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Simultaneous State and Input Reachability for Linear Time Invariant Systems

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Abstract

In this paper, we give an explicit solution to the behavioral reachability problem for linear time invariant systems, which amounts to finding an explicit control law that reaches a given final input-state pair (u_1, x_1) in a given finite time t_1 . We first tackle the case of state space realizations, and we then extend the obtained results to the case of implicit realizations. For this, we use the geometric approach and some results of the viability theory. Some complements are given about the existing relationships between reachability and pole placement, as well as some notions of unicity and existence of solution.

Keywords:

Linear systems, implicit systems, reachability, geometric approach.

Notation. Script capitals $\mathcal{V}, \mathcal{W}, \dots$, denote finite dimensional linear spaces with elements v, w, \dots ; the dimension of a space \mathcal{V} is denoted $\dim(\mathcal{V})$; $\mathcal{V} \approx \mathcal{W}$ stands for $\dim(\mathcal{V}) = \dim(\mathcal{W})$; when $\mathcal{V} \subset \mathcal{W}$, $\frac{\mathcal{W}}{\mathcal{V}}$ or \mathcal{W}/\mathcal{V} stand for the quotient

space \mathscr{W} modulo \mathscr{V} ; the direct sum of independent spaces is written as \oplus . $X^{-1}\mathscr{V}$, stands for the inverse image of the subspace \mathscr{V} by the linear transformation X . Given a linear transformation $X: \mathscr{V} \rightarrow \mathscr{W}$, $\text{Im } X = X\mathscr{V}$ denotes its image, and $\text{Ker } X$ denotes its kernel; when $\mathscr{V} \approx \mathscr{W}$, we write $X: \mathscr{V} \leftrightarrow \mathscr{W}$; when $\mathscr{U} \subset \mathscr{V}$, $X|_{\mathscr{U}}$ denotes the restriction of X to \mathscr{U} . Given the space $\mathscr{X} = \mathscr{S} \oplus \mathscr{T}$, the natural projection, $P: \mathscr{X} \rightarrow \mathscr{S}$, on \mathscr{S} along \mathscr{T} , is also written as $P: \mathscr{X} \rightarrow \mathscr{S} // \mathscr{T}$. The special subspaces $\text{Im } B$, $\text{Ker } E$ and $\text{Ker } C$, are denoted by \mathscr{B} , \mathscr{K}_E and \mathscr{K}_C , respectively. The zero dimension subspace is denoted $\{0\}$, and the identity operator is denoted I , namely $Ix = x$. Given the linear transformations $X: \mathscr{V} \rightarrow \mathscr{V}$ and $Y: \mathscr{W} \rightarrow \mathscr{V}$, $\langle X | \text{Im } Y \rangle$ stands for the subspace of \mathscr{V} : $\text{Im } Y + X\text{Im } Y + \dots + X^{\dim(\mathscr{V})-1}\text{Im } Y$. The notations A_{F_p} and E_{F_d} stand for $(A + BF_p)$ and $(E - BF_d)$, respectively.

$\text{BDM}\{X_1, \dots, X_k\}$ denotes a block diagonal matrix whose diagonal blocks are the matrices X_1, \dots, X_k , and $\text{DM}\{x_1, \dots, x_k\}$ denotes a diagonal matrix whose diagonal elements are x_1, \dots, x_k . The notation \mathbb{R}^k stands for the Euclidean space of dimension k . $\underline{e}_k^i \in \mathbb{R}^k$ stands for the vector whose i -th entry is equal to 1 and the other ones are equal to 0. $\text{T}_u\{v^T\}$ stands for the upper triangular Toeplitz matrix, whose first row is v^T . $*$ stands for some matrix which exact value has no importance.

\mathbb{R}^+ , \mathbb{R}^{+*} and \mathbb{Z}^+ , stand for the sets of non negative real numbers, positive real numbers and non negative integers, respectively. $\mathcal{C}^\infty(\mathbb{R}^+, \mathscr{V})$ and $\mathcal{L}^\infty(\mathbb{R}^+, \mathscr{V})$ are the space of infinitely differentiable functions and the space of bounded functions, $v: \mathbb{R}^+ \rightarrow \mathscr{V}$, respectively. $\mathcal{L}_1^{\text{loc}}(\mathbb{R}^+, \mathscr{V})$ stands for the locally integrable functions.

Geometric Algorithms. Given the linear transformations $X : \mathcal{V} \rightarrow \mathcal{W}$, $Y : \mathcal{T} \rightarrow \mathcal{W}$, and $Z : \mathcal{V} \rightarrow \mathcal{W}$, and the subspace $\mathcal{K} \subset \mathcal{V}$, we have the two following popular geometric algorithms (see mainly [Verghese, 1981](#), [Özçaldıran, 1986](#), [Malabre, 1987, 1989](#), [Lewis, 1992](#)):

$$\mathcal{V}_{[\mathcal{K}:X,Z,Y]}^0 = \mathcal{V}, \quad \mathcal{V}_{[\mathcal{K}:X,Z,Y]}^{\mu+1} = \mathcal{K} \cap X^{-1} \left(Z \mathcal{V}_{[\mathcal{K}:X,Z,Y]}^\mu + \text{Im } Y \right) \quad (\text{ALG-V})$$

$$\mathcal{S}_{[Z,X,Y]}^0 = \{0\}, \quad \mathcal{S}_{[Z,X,Y]}^{\mu+1} = Z^{-1} \left(X \mathcal{S}_{[Z,X,Y]}^\mu + \text{Im } Y \right) \quad (\text{ALG-S})$$

where $\mu \in \mathbb{Z}^+$. The limit of (ALG-V) is the *supremal* (X, Z, Y) invariant subspace contained in \mathcal{K} , $\mathcal{V}_{[\mathcal{K}:X,Z,Y]}^*$:= $\sup\{\mathcal{S} \subset \mathcal{K} \mid X\mathcal{S} \subset Z\mathcal{S} + \text{Im } Y\}$, and the limit of (ALG-S) is the *infimal* (Z, X, Y) invariant subspace related to $\text{Im } Y$, $\mathcal{S}_{[Z,X,Y]}^*$:= $\inf\{\mathcal{S} \subset \mathcal{V} \mid \mathcal{S} = Z^{-1}(X\mathcal{S} + \text{Im } Y)\}$.

We distinguish two cases.

- For the square brackets $[\mathcal{V} : X, Z, 0]$ and $[Z, X, 0]$, we write: $\mathcal{V}_{[X,Z]}^*$, $\mathcal{V}_{[X,Z]}^\mu$, $\mathcal{S}_{[Z,X]}^*$ and $\mathcal{S}_{[Z,X]}^\mu$, instead of: $\mathcal{V}_{[\mathcal{V}:X,Z,0]}^*$, $\mathcal{V}_{[\mathcal{V}:X,Z,0]}^\mu$, $\mathcal{S}_{[Z,X,0]}^*$ and $\mathcal{S}_{[Z,X,0]}^\mu$, respectively, where $\mu \in \mathbb{Z}^+$.
- For the square bracket $[\mathcal{K}_{\overline{C}} : \overline{A}, I, Y]$, we write: $\overline{\mathcal{V}}_Y^*$ and $\overline{\mathcal{V}}_Y^\mu$, instead of: $\mathcal{V}_{[\mathcal{K}_{\overline{C}}:\overline{A},I,Y]}^*$ and $\mathcal{V}_{[\mathcal{K}_{\overline{C}}:\overline{A},I,Y]}^\mu$, where $\mu \in \mathbb{Z}^+$.

Subspaces. Note that in the particular case $X = A : \mathcal{X} \rightarrow \mathcal{X}$, $Y = B : \mathcal{U} \rightarrow \mathcal{X}$, and $Z = I$, the equalities $\mathcal{V}_{[\mathcal{X}:A,I,B]}^* = \mathcal{X}$ and $\mathcal{S}_{[I,A,B]}^* = \langle A \mid \mathcal{B} \rangle$ hold true.

Given the linear transformations $X = A : \mathcal{X}_d \rightarrow \mathcal{X}_{eq}$, $Y = B : \mathcal{U} \rightarrow \mathcal{X}_{eq}$, and $Z = E : \mathcal{X}_d \rightarrow \mathcal{X}_{eq}$, it is observed that

- the *supremal* (A, E, B) invariant subspace contained in \mathcal{X}_d and the *infimal* (E, A, B) invariant subspace related to \mathcal{B} , $\mathcal{V}_{[\mathcal{X}_d:A,E,B]}^*$ and $\mathcal{S}_{[E,A,B]}^*$, are

identified by $\mathcal{V}_{\mathcal{X}_d}^*$ and $\mathcal{S}_{\mathcal{X}_d}^*$, respectively, and the respective subspaces of their algorithms (ALG–V) and (ALG–S) are identified by $\mathcal{V}_{\mathcal{X}_d}^\mu$ and $\mathcal{S}_{\mathcal{X}_d}^\mu$ ($\mu \in \mathbb{Z}^+$), respectively;

- the supremal (A, E, B) invariant subspace contained in \mathcal{K}_C , $\mathcal{V}_{[\mathcal{K}_C: A, E, B]}^*$, is identified by \mathcal{V}^* , and the respective subspaces of its algorithm (ALG–V) are identified by \mathcal{V}^μ ($\mu \in \mathbb{Z}^+$);
- the unobservable space $\mathcal{V}_{[\mathcal{K}_C: A, E, 0]}^*$ is identified by \mathcal{N} ; and the closed loop unobservable space $\mathcal{V}_{[\mathcal{K}_C: A_{F_p}, E_{F_d}, 0]}^*$ is identified by $\mathcal{N}_{(F_p, F_d)}$.

Let us note that:

- (i) $\mathcal{V}_{[\mathcal{K}: A, E, B]}^* = \mathcal{V}_{[\mathcal{K}: A_{F_p}, E_{F_d}, B]}^*$,
 - (ii) $\mathcal{S}_{[\mathcal{K}: E, A, B]}^* = \mathcal{S}_{[\mathcal{K}: E_{F_d}, A_{F_p}, B]}^*$, and
 - (iii) for any F_d , there exists F_p such that: $A_{F_p} \mathcal{V}_{[\mathcal{K}: A_{F_p}, E_{F_d}, B]}^* \subset E_{F_d} \mathcal{V}_{[\mathcal{K}: A_{F_p}, E_{F_d}, B]}^*$.
- The set of such pairs (F_p, F_d) is identified by $\mathbf{F}(\mathcal{V}_{[\mathcal{K}: A, E, B]}^*)$.

1. INTRODUCTION

One of the most studied concepts in System Theory is the one of *reachability*. This concept is normally associated with the set of vectors that can be reached from the origin in a finite time, following trajectories solutions of the system, generated by the input system. Here, the term *input system* refers to an exogenous signal which is available for controlling the output system.

1.1. State Space Representations

For the case of state space representations $\mathfrak{R}^{ss}(A, B)$,

$$dx/dt = Ax + Bu, \tag{1.1}$$

9 where $u \in \mathcal{U} \approx \mathbb{R}^m$ is the input variable, $x \in \mathcal{X} \approx \mathbb{R}^n$ is the state variable, and
10 with the usual assumption $\text{Ker } B = \{0\}$, Kalman (1960, 1963) introduced his
11 famous reachability matrix: $\mathcal{R}_{[A, B]} = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$. He showed that
12 given any $x_0, x_1 \in \mathcal{X}$, there exists a control law¹ $u(\cdot) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{U})$, generating
13 a trajectory $x(\cdot) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{X})$ solution of (1.1), starting from the given initial
14 condition $x(0) = x_0 \in \mathcal{X}$, and reaching the desired final state $x(t_1) = x_1 \in \mathcal{X}$,
15 in a finite time $t_1 \in \mathbb{R}^+$, iff, $\text{rank}(\mathcal{R}_{[A, B]}) = n$; in this case, the representation
16 (1.1) is called *reachable*. This concept is known as *state reachability*², and
17 when the pair (A, B) satisfies such a rank condition, we identify it as a *state*
18 *reachable pair*.

19 *A. State reachability.* Brunovsky (1970) showed that for a given *reachable*
20 state space representation (1.1), there exist a linear map $F_B : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and
21 isomorphisms $T_B : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $G_B : \mathbb{R}^m \rightarrow \mathbb{R}^m$, such that the pair (A_B, B_B) ,
22 where $A_B = T_B^{-1}(A + BF_B)T_B$ and $B_B = T_B^{-1}BG_B$, is expressed in the Brunovsky

¹ In this paper, we restrain our discussion to infinitely differentiable functions. This is not restrictive since $\mathcal{C}^\infty(\mathbb{R}^+, \mathcal{V})$ is dense in $\mathcal{L}_1^{\text{loc}}(\mathbb{R}^+, \mathcal{V})$ (see Polderman & Willems, 1998, Corollary 2.4.12).

² In many text books, this property is called *state controllability*, or simply *controllability*. Let us note that *controllability* only characterizes the system's property of reaching the origin $x_1 = 0$, from any state $x_0 \neq 0$, in a finite time t_1 . Since in the continuous time-invariant linear systems case both properties, *reachability* and *controllability*, are mutually implied, they are often treated indistinguishably, but in the general case of the implicit representations, this is no longer the case; for example the implicit representation, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} dx/dt = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$, is trivially controllable but not reachable.

23 canonical form, namely:

$$\begin{aligned} A_{\mathcal{B}} &= \text{BDM} \{A_{\mathcal{B}_1}, \dots, A_{\mathcal{B}_m}\}, \quad B_{\mathcal{B}} = \text{BDM} \{b_{\mathcal{B}_1}, \dots, b_{\mathcal{B}_m}\}, \\ [A_{\mathcal{B}_i} | b_{\mathcal{B}_i}] &= [\text{T}_u \{(\underline{e}_{\kappa_i}^2)^T\} | \underline{e}_{\kappa_i}^{\kappa_i}], \quad i \in \{1, \dots, m\}, \end{aligned} \quad (1.2)$$

24 where the set

$$S_{\kappa} = \left\{ \{\kappa_1, \kappa_2, \dots, \kappa_m\} \subset \mathbb{Z} \mid \kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_m \geq 1 \ \& \ \sum_{i=1}^m \kappa_i = n \right\} \quad (1.3)$$

25 is known as the set of *reachability indices*. They are also geometrically char-
26 acterized as follows

$$\text{card} \{\kappa_i \geq 1\} = \dim(\mathcal{B}) \quad \text{and} \quad \text{card} \{\kappa_i \geq \mu\} = \dim \left(\frac{\sum_{i=0}^{\mu-1} A^i \mathcal{B}}{\sum_{i=0}^{\mu-2} A^i \mathcal{B}} \right), \quad \forall \mu \geq 2.$$

27 Another important success was the introduction of the reachable space
28 $\langle A | \mathcal{B} \rangle$. In (Wonham, 1985) is showed that a pair (A, B) is reachable iff:

$$\langle A | \mathcal{B} \rangle = \mathcal{X}. \quad (1.4)$$

29 Note that $\langle A | \mathcal{B} \rangle = \mathcal{X}$ iff $\text{rank}(\mathcal{R}_{[A, B]}) = n$. Wonham (1985) showed that the
30 reachability Gramian $W_{t_1} = \int_0^{t_1} \exp(\tau A) B B^T \exp(\tau A^T) d\tau$, with $t \in [0, t_1]$, is non-
31 singular iff (1.4) is satisfied. Thus, with the control law

$$u(t) = B^T \exp((t_1 - t)A^T) W_{t_1}^{-1} (x_1 - \exp(t_1 A) x_0), \quad (1.5)$$

32 we get a trajectory $x(\cdot) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{X})$ solution of (1.1), such that $x(0) = x_0 \in \mathcal{X}$
33 and $x(t_1) = x_1 \in \mathcal{X}$.

34 Another well known result concerning state reachability is the one related
35 with pole assignment. Indeed, *the pair (A, B) is reachable iff for every sym-*
36 *metric (with respect to the real line) set of complex numbers Λ , of cardinality*
37 *n , there exists a proportional state feedback $u = Fx$ such that the spectrum of*
38 *$(\lambda I - A_F)$ is Λ (see for example Theorems 2.1 and 9.3.1 of Wonham (1985)*
39 *and Polderman & Willems (1998), respectively).*

40 *B. Behavioral reachability.* Willems (1983, 1991) defined an *input/state sys-*
 41 *tem* as the triple $\Sigma_{i/s} = (\mathbb{R}^+, \mathcal{U} \times \mathcal{X}, \mathfrak{B}_{[A,B]})$, with behavior³

$$\mathfrak{B}_{[A,B]} = \left\{ (u, x) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{U} \times \mathcal{X}) \mid \begin{bmatrix} (I \frac{d}{dt} - A) & -B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0 \right\}. \quad (1.6)$$

42 In the behavioral framework of Willems (1983, 1991), the system $\Sigma_{i/s} =$
 43 $(\mathbb{R}^+, \mathcal{U} \times \mathcal{X}, \mathfrak{B}_{[A,B]})$ is called⁴ *reachable* if *for any given* $(u_0, x_0), (u_1, x_1) \in$
 44 $\mathcal{U} \times \mathcal{X}$ *and* $t_1 > 0$, *it is possible to find a trajectory* $(u, x) \in \mathfrak{B}_{[A,B]}$, *such that*
 45 $(u(0), x(0)) = (u_0, x_0)$ *and* $(u(t_1), x(t_1)) = (u_1, x_1)$ (c.f. Polderman & Willems,
 46 1998, Definition 5.2.2). In the following, this *reachability* concept is called
 47 *behavioral reachability*.

48 In (Polderman & Willems, 1998, Theorem 5.2.27) is proved that for the
 49 case of state space representations $\mathfrak{R}^{ss}(A, B)$, *state reachability* is equivalent to
 50 *behavioral reachability*. Although the *behavioral reachability* is well charac-
 51 terized, it could be interesting to find an explicit control law $u(\cdot) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{U})$,
 52 similar to (1.5), which ensures $x(t_1) = x_1$ and $u(t_1) = u_1$. This will be done in
 53 Section 2.

54 1.2. Implicit Representations

55 As a generalization of proper linear systems, Rosenbrock (1970) intro-
 56 duced the *implicit representations* $\mathfrak{R}^{imp}(E, A, B)$, which are a set of differential

³ The original definition given by Willems (1983, 1991) is $\mathfrak{B}_{[A,B]} = \left\{ (u, x) \in \mathcal{L}_1^{loc}(\mathbb{R}^+, \mathcal{U} \times \mathcal{X}) \mid \exists x_0 \in \mathcal{X} \text{ s.t. } x(t) = \exp(At)x_0 + \int_0^t \exp(A(t-\tau))Bu(\tau)d\tau \right\}$. But since we restrict our attention to infinitely differentiable functions (see footnote ¹), weak and strong solutions coincide (see Polderman & Willems, 1998, Theorem 2.3.11).

⁴ For consistency of this paper, we say *reachable* instead of *controllable*, as is stated in (Polderman & Willems, 1998) (see also footnote ²).

57 and algebraic equations (Brenan *et al*, 1996) of the following form (see also
58 Lewis, 1992)

$$E dx/dt = Ax + Bu, \quad (1.7)$$

59 where $E : \mathcal{X}_d \rightarrow \mathcal{X}_{eq}$, $A : \mathcal{X}_d \rightarrow \mathcal{X}_{eq}$ and $B : \mathcal{U} \rightarrow \mathcal{X}_{eq}$ are linear maps. The lin-
60 ear spaces $\mathcal{X}_d \approx \mathbb{R}^{n_d}$, $\mathcal{X}_{eq} \approx \mathbb{R}^{n_{eq}}$, and $\mathcal{U} \approx \mathbb{R}^m$ are called the descriptor, the
61 equation, and the input spaces, respectively. In order to avoid redundant
62 components in the input variable u , and linear dependence on the descriptor
63 equations (1.7), as usually, we assume throughout the paper that the following
64 hypotheses are verified:

65 **[H1]** $\text{Ker } B = 0$, and

66 **[H2]** $\text{Im } E + \text{Im } A + \mathcal{B} = \mathcal{X}_{eq}$.

67 For the case of regular implicit representations, *i.e.* representations where
68 the linear transformations E and A are square and the pencil $[\lambda E - A]$ is in-
69 vertible (Gantmacher, 1977), the *reachability* was studied by Verghese, Lévy
70 and Kailath (1981) from a transfer function point of view, Yip and Sincovec
71 (1981) in the time domain, Cobb (1984) from a distributional point of view,
72 and by Özçaldiran (1985) from a geometric point of view.

73 In the case of implicit representations, where the linear transformations
74 E and A are square and the pencil $[\lambda E - A]$ is not necessarily invertible,
75 Özçaldiran (1986) extended his *reachability* geometric characterization for
76 the case of regular implicit representations (Özçaldiran, 1985), by means of
77 the supremal (A, E, B) reachability subspace contained in \mathcal{X}_d , defined as

$$\mathcal{R}_{\mathcal{X}_d}^* = \mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{S}_{\mathcal{X}_d}^*. \quad (1.8)$$

78 This is a nice generalization of the classical case, $\mathfrak{R}^{ss}(A, B) = \mathfrak{R}^{imp}(I, A, B)$,
79 where the reachable space $\mathcal{R}_{\mathcal{X}_d}^*$ is equal to $\langle A | \mathcal{B} \rangle$, namely equal to $\mathcal{V}_{[\mathcal{X}: A, I, B]}^* \cap$

80 $\mathcal{S}_{[I,A,B]}^*$. Thus, for representations $\mathfrak{R}^{imp}(E, A, B)$, with E and A not necessarily
81 square, it was natural to associate its reachability with $\mathcal{R}_{x_d}^*$.

82 [Frankowska \(1990\)](#) firmly established the pertinence of this reachability
83 concept, using differential inclusions to relate it with behavioral properties.

84 One major difficulty when studying reachability for implicit systems (1.7)
85 is that their solution set does not only depend on the initial conditions $x(0)$
86 and on the external control input u , but also depends on a possible internal
87 free variable (degree of freedom), which is completely unknown.

88 1.3. Outline

89 In this paper, we study the reachability notion in the sense of [Frankowska](#)
90 (1990), showing some connections with the important works of [Willems](#)
91 (1991) and [Geerts \(1993\)](#), and we consider the relationships between the
92 reachability property and the complete pole assignment ability.

93 The paper is organized as follows: In Section 2, we consider the behav-
94 ioral reachability problem for state space representations, namely the ability
95 of reaching the input-state pair $(u(\cdot), x(\cdot))$. In Section 3, we formalize the
96 notion of *implicit systems*, following the behavioral point of view, and we
97 also study the equivalences between the notions of existence of solution and
98 impulse controllability. In Section 4, we study the reachability notion of
99 [Frankowska \(1990\)](#) for implicit systems. In Section 5, we consider the exis-
100 tent relationships between the reachability property and the complete pole
101 assignment ability, and in Section 6, we conclude the paper.

102 2. BEHAVIORAL REACHABILITY PROBLEM

103 We consider the following problem.

104 **Problem 1.** Let us consider an *input/state system* $\Sigma_{i/s} = (\mathbb{R}^+, \mathcal{U} \times \mathcal{X}, \mathfrak{B}_{[A,B]})$
 105 represented by (1.1), and with the behavior (1.6). Given $(u_0, x_0), (u_1, x_1) \in$
 106 $B^{-1}\langle A | \mathcal{B} \rangle \times \langle A | \mathcal{B} \rangle$ and $t_1 > 0$, find a trajectory $(u, x) \in \mathfrak{B}_{[A,B]}$, such that
 107 $(u(0), x(0)) = (u_0, x_0)$ and $(u(t_1), x(t_1)) = (u_1, x_1)$.

108 This is the *behavioral reachability* problem, and in (Polderman & Willems,
 109 1998, Theorem 5.2.27) is proved that for the case of *state space representa-*
 110 *tions, state reachability* is equivalent to *behavioral reachability*. So, condition
 111 (1.4) guarantees the existence of a solution for Problem 1.

112 One could think that the control law (1.5), proposed by Wonham (1985),
 113 solves Problem 1, but this proposition only guarantees the *reachability* of the
 114 state variable, $x(0) = x_0$ and $x(t_1) = x_1$, and nothing about the input variable u ,
 115 which is let completely free at the end points $u(0)$ and $u(t_1)$. An intermediary
 116 step towards the solution of Problem 1 is given by the next result proved in
 117 Appendix A.

118 **Lemma 1.** *Let the state space representation (1.1) be reachable, with the*
 119 *reachability indices set (1.3). Let the linear map $F_{\mathcal{B}} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and the iso-*
 120 *morphisms $T_{\mathcal{B}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $G_{\mathcal{B}} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be such that the pair $(A_{\mathcal{B}}, B_{\mathcal{B}})$,*
 121 *where $A_{\mathcal{B}} = T_{\mathcal{B}}^{-1}(A + BF_{\mathcal{B}})T_{\mathcal{B}}$ and $B_{\mathcal{B}} = T_{\mathcal{B}}^{-1}BG_{\mathcal{B}}$, is expressed in the Brunovsky*
 122 *canonical form (1.2). Let the reachability matrices, $\mathcal{R}_{[A_{\mathcal{B}}, B_{\mathcal{B}}]}$ and $\mathcal{R}_{[A_{\mathcal{B}_i}, b_{\mathcal{B}_i}]}$,*
 123 *$i \in \{1, \dots, m\}$, of the pair $(A_{\mathcal{B}}, B_{\mathcal{B}})$ and the pairs $(A_{\mathcal{B}_i}, b_{\mathcal{B}_i})$, $i \in \{1, \dots, m\}$, respec-*
 124 *tively, be defined as follows:*

$$\begin{aligned} \mathcal{R}_{[A_{\mathcal{B}}, B_{\mathcal{B}}]} &= \text{BDM} \left\{ \mathcal{R}_{[A_{\mathcal{B}_1}, b_{\mathcal{B}_1}]}, \dots, \mathcal{R}_{[A_{\mathcal{B}_m}, b_{\mathcal{B}_m}]} \right\}, \\ \mathcal{R}_{[A_{\mathcal{B}_i}, b_{\mathcal{B}_i}]} &= \begin{bmatrix} b_{\mathcal{B}_i} & A_{\mathcal{B}_i} b_{\mathcal{B}_i} & \cdots & A_{\mathcal{B}_i}^{\kappa_i - 1} b_{\mathcal{B}_i} \end{bmatrix}. \end{aligned} \quad (2.1)$$

125 *Let us assume that we have found trajectories $f_i \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^1)$, satisfying:*

126 (i) for $j = 0, 1$

$$\mathfrak{D}\left(\frac{d}{dt}\right)f(t_j) = G_{\mathcal{B}}^{-1}(u_j - F_{\mathcal{B}}x_j), \quad (2.2)$$

127 where $\mathfrak{D}(d/dt) = \text{DM}\{d^{\kappa_1}/dt^{\kappa_1}, \dots, d^{\kappa_m}/dt^{\kappa_m}\}$, $f(t) = \begin{bmatrix} f_1(t) & \dots & f_m(t) \end{bmatrix}^T$
 128 and $t_0 = 0$.

129 (ii) If $\bar{w}_i(t) = \begin{bmatrix} \frac{d^{\kappa_i-1}f_i(t)}{dt^{\kappa_i-1}} & \dots & \frac{df_i(t)}{dt} & f_i(t) \end{bmatrix}^T$, $1 \leq i \leq m$, and $\bar{w}(t) = \begin{bmatrix} \bar{w}_1^T(t) & \dots & \bar{w}_m^T(t) \end{bmatrix}^T$
 130 then, for $j = 0, 1$,

$$\bar{w}(t_j) = \mathcal{R}_{[A_{\mathcal{B}}, B_{\mathcal{B}}]}^{-1} T_{\mathcal{B}}^{-1} x_j. \quad (2.3)$$

131 Then, applying the control law,

$$u(t) = F_{\mathcal{B}}x(t) + G_{\mathcal{B}}\mathfrak{D}(d/dt)f(t), \quad (2.4)$$

132 to the system represented by (1.1), we get:

$$x(t) = T_{\mathcal{B}}\mathcal{R}_{[A_{\mathcal{B}}, B_{\mathcal{B}}]}\bar{w}(t), \quad (2.5)$$

133 with

$$(u(0), x(0)) = (u_0, x_0) \quad \text{and} \quad (u(t_1), x(t_1)) = (u_1, x_1). \quad (2.6)$$

134 Let us now propose the trajectories:⁵

$$f_i(t) = \begin{bmatrix} t^{2\kappa_i+1} & \dots & t^{\kappa_i+1} \end{bmatrix} \mathbf{a}_{i,1} + \begin{bmatrix} t^{\kappa_i} & \dots & 1 \end{bmatrix} \mathbf{a}_{i,0},$$

$$\mathbf{a}_{i,1} = \begin{bmatrix} a_{i,2\kappa_i+1} & \dots & a_{i,\kappa_i+1} \end{bmatrix}^T \in \mathbb{R}^{\kappa_i+1}, \quad \text{and} \quad \mathbf{a}_{i,0} = \begin{bmatrix} a_{i,\kappa_i} & \dots & a_{i,0} \end{bmatrix}^T \in \mathbb{R}^{\kappa_i+1}, \quad (2.7)$$

135 with $i \in \{1, \dots, m\}$, and let us define the following auxiliary matrices:

⁵ Lewis (1986) did a similar proposition when he introduced a “fast” input in his reachability consideration.

$$X_{(i,0)}(t) = \begin{bmatrix} \kappa_i!/0! & 0 & \cdots & 0 & 0 \\ (\kappa_i!/1!)t & (\kappa_i - 1)!/0! & 0 \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \cdot & \cdot \\ (\kappa_i!/ \kappa_i!)t^{\kappa_i} & ((\kappa_i - 1)!/(\kappa_i - 1)!)t^{\kappa_i - 1} & \cdots & (1!/1!)t & 0!/0! \end{bmatrix}, \quad (2.8)$$

136

$$X_{(i,1)}(t) = \begin{bmatrix} ((2\kappa_i + 1)!/(\kappa_i + 1)!)t^{\kappa_i + 1} & \cdots & ((\kappa_i + 1)!/1!)t \\ \vdots & \cdots & \vdots \\ ((2\kappa_i + 1)!/(2\kappa_i + 1)!)t^{2\kappa_i + 1} & \cdots & ((\kappa_i + 1)!/(\kappa_i + 1)!)t^{\kappa_i + 1} \end{bmatrix}. \quad (2.9)$$

137 The following Lemma gives a selection of the coefficient vectors $\mathbf{a}_{i,0}$ and $\mathbf{a}_{i,1}$,
 138 $i \in \{1, \dots, m\}$, for f in (2.7) to satisfy assumptions (2.2) and (2.3) of Lemma
 139 1 (see Appendix B for the proof).

140 **Lemma 2.** For $i \in \{1, \dots, m\}$, the determinants of the auxiliary matrices (2.8)
 141 and (2.9) satisfy

$$\det(X_{(i,0)}(t)) = \prod_{\ell=0}^{\kappa_i} \ell! \quad \text{and} \quad \det(X_{(i,1)}(t)) = t^{(\kappa_i + 1)^2} \prod_{\ell=0}^{\kappa_i} \ell! \quad (2.10)$$

142 Moreover, if we select the coefficient vectors $\mathbf{a}_{i,0}$ and $\mathbf{a}_{i,1}$, as follows:

$$\begin{aligned} \mathbf{a}_{i,0} &= X_{(i,0)}^{-1}(0)v_0, \quad \mathbf{a}_{i,1} = X_{(i,1)}^{-1}(t_1)(v_1 - X_{(i,0)}(t_1)\mathbf{a}_{i,0}), \\ v_j &= \left[\left((\underline{\mathbf{e}}_m^i)^T G_{\mathcal{B}}^{-1}(u_j - F_{\mathcal{B}}x_j) \right)^T \left(\mathcal{R}_{[A_{\mathcal{B}_i}, B_{\mathcal{B}_i}]}^{-1} P_i T_{\mathcal{B}}^{-1} x_j \right)^T \right]^T, \quad j \in \{0, 1\}, \end{aligned} \quad (2.11)$$

143 where:

$$P_i = \left[\underline{\mathbf{e}}_n^{\hat{n}_i + 1} \quad \cdots \quad \underline{\mathbf{e}}_n^{\hat{n}_i + \kappa_i} \right]^T, \quad \hat{n}_1 = 0 \quad \text{and} \quad \hat{n}_{i \geq 2} = \sum_{j=1}^{i-1} \kappa_j, \quad (2.12)$$

144 then the function f defined by (2.7) fulfills assumptions (2.2) and (2.3) of
 145 Lemma 1.

146 Let us note from Lemma 2 that the proposed solutions only depend on
 147 the set of reachability indices S_κ , and on the fixed final time t_1 . Hence,
 148 once S_κ and t_1 are given, the matrices $X_{i,0}(0)$, $X_{i,0}(t_1)$ and $X_{i,1}(t_1)$ are uniquely
 149 determined. And thus, the values of $\mathbf{a}_{i,0}$ and $\mathbf{a}_{i,1}$ only depend on the boundary
 150 points, (u_0, x_0) and (u_1, x_1) , of the trajectory $(u, x) \in \mathfrak{B}_{[A,B]}$.

151 From the above observation, it is possible to track a given trajectory
 152 $(\bar{u}, \bar{x}) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^{m+n})$, with a delayed time t_1 . Indeed, we only need to fix a
 153 sampling time $t_1 \in \mathbb{R}^{+*}$, and to apply iteratively Lemma 1 with the settings
 154 $(u_0, x_0) = (u(kt_1), x(kt_1))$ and $(u_1, x_1) = (\bar{u}(kt_1), \bar{x}(kt_1))$.

155 Otherwise written, in each sampling interval $[kt_1, (k+1)t_1)$, we find a tra-
 156 jectory $(u, x) \in \mathfrak{B}_{[A,B]} \cap \mathcal{C}^\infty(\mathbb{R}^+ \cap [kt_1, (k+1)t_1), \mathcal{U} \times \mathcal{X})$, such that $(u(kt_1), x(kt_1))$
 157 $= (u_0, x_0)$ and $\lim_{\sigma \rightarrow t_1} (u(kt_1 + \sigma), x(kt_1 + \sigma)) = (u_1, x_1)$.

158 We have proved in this way the following Theorem.

159 **Theorem 1.** *Let us consider an input/state system $\Sigma_{i/s} = (\mathbb{R}^+, \mathcal{U} \times \mathcal{X},$
 160 $\mathfrak{B}_{[A,B]})$, represented by (1.1). If (1.4) is satisfied, then for any sequence
 161 $(\bar{u}_k, \bar{x}_k) \in \mathbb{R}^{m+n}$, $k \in \mathbb{Z}^+$, and a given sampling time $t_1 \in \mathbb{R}^{+*}$, there exists a
 162 control law $u \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^m)$, such that $(u(kt_1), x(kt_1)) = (\bar{u}_{k-1}, \bar{x}_{k-1})$.*

163 3. IMPLICIT SYSTEMS

164 In this Section, we formalize the notion of *implicit system* following the
 165 behavioral point of view. For this, let us first state the following definition:

166 **Definition 1.** An implicit representation $\mathfrak{R}^{imp}(E, A, B)$ is called an *input/des-*
 167 *criptor system*, when for all initial condition $x_0 \in \mathcal{X}_d$, there exists at least one
 168 solution $(u, x) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{U} \times \mathcal{X}_d)$, such that $x(0) = x_0$. The *input/descriptor*

169 *system* is defined by the triple⁶ $\Sigma_{i/d} = (\mathbb{R}^+, \mathcal{U} \times \mathcal{X}_d, \mathfrak{B}_{[E,A,B]})$, with behavior:

$$\mathfrak{B}_{[E,A,B]} = \left\{ (u, x) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{U} \times \mathcal{X}) \mid \begin{bmatrix} (E \frac{d}{dt} - A) & -B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0 \right\} \quad (3.1)$$

170 At this point, it is important to clarify what exactly means the sentence
 171 “*there exists at least one solution*”. For this, we are going to recall hereafter
 172 the notions of *existence of solution* introduced by Geerts (1993) and Aubin
 173 & Frankowska (1991).

174 3.1. Existence of solution for every initial condition

175 Following (Hautus, 1976) and (Hautus & Silverman, 1983), Geerts (1993)
 176 generalized the solvability results of (Geerts & Mehrmann, 1990). One advan-
 177 tage of this generalization is that the solvability is introduced in a very nat-
 178 ural way, passing from the distributional framework (Schwartz, 1978) to the
 179 usual time domain with ordinary differential equations; this is precisely the
 180 starting point of the so called behavioral approach (Polderman & Willems,
 181 1998), chosen in this paper.

182 Geerts (1993) considered the linear combinations of impulsive and smooth
 183 distributions, with μ coordinates, denoted by $\mathcal{C}_{\text{imp}}^\mu$, as the signal sets. The
 184 set $\mathcal{C}_{\text{imp}}^\mu$ is a subalgebra and is also decomposed as $\mathcal{C}_{\text{p-imp}}^\mu \oplus \mathcal{C}_{\text{sm}}^\mu$, where $\mathcal{C}_{\text{p-imp}}^\mu$
 185 and $\mathcal{C}_{\text{sm}}^\mu$ denote the subalgebras of pure impulses⁷ and smooth distributions⁸,

⁶ See also Polderman & Willems (1998) and Kuijper (1992b).

⁷ The unit element of this subalgebra is the Dirac delta distribution δ . Any linear combination of δ and its distributional derivatives $\delta^{(\ell)}$, $\ell > 1$, is called impulsive.

⁸ The set of regular distributions are distributions that are functions; namely piecewise continuous integrable, or measurable functions. In those papers, they assume that the regular distributions $u(t)$ are smooth on $[0, \infty)$, *i.e.* that a function $v : [0, \infty) \rightarrow \mathbb{R}$ exists, ar-

186 respectively (Schwartz, 1978). He introduced the following definitions for
 187 the distributional version of the *implicit representation* (1.7) $\mathfrak{R}_{dist}^{imp}(E, A, B)$:⁹
 188 $pEx = Ax + Bu + Ex_0$ (c.f. Definitions 3.1 and 4.1, Geerts, 1993)¹⁰.

189 **Definition 2.** (Geerts, 1993) Given the *solution set* $S_C(x_0, u) := \{x \in \mathcal{C}_{imp}^{n_d} \mid$
 190 $[pE - A]x = Bu + Ex_0\}$, the *implicit representation* $\mathfrak{R}_{dist}^{imp}(E, A, B)$ is:

- 191 • *C-solvable* if $\forall x_0 \in \mathcal{X}_d \exists u \in \mathcal{C}_{imp}^m : S_C(x_0, u) \neq \emptyset$,
- 192 • *C-solvable in the function sense* if $\forall x_0 \in \mathcal{X}_d \exists u \in \mathcal{C}_{sm}^m : S_C(x_0, u) \cap \mathcal{C}_{sm}^n \neq \emptyset$.

193 Given the “*consistent initial conditions set*” $\mathcal{I}_C := \{z_0 \in \mathcal{X}_d \mid \exists u \in \mathcal{C}_{sm}^m \exists x \in$
 194 $S_C(z_0, u) \cap \mathcal{C}_{sm}^{n_d} : x(0^+) = z_0\}$, and the “*weakly consistent initial conditions set*”
 195 $\mathcal{I}_C^w := \{z_0 \in \mathcal{X}_d \mid \exists u \in \mathcal{C}_{sm}^m \exists x \in S_C(z_0, u) \cap \mathcal{C}_{sm}^{n_d}\}$, a point $x_0 \in \mathcal{X}_d$ is called *C-*
 196 *consistent* if $x_0 \in \mathcal{I}_C$, and *weakly C-consistent* if $x_0 \in \mathcal{I}_C^w$.

197 Let us note that:

- 198 (i) *C-solvability* is concerned with distributional solutions,
- 199 (ii) *C-solvability in the function sense* is concerned with solutions only com-
 200 posed by ordinary functions arbitrarily often differentiable,
- 201 (iii) the two notions of consistency, *C-consistent* and *weakly C-consistent*,
 202 lead to *smooth solutions*, namely with no impulsions, but

bitrarily often differentiable including at $t = 0$, such that $u(t) = 0$ for $t < 0$ and $u(t) = v(t)$, for
 $t \geq 0$ (Hautus & Silverman, 1983). These distributions are identified as ordinary functions
 with support on \mathbb{R}^+ .

⁹ Ex_0 stands for $Ex_0 \delta$, $x_0 \in \mathcal{X}_d$ being the initial condition, and pEx stands for
 $\delta^{(1)} * Ex$ ($*$ denotes convolution); if pEx is smooth and $E\dot{x}$ stands for the distribution
 that can be identified with the ordinary derivative Edx/dt , then $pEx = E\dot{x} + Ex_{0+}$.

¹⁰ He also considered the *B-free* case $\mathfrak{R}_{dist}^{imp}(E, A, f)$: $pEx = Ax + f + Ex_0$.

203 (iv) *C-consistency* avoids jumps at the origin, namely the *smooth solutions*
 204 are continuous on the left, and
 205 (iv) *weakly C-consistent* enables jumps at the origin, but they are piece-wise
 206 continuous *smooth solutions*.

207 **Geerts (1993)** characterized the existence of solutions for every initial
 208 condition in his Corollary 3.6, Proposition 4.2 and Theorem 4.5. Hereafter
 209 we summarize these results with their geometric equivalences.

210 **Theorem 2.** (*Geerts, 1993*) *If [H2] is fulfilled, then*

- 211 • $\mathfrak{R}_{dist}^{imp}(E, A, B)$ is C-solvable if and only if $[(\lambda E - A) \quad -B]$ is right invert-
 212 ible as a rational matrix, i.e. if and only if¹¹

$$E\mathcal{V}_{\mathcal{X}_d}^* + A\mathcal{S}_{\mathcal{X}_d}^* + \mathcal{B} = \mathcal{X}_{eq} . \quad (3.2)$$

- 213 • $\mathfrak{R}_{dist}^{imp}(E, A, B)$ is C-solvable in the function sense if and only if $\mathcal{I}_C^w = \mathcal{X}_d$,
 214 namely, if and only if $\text{Im } E + A\mathcal{K}_E + \mathcal{B} = \mathcal{X}_{eq}$, i.e. if and only if¹²

$$E\mathcal{V}_{\mathcal{X}_d}^* = \text{Im } E . \quad (3.3)$$

¹¹ $[\lambda[E \quad 0] - [A \quad B]]$ is right invertible iff (see [Loiseau \(1985\)](#) and [Armentano \(1986\)](#)) $\mathcal{X}_{eq} = [E \quad 0]\mathcal{V}_{[[A \quad B], [E \quad 0]]}^* + [A \quad B]\mathcal{S}_{[[E \quad 0], [A \quad B]]}^*$, namely iff $E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} + A\mathcal{S}_{\mathcal{X}_d}^* = \mathcal{X}_{eq}$ (from [\(ALG-V\)](#) and [\(ALG-S\)](#) we get $\mathcal{V}_{[[A \quad B], [E \quad 0]]}^* = \mathcal{V}_{\mathcal{X}_d}^* \oplus \mathcal{U}$ and $\mathcal{S}_{[[E \quad 0], [A \quad B]]}^* = \mathcal{S}_{\mathcal{X}_d}^* \oplus \mathcal{U}$).

¹² From [\(ALG-V\)](#) and [\[H2\]](#), one obtains the following sequence of implications:
 $\text{Im } E + \mathcal{B} + A\mathcal{K}_E = \mathcal{X}_{eq} \Rightarrow \mathcal{V}_{\mathcal{X}_d}^1 + \mathcal{K}_E = \mathcal{X}_d \Rightarrow E\mathcal{V}_{\mathcal{X}_d}^1 = \text{Im } E \Rightarrow E\mathcal{V}_{\mathcal{X}_d}^* = \text{Im } E$
 $\Rightarrow \mathcal{X}_d = \mathcal{V}_{\mathcal{X}_d}^* + \mathcal{K}_E = A^{-1}(E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) + \mathcal{K}_E \Rightarrow \text{Im } A = \text{Im } A \cap (E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) + A\mathcal{K}_E \Rightarrow$
 $\mathcal{X}_{eq} = \text{Im } E + \mathcal{B} + A\mathcal{K}_E.$

215 • $\mathcal{I}_C = \mathcal{X}_d$ if and only if $\text{Im } E + \mathcal{B} = \mathcal{X}_{eq}$ i.e. if and only if ¹³

$$E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} = \mathcal{X}_{eq}. \quad (3.4)$$

216 3.2. Existence of a viable solution

217 In order to study the reachability for implicit systems, Frankowska (1990)
 218 introduced the set-valued map (the set of all admissible velocities) $\mathbf{F}: \mathcal{X}_d \rightsquigarrow$
 219 \mathcal{X}_d , $\mathbf{F}(x) = E^{-1}(Ax + \mathcal{B}) = \{v \in \mathcal{X} \mid Ev \in Ax + \mathcal{B}\}$, and the differential inclusion

$$dx/dt \in \mathbf{F}(x), \quad \text{where } x(0) = x_0, \quad (3.5)$$

220 Frankowska (1990) showed that the solutions of (1.7) and the ones of (3.5)
 221 are the same. She also clarified the meaning of a viable solution and she
 222 characterized the largest subspace of such viable solutions.

223 **Definition 3.** (Frankowska, 1990, Aubin & Frankowska, 1991)

- 224 • An absolutely continuous function $x: \mathbb{R}^+ \rightarrow \mathcal{X}_d$ is called a *trajectory* of
 225 (3.5), if $x(0) = x_0$ and $dx/dt \in \mathbf{F}(x)$ for almost every $t \in \mathbb{R}^+$, that is to say,
 226 if there exists a measurable function $u: \mathbb{R}^+ \rightarrow \mathcal{U}$ such that $x(0) = x_0$ and
 227 $Edx/dt = Ax + Bu$, for almost every $t \in \mathbb{R}^+$.
- 228 • Let \mathcal{K} be a subspace¹⁴ of \mathcal{X}_d . A trajectory x of (3.5) is called *viable*
 229 *in* \mathcal{K} , if $x(t) \in \mathcal{K}$ for all $t \geq 0$. The set of such trajectories is called

¹³ Directly follows from (ALG-V) and [H2].

¹⁴ We restrict our discussion to subspaces of finite dimensional vector spaces. In (Frankowska, 1990) and in (Aubin & Frankowska, 1991) these definitions are stated in the more general framework of closed sets of normed vector spaces.

230 the set of viable solutions in \mathcal{X} . The subspace \mathcal{X} is called a *viability*
 231 *domain* of \mathbf{F} , if for all $x \in \mathcal{X} : \mathbf{F}(x) \cap \mathcal{X} \neq \emptyset$. The subspace \mathcal{X} is called
 232 *the viability kernel* of (3.5) when it is the largest viability domain of \mathbf{F} .

233 **Theorem 3.** (*Aubin & Frankowska, 1991*) *The supremal (A, E, B) -invariant*
 234 *subspace contained in $\mathcal{X}_d, \mathcal{V}_{\mathcal{X}_d}^*$, is the viability kernel of \mathcal{X}_d for the set-valued*
 235 *map $\mathbf{F} : \mathcal{X}_d \rightsquigarrow \mathcal{X}_d, \mathbf{F}(x) = E^{-1}(Ax + \mathcal{B})$. Moreover, for all $x_0 \in \mathcal{V}_{\mathcal{X}_d}^*$ there exists*
 236 *a trajectory $x \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{V}_{\mathcal{X}_d}^*)$ solution of (1.7) satisfying $x(0) = x_0$.*

237 *Frankowska (1990)* called a singular system “*strict*” when the viability
 238 kernel coincides with the whole descriptor space \mathcal{X}_d , namely

$$\mathcal{V}_{\mathcal{X}_d}^* = \mathcal{X}_d. \quad (3.6)$$

239 In order to clarify ideas, let us extract from (*Bonilla & Malabre, 1997,*
 240 *Section 2.1*) the following result:

241 **Result 1.** *There exists a subspace \mathcal{X}_1 such that:*

$$\mathcal{X}_d = \mathcal{V}_{\mathcal{X}_d}^* \oplus \mathcal{X}_1, \quad \mathcal{X}_{eq} = (E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) \oplus A\mathcal{X}_1, \quad \text{and} \quad \mathcal{X}_1 \approx A\mathcal{X}_1. \quad (3.7)$$

242 *Moreover, when projecting on \mathcal{X}_1 any trajectory $x \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{V}_{\mathcal{X}_d}^*)$ solution*
 243 *of (1.7), we always get a null trajectory.*

244 *Furthermore, for all $x_0 \in \mathcal{V}_{\mathcal{X}_d}^*$ there exists at least one trajectory $(u, x_\rho) \in$*
 245 *$\mathcal{C}^\infty(\mathbb{R}^+, \mathcal{U} \times \mathcal{V}_{\mathcal{X}_d}^*)$ solution of (1.7), satisfying $x_\rho(0) = x_0$.*

246 **PROOF OF RESULT 1.** From the algorithm shown in (Fig. 1, *Bonilla & Mal-*
 247 *abre, 1997*) and from [**H2**], we get the geometric decompositions (3.7), and
 248 under these decompositions, (1.7) takes the following form:

$$\left[\begin{array}{c|c} E_\rho & * \\ \hline 0 & \bar{X}_{\rho-1} \end{array} \right] \frac{d}{dt} \left[\begin{array}{c} x_\rho \\ \bar{x}_{\rho-1} \end{array} \right] = \left[\begin{array}{c|c} A_\rho & 0 \\ \hline 0 & I_1 \end{array} \right] \left[\begin{array}{c} x_\rho \\ \bar{x}_{\rho-1} \end{array} \right] + \left[\begin{array}{c} B_\rho \\ 0 \end{array} \right] u, \quad (3.8)$$

249 where $x_\rho \in \mathcal{V}_{\mathcal{X}_d}^*$, $\bar{x}_{\rho-1} \in \mathcal{X}_1$, $I_1 : \mathcal{X}_1 \leftrightarrow A\mathcal{X}_1$ is an isomorphism, $\bar{X}_{\rho-1}$ is a nilpo-
 250 tent matrix (an upper triangular matrix with zeros in its diagonal). Then
 251 $\bar{x}_{\rho-1} \equiv 0$.

252 If we now apply the following geometric decompositions:

$$E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} = E\mathcal{V}_{\mathcal{X}_d}^* \oplus \mathcal{B}_C, \quad \mathcal{B} = (\mathcal{B} \cap E\mathcal{V}_{\mathcal{X}_d}^*) \oplus \mathcal{B}_C, \quad \mathcal{U} = B^{-1}E\mathcal{V}_{\mathcal{X}_d}^* \oplus B^{-1}\mathcal{B}_C, \quad (3.9)$$

253 where \mathcal{B}_C is some complementary subspace of $\mathcal{B} \cap E\mathcal{V}_{\mathcal{X}_d}^*$, we get for $\mathfrak{R}^{imp}(E_\rho, A_\rho,$
 254 $B_\rho)$ (recall (3.8)):

$$\begin{bmatrix} \bar{E}_\rho \\ 0 \end{bmatrix} \frac{d}{dt} x_\rho = \begin{bmatrix} \bar{A}_\rho \\ \hat{A}_\rho \end{bmatrix} x_\rho + \left[\begin{array}{c|c} \bar{B}_\rho & 0 \\ \hline 0 & I \end{array} \right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \quad (3.10)$$

255 Since $\text{Im } \bar{E}_\rho = E\mathcal{V}_{\mathcal{X}_d}^*$, there exists $\bar{E}_\rho^r : E\mathcal{V}_{\mathcal{X}_d}^* \rightarrow \mathcal{V}_{\mathcal{X}_d}^*$ such that $\bar{E}_\rho \bar{E}_\rho^r = I$. Then,
 256 one solution of (3.10) is given by

$$\begin{aligned} x_\rho(t) &= \exp(\bar{E}_\rho^r \bar{A}_\rho t) x_0 + \int_0^t \exp(\bar{E}_\rho^r \bar{A}_\rho(t-\tau)) \bar{E}_\rho^r \bar{B}_\rho u_1(\tau) d\tau, \\ u_2(t) &= -\hat{A}_\rho x_\rho(t). \quad \square \end{aligned}$$

257 Thus, the subspaces $E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} \subset \mathcal{X}_{eq}$ and $\mathcal{V}_{\mathcal{X}_d}^* \subset \mathcal{X}_d$ characterize the set of
 258 all possible trajectories of (1.7) which are not identically zero for any input
 259 u . The projection of any trajectory solution of (1.7) on the quotient space
 260 $\mathcal{X}_d/\mathcal{V}_{\mathcal{X}_d}^*$, in correspondence with the projection on $\mathcal{X}_{eq}/(E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B})$ for the
 261 equation space, results in an identically null function (see Bonilla & Malabre,
 262 1995, Corollary 2.1). Let us note that when Assumption [H2] holds, the
 263 geometric conditions $E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} = \mathcal{X}_{eq}$ and $\mathcal{V}_{\mathcal{X}_d}^* = \mathcal{X}_d$ are equivalent¹⁵.

¹⁵ From (ALG-V) and [H2]: $E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} = \mathcal{X}_{eq} \Rightarrow \mathcal{V}_{\mathcal{X}_d}^* = A^{-1}(E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) = \mathcal{X}_d$; $\mathcal{V}_{\mathcal{X}_d}^* = \mathcal{X}_d$
 $\Rightarrow \text{Im } E = E\mathcal{V}_{\mathcal{X}_d}^* \ \& \ \text{Im } A = \text{Im } A \cap (E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) \Rightarrow \mathcal{X}_{eq} = \text{Im } E + \text{Im } A + \mathcal{B} = E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}$.

264 *3.3. Discussion about existence of solution*

265 An important contribution of Geerts (1993) is to give conditions under
266 which the distributional and time-domain frameworks lead to the same con-
267 clusions with respect to the shape of the resulting system's solution trajecto-
268 ries (*c.f.* (3.4) and (3.3)), namely the resulting distributions are identified as
269 ordinary functions, with support on \mathbb{R}^+ , and the generalized derivatives can
270 be identified with ordinary derivatives. Also, it is well connected with the
271 viability discussion of Frankowska (1990) and Aubin & Frankowska (1991);
272 indeed, a singular system is strict if and only if the consistent initial condi-
273 tion set \mathcal{I}_C coincides with the whole descriptor variable space \mathcal{X}_d (*c.f.* (3.6)
274 and (3.4), and recall Assumption [H2]).

275 Regarding the set of weakly consistent initial conditions Geerts (1993)
276 notes, in his abstract and conclusion, that the condition that this set equals
277 to the whole state space (under the Assumption [H2]) is equivalent to the
278 impulse controllability for regular systems (Cobb, 1984) (or controllability of
279 the infinite part in the sense of Verghese *et al* (1981)). This correspondence
280 has been generalized to non regular systems and one can note that the nowa-
281 days most commonly adopted definition for impulse controllability is the one
282 cited by Ishihara & Terra (2001)¹⁶: *a general singular system is impulse*
283 *controllable if for every initial condition there exists a smooth (impulse-free)*
284 *control $u(t)$, and a smooth (impulse-free, but with possible jumps, especially*
285 *at the origin) variable descriptor trajectory solution of the system.*

286 More generally, one can verify that the paper of Geerts (1993) is the

¹⁶Notice that in this paper is stated that the definition comes from Geerts (1993).

287 main reference on solvability properties, consistency of initial conditions, the
 288 ability to find control such that no impulsive phenomenon appears (see for
 289 examples [Hou & Müller \(1999\)](#), [Ishihara & Terra \(2001\)](#), [Hou \(2004\)](#) and
 290 [Zhang \(2006\)](#)).

291 However, one should also cite [Özçaldıran & Haliloğlu \(1993\)](#) who proved
 292 that there exists a pair of smooth distributions (without jumps), satisfying
 293 $\mathfrak{R}_{dist}^{imp}(E, A, B)$ if and only if $x(0_*) \in \mathcal{V}_{\mathcal{X}_d^*}$, namely $\mathcal{V}_{\mathcal{X}_d^*} = \mathcal{X}_d$ (see their Proposi-
 294 tion 1.3), and [Przyłuski & Sosnowski \(1994\)](#) who proved that the subspace
 295 $\mathcal{V}_{\mathcal{X}_d^*} + \mathcal{K}_E$ characterizes the set of initial conditions, for which there exists a
 296 pair of smooth distributions (with possible jumps) satisfying $\mathfrak{R}_{dist}^{imp}(E, A, B)$,
 297 namely $E\mathcal{V}_{\mathcal{X}_d^*} = \text{Im } E$ (see their Proposition 1).

298 In Figure 1, we summarize all the above discussion.

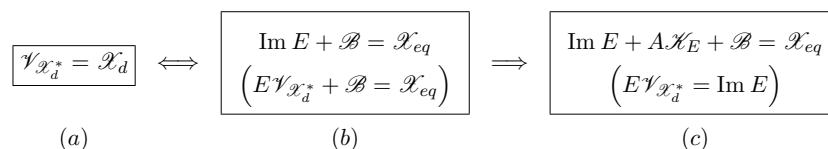


Figure 1: Connexions between the notions of existence of solution and impulse control-
 ability (under the Assumption **[H2]**).

299 Fig. 1(a) is the condition of viable solution of [Aubin & Frankowska \(1991\)](#)
 300 or smooth solution (without any jump) of [Özçaldıran & Haliloğlu \(1993\)](#). Fig.
 301 1(b) is the condition that the set of consistent initial condition equals the
 302 whole space of [Geerts \(1993\)](#). Fig. 1(c) is the condition of *C-solvability in the*
 303 *function sense* of [Geerts \(1993\)](#) or the condition of [Przyłuski & Sosnowski](#)
 304 [\(1994\)](#) that the set of initial conditions of smooth solutions (with possible
 305 jumps) equals the whole space, or the impulse controllability condition of
 306 [Ishihara & Terra \(2001\)](#), or the impulse-mode controllability with arbitrary

307 initial conditions of Hou (2004).

308 Finally let us note that if the notion of weakly consistent initial conditions
309 as defined by Geerts (1993) is associated to the notion of impulse controlla-
310 bility, the notion of consistent initial conditions as defined by Geerts (1993)
311 is associated to the notion of reachability of Frankowska (1990) (in the more
312 general non regular case) since the system must be strict to be reachable.
313 See also the controllability discussion found in Korotka *et al* (2011).

314 4. REACHABILITY FOR IMPLICIT SYSTEMS

315 For the case of implicit systems, Frankowska (1990) extended the classical
316 reachability definition as follows.

317 **Definition 4.** (Frankowska, 1990) The implicit representation (1.7) is called
318 *reachable* if for any pair of vectors $x_0, x_1 \in \mathcal{X}_d$ and for any pair of real numbers
319 $t_1 > t_0 \geq 0$, there exists a trajectory $x(\cdot)$ solution of (1.7), such that $x(t_0) = x_0$
320 and $x(t_1) = x_1$.

321 Frankowska (1990) has established in her Theorem 4.4 that $\mathcal{R}_{\mathcal{X}_d}^*$ (see (1.8))
322 is the reachable space of implicit systems like (1.7), with E and A not nec-
323 essarily square. Hereafter, we recall Corollary 2.4 of Aubin and Frankowska
324 (1991) which is *ad hoc* for our paper.

325 **Theorem 4.** (Aubin & Frankowska, 1991) For any $t_1 > 0$ and for a system
326 like (1.7), with E and A not necessarily square, the reachable space of (1.7)
327 at time t_1 from the initial descriptor variable $x(0)$ is equal to $\mathcal{R}_{\mathcal{X}_d}^*$. Moreover,
328 $\mathcal{R}_{\mathcal{X}_d}^*$ is the supremal subspace such that for all $x_0, x_1 \in \mathcal{R}_{\mathcal{X}_d}^*$ and $t_1 > 0$, there

329 exists a trajectory $x \in C^\infty(\mathbb{R}^+ \mathcal{R}_{\mathcal{X}_d}^*)$ solution of (1.7) satisfying $x(0) = x_0$ and
 330 $x(t_1) = x_1$.

331 In this Section we are interested in generalizing and solving Problem 1
 332 in the case of an *input/descriptor system* $\Sigma_{i/d} = (\mathbb{R}^+, \mathcal{U} \times \mathcal{X}_d, \mathfrak{B}_{[E,A,B]})$, with
 333 behavior (3.1).

334 **Problem 2.** Let us consider a *input/descriptor system* $\Sigma_{i/d} = (\mathbb{R}^+, \mathcal{U} \times \mathcal{X}_d,$
 335 $\mathfrak{B}_{[E,A,B]})$ represented by (1.7), and with the behavior (3.1). Given $(u_0, x_0),$
 336 $(u_1, x_1) \in B^{-1}E\mathcal{R}_{\mathcal{X}_d}^* \times \mathcal{R}_{\mathcal{X}_d}^*$ and $t_1 > 0$, find a trajectory $(u, x) \in \mathfrak{B}_{[E,A,B]}$, such
 337 that $(u(0), x(0)) = (u_0, x_0)$ and $(u(t_1), x(t_1)) = (u_1, x_1)$.

338 For answering this question, we proceed as follows.

339 (i) We first apply some geometric decompositions to the subspaces \mathcal{X}_d and
 340 \mathcal{X}_{eq} , inspired by Proposition 2.2 of Aubin and Frankowska (1991); the aim
 341 of these decompositions is to point out a part of the implicit representation,
 342 more or less explicit, which is expressed as a state space representation.

343 (ii) We next show that such a state space representation is reachable in the
 344 classical sense.

345 (iii) Finally, based on Section 2, we answer Problem 2.

346 4.1. State reachability

347 The following Lemma is proved in Appendix C.

348 **Lemma 3.** When $\mathcal{R}_{\mathcal{X}_d}^* = \mathcal{X}_d$, the implicit representation (1.7) can be re-
 349 stricted to $\mathcal{R}_{\mathcal{X}_d}^*$ in the domain, and to $A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}$ in the codomain.

350 Moreover, the spaces $\mathcal{R}_{\mathcal{X}_d}^*$, \mathcal{B} , $A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}$ and \mathcal{U} can be decomposed as fol-
 351 lows: $\mathcal{R}_{\mathcal{X}_d}^* = \mathcal{R}_C \oplus (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E)$, $\mathcal{B} = (\mathcal{B} \cap E\mathcal{R}_{\mathcal{X}_d}) \oplus \mathcal{B}_C$, $A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B} = E\mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{B}_C$

352 and $\mathcal{U} = B^{-1}E\mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{U}_C$, where \mathcal{R}_C and \mathcal{U}_C are complementary subspaces such
 353 that $\mathcal{R}_C \approx E\mathcal{R}_{\mathcal{X}_d}^*$ and $\mathcal{U}_C = B^{-1}\mathcal{B}_C \approx \mathcal{B}_C$. Under these decompositions, the im-
 354 plicit representation (1.7), restricted to $\mathcal{R}_{\mathcal{X}_d}^*$ in the domain and to $A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}$
 355 in the codomain, takes the following form:

$$\begin{aligned} \bar{E}dx/dt &= \bar{A}x + \bar{B}u, \\ \bar{E} &= \begin{bmatrix} I_C & 0 \\ 0 & 0 \end{bmatrix}, \bar{A} = \begin{bmatrix} \bar{A}_{1,1} & \bar{A}_{1,2} \\ \bar{A}_{2,1} & \bar{A}_{2,2} \end{bmatrix}, \bar{B} = \begin{bmatrix} \bar{B}_1 & 0 \\ 0 & I_{\mathcal{U}_C} \end{bmatrix}, \end{aligned} \quad (4.1)$$

356 where $I_C : \mathcal{R}_C \leftrightarrow E\mathcal{R}_{\mathcal{X}_d}^*$, and $I_{\mathcal{U}_C} : \mathcal{U}_C \leftrightarrow \mathcal{B}_C$ are isomorphisms.

357 In order to locate the state reachability part of (4.1), let us first define
 358 the natural projections:

$$\begin{aligned} P_C : \mathcal{R}_{\mathcal{X}_d}^* &\rightarrow \mathcal{R}_C // (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E), \quad P_l : \mathcal{R}_{\mathcal{X}_d}^* \rightarrow (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) // \mathcal{R}_C, \\ Q_1 : \mathcal{U} &\rightarrow B^{-1}E\mathcal{R}_{\mathcal{X}_d}^* // B^{-1}\mathcal{B}_C, \quad Q_2 : \mathcal{U} \rightarrow B^{-1}\mathcal{B}_C // B^{-1}E\mathcal{R}_{\mathcal{X}_d}^*. \end{aligned}$$

359 Let us next apply to $\mathfrak{R}^{imp}(\bar{E}, \bar{A}, \bar{B})$ the reachability algorithm of [Özçaldıran](#)
 360 (1985), $\bar{\mathcal{R}}^0 = \{0\}$, $\bar{\mathcal{R}}^{\mu+1} = \bar{E}^{-1} \left(\bar{A}\bar{\mathcal{R}}^\mu + (\bar{\mathcal{B}}_1 \oplus \mathcal{B}_C) \right)$, whose limit is $\mathcal{R}_{\mathcal{X}_d}^*$; namely:
 361 $\bar{\mathcal{R}}^1 = I_C^{-1}\bar{\mathcal{B}}_1 \oplus (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E)$ and $\bar{\mathcal{R}}^{\mu+1} = I_C^{-1} \left(\bar{A}_{1,1}P_C\bar{\mathcal{R}}^\mu + \text{Im} [\bar{A}_{1,2} \bar{B}_1] \right) \oplus (\mathcal{R}_{\mathcal{X}_d}^* \cap$
 362 $\mathcal{K}_E)$, for $\mu \geq 1$. We thus obtain $I_C P_C \bar{\mathcal{R}}^{\mu+1} = \bar{A}_{1,1}^\mu \text{Im} \bar{B}_1 + \sum_{i=0}^{\mu-1} \bar{A}_{1,1}^i \text{Im} [\bar{A}_{1,2} \bar{B}_1]$,
 363 which implies:

$$E\mathcal{R}_{\mathcal{X}_d}^* = \langle \bar{A}_{1,1} \mid \text{Im} [\bar{A}_{1,2} \bar{B}_1] \rangle. \quad (4.2)$$

364 Thus, $(\bar{A}_{1,1}, [\bar{A}_{1,2} \bar{B}_1])$ is a *state reachable pair*.

365 4.2. Behavioral reachability

366 Given any initial condition $x_0 \in \mathcal{R}_{\mathcal{X}_d}^*$, the solution set of (4.1) is charac-
 367 terized by the following behavior

$$\mathfrak{B}_{[\bar{E}, \bar{A}, \bar{B}]} = \left\{ (u, x) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{U} \times \mathcal{R}_{\mathcal{X}_d}^*) \mid \exists x_0 \in \mathcal{R}_{\mathcal{X}_d}^* \text{ s.t. } P_C x(t) = \exp(I_C^{-1} \bar{A}_{1,1} t) P_C x_0 + \int_0^t \exp(I_C^{-1} \bar{A}_{1,1}(t-\tau)) I_C^{-1} (\bar{A}_{1,2} P_\ell x(\tau) + \bar{B}_1 Q_1 u(\tau)) d\tau, \right. \\ \left. Q_2 u(t) = -I_{\mathcal{Q}_C}^{-1} (\bar{A}_{2,1} P_C x(t) + \bar{A}_{2,2} P_\ell x(t)) \right\}, \quad (4.3)$$

368 which behavioral equations are

$$\begin{aligned} \frac{d}{dt} I_C P_C x &= \bar{A}_{1,1} P_C x + \begin{bmatrix} \bar{A}_{1,2} & \bar{B}_1 \end{bmatrix} \begin{bmatrix} P_\ell x \\ Q_1 u \end{bmatrix}, \\ 0 &= \bar{A}_{2,1} P_C x + \bar{A}_{2,2} P_\ell x + I_{\mathcal{B}_C} Q_2 u. \end{aligned} \quad (4.4)$$

369 Let us note that

370 (i) the component $P_C x$ is the part of the descriptor variable which needs a
371 control law to reach the desired goal.

372 (ii) The component $P_\ell x$ is the free part of the descriptor variable which acts
373 as some kind of internal input variable, together with the component $Q_1 u$
374 which is the effective external control input variable.

375 (iii) The component $Q_2 u$ of the external control variable must be equal to
376 a component of the descriptor variable. This is because we have chosen a
377 purely integral description. This part of the input corresponds to algebraic
378 relationships linked with purely derivative actions.

379 From Lemmas 1 and 2, we get the following theorem which gives a solution
380 to Problem 2.

381 **Theorem 5.** Consider the reachable part (4.4) of the implicit representation
382 (1.7). Denote: $n = \dim(E\mathcal{R}_{\mathcal{X}_d}^*)$ and $m = \dim(\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) + \dim(B^{-1}E\mathcal{R}_{\mathcal{X}_d}^*)$.
383 Let $\{\kappa_1, \kappa_2, \dots, \kappa_m\} \subset \mathbb{Z}^+$ be the reachability indices of the pair $(\bar{A}_{1,1}, [\bar{A}_{1,2} \ \bar{B}_1])$,
384 with $\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_m \geq 1$ and $\kappa_1 + \kappa_2 + \dots + \kappa_m = n$.

385 Let the linear map $F: \mathcal{R}_C \rightarrow (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) \times (B^{-1}E\mathcal{R}_{\mathcal{X}_d}^*)$ and the isomor-
386 phisms $T: \mathcal{R}_C \leftrightarrow \mathcal{R}_C$ and $G: (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) \times (B^{-1}E\mathcal{R}_{\mathcal{X}_d}^*) \leftrightarrow (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) \times (B^{-1}$
387 $E\mathcal{R}_{\mathcal{X}_d}^*)$ be such that the pair $(\bar{A}_B, \bar{\Gamma}_B)$, where $\bar{A}_B = (I_C T)^{-1}(\bar{A}_{1,1} + [\bar{A}_{1,2} \bar{B}_1] F)$
388 $(I_C T)$ and $\bar{\Gamma}_B = (I_C T)^{-1}[\bar{A}_{1,2} \bar{B}_1] G$, is expressed in the Brunovsky canonical
389 form (1.2). The reachability matrix $\mathcal{R}_{[\bar{A}_B, \bar{\Gamma}_B]}$ is expressed in terms of the
390 reachability matrices $\mathcal{R}_{[\bar{A}_{B_i}, \bar{\gamma}_{B_i}]}$ as in (2.1).

391 Let $x_0, x_1 \in \mathcal{R}_{\mathcal{X}_d}^*$, $Q_1 u_0, Q_1 u_1 \in B^{-1}E\mathcal{R}_{\mathcal{X}_d}^*$, and $t_1 > 0$ be given. If we apply

$$\begin{bmatrix} P_t x(t) \\ Q_1 u(t) \end{bmatrix} = F P_C x(t) + G \mathfrak{D}(d/dt)f(t), \quad (4.5)$$

392 where $f(t) \in C^\infty(\mathbb{R}^+, \mathbb{R}^m)$ and $\mathfrak{D}(d/dt)$ are defined as in Lemmas 1 and 2, we
393 get

$$P_C x(t) = T \mathcal{R}_{[\bar{A}_B, \bar{\Gamma}_B]} \bar{w}(t), \quad (4.6)$$

394 and

$$(u(t_i), x(t_i)) = \left(\begin{bmatrix} Q_1 u_i \\ -I_{\mathcal{U}_C}^{-1} \begin{bmatrix} \bar{A}_{2,1} & \bar{A}_{2,2} \end{bmatrix} x_i \end{bmatrix}, x_i \right), \quad i \in \{0, 1\}, t_0 = 0. \quad (4.7)$$

395 4.3. Comments on the reachability

396 For the general case of implicit systems, represented by (1.7) with E and
397 A not necessarily square, Frankowska (1990) has been the first to give a
398 functional interpretation of reachability. For this, she has used the Viability
399 Theory. More precisely, she has shown that reachability is equivalent to
400 finding a trajectory $x \in C^\infty(\mathbb{R}^+, \mathcal{X}_d)$ solution of (1.7), starting from the initial
401 condition x_0 and reaching the desired x_1 in a given finite time t_1 , namely
402 $x(0) = x_0$ and $x(t_1) = x_1$ (see Theorem 4). Moreover, Frankowska (1990) has
403 shown that reachability is geometrically characterized by the well known

404 reachable space $\mathcal{R}_{x_d}^*$. Of course, $\mathcal{R}_{x_d}^*$ is contained in the viability kernel $\mathcal{V}_{x_d}^*$.
 405 This guarantees the existence of at least one trajectory solution of (1.7),
 406 leaving from x_0 . This is also clear from $\mathcal{R}_{x_d}^* = \mathcal{V}_{x_d}^* \cap \mathcal{S}_{x_d}^*$.

407 One interesting thing found in the proof of (Aubin & Frankowska, 1991,
 408 Proposition 2.2) was to put forward the importance of the state space rep-
 409 resentation (4.4) of the implicit equation(1.7). This fact has enabled us to
 410 apply systematically the results of the classical State Space Control The-
 411 ory. More precisely, thanks to the reachability of the pair $(\bar{A}_{1,1}, [\bar{A}_{1,2} \bar{B}_1])$
 412 (see (4.2)), it is possible to find trajectories $f_i \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^1)$ (see (2.7), (2.11),
 413 (2.12), (2.8), and (2.9)) for synthesizing the control law (4.5) (see also (2.4))
 414 which guarantees (4.7) (see Lemmas 1 and 2 and Theorem 5).

415 The aim of Theorem 5 was not to prove once more the sufficiency of The-
 416 orem 4, but to interpret the reachability of (1.7) in the classical state space
 417 framework. This interpretation allows us to have a better understanding of
 418 the existing mechanisms in the linear implicit systems reachability. Indeed,
 419 there exist two control actions. The first one is due to the free variable $P_\ell x$,
 420 and another one is due to the control input $Q_1 u$ (see (4.3)). The control input
 421 $Q_2 u$ is algebraically linked to the descriptor variable components, the state
 422 variable $P_C x$ and the free variable $P_\ell x$, by means of the algebraic restriction
 423 (4.4.b) (when it exists).

424 For systems composed by *infinite elementary divisors*¹⁷, the matrix Q_1

¹⁷ Kronecker showed that any pencil $[\lambda E - A]$, $\lambda \in \mathbb{C}$, is strictly equivalent to a canon-
 ical matrix, composed by four kind of blocks: (i) *finite elementary divisors* (integral ac-
 tions), e.g. $\begin{bmatrix} (\lambda - \alpha) & 1 \\ 0 & (\lambda - \alpha) \end{bmatrix}$, (ii) *infinite elementary divisors* (derivative actions), e.g.

425 is null and the square matrix Q_2 is invertible. In this case, the equations
 426 (4.5) and (4.6) describe the behavior of a system fed-back by the control law
 427 (4.4b). Indeed, from (4.5) and (4.6), we get:

$$x(t) = \begin{bmatrix} I \\ F \end{bmatrix} T\mathcal{R}_{[\bar{A}_B, \bar{\Gamma}_B]} \bar{w}(t) + \begin{bmatrix} 0 \\ G \end{bmatrix} \mathfrak{D}(d/dt)f(t).$$

428 And from (4.4b), (4.5) and (4.6), we have:

$$Q_2 u(t) = -I_{\mathcal{U}_C}^{-1} \left((\bar{A}_{2,1} + \bar{A}_{2,2}F) T\mathcal{R}_{[\bar{A}_B, \bar{\Gamma}_B]} \bar{w}(t) + \bar{A}_{2,2}G \mathfrak{D}(d/dt)f(t) \right).$$

429 It is remarkable that in the systems represented by *column minimal in-*
 430 *dices*, it is possible to have reachable systems without any control. This
 431 phenomenon is possible because of the existence of the free variable $P_\ell x$,
 432 which acts as an internal control signal.

433 5. POLE ASSIGNMENT

434 One of the most important features of the reachability of a state space
 435 representation (1.1) is the complete assignability of the closed loop spectrum
 436 by means of a state feedback. This equivalence is no longer the case when
 437 dealing with implicit representations (1.7). For the implicit description case,
 438 a geometric condition has to be added in order to guarantee such a *pole*
 439 *assignment ability*. In the sequel we give geometric conditions, which enable

$\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$, (iii) *column minimal indices* (internal variable structure), e.g. $\begin{bmatrix} \lambda & 1 \end{bmatrix}$, and
 (iv) *row minimal indices* (internal behavioral restrictions), e.g. $\begin{bmatrix} \lambda \\ 1 \end{bmatrix}$; see [Gantmacher](#)
 (1977).

440 us to assign the closed loop spectrum of: (i) a reachable implicit description
 441 (1.7), and (ii) a reachable and observable implicit description with output
 442 equation, $\mathfrak{R}^{imp}(E, A, B, C)$:

$$Edx/dt = Ax + Bu, \quad y = Cx, \quad (5.8)$$

443 where $C : \mathcal{X}_d \rightarrow \mathcal{Y}$ is a linear map, and the linear space \mathcal{Y} is the output space.

444 At this point, it is useful to clarify what we mean by spectrum and ob-
 445 servable part.

446 *A. Spectrum.* We distinguish between the finite spectrum, $\sigma_f(A, E) = \{\lambda \in \mathbb{C} \mid \exists$
 447 $v \neq 0 \text{ s.t. } Av = \lambda Ev\}$, and the infinite spectrum, $\sigma_\infty(E, A) = \{\mu \in \mathbb{C} \mid \exists w \neq 0 \text{ s.t.}$
 448 $Aw = \mu Ew\}$ (c.f. [Gantmacher \(1977\)](#), [Wong \(1974\)](#), [Armentano \(1986\)](#)); the
 449 elements of $\sigma_f(A, E)$ are called poles, and the elements of $\sigma_\infty(E, A)$ are called
 450 poles at infinity. Note that for the four kind of blocks of the Kronecker
 451 canonical form¹⁸: (i) $\sigma_f(A, E) = \emptyset$ and $\sigma_\infty(E, A) = \emptyset$ for its *row minimal in-*
 452 *dices* blocks, (ii) $\sigma_f(A, E) = \emptyset$ for its *infinite elementary divisors* blocks, (iii)
 453 $\text{card}\{\sigma_f(A, E)\} = \infty$ and $\text{card}\{\sigma_\infty(E, A)\} = \infty$ for its *column minimal indices*
 454 blocks.

455 *B. Observable part.* With respect to the observable part, let us recall that it
 456 was shown in ([Bonilla & Malabre, 1995](#)) that the third condition of [Kuijper](#)
 457 ([1992a](#)) – $\begin{bmatrix} sE - A \\ C \end{bmatrix}$ has full column rank for all $s \in \mathbb{C}$ – for getting a mini-
 458 mal implicit representation (among all externally equivalent¹⁹ representations

¹⁸ See footnote 17.

¹⁹ Two representations are called externally equivalent if the corresponding sets of all possible trajectories for the external variables, expressed in an *input/output partition* (u, y) , are the same ([Willems, 1983](#), [Polderman & Willems, 1998](#)).

459 of the same type), is equivalent to have a null unobservable space, namely:
 460 $\mathcal{N} = \{0\}$. Indeed, if we decompose the descriptor and equation spaces as:
 461 $\mathcal{X}_d = \mathcal{X}_{ob} \oplus \mathcal{N}$ and $\mathcal{X}_{eq} = \mathcal{W}_{ob} \oplus E\mathcal{N}$, where \mathcal{X}_{ob} and \mathcal{W}_{ob} are some complemen-
 462 tary subspaces, (5.8) takes the following form:

$$\begin{aligned} \begin{bmatrix} E_{ob} & 0 \\ Z & E_{\mathcal{N}} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x_{ob} \\ x_{\mathcal{N}} \end{bmatrix} &= \begin{bmatrix} A_{ob} & 0 \\ X & A_{\mathcal{N}} \end{bmatrix} \begin{bmatrix} x_{ob} \\ x_{\mathcal{N}} \end{bmatrix} + \begin{bmatrix} B_{ob} \\ Y \end{bmatrix} u \\ y &= \begin{bmatrix} C_{ob} & 0 \end{bmatrix} \begin{bmatrix} x_{ob} \\ x_{\mathcal{N}} \end{bmatrix} \end{aligned} \quad (5.9)$$

463 And the implicit descriptions $\mathfrak{R}^{imp}(E, A, B, C)$ and $\mathfrak{R}^{imp}(E_{ob}, A_{ob}, B_{ob}, C_{ob})$ are
 464 externally equivalents (c.f. [Bonilla & Malabre, 1995](#), Theorem 2.1). The point
 465 we want to enlighten here is that, since $E_{\mathcal{N}}$ is epic, there then exists $E_{\mathcal{N}}^r$ such
 466 that $E_{\mathcal{N}} E_{\mathcal{N}}^r = I$, which implies that all the homogeneous trajectories of (5.9),
 467 beginning at any initial condition $\begin{bmatrix} 0 \\ x_0 \end{bmatrix} \in \mathcal{N}$, $x_{\mathcal{N}}(t) = \exp(E_{\mathcal{N}}^r A_{\mathcal{N}} t) x_0$, al-
 468 ways remain inside $\mathcal{N} \subset \mathcal{K}_C$. Thus, like in the classical state representations,
 469 they are called unobservable trajectories; and since \mathcal{N} is the supremal (A, E)
 470 invariant subspace contained in \mathcal{K}_C with this property, $\mathfrak{R}^{imp}(E_{ob}, A_{ob}, B_{ob}, C_{ob})$
 471 is called the *observable part* of $\mathfrak{R}^{imp}(E, A, B, C)$.

472 5.1. Pole Assignment for a Reachable Implicit Description

473 **Theorem 6.** ([Bonilla & Malabre, 1993](#)) *Given an implicit system repre-*
 474 *sented by (1.7), for every finite symmetric (with respect to the real line) set*
 475 *of complex numbers Λ of cardinality $\dim(\mathcal{R}_{\mathcal{X}_d}^*)$, there exists a proportional and*
 476 *derivative descriptor feedback $u = F_p x + F_d dx/dt$, such that $\sigma_f(A_{F_p}, E_{F_d}) = \Lambda$, if*
 477 *and only if*

$$\mathcal{R}_{\mathcal{X}_d}^* = \mathcal{X}_d, \quad (5.10)$$

$$\dim(E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) \geq \dim(\mathcal{V}_{\mathcal{X}_d}^*). \quad (5.11)$$

479 [Bonilla & Malabre \(1993\)](#) named this property *external reachability*. In
480 that paper, condition (5.11) is expressed in its equivalent form:

$$\dim(\mathcal{B}/(\mathcal{B} \cap E\mathcal{V}_{\mathcal{X}_d}^*)) \geq \dim(\mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{K}_E). \quad (5.12)$$

481 Let us note that the geometric condition (5.10) is the *reachability* con-
482 dition of [Frankowska \(1990\)](#) (*c.f.* Theorem 4) and the geometric condi-
483 tion (5.11) is the *descriptor variable uniqueness* condition of [Lebret \(1991\)](#),
484 namely the closed loop left invertibility property, which enables us to assign
485 the poles by means of a proportional and derivative feedback.

486 **Lemma 4.** ([Lebret, 1991](#)) *There exists a proportional and derivative descrip-*
487 *tor feedback* $u = F_p x + F_d dx/dt + v$, *such that the fed-back implicit represen-*
488 *tation* $\mathfrak{R}^{imp}(E_{F_d}, A_{F_p}, B)$ *satisfies* $\text{Ker}(\lambda E_{F_d} - A_{F_p}) = \{0\}$ *iff (5.11) is satisfied.*

489 Let us also note that in the case of a strict singular system, the ge-
490 ometric condition (5.11) is translated to (*c.f.* (3.6), (3.4) and Fig. 1):
491 $\dim(\mathcal{X}_{eq}) \geq \dim(\mathcal{X}_d)$. In other words, it is not possible to assign all the spec-
492 trum of an implicit system having one degree of freedom, as for example the
493 ones considered in ([Bonilla & Malabre, 2003](#)).

494 We have the following Corollary of Theorem 6, proved in [Appendix D](#).

495 **Corollary 1.** *Let the implicit representation (1.7) satisfy the geometric con-*
496 *ditions (5.10) and (5.11). Then:*

- 497 1. *If* $\mathcal{U}_c = \{0\}$, *the implicit representation (4.1) reduces to the following*
498 *reachable state space representation* ($\overline{\mathcal{B}}_1 = \text{Im } \overline{B}_1$):

$$dx/dt = \overline{A}_{1,1}x + \overline{B}_1u \quad \text{with} \quad \langle \overline{A}_{1,1} \mid \overline{\mathcal{B}}_1 \rangle = \mathcal{X}_d. \quad (5.13)$$

499 2. If $\mathcal{U}_C \neq \{0\}$, there exists a map $\bar{V}_\ell : \mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E \rightarrow \mathcal{B}_C$ such that $\text{Ker } \bar{V}_\ell =$
500 $\{0\}$. Then, applying the proportional feedback

$$u = \begin{bmatrix} 0 & 0 \\ -I_{\mathcal{U}_C}^{-1} \bar{A}_{2,1} & -I_{\mathcal{U}_C}^{-1} (\bar{A}_{2,2} + \bar{V}_\ell) \end{bmatrix} x + v, \quad (5.14)$$

501 we get

$$\begin{bmatrix} I_C & 0 \\ 0 & 0 \end{bmatrix} dx/dt = \begin{bmatrix} \bar{A}_{1,1} & 0 \\ 0 & -I \end{bmatrix} x + \begin{bmatrix} \bar{B}_1 & \bar{A}_{1,2} \bar{V}_\ell^g I_{\mathcal{U}_C} \\ 0 & \bar{V}_\ell^g I_{\mathcal{U}_C} \end{bmatrix} v, \quad (5.15)$$

502 where $\bar{V}_\ell^g : \mathcal{B}_C \rightarrow \mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E$ is some left inverse of \bar{V}_ℓ , and

$$I_C \mathcal{R}_C = E \mathcal{R}_{\mathcal{X}_d}^* = \langle \bar{A}_{1,1} \mid \bar{\mathcal{B}}_1 + \bar{A}_{1,2} (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) \rangle \quad \text{and} \quad \mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E = \bar{V}_\ell^g I_{\mathcal{U}_C} \mathcal{U}_C. \quad (5.16)$$

503 Furthermore, applying the proportional and derivative feedback

$$u = \begin{bmatrix} 0 & 0 \\ -I_{\mathcal{U}_C}^{-1} \bar{A}_{2,1} & -I_{\mathcal{U}_C}^{-1} (\bar{A}_{2,2} + \bar{V}_\ell) \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & -I_{\mathcal{U}_C}^{-1} \bar{V}_\ell \end{bmatrix} dx/dt + v, \quad (5.17)$$

504 we get

$$dx/dt = \begin{bmatrix} \bar{A}_{1,1} & \bar{A}_{1,2} \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} \bar{B}_1 & 0 \\ 0 & \bar{V}_\ell^g I_{\mathcal{U}_C} \end{bmatrix} v, \quad (5.18)$$

505 with

$$\left\langle \begin{bmatrix} \bar{A}_{1,1} & \bar{A}_{1,2} \\ 0 & 0 \end{bmatrix} \middle| \text{Im} \begin{bmatrix} \bar{B}_1 & 0 \\ 0 & \bar{V}_\ell^g I_{\mathcal{U}_C} \end{bmatrix} \right\rangle = \langle \bar{A}_{1,1} \mid \bar{\mathcal{B}}_1 + \bar{A}_{1,2} (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) \rangle \oplus \mathcal{U}_C = \mathcal{X}_d. \quad (5.19)$$

506 From this Corollary, we realize that with a proportional feedback, we
507 can only modify the finite spectrum of $\bar{A}_{1,1} = R \bar{A} \mid (\mathcal{R}_{\mathcal{X}_d}^* / \mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E)$, where
508 $R : A \mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B} \rightarrow E \mathcal{R}_{\mathcal{X}_d}^* / \mathcal{B}_C$ is the natural projection. To assign all the finite
509 spectrum of \bar{A} , we need a proportional and derivative feedback.

510 *5.2. Pole Assignment for a Reachable and Observable Implicit Description*

511 In this section, we are going to consider the reachability of the observable
 512 part after feedback, of the implicit representation (5.8). For this, let us
 513 recall that the supremal (A, E, B) -invariant subspace contained in $\text{Ker } C$, \mathcal{V}^*
 514 $= \sup\{\mathcal{V} \subset \mathcal{H}_C \mid A\mathcal{V} \subset E\mathcal{V} + \text{Im } B\}$, that characterizes the biggest part of a given
 515 implicit representation $\mathfrak{R}^{imp}(E, A, B, C)$, can be made unobservable by means
 516 of a suitable proportional and derivative descriptor feedback (*c.f.* the early
 517 Geometric Algorithms Section).

518 Given a proportional and derivative descriptor feedback $u = F_p^*x + F_d^*dx/dt$,
 519 where $(F_p^*, F_d^*) \in \mathbf{F}(\mathcal{V}^*)$, let us consider the *quotient implicit representation*
 520 $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$, where the linear applications E_* , A_* , B_* and C_* are the
 521 induced maps uniquely defined by

$$E_*\Phi = \Pi E_{F_d^*}, \quad A_*\Phi = \Pi A_{F_p^*}, \quad B_* = \Pi B, \quad \text{and} \quad C = C_*\Phi, \quad (5.20)$$

522 where $\Phi : \mathcal{X}_d \rightarrow \mathcal{X}_d/\mathcal{V}^*$ and $\Pi : E\mathcal{X}_d \rightarrow E\mathcal{X}_d/E_{F_d^*}\mathcal{V}^*$ are the canonical projec-
 523 tions. In [Appendix E](#), we prove the following Theorem.²⁰

524 **Theorem 7.** *Given an implicit system represented by (5.8), for every sym-*
 525 *metric (with respect to the real line) set of complex numbers Λ of cardinal-*
 526 *ity $\dim((\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{V}^*)/\mathcal{V}^*)$, there exists a proportional and derivative descriptor*
 527 *feedback $u = F_p^*x + F_d^*dx/dt + v$, with $(F_p^*, F_d^*) \in \mathbf{F}(\mathcal{V}^*)$, such that $\sigma_f(A_*, E_*) = \Lambda$,*
 528 *where E_* and A_* are the induced maps (5.20), if and only if:*

$$(\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{V}^*)/\mathcal{V}^* = \mathcal{X}_d/\mathcal{V}^*, \quad (5.21)$$

529

$$\dim\left((E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B})/(E\mathcal{V}^* + \mathcal{B})\right) + \dim(\mathcal{B}) \geq \dim\left(\mathcal{V}_{\mathcal{X}_d}^*/\mathcal{V}^*\right). \quad (5.22)$$

²⁰For a related result for regular systems see [Schumacher \(1980\)](#).

530 Let us note that (5.22) is equivalent to:²¹

$$\dim \left(\mathcal{B} / (\mathcal{B} \cap E\mathcal{V}_{\mathcal{X}_d}^*) \right) \geq \dim \left(\mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{K}_E \right) - \dim \left(\mathcal{V}^* \cap E^{-1}\mathcal{B} \right). \quad (5.23)$$

531 For the implicit representations (5.8), satisfying Theorem 7, we will say that
 532 they have the *externally reachable output dynamics property*²². Theorem 7
 533 is important because it enables us to tackle systems having an internal vari-
 534 able structure (see for example Bonilla & Malabre (1991), Bonilla & Malabre
 535 (2003), and Bonilla & Malabre (2008)). Let us also note that the geomet-
 536 ric condition (5.22) is the *descriptor variable uniqueness property* notion of
 537 Lebret (1991), namely the closed loop left invertibility property of the ob-
 538 servable part of the system.

539 **Lemma 5.** (Lebret, 1991) *There exists a proportional and derivative de-*
 540 *scriptor feedback $u = F_p x + F_d dx/dt + v$, such that the fed-back implicit rep-*
 541 *resentation $\mathfrak{R}^{imp}(E_{F_d}, A_{F_p}, B)$ satisfies $\text{Ker}(\lambda E_{F_d} - A_{F_p}) \subset \mathcal{N}_{(F_p, F_d)}$ iff (5.22) is*
 542 *satisfied.*

543 Let us finally note that, when comparing (5.22) with (5.11), we realize
 544 that Theorem 7 is indeed establishing the external reachabilty of the observ-
 545 able part after feedback. Also note that in the case $\mathcal{V}^* = \{0\}$, (5.22) and

²¹ This equivalence follows from the equivalence between (5.11) and (5.12), and from the fact that $\mathcal{B} \cap E\mathcal{V}^* = E(\mathcal{V}^* \cap E^{-1}\mathcal{B})$ implies that $\dim(\mathcal{V}^* \cap E^{-1}\mathcal{B}) = \dim(\mathcal{V}^*) + \dim(\mathcal{B} \cap \text{Im } E) - \dim(E\mathcal{V}^* + \mathcal{B} \cap \text{Im } E)$.

²² The *externally reachable output dynamics* notion is a simplification of the one of *reachable with output dynamics assignment* (see Bonilla et al, 1994, Definition 6).

546 (5.11) are the same; and in the case $\mathcal{V}^* = \mathcal{V}_{\mathcal{X}_d}^*$, we get the trivial condition
 547 $\dim(\mathcal{B}) \geq 0$.

548 Let us finish this Section with an academic example.

549 *Academic Example.* Let us consider a perturbed linear system represented
 550 by the state space representation, $\mathfrak{R}^{ss}(\bar{A}, [\bar{B} \ \bar{S}], \bar{C})$:

$$d\bar{x}/dt = \bar{A}\bar{x} + \begin{bmatrix} \bar{B} & \bar{S} \end{bmatrix} \begin{bmatrix} u \\ q \end{bmatrix} \quad \text{and} \quad y = \bar{C}\bar{x}, \quad (5.24)$$

551 where $q \in \mathcal{Q} \approx \mathbb{R}^n$, $u \in \mathcal{U} \approx \mathbb{R}^m$, $y \in \mathcal{Y} \approx \mathbb{R}^p$ and $\bar{x} \in \bar{\mathcal{X}} \approx \mathbb{R}^{\bar{n}}$, are the distur-
 552 bance, the input, the output, and the state variables, respectively. We as-
 553 sume that the three following assumptions hold true:

554 **[H1]** $\text{Ker } \bar{B} = \{0\}$ and $\text{Ker } \bar{S} = \{0\}$,

555 **[H2]** $q(\cdot) \in \mathcal{C}^m(\mathbb{R}^+, \mathcal{Q})$, $q(t)$, $dq(t)/dt$, \dots , $d^m q(t)/dt^m \in \mathcal{L}^\infty$, $\forall t \geq 0$,

556 **[H3]** q is a measured disturbance.

557 We want to solve the Disturbance Decoupling Problem with a PD Feed-
 558 back (DDP-PDF).

559 **Problem 3 (DDP-PDF).** Under which conditions does there exist a pro-
 560 portional and derivative feedback $u = (\bar{F}_{p1} + \bar{F}_{d1}d/dt)\bar{x} + (\bar{F}_{p2} + \bar{F}_{d2}d/dt)q + v$,
 561 such that the closed-loop transfer function matrix between q and y is identi-
 562 cally zero, and the finite spectrum of the observable part of the closed loop
 563 system is assigned at will.

564 For solving this problem, let us rewrite (5.24) in the descriptor form (5.8)
 565 with

$$E = \begin{bmatrix} I_{\bar{n}} & | & 0 \end{bmatrix}, \quad A = \begin{bmatrix} \bar{A} & | & \bar{S} \end{bmatrix}, \quad B = \begin{bmatrix} \bar{B} \end{bmatrix}, \quad C = \begin{bmatrix} \bar{C} & | & 0 \end{bmatrix}, \quad (5.25)$$

566 where $x = \begin{bmatrix} \bar{x}^T & q^T \end{bmatrix}^T \in \mathcal{X}_d = \overline{\mathcal{X}} \oplus \mathcal{Q} \approx \mathbb{R}^{\bar{n}+\eta}$ and $\mathcal{X}_{eq} = \overline{\mathcal{X}} \approx \mathbb{R}^{\bar{n}}$. In this im-
567 plicit representation, the perturbation q is acting as the free part of the de-
568 scriptor variable x . Then from Theorem 7, the **DDP-PDF** is solvable if and
569 only if the implicit representation (5.8) and (5.25) satisfies (5.21) and (5.22),
570 namely if and only if both following conditions hold true (see Appendix F):

$$\langle \overline{A} \mid \text{Im} [\overline{B} \overline{S}] \rangle + \overline{\mathcal{V}}_{[\overline{B} \overline{S}]}^* = \overline{\mathcal{X}}, \quad (5.26)$$

571

$$\dim \left(\overline{\mathcal{V}}_{[\overline{B} \overline{S}]}^* \cap \overline{\mathcal{B}} \right) \geq \dim \left(\frac{\text{Im} \overline{S}}{\text{Im} \overline{S} \cap (\overline{\mathcal{V}}_{[\overline{B} \overline{S}]}^* + \overline{\mathcal{B}})} \right). \quad (5.27)$$

Let us consider for example: $\overline{A} = \text{Tr}_u \{e_3^2\}$, $\overline{S} = ae_3^1 + be_3^2$, with $|a| + |b| \neq 0$, $\overline{B} = e_3^3$
and $\overline{C} = (e_3^1)^T$. We have for this case $\text{Im} \overline{S} = \text{span} \{ae_3^1 + be_3^2\}$, $\overline{\mathcal{B}} = \text{span} \{e_3^3\}$,
and $\text{Im} [\overline{B} \overline{S}] = \text{span} \{ae_3^1 + be_3^2, e_3^3\}$, then $\langle \overline{A} \mid \text{Im} [\overline{B} \overline{S}] \rangle = \text{span} \{e_3^1, e_3^2, e_3^3\} = \overline{\mathcal{X}}$,
 $\overline{\mathcal{V}}_{[\overline{B} \overline{S}]}^* = \text{span} \{ae_3^2, e_3^3\}$, $\overline{\mathcal{V}}_{[\overline{B} \overline{S}]}^* \cap \overline{\mathcal{B}} = \text{span} \{e_3^3\}$, and $\text{Im} \overline{S} \cap (\overline{\mathcal{V}}_{[\overline{B} \overline{S}]}^* + \overline{\mathcal{B}}) = \{0\}$.
Therefore (5.26) and (5.27) are satisfied, and the DDP-PDF has solution.
Indeed, applying to (5.24) and (5.25) the PD feedback

$$u = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} d\bar{x}/dt + \begin{bmatrix} -1/\tau & 0 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 1/\tau \end{bmatrix} v,$$

we obtain the closed loop system described by:

$$\tau dy/dt + y = v, \quad \bar{x}_1 = y, \quad \bar{x}_2 = dy/dt - aq, \quad \text{and} \quad \bar{x}_3 = d^2y/dt^2 - adq/dt - bq.$$

572 Let us note that $\overline{\mathcal{V}}_{\overline{B}}^* = \{0\}$, and that $\text{Im} \overline{S} \cap (\overline{\mathcal{V}}_{\overline{B}}^* + \overline{\mathcal{B}}) = \{0\}$, so there is no
573 purely proportional solutions (see for example Wonham (1985)).

574 6. CONCLUDING REMARKS

575 The notion of reachability introduced by Frankowska (1990) generalizes
576 the property introduced by Yip & Sincovec (1981) in the regular case. Fur-
577 thermore, Cobb (1984) indicates that this last property is consistent with

578 that of [Rosenbrock \(1974\)](#) introduced in a purely structural framework. In
579 the same paper, [Cobb \(1984\)](#) enlightens with time domain characterizations
580 the difference between the reachability in the sense of [Rosenbrock \(1974\)](#)
581 and the reachability in the sense of [Verghese *et al* \(1981\)](#) based, once again,
582 on pure structural tools (Kronecker canonical forms and/or Smith canonical
583 forms). In the regular case, for which the system can be decomposed
584 into two parts, a finite or slow subsystem, and an infinite or fast subsystem,
585 [Cobb \(1984\)](#) showed that [Rosenbrock \(1974\)](#) reachability is equivalent
586 to the reachability of the finite part and controllability of the infinite part.
587 He also showed that [Verghese *et al* \(1981\)](#) reachability is equivalent to the
588 reachability of the finite part associated to the impulse controllability of the
589 infinite part. The impulse controllability as defined by [Cobb \(1984\)](#), or the
590 controllability of the infinite part in the sense of [Verghese *et al* \(1981\)](#) is not
591 any more defined by the idea to reach a desired descriptor variable but by the
592 ability of the system to generate a maximal class of impulses using piecewise
593 smooth, non impulsive controls.

594 One can deduce from this analysis that if a regular system is reachable
595 (reachability of the finite and controllability of the infinite part) in the sense
596 of [Cobb \(1984\)](#), [Yip & Sincovec \(1981\)](#), [Rosenbrock \(1974\)](#) or [Frankowska
597 \(1990\)](#) (the four notions are equivalent in this case) then any vector is a consistent
598 initial condition in the sense of [Geerts \(1993\)](#). The converse implication
599 is not true. In general, reachability is not a consequence of the fact that every
600 vector of the descriptor space defines a consistent initial condition. The
601 condition is necessary but not sufficient for reachability.

602 In this paper we have given a geometric interpretation of the implicit sys-

603 tems reachability Theorem of Frankowska (1990) and we have also found some
 604 interesting connections between the works (Frankowska, 1990) and (Geerts,
 605 1993). The geometric interpretation has enabled us to have a better under-
 606 standing of the existing mechanisms in the linear implicit systems reachabil-
 607 ity. For this, we have first interpreted the *viability* notion from a geometric
 608 point of view. We have next solved Problem 2, with Theorem 5, which is a
 609 generalization of Problem 1, solved with Theorem 1.

610 We have also studied the existing relationships, between the reachabil-
 611 ity property and the capability of the complete pole assignment ability. In
 612 Theorem 6, we have considered the pole assignment problem of a reachable
 613 implicit description, $\mathfrak{R}^{imp}(E, A, B)$; we have also shown in Corollary 1, that
 614 with a proportional feedback, we can only modify the spectrum of the re-
 615 striction to $\mathcal{R}_{\mathcal{X}_d}^*/\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E$ in the domain and $E\mathcal{R}_{\mathcal{X}_d}^*$ in the co-domain; to
 616 assign all the spectrum, we need a proportional and derivative feedback. In
 617 Theorem 7, we have considered the pole assignment problem of a reachable
 618 and observable implicit description with output equation, $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$.

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738 Appendix A. Proof of Lemma 1

739 For the existence of such F_B , T_B and G_B , see for example Theorems 5.9 and
740 5.10 and Corollary 5.3 of Wonham (1985). Doing the change of state variable:
741 $T_B^{-1}x = \xi = \begin{bmatrix} \xi_1^T & \dots & \xi_m^T \end{bmatrix}^T$, we obtain the following set of closed loop state
742 space representations (see (1.1), (1.2), and (2.4)): $d\xi_i/dt = A_{B,i}\xi_i + b_{B,i}d^{\kappa_i}f_i/dt^{\kappa_i}$,
743 $i \in \{1, \dots, m\}$, which solutions are (integrate by parts n_i times each solution):

$$\begin{aligned}
\xi_i(t) &= \exp(A_{B,i}t)\xi_i(0) + \int_0^t \exp(A_{B,i}(t-\tau))b_{B,i}\frac{d^{\kappa_i}f_i(\tau)}{d\tau^{\kappa_i}}d\tau \\
&= \exp(A_{B,i}t)\left(\xi_i(0) - \sum_{j=0}^{\kappa_i-1} A_{B,i}^j b_{B,i} \frac{d^{\kappa_i-(j+1)}f_i(0)}{dt^{\kappa_i-(j+1)}}\right) + \sum_{j=0}^{\kappa_i-1} A_{B,i}^j b_{B,i} \frac{d^{\kappa_i-(j+1)}f_i(t)}{dt^{\kappa_i-(j+1)}} \\
&= \exp(A_{B,i}t)\left(\xi_i(0) - \mathcal{R}_{[A_{B,i}, b_{B,i}]} \bar{w}_i(0)\right) + \mathcal{R}_{[A_{B,i}, b_{B,i}]} \bar{w}_i(t), \quad i \in \{1, \dots, m\}, \\
\xi(t) &= \exp(A_B t)\left(\xi(0) - \mathcal{R}_{[A_B, B_B]} \bar{w}(0)\right) + \mathcal{R}_{[A_B, B_B]} \bar{w}(t), \\
x(t) &= \exp((A + BF_B)t)\left(x(0) - T_B \mathcal{R}_{[A_B, B_B]} \bar{w}(0)\right) + T_B \mathcal{R}_{[A_B, B_B]} \bar{w}(t).
\end{aligned} \tag{A1}$$

744 Therefore, (A1), (2.2) and (2.3) imply (2.5) and (2.6). \square

745 Appendix B. Proof of Lemma 2

746 Let us first compute $\det(X_{(i,1)}(t))$, for $\kappa_i \geq 2$. For this, we first do the decom-
747 position $X_{(i,1)}(t) = D_{i,\ell}(t)\tilde{X}_{(i,\kappa_i+1)}D_{i,r}(t)$, $i \in \{1, \dots, m\}$, where $D_{i,\ell}(t) =$
748 $\text{DM}\left\{\frac{t}{(\kappa_i+1)!}, \dots, t^{\kappa_i+1}/(2\kappa_i+1)!\right\}$, $D_{i,r}(t) = \text{DM}\left\{(2\kappa_i+1)!t^{\kappa_i}, \dots, (\kappa_i+1)!\right\}$ and

$$\tilde{X}_{(i,\kappa_i+1)} = \begin{bmatrix} (\kappa_i+1)!/(\kappa_i+1)! & \dots & (\kappa_i+1)!/1! \\ \vdots & \dots & \vdots \\ (2\kappa_i+1)!/(2\kappa_i+1)! & \dots & (\kappa_i+1)!/(\kappa_i+1)! \end{bmatrix}. \tag{B1}$$

749 Defining the following column elementary matrices:

$$\begin{aligned}
750 \quad T_{i,1} &= \left[\begin{array}{c|c|c|c|c} \underline{e}_{\kappa_i+1}^1 & (\underline{e}_{\kappa_i+1}^2 - \underline{e}_{\kappa_i+1}^1) & (\underline{e}_{\kappa_i+1}^3 - \kappa_i \underline{e}_{\kappa_i+1}^2) & \cdots & (\underline{e}_{\kappa_i+1}^{\kappa_i+1} - 2\underline{e}_{\kappa_i+1}^{\kappa_i}) \end{array} \right], \quad T_{i,2} = \left[\right. \\
751 \quad & \left. \begin{array}{c|c|c|c|c} \underline{e}_{\kappa_i+1}^1 & \underline{e}_{\kappa_i+1}^2 & (\underline{e}_{\kappa_i+1}^3 - (\kappa_i + 2)\underline{e}_{\kappa_i+1}^2) & (\underline{e}_{\kappa_i+1}^4 - (\kappa_i + 1)\underline{e}_{\kappa_i+1}^3) & \cdots & (\underline{e}_{\kappa_i+1}^{\kappa_i+1} - 4\underline{e}_{\kappa_i+1}^{\kappa_i}) \end{array} \right. \\
752 \quad & \left. \right], \quad \dots, \quad T_{i,\kappa_i-1} = \left[\begin{array}{c|c|c|c|c} \underline{e}_{\kappa_i+1}^1 & \cdots & \underline{e}_{\kappa_i+1}^{\kappa_i-1} & (\underline{e}_{\kappa_i+1}^{\kappa_i} - (2\kappa_i - 1)\underline{e}_{\kappa_i+1}^{\kappa_i-1}) & (\underline{e}_{\kappa_i+1}^{\kappa_i+1} - (2\kappa_i - \\
753 \quad & \left. 2)\underline{e}_{\kappa_i+1}^{\kappa_i} \right)], \quad T_{i,\kappa_i} = \left[\begin{array}{c|c|c|c|c} \underline{e}_{\kappa_i+1}^1 & \cdots & \underline{e}_{\kappa_i+1}^{\kappa_i} & (\underline{e}_{\kappa_i+1}^{\kappa_i+1} - (2\kappa_i)\underline{e}_{\kappa_i+1}^{\kappa_i}) \end{array} \right], \text{ we then get:}
\end{aligned}$$

$$\tilde{X}_{(i,\kappa_i+1)} \prod_{j=1}^{\kappa_i} T_{i,j} = \begin{bmatrix} 0! & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1! & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 1 & \prod_{\ell=\kappa_i}^{\kappa_i} \ell & \prod_{\ell=\kappa_i-1}^{\kappa_i} \ell & \cdots & \prod_{\ell=3}^{\kappa_i} \ell & \prod_{\ell=2}^{\kappa_i} \ell & \kappa_i! \end{bmatrix}. \quad (\text{B2})$$

754 which implies (2.10.b).

755 For the second statement, let us first note that (2.7)-(2.9), (2.2) and (2.3),
756 imply:

$$X_{(i,1)}(t)\mathbf{a}_{i,1} + X_{(i,0)}(t)\mathbf{a}_{i,0} = \begin{bmatrix} d^{\kappa_i} f_i(t)/dt^{\kappa_i} \\ \bar{w}_i(t) \end{bmatrix}, \quad (\text{B3})$$

757 with $i \in \{1, \dots, m\}$. And let us next note that (2.2) and (2.3) are equivalent
758 to:

$$\frac{d^{\kappa_i} f(t_j)}{dt^{\kappa_i}} = (\underline{e}_m^i)^T G_B^{-1}(u(t_j) - F_B x(t_j)) \quad \text{and} \quad \bar{w}_i(t_j) = \mathcal{R}_{[A_B, B_B]}^{-1} P_i T_B^{-1} x(t_j), \quad (\text{B4})$$

759 with $i \in \{1, \dots, m\}$ and $j \in \{0, 1\}$, and where $t_0 = 0$, $u(t_0) = u_0$, $u(t_1) = u_1$, $x(t_0) =$
760 x_0 , and $x(t_1) = x_1$. Therefore, (2.8)-(2.10), (B3) and (2.11) imply (B4). \square

761 Appendix C. Proof of Lemma 3

762 Let us first prove that the spaces \mathcal{X}_d , \mathcal{X}_{eq} and \mathcal{U} can be decomposed as
763 follows:

$$\begin{aligned}
\mathcal{X}_d &= \mathcal{B}_C \oplus (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) \oplus \mathcal{X}_2 \oplus \mathcal{X}_1, \quad \mathcal{X}_{eq} = E\mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{B}_C \oplus E\mathcal{X}_2 \oplus A\mathcal{X}_1, \\
\mathcal{U} &= B^{-1}E\mathcal{R}_{\mathcal{X}_d}^* \oplus B^{-1}\mathcal{B}_C,
\end{aligned} \quad (\text{C1})$$

764 where:

$$\begin{aligned}
\mathcal{X}_d &= \mathcal{V}_{\mathcal{X}_d}^* \oplus \mathcal{X}_1, \quad \mathcal{V}_{\mathcal{X}_d}^* = \mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{X}_2, \quad \mathcal{R}_{\mathcal{X}_d}^* = \mathcal{R}_C \oplus (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E), \\
\mathcal{X}_{eq} &= (E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) \oplus A\mathcal{X}_1, \quad E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} = (A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \oplus E\mathcal{X}_2, \\
A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B} &= E\mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{B}_C, \quad \mathcal{B} = (\mathcal{B} \cap E\mathcal{R}_{\mathcal{X}_d}^*) \oplus \mathcal{B}_C.
\end{aligned} \tag{C2}$$

765 And also:

$$\mathcal{R}_C \approx E\mathcal{R}_{\mathcal{X}_d}^*, \quad \mathcal{X}_2 \approx E\mathcal{X}_2, \quad \mathcal{X}_1 \approx A\mathcal{X}_1, \quad \mathcal{B}_C \approx B^{-1}\mathcal{B}_C = \mathcal{U}_C, \tag{C3}$$

$$\mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{K}_E = \mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E, \quad \mathcal{B} \cap E\mathcal{V}_{\mathcal{X}_d}^* = \mathcal{B} \cap E\mathcal{R}_{\mathcal{X}_d}^*. \tag{C4}$$

766 **1.** From (1.8), (ALG-S) and (ALG-V), we get:

$$\mathcal{V}_{\mathcal{X}_d}^* = A^{-1}(E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) \quad \text{and} \quad \mathcal{R}_{\mathcal{X}_d}^* = \mathcal{V}_{\mathcal{X}_d}^* \cap E^{-1}(A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}). \tag{C5}$$

767 Indeed: $\mathcal{V}_{\mathcal{X}_d}^* \cap E^{-1}(A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) = \mathcal{V}_{\mathcal{X}_d}^* \cap E^{-1}(A((A^{-1}(E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B})) \cap \mathcal{S}_{\mathcal{X}_d}^*) + \mathcal{B})$
768 $= \mathcal{V}_{\mathcal{X}_d}^* \cap E^{-1}((E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) \cap A\mathcal{S}_{\mathcal{X}_d}^* + \mathcal{B}) = \mathcal{V}_{\mathcal{X}_d}^* \cap E^{-1}((E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) \cap (A\mathcal{S}_{\mathcal{X}_d}^* + \mathcal{B}))$
769 $= \mathcal{V}_{\mathcal{X}_d}^* \cap (\mathcal{V}_{\mathcal{X}_d}^* + E^{-1}\mathcal{B}) \cap \mathcal{S}_{\mathcal{X}_d}^* = \mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{S}_{\mathcal{X}_d}^* = \mathcal{R}_{\mathcal{X}_d}^*$ (see also [Özçaldıran, 1985](#),
770 [Malabre, 1987](#)). From (C5) and Result 1, we get:

$$\begin{aligned}
\mathcal{X}_d &= \mathcal{V}_{\mathcal{X}_d}^* \oplus \mathcal{X}_1, \quad \mathcal{X}_{eq} = (E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) \oplus A\mathcal{X}_1, \\
\mathcal{V}_{\mathcal{X}_d}^* &= \mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{X}_2, \quad \mathcal{R}_{\mathcal{X}_d}^* = \mathcal{R}_C \oplus (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E).
\end{aligned} \tag{C6}$$

$$771 \quad E\mathcal{R}_{\mathcal{X}_d}^* = E\mathcal{V}_{\mathcal{X}_d}^* \cap (A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \quad \text{and} \quad A\mathcal{R}_{\mathcal{X}_d}^* \subset A\mathcal{V}_{\mathcal{X}_d}^* \subset E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}. \tag{C7}$$

772 **2.** From (C5.b), we get (C4.a), which implies together with (C6.c):

$$E\mathcal{V}_{\mathcal{X}_d}^* = E\mathcal{R}_{\mathcal{X}_d}^* \oplus E\mathcal{X}_2. \tag{C8}$$

773 Indeed, the direct sum comes from the fact that $\mathcal{X}_2 \cap \mathcal{K}_E \subset \mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{K}_E =$
774 $\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E$ implies that $(\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{X}_2) \cap \mathcal{K}_E = (\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{X}_2) \cap (\mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) = (\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{X}_2)$
775 $\cap (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) = \mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E = \mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E + \mathcal{X}_2 \cap \mathcal{K}_E$.

776 Moreover, since: $\mathcal{X}_2 \cap \mathcal{K}_E = (\mathcal{X}_2 \cap \mathcal{V}_{\mathcal{X}_d}^*) \cap \mathcal{K}_E = \mathcal{X}_2 \cap (\mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) = \mathcal{X}_2$
777 $\cap (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) = (\mathcal{X}_2 \cap \mathcal{R}_{\mathcal{X}_d}^*) \cap \mathcal{K}_E = \{0\}$, we get: $\dim(E\mathcal{X}_2) = \dim(\mathcal{X}_2)$, thus
778 (C3.b) follows.

779 **3.** From (C8) and (C7), we get:

$$E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} = (E\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \oplus E\mathcal{X}_2. \quad (\text{C9})$$

780 Indeed, since: $\{0\} = (E\mathcal{R}_{\mathcal{X}_d}^*) \cap (E\mathcal{X}_2) = E\mathcal{V}_{\mathcal{X}_d}^* \cap (A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \cap (E\mathcal{X}_2) =$

781 $(A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \cap (E\mathcal{X}_2)$, we get: $E\mathcal{X}_2 \cap (E\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \subset E\mathcal{X}_2 \cap (A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) = \{0\}$.

782 Moreover, (C9), (C7) and (C8) imply:

$$\begin{aligned} E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} &= (E\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \oplus E\mathcal{X}_2 = (E\mathcal{V}_{\mathcal{X}_d}^* \cap (A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) + \mathcal{B}) \oplus E\mathcal{X}_2 \\ &= ((E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) \cap (A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B})) \oplus E\mathcal{X}_2 = (A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \oplus E\mathcal{X}_2. \end{aligned} \quad (\text{C10})$$

783 **4.** From (C7.a) and (3.9), there exist subspaces, \mathcal{W}_C and \mathcal{B}_C , such that:

$$A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B} = E\mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{W}_C, \quad \mathcal{B} = ((E\mathcal{R}_{\mathcal{X}_d}^*) \cap \mathcal{B}) \oplus \mathcal{B}_C, \quad \mathcal{W}_C \supset \mathcal{B}_C \quad (\text{C11})$$

784 From (C8), (C10), and (C11), we get: $E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} = (E\mathcal{R}_{\mathcal{X}_d}^* \oplus E\mathcal{X}_2) + \mathcal{B} =$

785 $E\mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{B}_C \oplus E\mathcal{X}_2 = (A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \oplus E\mathcal{X}_2$, that is to say: $E\mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{W}_C = A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}$

786 $\approx E\mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{B}_C$. Hence:

$$\mathcal{W}_C = \mathcal{B}_C \quad (\text{C12})$$

787 **5.** From the geometric decompositions (C6), (C10), (C11), and (C12), the

788 subspaces \mathcal{X}_d , \mathcal{X}_{eq} , and \mathcal{U} take the form (C1)-(C2).

789 **6.** From (C2.c,a) and since: $\text{Ker } A \subset \mathcal{V}_{\mathcal{X}_d}^*$ and $\text{Ker } B = \{0\}$, we get (C3.a,c,d).

790 **7.** To prove (C4.b), note first that (C8) and (C9) imply $\mathcal{B} \cap E\mathcal{V}_{\mathcal{X}_d}^* =$

791 $\mathcal{B} \cap (E\mathcal{R}_{\mathcal{X}_d}^* + E\mathcal{X}_2)$ and $(E\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \cap E\mathcal{X}_2 = \{0\}$. Let $x \in \mathcal{B} \cap (E\mathcal{R}_{\mathcal{X}_d}^* + E\mathcal{X}_2)$,

792 there then exist $z \in E\mathcal{R}_{\mathcal{X}_d}^*$, $y \in E\mathcal{X}_2$, and $b \in \mathcal{B}$ such that $x = z + y = b$,

793 which implies $y = b - z \in (E\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \cap E\mathcal{X}_2 = \{0\}$, *i.e.* $x \in \mathcal{B} \cap E\mathcal{R}_{\mathcal{X}_d}^*$. There-
794 fore: $\mathcal{B} \cap E\mathcal{V}_{\mathcal{X}_d}^* = \mathcal{B} \cap (E\mathcal{R}_{\mathcal{X}_d}^* + E\mathcal{X}_2) \subset \mathcal{B} \cap E\mathcal{R}_{\mathcal{X}_d}^* \subset \mathcal{B} \cap E\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B} \cap E\mathcal{X}_2 \subset$
795 $\mathcal{B} \cap (E\mathcal{R}_{\mathcal{X}_d}^* + E\mathcal{X}_2) = \mathcal{B} \cap E\mathcal{V}_{\mathcal{X}_d}^*$.

796 Let us next note that under the geometric decompositions, (C1)-(C3), the
797 implicit representation (1.7) takes the following form (recall (3.7) and (3.8)):

$$\begin{bmatrix} \bar{E} & 0 & * \\ 0 & I_2 & * \\ 0 & 0 & \bar{X}_{\rho-1} \end{bmatrix} \frac{d}{dt}x = \begin{bmatrix} \bar{A} & \hat{A} & 0 \\ 0 & \hat{A}_3 & 0 \\ 0 & 0 & I_1 \end{bmatrix} x + \begin{bmatrix} \bar{B} \\ 0 \\ 0 \end{bmatrix} u, \quad (\text{C13})$$

798 where $I_2 : \mathcal{X}_2 \leftrightarrow E\mathcal{X}_2$ is an isomorphism, and the matrices \bar{E} , \bar{A} and \bar{B} , are
799 the ones shown in (4.1). Then, when $\mathcal{R}_{\mathcal{X}_d} = \mathcal{X}_d$, we get (4.1). \square

800 Appendix D. Proof of Corollary 1

801 Let us first note that (5.10) implies that the implicit representation (C13)
802 is only composed by the linear transformations (4.1).

803 Let us next note that Lemma 3 and (5.11) imply that (see (C2)-(C4)):

$$\mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{K}_E = \mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E \quad \text{and} \quad \mathcal{B}/(\mathcal{B} \cap E\mathcal{V}_{\mathcal{X}_d}^*) = \mathcal{B}/(\mathcal{B} \cap E\mathcal{R}_{\mathcal{X}_d}^*) \approx \mathcal{B}_C \approx \mathcal{U}_C, \quad (\text{D1})$$

$$\dim(A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \geq \dim(\mathcal{R}_{\mathcal{X}_d}^*). \quad (\text{D2})$$

805 **Case 1.** If $\mathcal{U}_C = \{0\}$, then (5.12) and (D1) imply: $\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E = \{0\}$. Thus, the
806 blocks $\bar{A}_{1,2}$, $\bar{A}_{2,1}$, $\bar{A}_{2,2}$, and $I_{\mathcal{U}_C}$ actually disappear from (4.1), corresponding
807 to 0 row and 0 column. Moreover $\bar{B}_1 \neq 0$, because the pair $(\bar{A}_{1,1}, [\bar{A}_{1,2} \ \bar{B}_1])$ is
808 reachable (see (4.2)). Namely, we get (5.13).

809 **Case 2.** The existence of \bar{V}_ℓ is implied by (D2). From (5.14) and (4.1), we
 810 get (5.15). From (5.15) and (4.2), we get (5.16). From (5.17) and (4.1), we
 811 get (5.18). From (5.18) and (4.2), we get (5.19). \square

812 Appendix E. Proof of Theorem 7

813 Let us first propose a PD descriptor feedback $u = F_p^*x + F_d^*dx/dt + v$,
 814 where the pair of linear transformations (F_p, F_d) is chosen such that:

$$(F_p^*, F_d^*) \in \mathbf{F}(\mathcal{V}^*) \text{ and } \mathcal{B} \cap E_{F_d^*}\mathcal{V}^* = \{0\}. \quad (\text{E1})$$

815 Let us next, consider the *quotient implicit representation* $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$
 816 defined by (5.20). Let us note that $\text{Ker } B_* = B^{-1}E_{F_d^*}\mathcal{V}^* \approx \mathcal{B} \cap E_{F_d^*}\mathcal{V}^*$ implies
 817 $\text{Ker } B_* = \{0\}$, and that $\Phi \mathcal{N}_{(F_p^*, F_d^*)} = \Phi \sup \{ \mathcal{V} \subset \text{Ker } C \mid A_{F_p^*}\mathcal{V} \subset E_{F_d^*}\mathcal{V} \} = \Phi \mathcal{V}^* =$
 818 $\{0\}$ implies the observability of the *quotient implicit representation*
 819 $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$. The proof of Theorem 7 is done in 4 steps:

820 *i)* $\mathfrak{R}^{imp}(E_{F_d^*}, A_{F_p^*}, B, C)$ is externally equivalent to $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$. This
 821 fact follows from (Theorem 2.1, Bonilla & Malabre, 1995), which states,
 822 among others, the *external equivalency* between $\mathfrak{R}^{imp}(E_{F_d^*}, A_{F_p^*}, B, C)$ and
 823 $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$ (see also Kuijper & Schumacher, 1991).

824 *ii)* $\mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^* = \Phi \mathcal{V}_{\mathcal{X}_d}^*$ and $\mathcal{S}_{\mathcal{X}_d/\mathcal{V}^*}^* = \Phi \mathcal{S}_{\mathcal{X}_d}^*$. For the case of the *quotient im-*
 825 *PLICIT representation* $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$ the corresponding algorithms (ALG–
 826 V) and (ALG–S), for computing $\mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^*$ and $\mathcal{S}_{\mathcal{X}_d/\mathcal{V}^*}^*$, take the following form:

$$\begin{aligned} \mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^0 &= \mathcal{X}_d/\mathcal{V}_d^* , & \mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^{\mu+1} &= A_*^{-1}(E_*\mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^\mu + \mathcal{B}_*), \\ \mathcal{S}_{\mathcal{X}_d/\mathcal{V}^*}^0 &= \{0\} , & \mathcal{S}_{\mathcal{X}_d/\mathcal{V}^*}^{\mu+1} &= E_*^{-1}(A_*\mathcal{S}_{\mathcal{X}_d/\mathcal{V}^*}^\mu + \mathcal{B}_*). \end{aligned} \quad (\text{E2})$$

827 It is clear that: $\mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^0 = \Phi \mathcal{V}_{\mathcal{X}_d}^0$ and $\mathcal{S}_{\mathcal{X}_d/\mathcal{V}^*}^0 = \Phi \mathcal{S}_{\mathcal{X}_d}^0$. Let us assume that:
828 $\mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^\mu = \Phi \mathcal{V}_{\mathcal{X}_d}^\mu$ and $\mathcal{S}_{\mathcal{X}_d/\mathcal{V}^*}^\mu = \Phi \mathcal{S}_{\mathcal{X}_d}^\mu$, then from (E2) and from (5.20), we
829 get: $\mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^{\mu+1} = (\Phi \mathcal{X}_d) \cap A_*^{-1} (E_* \Phi \mathcal{V}_{\mathcal{X}_d}^\mu + \Pi \mathcal{B}) = \Phi \Phi^{-1} A_*^{-1} \Pi (E_{F_d^*} \mathcal{V}_{\mathcal{X}_d}^\mu + \mathcal{B}) = \Phi A_{F_d^*}^{-1}$
830 $\Pi^{-1} \Pi (E_{F_d^*} \mathcal{V}_{\mathcal{X}_d}^\mu + \mathcal{B}) = \Phi A_{F_d^*}^{-1} (E_{F_d^*} \mathcal{V}_{\mathcal{X}_d}^\mu + \mathcal{B} + E_{F_d^*} \mathcal{V}^*) = \Phi (\mathcal{X}_d \cap A_{F_d^*}^{-1} (E_{F_d^*} \mathcal{V}_{\mathcal{X}_d}^\mu + \mathcal{B}))$
831 $= \Phi \mathcal{V}_{\mathcal{X}_d}^{\mu+1}$, and $\mathcal{S}_{\mathcal{X}_d/\mathcal{V}^*}^{\mu+1} = (\Phi \mathcal{X}_d) \cap E_*^{-1} (A_* ((\Phi \mathcal{X}_d) \cap (\Phi \mathcal{S}_{\mathcal{X}_d}^\mu)) + \Pi \mathcal{B}) = \Phi \Phi^{-1} E_*^{-1}$
832 $\Pi (A_{F_d^*} (\mathcal{X}_d \cap \mathcal{S}_{\mathcal{X}_d}^\mu) + \mathcal{B}) = \Phi E_{F_d^*}^{-1} \Pi^{-1} \Pi (A_{F_d^*} (\mathcal{X}_d \cap \mathcal{S}_{\mathcal{X}_d}^\mu) + \mathcal{B}) = \Phi E_{F_d^*}^{-1} (A_{F_d^*}$
833 $(\mathcal{X}_d \cap \mathcal{S}_{\mathcal{X}_d}^\mu) + \mathcal{B} + E_{F_d^*} \mathcal{V}^*) = \Phi E_{F_d^*}^{-1} (A_{F_d^*} (\mathcal{X}_d \cap \mathcal{S}_{\mathcal{X}_d}^\mu) + \mathcal{B}) + \Phi \text{Ker } E_{F_d^*} =$
834 $\Phi (\mathcal{X}_d \cap E_{F_d^*}^{-1} (A_{F_d^*} (\mathcal{X}_d \cap \mathcal{S}_{\mathcal{X}_d}^\mu) + \mathcal{B})) = \Phi \mathcal{S}_{\mathcal{X}_d}^{\mu+1}$.

835 *iii) If (5.21) and (5.23) are satisfied, then $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$ satisfies The-*
836 *orem 6.* Since: $(\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{S}_{\mathcal{X}_d}^*) \cap \text{Ker } \Phi = (\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{S}_{\mathcal{X}_d}^*) \cap \mathcal{V}^* = \mathcal{V}^* = \mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{V}^* +$
837 $\mathcal{S}_{\mathcal{X}_d}^* \cap \mathcal{V}^*$, we get from (5.21): $\mathcal{R}_{\mathcal{X}_d/\mathcal{V}^*}^* = \mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^* \cap \mathcal{S}_{\mathcal{X}_d/\mathcal{V}^*}^* = \Phi \mathcal{V}_{\mathcal{X}_d}^* \cap \Phi \mathcal{S}_{\mathcal{X}_d}^* =$
838 $\Phi (\mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{S}_{\mathcal{X}_d}^*) = \Phi \mathcal{R}_{\mathcal{X}_d}^* = \Phi (\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{V}^*) = \Phi \mathcal{X}_d = \mathcal{X}_d/\mathcal{V}^*$, which is the first con-
839 dition of Theorem 6. On the other hand, since for any $F_d^* : \mathcal{X}_d \rightarrow \mathcal{U}$, $E^{-1} \mathcal{B} =$
840 $E_{F_d^*}^{-1} \mathcal{B}$, we have: $\dim(\mathcal{K}_E) + \dim(\text{Im } E \cap \mathcal{B}) = \dim(E_{F_d^*}^{-1} \mathcal{B})$, which together with
841 (5.23) imply:²³

$$842 \quad \dim(\mathcal{B}) \geq \dim(E_{F_d^*}^{-1} \mathcal{B} / (\mathcal{V}^* \cap E_{F_d^*}^{-1} \mathcal{B})) = \dim(\Phi E_{F_d^*}^{-1} \mathcal{B}) = \dim(E_*^{-1} \mathcal{B}_*),$$

843 then: $\dim(\mathcal{B}_*) = \dim(\Pi \mathcal{B}) \geq \dim(E_*^{-1} \mathcal{B}_*) - \dim(\mathcal{B} \cap \text{Ker } \Pi) = \dim(E_*^{-1} \mathcal{B}_*)$
844 $- \dim(B \text{Ker } B_*) = \dim(E_*^{-1} \mathcal{B}_*) - \dim(\text{Ker } B_*) = \dim(E_*^{-1} \mathcal{B}_*)$, that is to say:
845 $\dim(\mathcal{B}_*/(\mathcal{B}_* \cap \text{Im } E_*)) \geq \dim(\mathcal{K}_{E_*})$, which is the second condition²⁴ of Theorem
846 6.

847 *iv) If $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$ satisfies Theorem 6, then (5.21) and (5.23) are sat-*
848 *isfied.* From the first condition of Theorem 6, we have: $\mathcal{X}_d/\mathcal{V}^* = \mathcal{R}_{\mathcal{X}_d/\mathcal{V}^*}^* =$

²³ Note that: $\mathcal{X}_d = \mathcal{R}_{\mathcal{X}_d}^* + \mathcal{V}^* \subset \mathcal{V}_{\mathcal{X}_d}^* \subset \mathcal{X}_d$, and recall (5.20).

²⁴ Note that: $\mathcal{X}_d/\mathcal{V}^* = \mathcal{R}_{\mathcal{X}_d/\mathcal{V}^*}^* \subset \mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^* \subset \mathcal{X}_d/\mathcal{V}^*$.

849 $(\Phi\mathcal{V}_{\mathcal{X}_d}^*) \cap (\Phi\mathcal{S}_{\mathcal{X}_d}^*)$, which implies: $\mathcal{X}_d = \mathcal{V}_{\mathcal{X}_d}^* \cap (\mathcal{S}_{\mathcal{X}_d}^* + \mathcal{V}^*) = \mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{S}_{\mathcal{X}_d}^* + \mathcal{V}^*$
850 $= \mathcal{K}_{\mathcal{X}_d}^* + \mathcal{V}^*$, which is the first condition of Theorem 7. From the sec-
851 ond condition of Theorem 6, we have:²⁴ $\dim(\Pi\mathcal{B}) = \dim(\mathcal{B}_*) \geq \dim(\mathcal{K}_{E_*}) +$
852 $\dim(\mathcal{B}_* \cap \text{Im } E_*) = \dim(E_*^{-1}\mathcal{B}_*) = \dim(E_*^{-1}\Pi\mathcal{B}) = \dim(\Phi E_{F_d}^{-1}\mathcal{B})$. Then (recall
853 (E1)): $\dim(\mathcal{B}) \geq \dim(\Phi E_{F_d}^{-1}\mathcal{B}) + \dim(\mathcal{B} \cap \text{Ker } \Pi) = \dim(\Phi E^{-1}\mathcal{B}) + \dim(\mathcal{B} \cap$
854 $E_{F_d}^* \mathcal{V}^*) = \dim(E^{-1}\mathcal{B}) - \dim(\mathcal{V}^* \cap E^{-1}\mathcal{B}) = \dim(\mathcal{K}_E) + \dim(\mathcal{B} \cap \text{Im } E) - \dim(\mathcal{V}^* \cap$
855 $E^{-1}\mathcal{B})$, which is the second condition²³ of Theorem 7. \square

856 Appendix F. Geometric Inequalities (5.26) and (5.27)

857 From (ALG-V), (ALG-S) and (5.25), we obtain: $\mathcal{V}_{\mathcal{X}_d}^* = \overline{\mathcal{X}} \oplus \mathcal{Q}$ and $\mathcal{S}_{\mathcal{X}_d}^*$
858 $= \langle \overline{A} \mid \text{Im} [\overline{B} \ \overline{S}] \rangle \oplus \mathcal{Q}$, which imply: $E\mathcal{V}_{\mathcal{X}_d}^* = \overline{\mathcal{X}}$ and $\mathcal{K}_{\mathcal{X}_d}^* = \langle \overline{A} \mid \text{Im} [\overline{B} \ \overline{S}] \rangle \oplus \mathcal{Q}$.

859 From (ALG-V) and (5.25), we get: $\mathcal{V}^0 = E^{-1}\text{Im } E = E^{-1}\overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^0$ and \mathcal{V}^1
860 $= E^{-1}\mathcal{K}_{\overline{C}} = E^{-1}\overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^1$, then: $E\mathcal{V}^0 = \overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^0$ and $E\mathcal{V}^1 = \overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^1$. Let us as-
861 sume that: $E\mathcal{V}^\mu = \overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^\mu$, then: $\mathcal{V}^{\mu+1} = (E^{-1}\mathcal{K}_{\overline{C}}) \cap \left[\overline{A} \ \overline{S} \right]^{-1} \left(\overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^\mu + \overline{\mathcal{B}} \right)$,
862 which implies: $E\mathcal{V}^{\mu+1} = \mathcal{K}_{\overline{C}} \cap E \left[\overline{A} \ \overline{S} \right]^{-1} \left(\overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^\mu + \overline{\mathcal{B}} \right) = \mathcal{K}_{\overline{C}} \cap \overline{A}^{-1} \left(\overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^\mu \right.$
863 $\left. + \text{Im} [\overline{B} \ \overline{S}] \right) = \overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^{\mu+1}$. Thus: $E\mathcal{V}^* = \overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^*$.

864 From the previous paragraphs we have the following equivalences: $\mathcal{K}_{\mathcal{X}_d}^*$
865 $+ \mathcal{V}^* = \mathcal{X}_d \Leftrightarrow \langle \overline{A} \mid \text{Im} [\overline{B} \ \overline{S}] \rangle \oplus \mathcal{Q} + \mathcal{V}^* = \overline{\mathcal{X}} \oplus \mathcal{Q} \Leftrightarrow E^{-1} \langle \overline{A} \mid \text{Im} [\overline{B} \ \overline{S}] \rangle + \mathcal{V}^* =$
866 $\overline{\mathcal{X}} \oplus \mathcal{Q} \Rightarrow \text{Im } E \cap \langle \overline{A} \mid \text{Im} [\overline{B} \ \overline{S}] \rangle + E\mathcal{V}^* = \overline{\mathcal{X}} \Rightarrow \langle \overline{A} \mid \text{Im} [\overline{B} \ \overline{S}] \rangle + \overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^* = \overline{\mathcal{X}}$
867 $\Rightarrow E^{-1} (\langle \overline{A} \mid \text{Im} [\overline{B} \ \overline{S}] \rangle + E\mathcal{V}^*) = \overline{\mathcal{X}} \oplus \mathcal{Q} \Rightarrow E^{-1} \langle \overline{A} \mid \text{Im} [\overline{B} \ \overline{S}] \rangle + \mathcal{V}^* = \overline{\mathcal{X}} \oplus \mathcal{Q};$
868 which imply (5.26).

869 From the two first paragraphs, (5.22) takes the form:

$$\dim \left(\frac{\overline{\mathcal{X}}}{\overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^* + \overline{\mathcal{B}}} \right) + \dim(\overline{\mathcal{B}}) \geq \dim \left(\frac{\overline{\mathcal{X}}}{\overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^*} \right) + \dim \left(\frac{\{0\} \oplus \mathcal{Q}}{\mathcal{V}^* \cap \mathcal{K}_E} \right) \quad (\text{F1})$$

870 From (ALG-V), (5.25) and the second paragraph, we obtain:

$$\begin{aligned}
 \mathcal{K}_E \cap \mathcal{V}^* &= \mathcal{K}_E \cap (E^{-1} \mathcal{K}_{\bar{C}}) \cap \left[\begin{array}{cc} \bar{A} & \bar{S} \end{array} \right]^{-1} \left(\overline{\mathcal{V}}_{[\bar{B} \bar{S}]}^* + \overline{\mathcal{B}} \right) \\
 &= \mathcal{K}_E \cap \left[\begin{array}{cc} \bar{A} & \bar{S} \end{array} \right]^{-1} \left(\overline{\mathcal{V}}_{[\bar{B} \bar{S}]}^* + \overline{\mathcal{B}} \right) = \{0\} \oplus \bar{S}^{-1} \left(\overline{\mathcal{V}}_{[\bar{B} \bar{S}]}^* + \overline{\mathcal{B}} \right)
 \end{aligned}
 \tag{F2}$$

871 From (F1) and (F2) we get (5.27) (recall that $\text{Ker } \bar{S} = \{0\}$). □