathfrak{a}_{i,0}$ and $\mathfrak{a}_{i,1}$ only depend on the boundary points, (u_0, x_0) and (u_1, x_1), of the trajectory $(u, x) \in \mathfrak{B}_{[A,B]}$.

From the above observation, it is possible to track a given trajectory $(\bar{u}, \bar{x}) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^{m+n})$, with a delayed time t_1. Indeed, we only need to fix a sampling time $t_1 \in \mathbb{R}^{+*}$, and to apply iteratively Lemma 1 with the settings $(u_0, x_0) = (u(kt_1), x(kt_1))$ and $(u_1, x_1) = (\bar{u}(kt_1), \bar{x}(kt_1))$.

Otherwise written, in each sampling interval $[kt_1, (k+1)t_1)$, we find a trajectory $(u, x) \in \mathfrak{B}_{[A,B]} \cap \mathcal{C}^\infty\big(\mathbb{R}^+ \cap [kt_1, (k+1)t_1), \mathscr{U} \times \mathscr{X}\big)$, such that $(u(kt_1), x(kt_1))$ $= (u_0, x_0)$ and $\lim_{\sigma \to t_1} (u(kt_1 + \sigma), x(kt_1 + \sigma)) = (u_1, x_1)$.

We have proved in this way the following Theorem.

Theorem 1. *Let us consider an* input/state system $\Sigma_{i/s} = \Big(\mathbb{R}^+, \mathscr{U} \times \mathscr{X}, \mathfrak{B}_{[A,B]}\Big)$*, represented by (1.1). If (1.4) is satisfied, then for any sequence* $(\bar{u}_k, \bar{x}_k) \in \mathbb{R}^{m+n}$*,* $k \in \mathbb{Z}^+$*, and a given sampling time* $t_1 \in \mathbb{R}^{+*}$*, there exists a control law* $u \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^m)$*, such that* $(u(kt_1), x(kt_1)) = (\bar{u}_{k-1}, \bar{x}_{k-1})$.

3. IMPLICIT SYSTEMS

In this Section, we formalize the notion of *implicit system* following the behavioral point of view. For this, let us first state the following definition:

Definition 1. An implicit representation $\mathfrak{R}^{imp}(E, A, B)$ is called an *input/descriptor system*, when for all initial condition $x_0 \in \mathscr{X}_d$, there exists at least one solution $(u, x) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathscr{U} \times \mathscr{X}_d)$, such that $x(0) = x_0$. The *input/descriptor* *system* is defined by the triple[6] $\Sigma_{i/d} = (\mathbb{R}^+, \mathscr{U} \times \mathscr{X}_d, \mathfrak{B}_{[E,A,B]})$, with behavior:

$$\mathfrak{B}_{[E,A,B]} = \left\{ (u,x) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathscr{U} \times \mathscr{X}) \middle| \begin{bmatrix} (E\frac{\mathrm{d}}{\mathrm{d}t} - A) & -B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0 \right\} \quad (3.1)$$

At this point, it is important to clarify what exactly means the sentence "*there exists at least one solution*". For this, we are going to recall hereafter the notions of *existence of solution* introduced by Geerts (1993) and Aubin & Frankowska (1991).

3.1. Existence of solution for every initial condition

Following (Hautus, 1976) and (Hautus & Silverman, 1983), Geerts (1993) generalized the solvability results of (Geerts & Mehrmann, 1990). One advantage of this generalization is that the solvability is introduced in a very natural way, passing from the distributional framework (Schwartz, 1978) to the usual time domain with ordinary differential equations; this is precisely the starting point of the so called behavioral approach (Polderman & Willems, 1998), chosen in this paper.

Geerts (1993) considered the linear combinations of impulsive and smooth distributions, with μ coordinates, denoted by $\mathscr{C}^\mu_{\mathrm{imp}}$, as the signal sets. The set $\mathscr{C}^\mu_{\mathrm{imp}}$ is a subalgebra and is also decomposed as $\mathscr{C}^\mu_{\mathrm{p-imp}} \oplus \mathscr{C}^\mu_{\mathrm{sm}}$, where $\mathscr{C}^\mu_{\mathrm{p-imp}}$ and $\mathscr{C}^\mu_{\mathrm{sm}}$ denote the subalgebras of pure impulses[7] and smooth distributions[8],

[6] See also Polderman & Willems (1998) and Kuijper (1992b).

[7] The unit element of this subalgebra is the Dirac delta distribution δ. Any linear combination of δ and its distributional derivatives $\delta^{(\ell)}$, $\ell > 1$, is called impulsive.

[8] The set of regular distributions are distributions that are functions; namely piecewise continuous integrable, or measurable functions. In those papers, they assume that the regular distributions $u(t)$ are smooth on $[0, \infty)$, *i.e.* that a function $v : [0, \infty) \to \mathbb{R}$ exists, ar-

respectively (Schwartz, 1978). He introduced the following definitions for the distributional version of the *implicit representation* (1.7) $\mathfrak{R}^{imp}_{dist}(E, A, B)$:[9] $p\,Ex = Ax + Bu + Ex_0$ (*c.f.* Definitions 3.1 and 4.1, Geerts, 1993)[10].

Definition 2. (Geerts, 1993) Given the *solution set* $S_C(x_0, u) := \{x \in \mathscr{C}^{n_d}_{\text{imp}} | [p\,E - A]x = Bu + Ex_0\}$, the *implicit representation* $\mathfrak{R}^{imp}_{dist}(E, A, B)$ is:

- *C-solvable* if $\forall\, x_0 \in \mathscr{X}_d\ \exists\ u \in \mathscr{C}^m_{\text{imp}} : S_C(x_0, u) \neq \emptyset$,
- *C-solvable in the function sense* if $\forall\, x_0 \in \mathscr{X}_d\ \exists\, u \in \mathscr{C}^m_{\text{sm}} : S_C(x_0, u) \cap \mathscr{C}^n_{\text{sm}} \neq \emptyset$.

Given the *"consistent initial conditions set"* $\mathcal{I}_C := \{z_0 \in \mathscr{X}_d |\ \exists\, u \in \mathscr{C}^m_{\text{sm}}\ \exists\, x \in S_C(z_0, u) \cap \mathscr{C}^{n_d}_{\text{sm}} : x(0^+) = z_0\}$, and the *"weakly consistent initial conditions set"* $\mathcal{I}^w_C := \{z_0 \in \mathscr{X}_d |\ \exists\, u \in \mathscr{C}^m_{\text{sm}}\ \ \exists\, x \in S_C(z_0, u) \cap \mathscr{C}^{n_d}_{\text{sm}}\}$, a point $x_0 \in \mathscr{X}_d$ is called *C-consistent* if $x_0 \in \mathcal{I}_C$, and *weakly C-consistent* if $x_0 \in \mathcal{I}^w_C$.

Let us note that:

(i) C-solvability is concerned with distributional solutions,

(ii) C-solvability in the function sense is concerned with solutions only composed by ordinary functions arbitrarily often differentiable,

(iii) the two notions of consistency, *C-consistent* and *weakly C-consistent*, lead to *smooth solutions*, namely with no impulsions, but

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bitrarily often differentiable including at $t = 0$, such that $u(t) = 0$ for $t < 0$ and $u(t) = v(t)$, for $t \geq 0$ (Hautus & Silverman, 1983). These distributions are identified as ordinary functions with support on $\mathbb{R}^+$.

[9] Ex_0 stands for $Ex_0\,\delta$, $x_0 \in \mathscr{X}_d$ being the initial condition, and $p\,Ex$ stands for $\delta^{(1)} * Ex$ ($*$ denotes convolution); if $p\,Ex$ is smooth and $E\dot{x}$ stands for the distribution that can be identified with the ordinary derivative $E\mathrm{d}x/\mathrm{d}t$, then $p\,Ex = E\dot{x} + Ex_{0^+}$.

[10] He also considered the *B-free* case $\mathfrak{R}^{imp}_{dist}(E, A, f)$: $p\,Ex = Ax + f + Ex_0$.

(iv) C-consistency avoids jumps at the origin, namely the *smooth solutions* are continuous on the left, and

(iv) weakly C-consistent enables jumps at the origin, but they are piece-wise continuous *smooth solutions*.

Geerts (1993) characterized the existence of solutions for every initial condition in his Corollary 3.6, Proposition 4.2 and Theorem 4.5. Hereafter we summarize these results with their geometric equivalences.

Theorem 2. *(Geerts, 1993) If [**H2**] is fulfilled, then*

- $\mathfrak{R}^{imp}_{dist}(E,A,B)$ *is* C-solvable *if and only if* $[(\lambda E - A) \quad -B]]$ *is right invertible as a rational matrix, i.e. if and only if* [11]

$$E\mathscr{V}^*_{\mathscr{X}_d} + A\mathscr{S}^*_{\mathscr{X}_d} + \mathscr{B} = \mathscr{X}_{eq}\,. \tag{3.2}$$

- $\mathfrak{R}^{imp}_{dist}(E,A,B)$ *is* C-solvable in the function sense *if and only if* $\mathcal{I}^w_C = \mathscr{X}_d$*, namely, if and only if* $\operatorname{Im} E + A\mathscr{K}_E + \mathscr{B} = \mathscr{X}_{eq}$*, i.e. if and only if* [12]

$$E\mathscr{V}^*_{\mathscr{X}_d} = \operatorname{Im} E\,. \tag{3.3}$$

[11] $[\lambda[E \quad 0] - [A \quad B]]$ is right invertible iff (see Loiseau (1985) and Armentano (1986)) $\mathscr{X}_{eq} = [E \quad 0]\mathscr{V}^*_{[[A \quad B],\,[E \quad 0]]} + [A \quad B]\mathscr{S}^*_{[[E \quad 0],\,[A \quad B]]}$, namely iff $E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B} + A\mathscr{S}^*_{\mathscr{X}_d} = \mathscr{X}_{eq}$ (from (ALG–V) and (ALG–S) we get $\mathscr{V}^*_{[[A \quad B],\,[E \quad 0]]} = \mathscr{V}^*_{\mathscr{X}_d} \oplus \mathscr{U}$ and $\mathscr{S}^*_{[[E \quad 0],\,[A \quad B]]} = \mathscr{S}^*_{\mathscr{X}_d} \oplus \mathscr{U}$).

[12] From (ALG–V) and [**H2**], one obtains the following sequence of implications: $\operatorname{Im} E + \mathscr{B} + A\mathscr{K}_E = \mathscr{X}_{eq} \;\Rightarrow\; \mathscr{V}^1_{\mathscr{X}_d} + \mathscr{K}_E = \mathscr{X}_d \;\Rightarrow\; E\mathscr{V}^1_{\mathscr{X}_d} = \operatorname{Im} E \;\Rightarrow\; E\mathscr{V}^*_{\mathscr{X}_d} = \operatorname{Im} E$ $\Rightarrow\; \mathscr{X}_d = \mathscr{V}^*_{\mathscr{X}_d} + \mathscr{K}_E = A^{-1}(E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B}) + \mathscr{K}_E \;\Rightarrow\; \operatorname{Im} A = \operatorname{Im} A \cap (E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B}) + A\mathscr{K}_E \;\Rightarrow$ $\mathscr{X}_{eq} = \operatorname{Im} E + \mathscr{B} + A\mathscr{K}_E$.

- $\mathcal{I}_C = \mathscr{X}_d$ *if and only if* $\operatorname{Im} E + \mathscr{B} = \mathscr{X}_{eq}$ *i.e. if and only if* [13]

$$E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B} = \mathscr{X}_{eq}\ . \tag{3.4}$$

3.2. Existence of a viable solution

In order to study the reachability for implicit systems, Frankowska (1990) introduced the set–valued map (the set of all admissible velocities) $\mathbf{F}:\ \mathscr{X}_d \rightsquigarrow \mathscr{X}_d$, $\mathbf{F}(x) = E^{-1}\big(Ax + \mathscr{B}\big) = \big\{v \in \mathscr{X} \,|\, Ev \in Ax + \mathscr{B}\big\}$, and the differential inclusion

$$\mathrm{d}x/\mathrm{d}t \in \mathbf{F}(x), \qquad \text{where} \ \ x(0) = x_0, \tag{3.5}$$

Frankowska (1990) showed that the solutions of (1.7) and the ones of (3.5) are the same. She also clarified the meaning of a viable solution and she characterized the largest subspace of such viable solutions.

Definition 3. (Frankowska, 1990, Aubin & Frankowska, 1991)

- An absolutely continuous function $x:\ \mathbb{R}^+ \to \mathscr{X}_d$ is called a *trajectory* of (3.5), if $x(0) = x_0$ and $\mathrm{d}x/\mathrm{d}t \in \mathbf{F}(x)$ for almost every $t \in \mathbb{R}^+$, that is to say, if there exists a measurable function $u:\ \mathbb{R}^+ \to \mathscr{U}$ such that $x(0) = x_0$ and $E\mathrm{d}x/\mathrm{d}t = Ax + Bu$, for almost every $t \in \mathbb{R}^+$.

- Let $\mathscr{K}$ be a subspace[14] of $\mathscr{X}_d$. A trajectory x of (3.5) is called *viable in* $\mathscr{K}$, if $x(t) \in \mathscr{K}$ for all $t \geq 0$. The set of such trajectories is called

[13] Directly follows from (ALG–V) and [**H2**].

[14] We restrict our discussion to subspaces of finite dimensional vector spaces. In (Frankowska, 1990) and in (Aubin & Frankowska, 1991) these definitions are stated in the more general framework of closed sets of normed vector spaces.

the set of viable solutions in $\mathscr{K}$. The subspace $\mathscr{K}$ is called *a viability domain of* $\mathbf{F}$, if for all $x \in \mathscr{K} : \mathbf{F}(x) \cap \mathscr{K} \neq \emptyset$. The subspace $\mathscr{K}$ is called *the viability kernel of* (3.5) when it is the largest viability domain of $\mathbf{F}$.

Theorem 3. *(Aubin & Frankowska, 1991) The supremal (A, E, B)–invariant subspace contained in $\mathscr{X}_d$, $\mathscr{V}^*_{\mathscr{X}_d}$, is the viability kernel of $\mathscr{X}_d$ for the set-valued map $\mathbf{F} : \mathscr{X}_d \rightsquigarrow \mathscr{X}_d$, $\mathbf{F}(x) = E^{-1}(Ax + \mathscr{B})$. Moreover, for all $x_0 \in \mathscr{V}^*_{\mathscr{X}_d}$ there exists a trajectory $x \in \mathcal{C}^\infty(\mathbb{R}^+, \mathscr{V}^*_{\mathscr{X}_d})$ solution of (1.7) satisfying $x(0) = x_0$.*

Frankowska (1990) called a singular system *"strict"* when the viability kernel coincides with the whole descriptor space $\mathscr{X}_d$, namely

$$\mathscr{V}^*_{\mathscr{X}_d} = \mathscr{X}_d \; . \tag{3.6}$$

In order to clarify ideas, let us extract from (Bonilla & Malabre, 1997, Section 2.1) the following result:

Result 1. *There exists a subspace $\mathscr{X}_1$ such that:*

$$\mathscr{X}_d = \mathscr{V}^*_{\mathscr{X}_d} \oplus \mathscr{X}_1, \quad \mathscr{X}_{eq} = (E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B}) \oplus A\mathscr{X}_1 \; , \quad \textit{and} \quad \mathscr{X}_1 \approx A\mathscr{X}_1. \tag{3.7}$$

Moreover, when projecting on $\mathscr{X}_1$ any trajectory $x \in \mathcal{C}^\infty(\mathbb{R}^+, \mathscr{V}^*_{\mathscr{X}_d})$ solution of (1.7), we always get a null trajectory.

Furthermore, for all $x_0 \in \mathscr{V}^*_{\mathscr{X}_d}$ there exists at least one trajectory $(u, x_\rho) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathscr{U} \times \mathscr{V}^*_{\mathscr{X}_d})$ solution of (1.7), satisfying $x_\rho(0) = x_0$.

Proof of Result 1. From the algorithm shown in (Fig. 1, Bonilla & Malabre, 1997) and from [**H2**], we get the geometric decompositions (3.7), and under these decompositions, (1.7) takes the following form:

$$\left[\begin{array}{c|c} E_\rho & * \\ \hline 0 & \overline{X}_{\rho-1} \end{array}\right] \frac{\mathrm{d}}{\mathrm{d}t} \left[\begin{array}{c} x_\rho \\ \bar{x}_{\rho-1} \end{array}\right] = \left[\begin{array}{c|c} A_\rho & 0 \\ \hline 0 & I_1 \end{array}\right] \left[\begin{array}{c} x_\rho \\ \bar{x}_{\rho-1} \end{array}\right] + \left[\begin{array}{c} B_\rho \\ \hline 0 \end{array}\right] u \, , \tag{3.8}$$

where $x_\rho \in \mathscr{V}^*_{\mathscr{X}_d}$, $\bar{x}_{\rho-1} \in \mathscr{X}_1$, $I_1 : \mathscr{X}_1 \leftrightarrow A\mathscr{X}_1$ is an isomorphism, $\overline{X}_{\rho-1}$ is a nilpotent matrix (an upper triangular matrix with zeros in its diagonal). Then $\bar{x}_{\rho-1} \equiv 0$.

If we now apply the following geometric decompositions:

$$E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B} = E\mathscr{V}^*_{\mathscr{X}_d} \oplus \mathscr{B}_C,\ \mathscr{B} = (\mathscr{B} \cap E\mathscr{V}^*_{\mathscr{X}_d}) \oplus \mathscr{B}_C,\ \mathscr{U} = B^{-1}E\mathscr{V}^*_{\mathscr{X}_d} \oplus B^{-1}\mathscr{B}_C, \tag{3.9}$$

where $\mathscr{B}_C$ is some complementary subspace of $\mathscr{B} \cap E\mathscr{V}^*_{\mathscr{X}_d}$, we get for $\mathfrak{R}^{imp}(E_\rho, A_\rho, B_\rho)$ (recall (3.8)):

$$\left[\begin{array}{c} \overline{E}_\rho \\ \hline 0 \end{array}\right] \tfrac{\mathrm{d}}{\mathrm{d}t} x_\rho = \left[\begin{array}{c} \overline{A}_\rho \\ \hline \widehat{A}_\rho \end{array}\right] x_\rho + \left[\begin{array}{c|c} \overline{B}_\rho & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c} u_1 \\ u_2 \end{array}\right]. \tag{3.10}$$

Since $\operatorname{Im} \overline{E}_\rho = E\mathscr{V}^*_{\mathscr{X}_d}$, there exists $\overline{E}^r_\rho : E\mathscr{V}^*_{\mathscr{X}_d} \to \mathscr{V}^*_{\mathscr{X}_d}$ such that $\overline{E}_\rho \overline{E}^r_\rho = I$. Then, one solution of (3.10) is given by

$$\begin{array}{rcl} x_\rho(t) &=& \exp\left(\overline{E}^r_\rho \overline{A}_\rho t\right) x_0 + \displaystyle\int_0^t \exp\left(\overline{E}^r_\rho \overline{A}_\rho (t-\tau)\right) \overline{E}^r_\rho \overline{B}_\rho u_1(\tau) \mathrm{d}\tau, \\ u_2(t) &=& -\widehat{A}_\rho x_\rho(t). \end{array} \qquad \square$$

Thus, the subspaces $E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B} \subset \mathscr{X}_{eq}$ and $\mathscr{V}^*_{\mathscr{X}_d} \subset \mathscr{X}_d$ characterize the set of all possible trajectories of (1.7) which are not identically zero for any input u. The projection of any trajectory solution of (1.7) on the quotient space $\mathscr{X}_d / \mathscr{V}^*_{\mathscr{X}_d}$, in correspondence with the projection on $\mathscr{X}_{eq}/(E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B})$ for the equation space, results in an identically null function (see Bonilla & Malabre, 1995, Corollary 2.1). Let us note that when Assumption [**H2**] holds, the geometric conditions $E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B} = \mathscr{X}_{eq}$ and $\mathscr{V}^*_{\mathscr{X}_d} = \mathscr{X}_d$ are equivalent[15].

[15] From (ALG–V) and [**H2**]: $E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B} = \mathscr{X}_{eq} \Rightarrow \mathscr{V}^*_{\mathscr{X}_d} = A^{-1}(E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B}) = \mathscr{X}_d$; $\mathscr{V}^*_{\mathscr{X}_d} = \mathscr{X}_d$ $\Rightarrow \operatorname{Im} E = E\mathscr{V}^*_{\mathscr{X}_d}$ & $\operatorname{Im} A = \operatorname{Im} A \cap (E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B}) \Rightarrow \mathscr{X}_{eq} = \operatorname{Im} E + \operatorname{Im} A + \mathscr{B} = E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B}$.

3.3. Discussion about existence of solution

An important contribution of Geerts (1993) is to give conditions under which the distributional and time-domain frameworks lead to the same conclusions with respect to the shape of the resulting system's solution trajectories (*c.f.* (3.4) and (3.3)), namely the resulting distributions are identified as ordinary functions, with support on $\mathbb{R}^+$, and the generalized derivatives can be identified with ordinary derivatives. Also, it is well connected with the viability discussion of Frankowska (1990) and Aubin & Frankowska (1991); indeed, a singular system is strict if and only if the consistent initial condition set $\mathcal{I}_C$ coincides with the whole descriptor variable space $\mathscr{X}_d$ (*c.f.* (3.6) and (3.4), and recall Assumption [**H2**]).

Regarding the set of weakly consistent initial conditions Geerts (1993) notes, in his abstract and conclusion, that the condition that this set equals to the whole state space (under the Assumption [**H2**]) is equivalent to the impulse controllability for regular systems (Cobb, 1984) (or controllability of the infinite part in the sense of Verghese *et al* (1981)). This correspondence has been generalized to non regular systems and one can note that the nowadays most commonly adopted definition for impulse controllability is the one cited by Ishihara & Terra (2001)[16]: *a general singular system is impulse controllable if for every initial condition there exists a smooth (impulse-free) control $u(t)$, and a smooth (impulse-free, but with possible jumps, especially at the origin) variable descriptor trajectory solution of the system.*

More generally, one can verify that the paper of Geerts (1993) is the

[16]Notice that in this paper is stated that the definition comes from Geerts (1993).

main reference on solvability properties, consistency of initial conditions, the ability to find control such that no impulsive phenomenon appears (see for examples Hou & Müller (1999), Ishihara & Terra (2001), Hou (2004) and Zhang (2006)).

However, one should also cite Özçaldiran & Haliločlu (1993) who proved that there exists a pair of smooth distributions (without jumps), satisfying $\mathfrak{R}^{imp}_{dist}(E, A, B)$ if and only if $x(0_*) \in \mathscr{V}_{\mathscr{X}_d^*}$, namely $\mathscr{V}_{\mathscr{X}_d^*} = \mathscr{X}_d$ (see their Proposition 1.3), and Przyluski & Sosnowski (1994) who proved that the subspace $\mathscr{V}_{\mathscr{X}_d^*} + \mathscr{K}_E$ characterizes the set of initial conditions, for which there exists a pair of smooth distributions (with possible jumps) satisfying $\mathfrak{R}^{imp}_{dist}(E,\ A, B)$, namely $E\mathscr{V}_{\mathscr{X}_d^*} = \operatorname{Im} E$ (see their Proposition 1).

In Figure 1, we summarize all the above discussion.

$$\underset{(a)}{\boxed{\mathscr{V}_{\mathscr{X}_d^*} = \mathscr{X}_d}} \iff \underset{(b)}{\boxed{\begin{array}{c}\operatorname{Im} E + \mathscr{B} = \mathscr{X}_{eq} \\ \left(E\mathscr{V}_{\mathscr{X}_d^*} + \mathscr{B} = \mathscr{X}_{eq}\right)\end{array}}} \implies \underset{(c)}{\boxed{\begin{array}{c}\operatorname{Im} E + A\mathscr{K}_E + \mathscr{B} = \mathscr{X}_{eq} \\ \left(E\mathscr{V}_{\mathscr{X}_d^*} = \operatorname{Im} E\right)\end{array}}}$$

Figure 1: Connexions between the notions of existence of solution and impulse controllability (under the Asumption [**H2**]).

Fig. 1(a) is the condition of viable solution of Aubin & Frankowska (1991) or smooth solution (without any jump) of Özçaldiran & Haliločlu (1993). Fig. 1(b) is the condition that the set of consistent initial condition equals the whole space of Geerts (1993). Fig. 1(c) is the condition of *C-solvability in the function sense* of Geerts (1993) or the condition of Przyluski & Sosnowski (1994) that the set of initial conditions of smooth solutions (with possible jumps) equals the whole space, or the impulse controllability condition of Ishihara & Terra (2001), or the impulse-mode controllability with arbitrary initial conditions of Hou (2004).

Finally let us note that if the notion of weakly consistent initial conditions as defined by Geerts (1993) is associated to the notion of impulse controllability, the notion of consistent initial conditions as defined by Geerts (1993) is associated to the notion of reachability of Frankowska (1990) (in the more general non regular case) since the system must be strict to be reachable. See also the controllability discussion found in Korotka *et al* (2011).

4. REACHABILITY FOR IMPLICIT SYSTEMS

For the case of implicit systems, Frankowska (1990) extended the classical reachability definition as follows.

Definition 4. (Frankowska, 1990) The implicit representation (1.7) is called *reachable* if for any pair of vectors x_0, $x_1 \in \mathscr{X}_d$ and for any pair of real numbers $t_1 > t_0 \geq 0$, there exists a trajectory $x(\cdot)$ solution of (1.7), such that $x(t_0) = x_0$ and $x(t_1) = x_1$.

Frankowska (1990) has established in her Theorem 4.4 that $\mathscr{R}^*_{\mathscr{X}_d}$ (see (1.8)) is the reachable space of implicit systems like (1.7), with E and A not necessarily square. Hereafter, we recall Corollary 2.4 of Aubin and Frankowska (1991) which is *ad hoc* for our paper.

Theorem 4. *(Aubin & Frankowska, 1991) For any $t_1 > 0$ and for a system like (1.7), with E and A not necessarily square, the reachable space of (1.7) at time t_1 from the initial descriptor variable $x(0)$ is equal to $\mathscr{R}^*_{\mathscr{X}_d}$. Moreover, $\mathscr{R}^*_{\mathscr{X}_d}$ is the supremal subspace such that for all $x_0, x_1 \in \mathscr{R}^*_{\mathscr{X}_d}$ and $t_1 > 0$, there* *exists a trajectory* $x \in \mathcal{C}^\infty(\mathbb{R}^+\mathscr{R}^*_{\mathscr{X}_d})$ *solution of (1.7) satisfying* $x(0) = x_0$ *and* $x(t_1) = x_1$.

In this Section we are interested in generalizing and solving Problem 1 in the case of an *input/descriptor system* $\Sigma_{i/d} = (\mathbb{R}^+, \mathscr{U} \times \mathscr{X}_d,\ \mathfrak{B}_{[E,A,B]})$, with behavior (3.1).

Problem 2. Let us consider a *input/descriptor system* $\Sigma_{i/d} = (\mathbb{R}^+, \mathscr{U} \times \mathscr{X}_d,$ $\mathfrak{B}_{[E,A,B]})$ represented by (1.7), and with the behavior (3.1). Given (u_0, x_0), $(u_1, x_1) \in B^{-1}E\mathscr{R}^*_{\mathscr{X}_d} \times \mathscr{R}^*_{\mathscr{X}_d}$ and $t_1 > 0$, find a trajectory $(u, x) \in \mathfrak{B}_{[E,A,B]}$, such that $(u(0), x(0)) = (u_0, x_0)$ and $(u(t_1), x(t_1)) = (u_1, x_1)$.

For answering this question, we proceed as follows.

(i) We first apply some geometric decompositions to the subspaces $\mathscr{X}_d$ and $\mathscr{X}_{eq}$, inspired by Proposition 2.2 of Aubin and Frankowska (1991); the aim of these decompositions is to point out a part of the implicit representation, more or less explicit, which is expressed as a state space representation.

(ii) We next show that such a state space representation is reachable in the classical sense.

(iii) Finally, based on Section 2, we answer Problem 2.

4.1. State reachability

The following Lemma is proved in Appendix C.

Lemma 3. *When* $\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{X}_d$, *the* implicit representation *(1.7) can be restricted to* $\mathscr{R}^*_{\mathscr{X}_d}$ *in the domain, and to* $A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}$ *in the codomain.*

Moreover, the spaces $\mathscr{R}^*_{\mathscr{X}_d}$, $\mathscr{B}$, $A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}$ *and* $\mathscr{U}$ *can be decomposed as follows:* $\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{R}_{\mathcal{C}} \oplus (\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E)$, $\mathscr{B} = (\mathscr{B} \cap E\mathscr{R}_{\mathscr{X}_d}) \oplus \mathscr{B}_{\mathcal{C}}$, $A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B} = E\mathscr{R}^*_{\mathscr{X}_d} \oplus \mathscr{B}_{\mathcal{C}}$

and $\mathscr{U} = B^{-1}E\mathscr{R}^*_{\mathscr{X}_d} \oplus \mathscr{U}_\mathcal{C}$*, where* $\mathscr{R}_\mathcal{C}$ *and* $\mathscr{U}_\mathcal{C}$ *are complementary subspaces such that* $\mathscr{R}_\mathcal{C} \approx E\mathscr{R}^*_{\mathscr{X}_d}$ *and* $\mathscr{U}_\mathcal{C} = B^{-1}\mathscr{B}_\mathcal{C} \approx \mathscr{B}_\mathcal{C}$*. Under these decompositions, the* implicit representation *(1.7), restricted to* $\mathscr{R}^*_{\mathscr{X}_d}$ *in the domain and to* $A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}$ *in the codomain, takes the following form:*

$$\begin{gathered}\overline{E}\mathrm{d}x/\mathrm{d}t = \overline{A}x + \overline{B}u,\\ \overline{E} = \begin{bmatrix} I_\mathcal{C} & 0 \\ 0 & 0 \end{bmatrix},\ \overline{A} = \begin{bmatrix} \overline{A}_{1,1} & \overline{A}_{1,2} \\ \overline{A}_{2,1} & \overline{A}_{2,2} \end{bmatrix},\ \overline{B} = \begin{bmatrix} \overline{B}_1 & 0 \\ 0 & I_{\mathscr{U}_\mathcal{C}} \end{bmatrix},\end{gathered} \tag{4.1}$$

where $I_\mathcal{C} : \mathscr{R}_\mathcal{C} \leftrightarrow E\mathscr{R}^*_{\mathscr{X}_d}$*, and* $I_{\mathscr{U}_\mathcal{C}} : \mathscr{U}_\mathcal{C} \leftrightarrow \mathscr{B}_\mathcal{C}$ *are isomorphisms.*

In order to locate the state reachability part of (4.1), let us first define the natural projections:

$$\begin{aligned} &P_\mathcal{C} : \mathscr{R}^*_{\mathscr{X}_d} \to \mathscr{R}_\mathcal{C}//(\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E),\ P_\ell : \mathscr{R}^*_{\mathscr{X}_d} \to (\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E)//\mathscr{R}_\mathcal{C},\\ &Q_1 : \mathscr{U} \to B^{-1}E\mathscr{R}^*_{\mathscr{X}_d}//B^{-1}\mathscr{B}_\mathcal{C},\ Q_2 : \mathscr{U} \to B^{-1}\mathscr{B}_\mathcal{C}//B^{-1}E\mathscr{R}^*_{\mathscr{X}_d}. \end{aligned}$$

Let us next apply to $\mathfrak{R}^{imp}(\overline{E}, \overline{A}, \overline{B})$ the reachability algorithm of Özçaldiran (1985), $\overline{\mathscr{R}}^0 = \{0\}$, $\overline{\mathscr{R}}^{\mu+1} = \overline{E}^{-1}\left(\overline{A\mathscr{R}}^\mu + (\overline{\mathscr{B}}_1 \oplus \mathscr{B}_\mathcal{C})\right)$, whose limit is $\mathscr{R}^*_{\mathscr{X}_d}$; namely: $\overline{\mathscr{R}}^1 = I_\mathcal{C}^{-1}\overline{\mathscr{B}}_1 \oplus (\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E)$ and $\overline{\mathscr{R}}^{\mu+1} = I_\mathcal{C}^{-1}\left(\overline{A}_{1,1}P_\mathcal{C}\overline{\mathscr{R}}^\mu + \mathrm{Im}\,[\,\overline{A}_{1,2}\ \overline{B}_1\,]\right) \oplus (\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E)$, for $\mu \geq 1$. We thus obtain $I_\mathcal{C}P_\mathcal{C}\overline{\mathscr{R}}^{\mu+1} = \overline{A}^\mu_{1,1}\mathrm{Im}\,\overline{B}_1 + \sum_{i=0}^{\mu-1}\overline{A}^i_{1,1}\mathrm{Im}\,[\,\overline{A}_{1,2}\,\overline{B}_1\,]$, which implies:

$$E\mathscr{R}^*_{\mathscr{X}_d} = \left\langle \overline{A}_{1,1} \mid \mathrm{Im}\,[\overline{A}_{1,2}\ \ \overline{B}_1] \right\rangle. \tag{4.2}$$

Thus, $(\overline{A}_{1,1}, [\overline{A}_{1,2}\,\overline{B}_1])$ is a *state reachable pair.*

4.2. Behavioral reachability

Given any initial condition $x_0 \in \mathscr{R}^*_{\mathscr{X}_d}$, the solution set of (4.1) is characterized by the following behavior

$$\mathfrak{B}_{[\overline{E},\overline{A},\overline{B}]} = \Big\{ (u,x) \in \mathcal{C}^\infty \left(\mathbb{R}^+, \mathscr{U} \times \mathscr{R}^*_{\mathscr{X}_d}\right) \ \Big| \ \exists\, x_0 \in \mathscr{R}^*_{\mathscr{X}_d} \text{ s.t. } P_{\mathcal{C}} x(t) =$$

$$\exp\left(I_{\mathcal{C}}^{-1}\overline{A}_{1,1}t\right) P_{\mathcal{C}}x_0 + \int_0^t \exp\left(I_{\mathcal{C}}^{-1}\overline{A}_{1,1}(t-\tau)\right) I_{\mathcal{C}}^{-1}\left(\overline{A}_{1,2}P_\ell x(\tau) + \overline{B}_1 Q_1 u(\tau)\right) \mathrm{d}\tau,$$

$$Q_2 u(t) = -I_{\mathscr{U}_{\mathcal{C}}}^{-1}\left(\overline{A}_{2,1}P_{\mathcal{C}}x(t) + \overline{A}_{2,2}P_\ell x(t)\right) \Big\}, \tag{4.3}$$

which behavioral equations are

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} I_{\mathcal{C}} P_{\mathcal{C}} x &= \overline{A}_{1,1} P_{\mathcal{C}} x + \begin{bmatrix} \overline{A}_{1,2} & \overline{B}_1 \end{bmatrix} \begin{bmatrix} P_\ell x \\ Q_1 u \end{bmatrix}, \\ 0 &= \overline{A}_{2,1} P_{\mathcal{C}} x + \overline{A}_{2,2} P_\ell x + I_{\mathcal{B}_{\mathcal{C}}} Q_2 u. \end{aligned} \tag{4.4}$$

Let us note that

(i) the component $P_{\mathcal{C}}x$ is the part of the descriptor variable which needs a control law to reach the desired goal.

(ii) The component $P_\ell x$ is the free part of the descriptor variable which acts as some kind of internal input variable, together with the component $Q_1 u$ which is the effective external control input variable.

(iii) The component $Q_2 u$ of the external control variable must be equal to a component of the descriptor variable. This is because we have chosen a purely integral description. This part of the input corresponds to algebraic relationships linked with purely derivative actions.

From Lemmas 1 and 2, we get the following theorem which gives a solution to Problem 2.

Theorem 5. *Consider the reachable part (4.4) of the implicit representation (1.7). Denote:* $n = \dim\left(E\mathscr{R}^*_{\mathscr{X}_d}\right)$ *and* $m = \dim\left(\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E\right) + \dim\left(B^{-1}E\mathscr{R}^*_{\mathscr{X}_d}\right)$*. Let* $\{\kappa_1, \kappa_2, \ldots, \kappa_m\} \subset \mathbb{Z}^+$ *be the reachability indices of the pair* $\left(\overline{A}_{1,1}, \begin{bmatrix} \overline{A}_{1,2} & \overline{B}_1 \end{bmatrix}\right)$*, with* $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_m \geq 1$ *and* $\kappa_1 + \kappa_2 + \cdots + \kappa_m = n$*.*

Let the linear map $F: \mathscr{R}_{\mathcal{C}} \to (\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E) \times (B^{-1}E\mathscr{R}^*_{\mathscr{X}_d})$ and the isomorphisms $T: \mathscr{R}_{\mathcal{C}} \leftrightarrow \mathscr{R}_{\mathcal{C}}$ and $G: (\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E) \times (B^{-1}E\mathscr{R}^*_{\mathscr{X}_d}) \leftrightarrow (\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E) \times (B^{-1} E\mathscr{R}^*_{\mathscr{X}_d})$ be such that the pair $(\overline{A}_{\mathcal{B}},\, \overline{\Gamma}_{\mathcal{B}})$, where $\overline{A}_{\mathcal{B}} = (I_{\mathcal{C}}T)^{-1}(\overline{A}_{1,1} + \left[\,\overline{A}_{1,2}\ \overline{B}_1\,\right] F)(I_{\mathcal{C}}T)$ and $\overline{\Gamma}_{\mathcal{B}} = (I_{\mathcal{C}}T)^{-1}\left[\,\overline{A}_{1,2}\ \overline{B}_1\,\right] G$, is expressed in the Brunovsky canonical form (1.2). The reachability matrix $\mathcal{R}_{[\overline{A}_{\mathcal{B}},\overline{\Gamma}_{\mathcal{B}}]}$ is expressed in terms of the reachabillity matrices $\mathcal{R}_{[\overline{A}_{\mathcal{B}_i},\overline{\gamma}_{\mathcal{B}_i}]}$ as in (2.1).

Let $x_0,\, x_1 \in \mathscr{R}^*_{\mathscr{X}_d}$, $Q_1u_0,\, Q_1u_1 \in B^{-1}E\mathscr{R}^*_{\mathscr{X}_d}$, and $t_1 > 0$ be given. If we apply

$$\begin{bmatrix} P_\ell x(t) \\ Q_1 u(t) \end{bmatrix} = FP_{\mathcal{C}}x(t) + G\mathfrak{D}(\mathrm{d/dt})f(t)\,, \tag{4.5}$$

where $f(t) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^m)$ and $\mathfrak{D}(\mathrm{d/dt})$ are defined as in Lemmas 1 and 2, we get

$$P_{\mathcal{C}}x(t) = T\mathcal{R}_{[\overline{A}_{\mathcal{B}},\overline{\Gamma}_{\mathcal{B}}]}\bar{w}(t)\,, \tag{4.6}$$

and

$$(u(t_i),\, x(t_i)) = \left(\begin{bmatrix} Q_1 u_i \\ -I^{-1}_{\mathscr{U}_{\mathcal{C}}} \left[\, \overline{A}_{2,1} \quad \overline{A}_{2,2} \,\right] x_i \end{bmatrix},\, x_i \right),\ i \in \{0,\,1\},\, t_0 = 0\,. \tag{4.7}$$

4.3. Comments on the reachability

For the general case of implicit systems, represented by (1.7) with E and A not necessarily square, Frankowska (1990) has been the first to give a functional interpretation of reachability. For this, she has used the Viability Theory. More precisely, she has shown that reachability is equivalent to finding a trajectory $x \in \mathcal{C}^\infty(\mathbb{R}^+, \mathscr{X}_d)$ solution of (1.7), starting from the initial condition x_0 and reaching the desired x_1 in a given finite time t_1, namely $x(0) = x_0$ and $x(t_1) = x_1$ (see Theorem 4). Moreover, Frankowska (1990) has shown that reachability is geometrically characterized by the well known reachable space $\mathscr{R}^*_{\mathscr{X}_d}$. Of course, $\mathscr{R}^*_{\mathscr{X}_d}$ is contained in the viability kernel $\mathscr{V}^*_{\mathscr{X}_d}$. This guarantees the existence of at least one trajectory solution of (1.7), leaving from x_0. This is also clear from $\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{V}^*_{\mathscr{X}_d} \cap \mathscr{S}^*_{\mathscr{X}_d}$.

One interesting thing found in the proof of (Aubin & Frankowska, 1991, Proposition 2.2) was to put forward the importance of the state space representation (4.4) of the implicit equation(1.7). This fact has enabled us to apply systematically the results of the classical State Space Control Theory. More precisely, thanks to the reachability of the pair $(\overline{A}_{1,1},\ [\,\overline{A}_{1,2}\ \overline{B}_1\,])$ (see (4.2)), it is possible to find trajectories $f_i \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^1)$ (see (2.7), (2.11), (2.12), (2.8), and (2.9)) for synthesizing the control law (4.5) (see also (2.4)) which guarantees (4.7) (see Lemmas 1 and 2 and Theorem 5).

The aim of Theorem 5 was not to prove once more the sufficiency of Theorem 4, but to interpret the reachability of (1.7) in the classical state space framework. This interpretation allows us to have a better understanding of the existing mechanisms in the linear implicit systems reachability. Indeed, there exist two control actions. The first one is due to the free variable $P_\ell x$, and another one is due to the control input $Q_1 u$ (see (4.3)). The control input $Q_2 u$ is algebraically linked to the descriptor variable components, the state variable $P_{\mathcal{C}} x$ and the free variable $P_\ell x$, by means of the algebraic restriction (4.4.b) (when it exists).

For systems composed by *infinite elementary divisors*[17], the matrix Q_1

[17] Kronecker showed that any pencil $[\lambda E - A]$, $\lambda \in \mathbb{C}$, is strictly equivalent to a canonical matrix, composed by four kind of blocks: *(i) finite elementary divisors* (integral actions), *e.g.* $\begin{bmatrix} (\lambda-\alpha) & 1 \\ 0 & (\lambda-\alpha) \end{bmatrix}$, *(ii) infinite elementary divisors* (derivative actions), *e.g.*

is null and the square matrix Q_2 is invertible. In this case, the equations (4.5) and (4.6) describe the behavior of a system fed-back by the control law (4.4b). Indeed, from (4.5) and (4.6), we get:

$$x(t) = \begin{bmatrix} I \\ F \end{bmatrix} T\mathcal{R}_{[\overline{A}_{\mathcal{B}},\overline{\Gamma}_{\mathcal{B}}]}\overline{w}(t) + \begin{bmatrix} 0 \\ G \end{bmatrix} \mathfrak{D}(\mathrm{d}/\mathrm{d}t)f(t).$$

And from (4.4b), (4.5) and (4.6), we have:

$$Q_2 u(t) = -I_{\mathscr{U}_{\mathcal{C}}}^{-1}\left(\left(\overline{A}_{2,1} + \overline{A}_{2,2}F\right) T\mathcal{R}_{[\overline{A}_{\mathcal{B}},\overline{\Gamma}_{\mathcal{B}}]}\bar{w}(t) + \overline{A}_{2,2}G\mathfrak{D}(\mathrm{d}/\mathrm{d}t)f(t)\right).$$

It is remarkable that in the systems represented by *column minimal indices*, it is possible to have reachable systems without any control. This phenomenon is possible because of the existence of the free variable $P_\ell x$, which acts as an internal control signal.

5. POLE ASSIGNMENT

One of the most important features of the reachability of a state space representation (1.1) is the complete assignability of the closed loop spectrum by means of a state feedback. This equivalence is no longer the case when dealing with implicit representations (1.7). For the implicit description case, a geometric condition has to be added in order to guarantee such a *pole assignment ability*. In the sequel we give geometric conditions, which enable

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$\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$, *(iii) column minimal indices* (internal variable structure), *e.g.* $\begin{bmatrix} \lambda & 1 \end{bmatrix}$, and *(iv) row minimal indices* (internal behavioral restrictions), *e.g.* $\begin{bmatrix} \lambda \\ 1 \end{bmatrix}$; see Gantmacher (1977).

us to assign the closed loop spectrum of: *(i)* a reachable implicit description (1.7), and *(ii)* a reachable and observable implicit description with output equation, $\mathfrak{R}^{imp}(E, A, B, C)$:

$$E\mathrm{d}x/\mathrm{d}t = Ax + Bu, \quad y = Cx, \tag{5.8}$$

where $C : \mathscr{X}_d \to \mathscr{Y}$ is a linear map, and the linear space $\mathscr{Y}$ is the output space.

At this point, it is useful to clarify what we mean by spectrum and observable part.

A. Spectrum. We distinguish between the finite spectrum, $\sigma_f(A, E) = \{\lambda \in \mathbb{C} \mid \exists\, v \neq 0 \text{ s.t. } Av = \lambda E v\}$, and the infinite spectrum, $\sigma_\infty(E, A) = \{\mu \in \mathbb{C} \mid \exists\, w \neq 0 \text{ s.t. } Ew = \mu A w\}$ (*c.f.* Gantmacher (1977), Wong (1974), Armentano (1986)); the elements of $\sigma_f(A, E)$ are called poles, and the elements of $\sigma_\infty(E, A)$ are called poles at infinity. Note that for the four kind of blocks of the Kronecker canonical form[18]: *(i)* $\sigma_f(A, E) = \emptyset$ and $\sigma_\infty(E, A) = \emptyset$ for its *row minimal indices* blocks, *(ii)* $\sigma_f(A, E) = \emptyset$ for its *infinite elementary divisors* blocks, *(iii)* $\mathrm{card}\,\{\sigma_f(A, E)\} = \infty$ and $\mathrm{card}\,\{\sigma_\infty(E, A)\} = \infty$ for its *column minimal indices* blocks.

B. Observable part. With respect to the observable part, let us recall that it was shown in (Bonilla & Malabre, 1995) that the third condition of Kuijper (1992a) – $\begin{bmatrix} \mathrm{s}E - A \\ C \end{bmatrix}$ *has full column rank for all* $\mathrm{s} \in \mathbb{C}$ – for getting a minimal implicit representation (among all externally equivalent[19] representations

[18] See footnote 17.

[19] Two representations are called externally equivalent if the corresponding sets of all possible trajectories for the external variables, expressed in an *input/output partition* (u, y), are the same (Willems, 1983, Polderman & Willems, 1998).

of the same type), is equivalent to have a null unobservable space, namely: $\mathscr{N} = \{0\}$. Indeed, if we decompose the descriptor and equation spaces as: $\mathscr{X}_d = \mathscr{X}_{ob} \oplus \mathscr{N}$ and $\mathscr{X}_{eq} = \mathscr{W}_{ob} \oplus E\mathscr{N}$, where $\mathscr{X}_{ob}$ and $\mathscr{W}_{ob}$ are some complementary subspaces, (5.8) takes the following form:

$$\begin{aligned} \begin{bmatrix} E_{ob} & 0 \\ Z & E_{\mathscr{N}} \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} x_{ob} \\ x_{\mathscr{N}} \end{bmatrix} &= \begin{bmatrix} A_{ob} & 0 \\ X & A_{\mathscr{N}} \end{bmatrix} \begin{bmatrix} x_{ob} \\ x_{\mathscr{N}} \end{bmatrix} + \begin{bmatrix} B_{ob} \\ Y \end{bmatrix} u \\ y &= \begin{bmatrix} C_{ob} & 0 \end{bmatrix} \begin{bmatrix} x_{ob} \\ x_{\mathscr{N}} \end{bmatrix} \end{aligned} \tag{5.9}$$

And the implicit descriptions $\mathfrak{R}^{imp}(E, A, B, C)$ and $\mathfrak{R}^{imp}(E_{ob}, A_{ob}, B_{ob}, C_{ob})$ are externally equivalents (*c.f.* Bonilla & Malabre, 1995, Theorem 2.1). The point we want to enlighten here is that, since $E_{\mathscr{N}}$ is epic, there then exists $E^r_{\mathscr{N}}$ such that $E_{\mathscr{N}} E^r_{\mathscr{N}} = I$, which implies that all the homogeneous trajectories of (5.9), beginning at any initial condition $\begin{bmatrix} 0 \\ x_0 \end{bmatrix} \in \mathscr{N}$, $x_{\mathscr{N}}(t) = \exp\left(E^r_{\mathscr{N}} A_{\mathscr{N}} t\right) x_0$, always remain inside $\mathscr{N} \subset \mathscr{K}_C$. Thus, like in the classical state representations, they are called unobservable trajectories; and since $\mathscr{N}$ is the supremal (A, E) invariant subspace contained in $\mathscr{K}_C$ with this property, $\mathfrak{R}^{imp}(E_{ob}, A_{ob}, B_{ob}, C_{ob})$ is called the *observable part* of $\mathfrak{R}^{imp}(E, A, B, C)$.

5.1. Pole Assignment for a Reachable Implicit Description

Theorem 6. *(Bonilla & Malabre, 1993) Given an implicit system represented by (1.7), for every finite symmetric (with respect to the real line) set of complex numbers* Λ *of cardinality* $\dim(\mathscr{R}^*_{\mathscr{X}_d})$*, there exists a proportional and derivative descriptor feedback* $u = F_p x + F_d \mathrm{d}x/\mathrm{d}t$*, such that* $\sigma_f(A_{F_p}, E_{F_d}) = \Lambda$*, if and only if*

$$\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{X}_d \, , \tag{5.10}$$

$$\dim(E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B}) \geq \dim(\mathscr{V}^*_{\mathscr{X}_d}) \,. \tag{5.11}$$

Bonilla & Malabre (1993) named this property *external reachability*. In that paper, condition (5.11) is expressed in its equivalent form:

$$\dim(\mathscr{B}/(\mathscr{B} \cap E\mathscr{V}^*_{\mathscr{X}_d})) \geq \dim(\mathscr{V}^*_{\mathscr{X}_d} \cap \mathscr{K}_E) \,. \tag{5.12}$$

Let us note that the geometric condition (5.10) is the *reachability* condition of Frankowska (1990) (*c.f.* Theorem 4) and the geometric condition (5.11) is the *descriptor variable uniqueness* condition of Lebret (1991), namely the closed loop left invertibility property, which enables us to assign the poles by means of a proportional and derivative feedback.

Lemma 4. *(Lebret, 1991) There exists a proportional and derivative descriptor feedback $u = F_p x + F_d \mathrm{d}x/\mathrm{d}t + v$, such that the fed-back implicit representation $\mathfrak{R}^{imp}(E_{F_d}, A_{F_p}, B)$ satisfies $\mathrm{Ker}\,(\lambda E_{F_d} - A_{F_p}) = \{0\}$ iff (5.11) is satisfied.*

Let us also note that in the case of a strict singular system, the geometric condition (5.11) is translated to (*c.f.* (3.6), (3.4) and Fig. 1): $\dim(\mathscr{X}_{eq}) \geq \dim(\mathscr{X}_d)$. In other words, it is not possible to assign all the spectrum of an implicit system having one degree of freedom, as for example the ones considered in (Bonilla & Malabre, 2003).

We have the following Corollary of Theorem 6, proved in Appendix D.

Corollary 1. *Let the implicit representation (1.7) satisfy the geometric conditions (5.10) and (5.11). Then:*

1. *If $\mathscr{U}_{\mathcal{C}} = \{0\}$, the implicit representation (4.1) reduces to the following reachable state space representation ($\overline{\mathscr{B}}_1 = \mathrm{Im}\,\overline{B}_1$):*

$$\mathrm{d}x/\mathrm{d}t = \overline{A}_{1,1}x + \overline{B}_1 u \quad \textit{with} \quad \langle \overline{A}_{1,1} \mid \overline{\mathscr{B}}_1 \rangle = \mathscr{X}_d \,. \tag{5.13}$$

2. *If* $\mathscr{U}_{\mathcal{C}} \neq \{0\}$*, there exists a map* $\overline{V}_\ell : \mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E \to \mathscr{B}_{\mathcal{C}}$ *such that* $\operatorname{Ker} \overline{V}_\ell = \{0\}$*. Then, applying the proportional feedback*

$$u = \begin{bmatrix} 0 & 0 \\ -I_{\mathscr{U}_{\mathcal{C}}}^{-1}\overline{A}_{2,1} & -I_{\mathscr{U}_{\mathcal{C}}}^{-1}(\overline{A}_{2,2} + \overline{V}_\ell) \end{bmatrix} x + v \,, \tag{5.14}$$

we get

$$\begin{bmatrix} I_{\mathcal{C}} & 0 \\ 0 & 0 \end{bmatrix} \mathrm{d}x/\mathrm{d}t = \begin{bmatrix} \overline{A}_{1,1} & 0 \\ 0 & -I \end{bmatrix} x + \begin{bmatrix} \overline{B}_1 & \overline{A}_{1,2}\overline{V}_\ell^g I_{\mathscr{U}_{\mathcal{C}}} \\ 0 & \overline{V}_\ell^g I_{\mathscr{U}_{\mathcal{C}}} \end{bmatrix} v, \tag{5.15}$$

where $\overline{V}_\ell^g : \mathscr{B}_{\mathcal{C}} \to \mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E$ *is some left inverse of* $\overline{V}_\ell$*, and*

$$I_{\mathcal{C}}\mathscr{R}_{\mathcal{C}} = E\mathscr{R}^*_{\mathscr{X}_d} = \left\langle \overline{A}_{1,1} \,\middle|\, \overline{\mathscr{B}}_1 + \overline{A}_{1,2}(\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E) \right\rangle \quad \textit{and} \quad \mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E = \overline{V}_\ell^g I_{\mathscr{U}_{\mathcal{C}}} \mathscr{U}_{\mathcal{C}}. \tag{5.16}$$

Furthermore, applying the proportional and derivative feedback

$$u = \begin{bmatrix} 0 & 0 \\ -I_{\mathscr{U}_{\mathcal{C}}}^{-1}\overline{A}_{2,1} & -I_{\mathscr{U}_{\mathcal{C}}}^{-1}(\overline{A}_{2,2} + \overline{V}_\ell) \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & -I_{\mathscr{U}_{\mathcal{C}}}^{-1}\overline{V}_\ell \end{bmatrix} \mathrm{d}x/\mathrm{d}t + v \,, \tag{5.17}$$

we get

$$\mathrm{d}x/\mathrm{d}t = \begin{bmatrix} \overline{A}_{1,1} & \overline{A}_{1,2} \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} \overline{B}_1 & 0 \\ 0 & \overline{V}_\ell^g I_{\mathscr{U}_{\mathcal{C}}} \end{bmatrix} v, \tag{5.18}$$

with

$$\begin{aligned} &\left\langle \begin{bmatrix} \overline{A}_{1,1} & \overline{A}_{1,2} \\ 0 & 0 \end{bmatrix} \,\middle|\, \operatorname{Im} \begin{bmatrix} \overline{B}_1 & 0 \\ 0 & \overline{V}_\ell^g I_{\mathscr{U}_{\mathcal{C}}} \end{bmatrix} \right\rangle = \\ &\qquad \left\langle \overline{A}_{1,1} \,\middle|\, \overline{\mathscr{B}}_1 + \overline{A}_{1,2}(\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E) \right\rangle \oplus \mathscr{U}_{\mathcal{C}} = \mathscr{X}_d \,. \end{aligned} \tag{5.19}$$

From this Corollary, we realize that with a proportional feedback, we can only modify the finite spectrum of $\overline{A}_{1,1} = R\,\overline{A}|_{(\mathscr{R}^*_{\mathscr{X}_d}/\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E)}$, where $R : A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B} \to E\mathscr{R}^*_{\mathscr{X}_d}//\mathscr{B}_{\mathcal{C}}$ is the natural projection. To assign all the finite spectrum of $\overline{A}$, we need a proportional and derivative feedback.

5.2. Pole Assignment for a Reachable and Observable Implicit Description

In this section, we are going to consider the reachability of the observable part after feedback, of the implicit representation (5.8). For this, let us recall that the supremal (A, E, B)–invariant subspace contained in $\operatorname{Ker} C$, $\mathscr{V}^*$ $= \sup\{\mathscr{V} \subset \mathscr{K}_C |\ A\mathscr{V} \subset E\mathscr{V} + \operatorname{Im} B\}$, that characterizes the biggest part of a given implicit representation $\mathfrak{R}^{imp}(E,\ A,\ B,\ C)$, can be made unobservable by means of a suitable proportional and derivative descriptor feedback (*c.f.* the early Geometric Algorithms Section).

Given a proportional and derivative descriptor feedback $u = F_p^* x + F_d^* \mathrm{d}x/\mathrm{d}t$, where $(F_p^*, F_d^*) \in \mathbf{F}(\mathscr{V}^*)$, let us consider the *quotient implicit representation* $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$, where the linear applications E_*, A_*, B_* and C_* are the induced maps uniquely defined by

$$E_*\Phi = \Pi E_{F_d^*}, \quad A_*\Phi = \Pi A_{F_p^*}, \quad B_* = \Pi B \ \ , \text{ and } \ C = C_*\Phi \ , \tag{5.20}$$

where $\Phi : \mathscr{X}_d \to \mathscr{X}_d/\mathscr{V}^*$ and $\Pi : E\mathscr{X}_d \to E\mathscr{X}_d/E_{F_d^*}\mathscr{V}^*$ are the canonical projections. In Appendix E, we prove the following Theorem.[20]

Theorem 7. *Given an implicit system represented by (5.8), for every symmetric (with respect to the real line) set of complex numbers Λ of cardinality $\dim\left((\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{V}^*)/\mathscr{V}^*\right)$, there exists a proportional and derivative descriptor feedback $u = F_p^* x + F_d^* \mathrm{d}x/\mathrm{d}t + v$, with $(F_p^*, F_d^*) \in \mathbf{F}(\mathscr{V}^*)$, such that $\sigma_f(A_*, E_*) = \Lambda$, where E_* and A_* are the induced maps (5.20), if and only if:*

$$(\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{V}^*)/\mathscr{V}^* = \mathscr{X}_d/\mathscr{V}^*, \tag{5.21}$$

$$\dim\left((E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B})/(E\mathscr{V}^* + \mathscr{B})\right) + \dim(\mathscr{B}) \geq \dim\left(\mathscr{V}^*_{\mathscr{X}_d}/\mathscr{V}^*\right). \tag{5.22}$$

[20]For a related result for regular systems see Schumacher (1980).

Let us note that (5.22) is equivalent to:[21]

$$\dim\left(\mathscr{B}/(\mathscr{B}\cap E\mathscr{V}^*_{\mathscr{X}_d})\right) \geq \dim\left(\mathscr{V}^*_{\mathscr{X}_d}\cap\mathscr{K}_E\right) - \dim\left(\mathscr{V}^*\cap E^{-1}\mathscr{B}\right). \tag{5.23}$$

For the implicit representations (5.8), satisfying Theorem 7, we will say that they have the *externally reachable output dynamics property*[22]. Theorem 7 is important because it enables us to tackle systems having an internal variable structure (see for example Bonilla & Malabre (1991), Bonilla & Malabre (2003), and Bonilla & Malabre (2008)). Let us also note that the geometric condition (5.22) is the *descriptor variable uniqueness property* notion of Lebret (1991), namely the closed loop left invertibility property of the observable part of the system.

Lemma 5. *(Lebret, 1991) There exists a proportional and derivative descriptor feedback* $u = F_p x + F_d \mathrm{d}x/\mathrm{d}t + v$*, such that the fed-back implicit representation* $\mathfrak{R}^{imp}(E_{F_d}, A_{F_p}, B)$ *satisfies* $\mathrm{Ker}\left(\lambda E_{F_d} - A_{F_p}\right) \subset \mathscr{N}_{(F_p,F_d)}$ *iff (5.22) is satisfied.*

Let us finally note that, when comparing (5.22) with (5.11), we realize that Theorem 7 is indeed establishing the external reachabilty of the observable part after feedback. Also note that in the case $\mathscr{V}^* = \{0\}$, (5.22) and

[21] This equivalence follows from the equivalence between (5.11) and (5.12), and from the fact that $\mathscr{B}\cap E\mathscr{V}^* = E(\mathscr{V}^*\cap E^{-1}\mathscr{B})$ implies that $\dim\left(\mathscr{V}^*\cap E^{-1}\mathscr{B}\right) = \dim\left(\mathscr{V}^*\right) + \dim\left(\mathscr{B}\cap\mathrm{Im}\,E\right) - \dim\left(E\mathscr{V}^* + \mathscr{B}\cap\mathrm{Im}\,E\right)$.

[22] The *externally reachable output dynamics* notion is a simplification of the one of *reachable with output dynamics assignment* (see Bonilla *et al*, 1994, Definition 6).

(5.11) are the same; and in the case $\mathscr{V}^* = \mathscr{V}^*_{\mathscr{X}_d}$, we get the trivial condition $\dim(\mathscr{B}) \geq 0$.

Let us finish this Section with an academic example.

Academic Example. Let us consider a perturbed linear system represented by the state space representation, $\mathfrak{R}^{ss}(\overline{A},\, [\, \overline{B} \;\; \overline{S} \,]\, , \overline{C})$:

$$\mathrm{d}\bar{x}/\mathrm{d}t = \overline{A}\bar{x} + \begin{bmatrix} \overline{B} & \overline{S} \end{bmatrix} \begin{bmatrix} u \\ q \end{bmatrix} \quad \text{and} \quad y = \overline{C}\bar{x}, \tag{5.24}$$

where $q \in \mathscr{Q} \approx \mathbb{R}^{\eta}$, $u \in \mathscr{U} \approx \mathbb{R}^{m}$, $y \in \mathscr{Y} \approx \mathbb{R}^{p}$ and $\bar{x} \in \overline{\mathscr{X}} \approx \mathbb{R}^{\bar{n}}$, are the disturbance, the input, the output, and the state variables, respectively. We assume that the three following assumptions hold true:

[H1] $\mathrm{Ker}\, \overline{B} = \{0\}$ and $\mathrm{Ker}\, \overline{S} = \{0\}$,

[H2] $q(\cdot) \in \mathcal{C}^m(\mathbb{R}^+,\, \mathscr{Q})$, $q(t)$, $\mathrm{d}q(t)/\mathrm{d}t$, $\ldots$, $\mathrm{d}^m q(t)/\mathrm{d}t^m \in \mathcal{L}^{\infty}$, $\forall\ t \geq 0$,

[H3] q is a measured disturbance.

We want to solve the Disturbance Decoupling Problem with a PD Feedback (DDP-PDF).

Problem 3 (DDP-PDF). Under which conditions does there exist a proportional and derivative feedback $u = (\overline{F}_{p1} + \overline{F}_{d1}\mathrm{d}/\mathrm{d}t)\bar{x} + (\overline{F}_{p2} + \overline{F}_{d2}\mathrm{d}/\mathrm{d}t)q + v$, such that the closed-loop transfer function matrix between q and y is identically zero, and the finite spectrum of the observable part of the closed loop system is assigned at will.

For solving this problem, let us rewrite (5.24) in the descriptor form (5.8) with

$$E = \left[\begin{array}{c|c} I_{\bar{n}} & 0 \end{array}\right], \;\; A = \left[\begin{array}{c|c} \overline{A} & \overline{S} \end{array}\right], \;\; B = \left[\begin{array}{c} \overline{B} \end{array}\right], \;\; C = \left[\begin{array}{c|c} \overline{C} & 0 \end{array}\right], \tag{5.25}$$

where $x = \begin{bmatrix} \bar{x}^T & q^T \end{bmatrix}^T \in \mathscr{X}_d = \overline{\mathscr{X}} \oplus \mathscr{Q} \approx \mathbb{R}^{\bar{n}+\eta}$ and $\mathscr{X}_{eq} = \overline{\mathscr{X}} \approx \mathbb{R}^{\bar{n}}$. In this implicit representation, the perturbation q is acting as the free part of the descriptor variable x. Then from Theorem 7, the **DDP-PDF** is solvable if and only if the implicit representation (5.8) and (5.25) satisfies (5.21) and (5.22), namely if and only if both following conditions hold true (see Appendix F):

$$\langle \overline{A} \,|\, \mathrm{Im}\,[\,\overline{B}\;\overline{S}\,] \,\rangle + \overline{\mathscr{V}}^*_{[\overline{B}\;\overline{S}]} = \overline{\mathscr{X}}, \tag{5.26}$$

$$\dim\left(\overline{\mathscr{V}}^*_{[\overline{B}\;\overline{S}]} \cap \overline{\mathscr{B}}\right) \geq \dim\left(\frac{\mathrm{Im}\,\overline{S}}{\mathrm{Im}\,\overline{S} \cap \left(\overline{\mathscr{V}}^*_{[\overline{B}\;\overline{S}]} + \overline{\mathscr{B}}\right)}\right). \tag{5.27}$$

Let us consider for example: $\overline{A} = \mathrm{T}_u\,\{\underline{e}_3^2\}$, $\overline{S} = a\underline{e}_3^1 + b\underline{e}_3^2$, with $|a| + |b| \neq 0$, $\overline{B} = \underline{e}_3^3$ and $\overline{C} = (\underline{e}_3^1)^T$. We have for this case $\mathrm{Im}\,\overline{S} = \mathrm{span}\,\{a\underline{e}_3^1 + b\underline{e}_3^2\}$, $\overline{\mathscr{B}} = \mathrm{span}\,\{\underline{e}_3^3\}$, and $\mathrm{Im}\,[\,\overline{B}\;\overline{S}\,] = \mathrm{span}\,\{a\underline{e}_3^1 + b\underline{e}_3^2,\, \underline{e}_3^3\}$, then $\langle \overline{A} \,|\, \mathrm{Im}\,[\,\overline{B}\;\overline{S}\,] \,\rangle = \mathrm{span}\,\{\underline{e}_3^1,\, \underline{e}_3^2,\, \underline{e}_3^3\} = \overline{\mathscr{X}}$, $\overline{\mathscr{V}}^*_{[\overline{B}\;\overline{S}]} = \mathrm{span}\,\{a\underline{e}_3^2,\, \underline{e}_3^3\}$, $\overline{\mathscr{V}}^*_{[\overline{B}\;\overline{S}]} \cap \overline{\mathscr{B}} = \mathrm{span}\,\{\underline{e}_3^3\}$, and $\mathrm{Im}\,\overline{S} \cap \left(\overline{\mathscr{V}}^*_{[\overline{B}\;\overline{S}]} + \overline{\mathscr{B}}\right) = \{0\}$. Therefore (5.26) and (5.27) are satisfied, and the DDP-PDF has solution. Indeed, applying to (5.24) and (5.25) the PD feedback

$$u = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \mathrm{d}\bar{x}/\mathrm{d}t + \begin{bmatrix} -1/\tau & 0 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 1/\tau \end{bmatrix} v\,,$$

we obtain the closed loop system described by:

$$\tau \mathrm{d}y/\mathrm{d}t + y = v,\; \bar{x}_1 = y,\; \bar{x}_2 = \mathrm{d}y/\mathrm{d}t - aq, \;\text{ and }\; \bar{x}_3 = \mathrm{d}^2y/\mathrm{d}t^2 - a\mathrm{d}q/\mathrm{d}t - bq.$$

Let us note that $\overline{\mathscr{V}}^*_{\overline{B}} = \{0\}$, and that $\mathrm{Im}\,\overline{S} \cap \left(\overline{\mathscr{V}}^*_{\overline{B}} + \overline{\mathscr{B}}\right) = \{0\}$, so there is no purely proportional solutions (see for example Wonham (1985)).

6. CONCLUDING REMARKS

The notion of reachability introduced by Frankowska (1990) generalizes the property introduced by Yip & Sincovec (1981) in the regular case. Furthermore, Cobb (1984) indicates that this last property is consistent with that of Rosenbrock (1974) introduced in a purely structural framework. In the same paper, Cobb (1984) enlightens with time domain characterizations the difference between the reachability in the sense of Rosenbrock (1974) and the reachability in the sense of Verghese *et al* (1981) based, once again, on pure structural tools (Kronecker canonical forms and/or Smith canonical forms). In the regular case, for which the system can be decomposed into two parts, a finite or slow subsystem, and an infinite or fast subsystem, Cobb (1984) showed that Rosenbrock (1974) reachability is equivalent to the reachability of the finite part and controllability of the infinite part. He also showed that Verghese *et al* (1981) reachability is equivalent to the reachability of the finite part associated to the impulse controllability of the infinite part. The impulse controllability as defined by Cobb (1984), or the controllability of the infinite part in the sense of Verghese *et al* (1981) is not any more defined by the idea to reach a desired descriptor variable but by the ability of the system to generate a maximal class of impulses using piecewise smooth, non impulsive controls.

One can deduce from this analysis that if a regular system is reachable (reachability of the finite and controllability of the infinite part) in the sense of Cobb (1984), Yip & Sincovec (1981), Rosenbrock (1974) or Frankowska (1990) (the four notion are equivalent in this case) then any vector is a consistent initial condition in the sense of Geerts (1993). The converse implication is not true. In general, reachability is not a consequence of the fact that every vector of the descriptor space defines a consistent initial condition. The condition is necessary but not sufficient for reachability.

In this paper we have given a geometric interpretation of the implicit systems reachability Theorem of Frankowska (1990) and we have also found some interesting connections between the works (Frankowska, 1990) and (Geerts, 1993). The geometric interpretation has enabled us to have a better understanding of the existing mechanisms in the linear implicit systems reachability. For this, we have first interpreted the *viability* notion from a geometric point of view. We have next solved Problem 2, with Theorem 5, which is a generalization of Problem 1, solved with Theorem 1.

We have also studied the existing relationships, between the reachability property and the capability of the complete pole assignment ability. In Theorem 6, we have considered the pole assignment problem of a reachable implicit description, $\mathfrak{R}^{imp}(E, A, B)$; we have also shown in Corollary 1, that with a proportional feedback, we can only modify the spectrum of the restriction to $\mathscr{R}^*_{\mathscr{X}_d}/\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E$ in the domain and $E\mathscr{R}^*_{\mathscr{X}_d}$ in the co-domain; to assign all the spectrum, we need a proportional and derivative feedback. In Theorem 7, we have considered the pole assignment problem of a reachable and observable implicit description with output equation, $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$.

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Appendix A. Proof of Lemma 1

For the existence of such $F_{\mathcal{B}}$, $T_{\mathcal{B}}$ and $G_{\mathcal{B}}$, see for example Theorems 5.9 and 5.10 and Corollary 5.3 of Wonham (1985). Doing the change of state variable: $T_{\mathcal{B}}^{-1}x = \xi = \begin{bmatrix} \xi_1^T & \cdots & \xi_m^T \end{bmatrix}^T$, we obtain the following set of closed loop state space representations (see (1.1), (1.2), and (2.4)): $\mathrm{d}\xi_i/\mathrm{d}t = A_{\mathcal{B},i}\xi_i + b_{\mathcal{B},i}\mathrm{d}^{\kappa_i} f_i/\mathrm{d}t^{\kappa_i}$, $i \in \{1, \dots, m\}$, which solutions are (integrate by parts n_i times each solution):

$$\begin{aligned}
\xi_i(t) &= \exp\left(A_{\mathcal{B},i}t\right)\xi_i(0) + \textstyle\int_0^t \exp\left(A_{\mathcal{B},i}(t-\tau)\right) b_{\mathcal{B},i}\frac{\mathrm{d}^{\kappa_i} f_i(\tau)}{\mathrm{d}\tau^{\kappa_i}}\mathrm{d}\tau \\
&= \exp\left(A_{\mathcal{B},i}t\right)\left(\xi_i(0) - \sum_{j=0}^{\kappa_i-1} A_{\mathcal{B},i}^j b_{\mathcal{B},i}\frac{\mathrm{d}^{\kappa_i-(j+1)} f_i(0)}{\mathrm{d}t^{\kappa_i-(j+1)}}\right) + \sum_{j=0}^{\kappa_i-1} A_{\mathcal{B},i}^j b_{\mathcal{B},i}\frac{\mathrm{d}^{\kappa_i-(j+1)} f_i(t)}{\mathrm{d}t^{\kappa_i-(j+1)}} \\
&= \exp\left(A_{\mathcal{B},i}t\right)\left(\xi_i(0) - \mathcal{R}_{\left[A_{\mathcal{B}_i},\, b_{\mathcal{B}_i}\right]}\overline{w}_i(0)\right) + \mathcal{R}_{\left[A_{\mathcal{B}_i},\, b_{\mathcal{B}_i}\right]}\overline{w}_i(t), \quad i \in \{1, \dots, m\}, \\
\xi(t) &= \exp\left(A_{\mathcal{B}}t\right)\left(\xi(0) - \mathcal{R}_{[A_{\mathcal{B}},\, B_{\mathcal{B}}]}\overline{w}(0)\right) + \mathcal{R}_{[A_{\mathcal{B}},\, B_{\mathcal{B}}]}\overline{w}(t), \\
x(t) &= \exp\left((A + BF_{\mathcal{B}})t\right)\left(x(0) - T_{\mathcal{B}}\mathcal{R}_{[A_{\mathcal{B}},\, B_{\mathcal{B}}]}\overline{w}(0)\right) + T_{\mathcal{B}}\mathcal{R}_{[A_{\mathcal{B}},\, B_{\mathcal{B}}]}\overline{w}(t).
\end{aligned} \tag{A1}$$

Therefore, (A1), (2.2) and (2.3) imply (2.5) and (2.6). $\square$

Appendix B. Proof of Lemma 2

Let us first compute $\det\left(X_{(i,1)}(t)\right)$, for $\kappa_i \geq 2$. For this, we first do the decomposition $X_{(i,1)}(t) = D_{i,\ell}(t)\widetilde{X}_{(i,\kappa_i+1)}D_{i,r}(t)$, $i \in \{1, \dots, m\}$, where $D_{i,\ell}(t) = \mathrm{DM}\left\{\frac{t}{(\kappa_i+1)!}, \cdots, t^{\kappa_i+1}/(2\kappa_i+1)!\right\}$, $D_{i,r}(t) = \mathrm{DM}\left\{(2\kappa_i+1)!t^{\kappa_i}, \cdots, (\kappa_i+1)!\right\}$ and

$$\widetilde{X}_{(i,\kappa_i+1)} = \begin{bmatrix} (\kappa_i+1)!/(\kappa_i+1)! & \cdots & (\kappa_i+1)!/1! \\ \vdots & \cdots & \vdots \\ (2\kappa_i+1)!/(2\kappa_i+1)! & \cdots & (2\kappa_i+1)!/(\kappa_i+1)! \end{bmatrix}. \tag{B1}$$

Defining the following column elementary matrices: $T_{i,1} = \left[\ \underline{e}^1_{\kappa_i+1}\ \middle|\ (\underline{e}^2_{\kappa_i+1} - \underline{e}^1_{\kappa_i+1})\ \middle|\ (\underline{e}^3_{\kappa_i+1} - \kappa_i\underline{e}^2_{\kappa_i+1})\ \middle|\ \cdots\ \middle|\ (\underline{e}^{\kappa_i+1}_{\kappa_i+1} - 2\underline{e}^{\kappa_i}_{\kappa_i+1})\ \right]$, $T_{i,2} = \left[\ \underline{e}^1_{\kappa_i+1}\ \middle|\ \underline{e}^2_{\kappa_i+1}\ \middle|\ (\underline{e}^3_{\kappa_i+1} - (\kappa_i+2)\underline{e}^2_{\kappa_i+1})\ \middle|\ (\underline{e}^4_{\kappa_i+1} - (\kappa_i+1)\underline{e}^3_{\kappa_i+1})\ \middle|\ \cdots\ \middle|\ (\underline{e}^{\kappa_i+1}_{\kappa_i+1} - 4\underline{e}^{\kappa_i}_{\kappa_i+1})\ \right]$, $\ldots$, $T_{i,\kappa_i-1} = \left[\ \underline{e}^1_{\kappa_i+1}\ \middle|\ \cdots\ \middle|\ \underline{e}^{\kappa_i-1}_{\kappa_i+1}\ \middle|\ (\underline{e}^{\kappa_i}_{\kappa_i+1} - (2\kappa_i-1)\underline{e}^{\kappa_i-1}_{\kappa_i+1})\ \middle|\ (\underline{e}^{\kappa_i+1}_{\kappa_i+1} - (2\kappa_i-2)\underline{e}^{\kappa_i}_{\kappa_i+1})\right]$, $T_{i,\kappa_i} = \left[\ \underline{e}^1_{\kappa_i+1}\ \middle|\ \cdots\ \middle|\ \underline{e}^{\kappa_i}_{\kappa_i+1}\ \middle|\ (\underline{e}^{\kappa_i+1}_{\kappa_i+1} - (2\kappa_i)\underline{e}^{\kappa_i}_{\kappa_i+1})\ \right]$, we then get:

$$\widetilde{X}_{(i,\kappa_i+1)}\prod_{j=1}^{\kappa_i} T_{i,j} = \begin{bmatrix} 0! & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1! & 0 & \cdots & 0 & 0 & 0 \\ . & . & . & \cdots & . & . & . \\ 1 & \prod_{\ell=\kappa_i}^{\kappa_i}\ell & \prod_{\ell=\kappa_i-1}^{\kappa_i}\ell & \cdots & \prod_{\ell=3}^{\kappa_i}\ell & \prod_{\ell=2}^{\kappa_i}\ell & \kappa_i! \end{bmatrix}. \tag{B2}$$

which implies (2.10.b).

For the second statement, let us first note that (2.7)-(2.9), (2.2) and (2.3), imply:

$$X_{(i,1)}(t)\mathfrak{a}_{i,1} + X_{(i,0)}(t)\mathfrak{a}_{i,0} = \begin{bmatrix} \mathrm{d}^{\kappa_i} f_i(t)/\mathrm{d}t^{\kappa_i} \\ \overline{w}_i(t) \end{bmatrix}, \tag{B3}$$

with $i \in \{1, \ldots, m\}$. And let us next note that (2.2) and (2.3) are equivalent to:

$$\frac{\mathrm{d}^{\kappa_i} f(t_j)}{\mathrm{d}t^{\kappa_i}} = (\underline{e}^i_m)^T G_{\mathcal{B}}^{-1}(u(t_j) - F_{\mathcal{B}}x(t_j)) \ \text{ and } \ \overline{w}_i(t_j) = \mathcal{R}^{-1}_{[A_{\mathcal{B}},\, B_{\mathcal{B}}]} P_i T_{\mathcal{B}}^{-1} x(t_j), \tag{B4}$$

with $i \in \{1, \ldots, m\}$ and $j \in \{0, 1\}$, and where $t_0 = 0$, $u(t_0) = u_0$, $u(t_1) = u_1$, $x(t_0) = x_0$, and $x(t_1) = x_1$. Therefore, (2.8)-(2.10), (B3) and (2.11) imply (B4). $\square$

Appendix C. Proof of Lemma 3

Let us first prove that the spaces $\mathscr{X}_d$, $\mathscr{X}_{eq}$ and $\mathscr{U}$ can be decomposed as follows:

$$\begin{gathered} \mathscr{X}_d = \mathscr{R}_{\mathcal{C}} \oplus (\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E) \oplus \mathscr{X}_2 \oplus \mathscr{X}_1, \quad \mathscr{X}_{eq} = E\mathscr{R}^*_{\mathscr{X}_d} \oplus \mathscr{B}_{\mathcal{C}} \oplus E\mathscr{X}_2 \oplus A\mathscr{X}_1, \\ \mathscr{U} = B^{-1}E\mathscr{R}^*_{\mathscr{X}_d} \oplus B^{-1}\mathscr{B}_{\mathcal{C}}, \end{gathered} \tag{C1}$$

where:

$$\begin{gathered}\mathscr{X}_d = \mathscr{V}^*_{\mathscr{X}_d} \oplus \mathscr{X}_1, \quad \mathscr{V}^*_{\mathscr{X}_d} = \mathscr{R}^*_{\mathscr{X}_d} \oplus \mathscr{X}_2, \quad \mathscr{R}^*_{\mathscr{X}_d} = \mathscr{R}_{\mathcal{C}} \oplus (\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E),\\ \mathscr{X}_{eq} = (E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B}) \oplus A\mathscr{X}_1, \quad E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B} = (A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}) \oplus E\mathscr{X}_2,\\ A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B} = E\mathscr{R}^*_{\mathscr{X}_d} \oplus \mathscr{B}_{\mathcal{C}}, \quad \mathscr{B} = (\mathscr{B} \cap E\mathscr{R}^*_{\mathscr{X}_d}) \oplus \mathscr{B}_{\mathcal{C}}.\end{gathered} \tag{C2}$$

And also:

$$\mathscr{R}_{\mathcal{C}} \approx E\mathscr{R}^*_{\mathscr{X}_d}, \quad \mathscr{X}_2 \approx E\mathscr{X}_2, \quad \mathscr{X}_1 \approx A\mathscr{X}_1, \quad \mathscr{B}_{\mathcal{C}} \approx B^{-1}\mathscr{B}_{\mathcal{C}} = \mathscr{U}_{\mathcal{C}}, \tag{C3}$$

$$\mathscr{V}^*_{\mathscr{X}_d} \cap \mathscr{K}_E = \mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E, \quad \mathscr{B} \cap E\mathscr{V}^*_{\mathscr{X}_d} = \mathscr{B} \cap E\mathscr{R}^*_{\mathscr{X}_d}. \tag{C4}$$

1. From (1.8), (ALG–S) and (ALG–V), we get:

$$\mathscr{V}^*_{\mathscr{X}_d} = A^{-1}(E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B}) \ \text{ and } \ \mathscr{R}^*_{\mathscr{X}_d} = \mathscr{V}^*_{\mathscr{X}_d} \cap E^{-1}(A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}). \tag{C5}$$

Indeed: $\mathscr{V}^*_{\mathscr{X}_d} \cap E^{-1}\left(A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}\right) = \mathscr{V}^*_{\mathscr{X}_d} \cap E^{-1}\big(A\left(\left(A^{-1}\left(E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B}\right)\right) \cap \mathscr{S}^*_{\mathscr{X}_d}\right) + \mathscr{B}\big)$ $= \mathscr{V}^*_{\mathscr{X}_d} \cap E^{-1}\big(\left(E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B}\right) \cap A\mathscr{S}^*_{\mathscr{X}_d} + \mathscr{B}\big) = \mathscr{V}^*_{\mathscr{X}_d} \cap E^{-1}\big(\left(E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B}\right) \cap \left(A\mathscr{S}^*_{\mathscr{X}_d} + \mathscr{B}\right)\big)$ $= \mathscr{V}^*_{\mathscr{X}_d} \cap \left(\mathscr{V}^*_{\mathscr{X}_d} + E^{-1}\mathscr{B}\right) \cap \mathscr{S}^*_{\mathscr{X}_d} = \mathscr{V}^*_{\mathscr{X}_d} \cap \mathscr{S}^*_{\mathscr{X}_d} = \mathscr{R}^*_{\mathscr{X}_d}$ (see also Özçaldiran, 1985, Malabre, 1987). From (C5) and Result 1, we get:

$$\begin{gathered}\mathscr{X}_d = \mathscr{V}^*_{\mathscr{X}_d} \oplus \mathscr{X}_1, \quad \mathscr{X}_{eq} = (E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B}) \oplus A\mathscr{X}_1,\\ \mathscr{V}^*_{\mathscr{X}_d} = \mathscr{R}^*_{\mathscr{X}_d} \oplus \mathscr{X}_2, \quad \mathscr{R}^*_{\mathscr{X}_d} = \mathscr{R}_{\mathcal{C}} \oplus (\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E).\end{gathered} \tag{C6}$$

$$E\mathscr{R}^*_{\mathscr{X}_d} = E\mathscr{V}^*_{\mathscr{X}_d} \cap (A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}) \ \text{ and } \ A\mathscr{R}^*_{\mathscr{X}_d} \subset A\mathscr{V}^*_{\mathscr{X}_d} \subset E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B}. \tag{C7}$$

2. From (C5.b), we get (C4.a), which implies together with (C6.c):

$$E\mathscr{V}^*_{\mathscr{X}_d} = E\mathscr{R}^*_{\mathscr{X}_d} \oplus E\mathscr{X}_2. \tag{C8}$$

Indeed, the direct sum comes from the fact that $\mathscr{X}_2 \cap \mathscr{K}_E \subset \mathscr{V}^*_{\mathscr{X}_d} \cap \mathscr{K}_E = \mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E$ implies that $(\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{X}_2) \cap \mathscr{K}_E = (\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{X}_2) \cap (\mathscr{V}^*_{\mathscr{X}_d} \cap \mathscr{K}_E) = (\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{X}_2)$ $\cap\ (\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E) = \mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E = \mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E + \mathscr{X}_2 \cap \mathscr{K}_E$.

Moreover, since: $\mathscr{X}_2 \cap \mathscr{K}_E = (\mathscr{X}_2 \cap \mathscr{V}^*_{\mathscr{X}_d}) \cap \mathscr{K}_E = \mathscr{X}_2 \cap (\mathscr{V}^*_{\mathscr{X}_d} \cap \mathscr{K}_E) = \mathscr{X}_2$ $\cap\ (\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E) = (\mathscr{X}_2 \cap \mathscr{R}^*_{\mathscr{X}_d}) \cap \mathscr{K}_E = \{0\}$, we get: $\dim(E\mathscr{X}_2) = \dim(\mathscr{X}_2)$, thus (C3.b) follows.

3. From (C8) and (C7), we get:

$$E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B} = (E\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}) \oplus E\mathscr{X}_2. \tag{C9}$$

Indeed, since: $\{0\} = (E\mathscr{R}^*_{\mathscr{X}_d}) \cap (E\mathscr{X}_2) = E\mathscr{V}^*_{\mathscr{X}_d} \cap (A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}) \cap (E\mathscr{X}_2) = (A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}) \cap (E\mathscr{X}_2)$, we get: $E\mathscr{X}_2 \cap (E\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}) \subset E\mathscr{X}_2 \cap (A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}) = \{0\}$. Moreover, (C9), (C7) and (C8) imply:

$$\begin{aligned} E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B} &= (E\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}) \oplus E\mathscr{X}_2 = (E\mathscr{V}^*_{\mathscr{X}_d} \cap (A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}) + \mathscr{B}) \oplus E\mathscr{X}_2 \\ &= ((E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B}) \cap (A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B})) \oplus E\mathscr{X}_2 = (A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}) \oplus E\mathscr{X}_2. \end{aligned} \tag{C10}$$

4. From (C7.a) and (3.9), there exist subspaces, $\mathscr{W}_{\mathcal{C}}$ and $\mathscr{B}_{\mathcal{C}}$, such that:

$$A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B} = E\mathscr{R}^*_{\mathscr{X}_d} \oplus \mathscr{W}_{\mathcal{C}}, \quad \mathscr{B} = ((E\mathscr{R}^*_{\mathscr{X}_d}) \cap \mathscr{B}) \oplus \mathscr{B}_{\mathcal{C}}, \quad \mathscr{W}_{\mathcal{C}} \supset \mathscr{B}_{\mathcal{C}} \tag{C11}$$

From (C8), (C10), and (C11), we get: $E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B} = (E\mathscr{R}^*_{\mathscr{X}_d} \oplus E\mathscr{X}_2) + \mathscr{B} = E\mathscr{R}^*_{\mathscr{X}_d} \oplus \mathscr{B}_{\mathcal{C}} \oplus E\mathscr{X}_2 = (A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}) \oplus E\mathscr{X}_2$, that is to say: $E\mathscr{R}^*_{\mathscr{X}_d} \oplus \mathscr{W}_{\mathcal{C}} = A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B} \approx E\mathscr{R}^*_{\mathscr{X}_d} \oplus \mathscr{B}_{\mathcal{C}}$. Hence:

$$\mathscr{W}_{\mathcal{C}} = \mathscr{B}_{\mathcal{C}} \tag{C12}$$

5. From the geometric decompositions (C6), (C10), (C11), and (C12), the subspaces $\mathscr{X}_d$, $\mathscr{X}_{eq}$, and $\mathscr{U}$ take the form (C1)-(C2).

6. From (C2.c,a) and since: $\operatorname{Ker} A \subset \mathscr{V}^*_{\mathscr{X}_d}$ and $\operatorname{Ker} B = \{0\}$, we get (C3.a,c,d).

7. To prove (C4.b), note first that (C8) and (C9) imply $\mathscr{B} \cap E\mathscr{V}^*_{\mathscr{X}_d} = \mathscr{B} \cap (E\mathscr{R}^*_{\mathscr{X}_d} + E\mathscr{X}_2)$ and $(E\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}) \cap E\mathscr{X}_2 = \{0\}$. Let $x \in \mathscr{B} \cap (E\mathscr{R}^*_{\mathscr{X}_d} + E\mathscr{X}_2)$, there then exist $z \in E\mathscr{R}^*_{\mathscr{X}_d}$, $y \in E\mathscr{X}_2$, and $b \in \mathscr{B}$ such that $x = z + y = b$, which implies $y = b - z \in (E\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}) \cap E\mathscr{X}_2 = \{0\}$, *i.e.* $x \in \mathscr{B} \cap E\mathscr{R}^*_{\mathscr{X}_d}$. Therefore: $\mathscr{B} \cap E\mathscr{V}^*_{\mathscr{X}_d} = \mathscr{B} \cap (E\mathscr{R}^*_{\mathscr{X}_d} + E\mathscr{X}_2) \subset \mathscr{B} \cap E\mathscr{R}^*_{\mathscr{X}_d} \subset \mathscr{B} \cap E\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B} \cap E\mathscr{X}_2 \subset \mathscr{B} \cap (E\mathscr{R}^*_{\mathscr{X}_d} + E\mathscr{X}_2) = \mathscr{B} \cap E\mathscr{V}^*_{\mathscr{X}_d}$.

Let us next note that under the geometric decompositions, (C1)-(C3), the implicit representation (1.7) takes the following form (recall (3.7) and (3.8)):

$$\begin{bmatrix} \overline{E} & 0 & * \\ 0 & I_2 & * \\ 0 & 0 & \overline{X}_{\rho-1} \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} x = \begin{bmatrix} \overline{A} & \widehat{A} & 0 \\ 0 & \widehat{A}_3 & 0 \\ 0 & 0 & I_1 \end{bmatrix} x + \begin{bmatrix} \overline{B} \\ 0 \\ 0 \end{bmatrix} u, \tag{C13}$$

where $I_2 : \mathscr{X}_2 \leftrightarrow E\mathscr{X}_2$ is an isomorphism, and the matrices $\overline{E}$, $\overline{A}$ and $\overline{B}$, are the ones shown in (4.1). Then, when $\mathscr{R}_{\mathscr{X}_d} = \mathscr{X}_d$, we get (4.1). □

Appendix D. Proof of Corollary 1

Let us first note that (5.10) implies that the implicit representation (C13) is only composed by the linear transformations (4.1).

Let us next note that Lemma 3 and (5.11) imply that (see (C2)-(C4)):

$$\mathscr{V}^*_{\mathscr{X}_d} \cap \mathscr{K}_E = \mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E \quad \text{and} \quad \mathscr{B}/(\mathscr{B} \cap E\mathscr{V}^*_{\mathscr{X}_d}) = \mathscr{B}/(\mathscr{B} \cap E\mathscr{R}^*_{\mathscr{X}_d}) \approx \mathscr{B}_{\mathcal{C}} \approx \mathscr{U}_{\mathcal{C}}\,, \tag{D1}$$

$$\dim(A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}) \geq \dim(\mathscr{R}^*_{\mathscr{X}_d})\,. \tag{D2}$$

Case 1. If $\mathscr{U}_{\mathcal{C}} = \{0\}$, then (5.12) and (D1) imply: $\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E = \{0\}$. Thus, the blocks $\overline{A}_{1,2}$, $\overline{A}_{2,1}$, $\overline{A}_{2,2}$, and $I_{\mathscr{U}_c}$ actually disappear from (4.1), corresponding to 0 row and 0 column. Moreover $\overline{B}_1 \neq 0$, because the pair $(\overline{A}_{1,1},\ [\overline{A}_{1,2}\ \overline{B}_1])$ is reachable (see (4.2)). Namely, we get (5.13).

Case 2. The existence of $\overline{V}_\ell$ is implied by (D2). From (5.14) and (4.1), we get (5.15). From (5.15) and (4.2), we get (5.16). From (5.17) and (4.1), we get (5.18). From (5.18) and (4.2), we get (5.19). □

Appendix E. Proof of Theorem 7

Let us first propose a PD descriptor feedback $u = F_p^* x + F_d^* \mathrm{d}x/\mathrm{d}t + v$, where the pair of linear transformations (F_p, F_d) is chosen such that:

$$(F_p^*, F_d^*) \in \mathbf{F}(\mathscr{V}^*) \text{ and } \mathscr{B} \cap E_{F_d^*}\mathscr{V}^* = \{0\}\,. \tag{E1}$$

Let us next, consider the *quotient implicit representation* $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$ defined by (5.20). Let us note that $\operatorname{Ker} B_* = B^{-1}E_{F_d^*}\mathscr{V}^* \approx \mathscr{B} \cap E_{F_d^*}\mathscr{V}^*$ implies $\operatorname{Ker} B_* = \{0\}$, and that $\Phi\mathscr{N}_{(F_p^*, F_d^*)} = \Phi \sup\left\{\mathscr{V} \subset \operatorname{Ker} C \,|\, A_{F_p^*}\mathscr{V} \subset E_{F_d^*}\mathscr{V}\right\} = \Phi\mathscr{V}^* = \{0\}$ implies the observability of the *quotient implicit representation* $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$. The proof of Theorem 7 is done in 4 steps:

i) $\mathfrak{R}^{imp}(E_{F_d^*}, A_{F_p^*}, B, C)$ *is externally equivalent to* $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$. This fact follows from (Theorem 2.1, Bonilla & Malabre, 1995), which states, among others, the *external equivalency* between $\mathfrak{R}^{imp}(E_{F_d^*}, A_{F_p^*}, B, C)$ and $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$ (see also Kuijper & Schumacher, 1991).

ii) $\mathscr{V}^*_{\mathscr{X}_d/\mathscr{V}^*} = \Phi\mathscr{V}^*_{\mathscr{X}_d}$ *and* $\mathscr{S}^*_{\mathscr{X}_d/\mathscr{V}^*} = \Phi\mathscr{S}^*_{\mathscr{X}_d}$. For the case of the *quotient implicit representation* $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$ the corresponding algorithms (ALG–V) and (ALG–S), for computing $\mathscr{V}^*_{\mathscr{X}_d/\mathscr{V}^*}$ and $\mathscr{S}^*_{\mathscr{X}_d/\mathscr{V}^*}$, take the following form:

$$\begin{aligned} \mathscr{V}^0_{\mathscr{X}_d/\mathscr{V}^*} &= \mathscr{X}_d/\mathscr{V}_d^* \;, \quad \mathscr{V}^{\mu+1}_{\mathscr{X}_d/\mathscr{V}^*} = A_*^{-1}(E_*\mathscr{V}^{\mu}_{\mathscr{X}_d/\mathscr{V}^*} + \mathscr{B}_*), \\ \mathscr{S}^0_{\mathscr{X}_d/\mathscr{V}^*} &= \{0\} \;, \quad \mathscr{S}^{\mu+1}_{\mathscr{X}_d/\mathscr{V}^*} = E_*^{-1}(A_*\mathscr{S}^{\mu}_{\mathscr{X}_d/\mathscr{V}^*} + \mathscr{B}_*). \end{aligned} \tag{E2}$$

It is clear that: $\mathscr{V}^0_{\mathscr{X}_d/\mathscr{V}^*} = \Phi\mathscr{V}^0_{\mathscr{X}_d}$ and $\mathscr{S}^0_{\mathscr{X}_d/\mathscr{V}^*} = \Phi\mathscr{S}^0_{\mathscr{X}_d}$. Let us assume that: $\mathscr{V}^\mu_{\mathscr{X}_d/\mathscr{V}^*} = \Phi\mathscr{V}^\mu_{\mathscr{X}_d}$ and $\mathscr{S}^\mu_{\mathscr{X}_d/\mathscr{V}^*} = \Phi\mathscr{S}^\mu_{\mathscr{X}_d}$, then from (E2) and from (5.20), we get: $\mathscr{V}^{\mu+1}_{\mathscr{X}_d/\mathscr{V}^*} = (\Phi\mathscr{X}_d)\cap A_*^{-1}\left(E_*\Phi\mathscr{V}^\mu_{\mathscr{X}_d} + \Pi\mathscr{B}\right) = \Phi\Phi^{-1}A_*^{-1}\Pi\left(E_{F_d^*}\mathscr{V}^\mu_{\mathscr{X}_d} + \mathscr{B}\right) = \Phi A^{-1}_{F_d^*}\Pi^{-1}\Pi\left(E_{F_d^*}\mathscr{V}^\mu_{\mathscr{X}_d} + \mathscr{B}\right) = \Phi A^{-1}_{F_d^*}\left(E_{F_d^*}\mathscr{V}^\mu_{\mathscr{X}_d} + \mathscr{B} + E_{F_d^*}\mathscr{V}^*\right) = \Phi\Big(\mathscr{X}_d\cap A^{-1}_{F_d^*}\big(E_{F_d^*}\mathscr{V}^\mu_{\mathscr{X}_d} + \mathscr{B}\big)\Big) = \Phi\mathscr{V}^{\mu+1}_{\mathscr{X}_d}$, and $\mathscr{S}^{\mu+1}_{\mathscr{X}_d/\mathscr{V}^*} = (\Phi\mathscr{X}_d)\cap E_*^{-1}\left(A_*\left((\Phi\mathscr{X}_d)\cap(\Phi\mathscr{S}^\mu_{\mathscr{X}_d})\right) + \Pi\mathscr{B}\right) = \Phi\Phi^{-1}E_*^{-1}\Pi\left(A_{F_d^*}\left(\mathscr{X}_d\cap\mathscr{S}^\mu_{\mathscr{X}_d}\right) + \mathscr{B}\right) = \Phi E^{-1}_{F_d^*}\Pi^{-1}\Pi\left(A_{F_d^*}\left(\mathscr{X}_d\cap\mathscr{S}^\mu_{\mathscr{X}_d}\right) + \mathscr{B}\right) = \Phi E^{-1}_{F_d^*}\left(A_{F_d^*}\left(\mathscr{X}_d\cap\mathscr{S}^\mu_{\mathscr{X}_d}\right) + \mathscr{B} + E_{F_d^*}\mathscr{V}^*\right) = \Phi E^{-1}_{F_d^*}\left(A_{F_d^*}\left(\mathscr{X}_d\cap\mathscr{S}^\mu_{\mathscr{X}_d}\right) + \mathscr{B}\right) + \Phi\mathrm{Ker}\,E_{F_d^*} = \Phi\Big(\mathscr{X}_d\cap E^{-1}_{F_d^*}\left(A_{F_d^*}\left(\mathscr{X}_d\cap\mathscr{S}^\mu_{\mathscr{X}_d}\right) + \mathscr{B}\right)\Big) = \Phi\mathscr{S}^{\mu+1}_{\mathscr{X}_d}$.

iii) If (5.21) and (5.23) are satisfied, then $\mathfrak{R}^{imp}(E_*,\, A_*, B_*, C_*)$ satisfies Theorem 6. Since: $\left(\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{S}^*_{\mathscr{X}_d}\right)\cap\mathrm{Ker}\,\Phi = \left(\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{S}^*_{\mathscr{X}_d}\right)\cap\mathscr{V}^* = \mathscr{V}^* = \mathscr{V}^*_{\mathscr{X}_d}\cap\mathscr{V}^* + \mathscr{S}^*_{\mathscr{X}_d}\cap\mathscr{V}^*$, we get from (5.21): $\mathscr{R}^*_{\mathscr{X}_d/\mathscr{V}^*} = \mathscr{V}^*_{\mathscr{X}_d/\mathscr{V}^*}\cap\mathscr{S}^*_{\mathscr{X}_d/\mathscr{V}^*} = \Phi\mathscr{V}^*_{\mathscr{X}_d}\cap\Phi\mathscr{S}^*_{\mathscr{X}_d} = \Phi\left(\mathscr{V}^*_{\mathscr{X}_d}\cap\mathscr{S}^*_{\mathscr{X}_d}\right) = \Phi\mathscr{R}^*_{\mathscr{X}_d} = \Phi\left(\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{V}^*\right) = \Phi\mathscr{X}_d = \mathscr{X}_d/\mathscr{V}^*$, which is the first condition of Theorem 6. On the other hand, since for any $F_d^*:\mathscr{X}_d\to\mathscr{U}$, $E^{-1}\mathscr{B} = E^{-1}_{F_d^*}\mathscr{B}$, we have: $\dim(\mathscr{K}_E) + \dim(\mathrm{Im}\,E\cap\mathscr{B}) = \dim\left(E^{-1}_{F_d^*}\mathscr{B}\right)$, which together with (5.23) imply:[23]

$$\dim(\mathscr{B}) \geq \dim\left(E^{-1}_{F_d^*}\mathscr{B}\Big/\left(\mathscr{V}^*\cap E^{-1}_{F_d^*}\mathscr{B}\right)\right) = \dim\left(\Phi E^{-1}_{F_d^*}\mathscr{B}\right) = \dim\left(E_*^{-1}\mathscr{B}_*\right),$$

then: $\dim(\mathscr{B}_*) = \dim(\Pi\mathscr{B}) \geq \dim\left(E_*^{-1}\mathscr{B}_*\right) - \dim(\mathscr{B}\cap\mathrm{Ker}\,\Pi) = \dim\left(E_*^{-1}\mathscr{B}_*\right) - \dim(B\mathrm{Ker}\,B_*) = \dim\left(E_*^{-1}\mathscr{B}_*\right) - \dim(\mathrm{Ker}\,B_*) = \dim\left(E_*^{-1}\mathscr{B}_*\right)$, that is to say: $\dim(\mathscr{B}_*/(\mathscr{B}_*\cap\mathrm{Im}\,E_*)) \geq \dim(\mathscr{K}_{E_*})$, which is the second condition[24] of Theorem 6.

iv) If $\mathfrak{R}^{imp}(E_*,\, A_*, B_*, C_*)$ satisfies Theorem 6, then (5.21) and (5.23) are satisfied. From the first condition of Theorem 6, we have: $\mathscr{X}_d/\mathscr{V}^* = \mathscr{R}^*_{\mathscr{X}_d/\mathscr{V}^*} =$

[23] Note that: $\mathscr{X}_d = \mathscr{R}^*_{\mathscr{X}_d} + \mathscr{V}^* \subset \mathscr{V}^*_{\mathscr{X}_d} \subset \mathscr{X}_d$, and recall (5.20).

[24] Note that: $\mathscr{X}_d/\mathscr{V}^* = \mathscr{R}^*_{\mathscr{X}_d/\mathscr{V}^*} \subset \mathscr{V}^*_{\mathscr{X}_d/\mathscr{V}^*} \subset \mathscr{X}_d/\mathscr{V}^*$.

$(\Phi\mathscr{V}^*_{\mathscr{X}_d}) \cap (\Phi\mathscr{S}^*_{\mathscr{X}_d})$, which implies: $\mathscr{X}_d = \mathscr{V}^*_{\mathscr{X}_d} \cap (\mathscr{S}^*_{\mathscr{X}_d} + \mathscr{V}^*) = \mathscr{V}^*_{\mathscr{X}_d} \cap \mathscr{S}^*_{\mathscr{X}_d} + \mathscr{V}^* = \mathscr{R}^*_{\mathscr{X}_d} + \mathscr{V}^*$, which is the first condition of Theorem 7. From the second condition of Theorem 6, we have:[24] $\dim(\Pi\mathscr{B}) = \dim(\mathscr{B}_*) \geq \dim(\mathscr{K}_{E_*}) + \dim(\mathscr{B}_* \cap \operatorname{Im} E_*) = \dim\left(E_*^{-1}\mathscr{B}_*\right) = \dim\left(E_*^{-1}\Pi\mathscr{B}\right) = \dim\left(\Phi E^{-1}_{F_d^*}\,\mathscr{B}\right)$. Then (recall (E1)): $\dim(\mathscr{B}) \geq \dim\left(\Phi E^{-1}_{F_d^*}\mathscr{B}\right) + \dim(\mathscr{B} \cap \operatorname{Ker}\Pi) = \dim\left(\Phi E^{-1}\mathscr{B}\right) + \dim\left(\mathscr{B} \cap E_{F_d^*}\mathscr{V}^*\right) = \dim\left(E^{-1}\mathscr{B}\right) - \dim\left(\mathscr{V}^* \cap E^{-1}\mathscr{B}\right) = \dim(\mathscr{K}_E) + \dim(\mathscr{B} \cap \operatorname{Im} E) - \dim\left(\mathscr{V}^* \cap E^{-1}\mathscr{B}\right)$, which is the second condition[23] of Theorem 7. □

Appendix F. Geometric Inequalities (5.26) and (5.27)

From (ALG–V), (ALG–S) and (5.25), we obtain: $\mathscr{V}^*_{\mathscr{X}_d} = \overline{\mathscr{X}} \oplus \mathscr{Q}$ and $\mathscr{S}^*_{\mathscr{X}_d} = \langle \overline{A} \,|\, \operatorname{Im}[\,\overline{B}\ \overline{S}\,]\rangle \oplus \mathscr{Q}$, which imply: $E\mathscr{V}^*_{\mathscr{X}_d} = \overline{\mathscr{X}}$ and $\mathscr{R}^*_{\mathscr{X}_d} = \langle \overline{A} \,|\, \operatorname{Im}[\,\overline{B}\ \overline{S}\,]\rangle \oplus \mathscr{Q}$.

From (ALG–V) and (5.25), we get: $\mathscr{V}^0 = E^{-1}\operatorname{Im} E = E^{-1}\overline{\mathscr{V}}^0_{[\overline{B}\ \overline{S}]}$ and $\mathscr{V}^1 = E^{-1}\mathscr{K}_{\overline{C}} = E^{-1}\overline{\mathscr{V}}^1_{[\overline{B}\ \overline{S}]}$, then: $E\mathscr{V}^0 = \overline{\mathscr{V}}^0_{[\overline{B}\ \overline{S}]}$ and $E\mathscr{V}^1 = \overline{\mathscr{V}}^1_{[\overline{B}\ \overline{S}]}$. Let us assume that: $E\mathscr{V}^\mu = \overline{\mathscr{V}}^\mu_{[\overline{B}\ \overline{S}]}$, then: $\mathscr{V}^{\mu+1} = (E^{-1}\mathscr{K}_{\overline{C}}) \cap \begin{bmatrix} \overline{A} & \overline{S} \end{bmatrix}^{-1} \left(\overline{\mathscr{V}}^\mu_{[\overline{B}\ \overline{S}]} + \overline{\mathscr{B}}\right)$, which implies: $E\mathscr{V}^{\mu+1} = \mathscr{K}_{\overline{C}} \cap E \begin{bmatrix} \overline{A} & \overline{S} \end{bmatrix}^{-1} \left(\overline{\mathscr{V}}^\mu_{[\overline{B}\ \overline{S}]} + \overline{\mathscr{B}}\right) = \mathscr{K}_{\overline{C}} \cap \overline{A}^{-1}\left(\overline{\mathscr{V}}^\mu_{[\overline{B}\ \overline{S}]} + \operatorname{Im}[\,\overline{B}\ \overline{S}\,]\right) = \overline{\mathscr{V}}^{\mu+1}_{[\overline{B}\ \overline{S}]}$. Thus: $E\mathscr{V}^* = \overline{\mathscr{V}}^*_{[\overline{B}\ \overline{S}]}$.

From the previous paragraphs we have the following equivalences: $\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{V}^* = \mathscr{X}_d \Leftrightarrow \langle \overline{A} \,|\, \operatorname{Im}[\,\overline{B}\ \overline{S}\,]\rangle \oplus \mathscr{Q} + \mathscr{V}^* = \overline{\mathscr{X}} \oplus \mathscr{Q} \Leftrightarrow E^{-1}\langle \overline{A} \,|\, \operatorname{Im}[\,\overline{B}\ \overline{S}\,]\rangle + \mathscr{V}^* = \overline{\mathscr{X}} \oplus \mathscr{Q} \Rightarrow \operatorname{Im} E \cap \langle \overline{A} \,|\, \operatorname{Im}[\,\overline{B}\ \overline{S}\,]\rangle + E\mathscr{V}^* = \overline{\mathscr{X}} \Rightarrow \langle \overline{A} \,|\, \operatorname{Im}[\,\overline{B}\ \overline{S}\,]\rangle + \overline{\mathscr{V}}^*_{[\overline{B}\ \overline{S}]} = \overline{\mathscr{X}} \Rightarrow E^{-1}\left(\langle \overline{A} \,|\, \operatorname{Im}[\,\overline{B}\ \overline{S}\,]\rangle + E\mathscr{V}^*\right) = \overline{\mathscr{X}} \oplus \mathscr{Q} \Rightarrow E^{-1}\langle \overline{A} \,|\, \operatorname{Im}[\,\overline{B}\ \overline{S}\,]\rangle + \mathscr{V}^* = \overline{\mathscr{X}} \oplus \mathscr{Q}$; which imply (5.26).

From the two first paragraphs, (5.22) takes the form:

$$\dim\left(\frac{\overline{\mathscr{X}}}{\overline{\mathscr{V}}^*_{[\overline{B}\ \overline{S}]} + \overline{\mathscr{B}}}\right) + \dim(\overline{\mathscr{B}}) \geq \dim\left(\frac{\overline{\mathscr{X}}}{\overline{\mathscr{V}}^*_{[\overline{B}\ \overline{S}]}}\right) + \dim\left(\frac{\{0\} \oplus \mathscr{Q}}{\mathscr{V}^* \cap \mathscr{K}_E}\right) \tag{F1}$$

870 From (ALG–V), (5.25) and the second paragraph, we obtain:

$$\begin{aligned}\mathscr{K}_E \cap \mathscr{V}^* &= \mathscr{K}_E \cap (E^{-1}\mathscr{K}_{\overline{C}}) \cap \left[\begin{array}{cc} \overline{A} & \overline{S} \end{array}\right]^{-1} \left(\overline{\mathscr{V}}^*_{[\overline{B}\ \overline{S}]} + \overline{\mathscr{B}}\right) \\ &= \mathscr{K}_E \cap \left[\begin{array}{cc} \overline{A} & \overline{S} \end{array}\right]^{-1} \left(\overline{\mathscr{V}}^*_{[\overline{B}\ \overline{S}]} + \overline{\mathscr{B}}\right) = \{0\} \oplus \overline{S}^{-1} \left(\overline{\mathscr{V}}^*_{[\overline{B}\ \overline{S}]} + \overline{\mathscr{B}}\right)\end{aligned} \tag{F2}$$

871 From (F1) and (F2) we get (5.27) (recall that $\operatorname{Ker} \overline{S} = \{0\}$). $\square$