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# Simultaneous State and Input Reachability for Linear Time Invariant Systems

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## Abstract

In this paper, we give an explicit solution to the behavioral reachability problem for linear time invariant systems, which amounts to finding an explicit control law that reaches a given final input-state pair  $(u_1, x_1)$  in a given finite time  $t_1$ . We first tackle the case of state space realizations, and we then extend the obtained results to the case of implicit realizations. For this, we use the geometric approach and some results of the viability theory. Some complements are given about the existing relationships between reachability and pole placement, as well as some notions of unicity and existence of solution.

*Keywords:*

Linear systems, implicit systems, reachability, geometric approach.

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**Notation.** Script capitals  $\mathcal{V}, \mathcal{W}, \dots$ , denote finite dimensional linear spaces with elements  $v, w, \dots$ ; the dimension of a space  $\mathcal{V}$  is denoted  $\dim(\mathcal{V})$ ;  $\mathcal{V} \approx \mathcal{W}$  stands for  $\dim(\mathcal{V}) = \dim(\mathcal{W})$ ; when  $\mathcal{V} \subset \mathcal{W}$ ,  $\frac{\mathcal{W}}{\mathcal{V}}$  or  $\mathcal{W}/\mathcal{V}$  stand for the quotient

space  $\mathcal{W}$  modulo  $\mathcal{V}$ ; the direct sum of independent spaces is written as  $\oplus$ .  $X^{-1}\mathcal{V}$ , stands for the inverse image of the subspace  $\mathcal{V}$  by the linear transformation  $X$ . Given a linear transformation  $X: \mathcal{V} \rightarrow \mathcal{W}$ ,  $\text{Im } X = X\mathcal{V}$  denotes its image, and  $\text{Ker } X$  denotes its kernel; when  $\mathcal{V} \approx \mathcal{W}$ , we write  $X: \mathcal{V} \leftrightarrow \mathcal{W}$ ; when  $\mathcal{U} \subset \mathcal{V}$ ,  $X|_{\mathcal{U}}$  denotes the restriction of  $X$  to  $\mathcal{U}$ . Given the space  $\mathcal{X} = \mathcal{S} \oplus \mathcal{T}$ , the natural projection,  $P: \mathcal{X} \rightarrow \mathcal{S}$ , on  $\mathcal{S}$  along  $\mathcal{T}$ , is also written as  $P: \mathcal{X} \rightarrow \mathcal{S} // \mathcal{T}$ . The special subspaces  $\text{Im } B$ ,  $\text{Ker } E$  and  $\text{Ker } C$ , are denoted by  $\mathcal{B}$ ,  $\mathcal{K}_E$  and  $\mathcal{K}_C$ , respectively. The zero dimension subspace is denoted  $\{0\}$ , and the identity operator is denoted  $I$ , namely  $Ix = x$ . Given the linear transformations  $X: \mathcal{V} \rightarrow \mathcal{V}$  and  $Y: \mathcal{W} \rightarrow \mathcal{V}$ ,  $\langle X | \text{Im } Y \rangle$  stands for the subspace of  $\mathcal{V}$ :  $\text{Im } Y + X\text{Im } Y + \dots + X^{\dim(\mathcal{V})-1}\text{Im } Y$ . The notations  $A_{F_p}$  and  $E_{F_d}$  stand for  $(A + BF_p)$  and  $(E - BF_d)$ , respectively.

$\text{BDM}\{X_1, \dots, X_k\}$  denotes a block diagonal matrix whose diagonal blocks are the matrices  $X_1, \dots, X_k$ , and  $\text{DM}\{x_1, \dots, x_k\}$  denotes a diagonal matrix whose diagonal elements are  $x_1, \dots, x_k$ . The notation  $\mathbb{R}^k$  stands for the Euclidean space of dimension  $k$ .  $\underline{e}_k^i \in \mathbb{R}^k$  stands for the vector whose  $i$ -th entry is equal to 1 and the other ones are equal to 0.  $\text{T}_u\{v^T\}$  stands for the upper triangular Toeplitz matrix, whose first row is  $v^T$ .  $*$  stands for some matrix which exact value has no importance.

$\mathbb{R}^+$ ,  $\mathbb{R}^{+*}$  and  $\mathbb{Z}^+$ , stand for the sets of non negative real numbers, positive real numbers and non negative integers, respectively.  $\mathcal{C}^\infty(\mathbb{R}^+, \mathcal{V})$  and  $\mathcal{L}^\infty(\mathbb{R}^+, \mathcal{V})$  are the space of infinitely differentiable functions and the space of bounded functions,  $v: \mathbb{R}^+ \rightarrow \mathcal{V}$ , respectively.  $\mathcal{L}_1^{\text{loc}}(\mathbb{R}^+, \mathcal{V})$  stands for the locally integrable functions.

**Geometric Algorithms.** Given the linear transformations  $X : \mathcal{V} \rightarrow \mathcal{W}$ ,  $Y : \mathcal{T} \rightarrow \mathcal{W}$ , and  $Z : \mathcal{V} \rightarrow \mathcal{W}$ , and the subspace  $\mathcal{K} \subset \mathcal{V}$ , we have the two following popular geometric algorithms (see mainly [Verghese, 1981](#), [Özcaldiran, 1986](#), [Malabre, 1987, 1989](#), [Lewis, 1992](#)):

$$\mathcal{V}_{[\mathcal{K}:X,Z,Y]}^0 = \mathcal{V}, \quad \mathcal{V}_{[\mathcal{K}:X,Z,Y]}^{\mu+1} = \mathcal{K} \cap X^{-1} \left( Z \mathcal{V}_{[\mathcal{K}:X,Z,Y]}^\mu + \text{Im } Y \right) \quad (\text{ALG-V})$$

$$\mathcal{S}_{[Z,X,Y]}^0 = \{0\}, \quad \mathcal{S}_{[Z,X,Y]}^{\mu+1} = Z^{-1} \left( X \mathcal{S}_{[Z,X,Y]}^\mu + \text{Im } Y \right) \quad (\text{ALG-S})$$

where  $\mu \in \mathbb{Z}^+$ . The limit of (ALG-V) is the *supremal*  $(X, Z, Y)$  *invariant subspace* contained in  $\mathcal{K}$ ,  $\mathcal{V}_{[\mathcal{K}:X,Z,Y]}^* := \sup\{\mathcal{S} \subset \mathcal{K} \mid X\mathcal{S} \subset Z\mathcal{S} + \text{Im } Y\}$ , and the limit of (ALG-S) is the *infimal*  $(Z, X, Y)$  *invariant subspace related to*  $\text{Im } Y$ ,  $\mathcal{S}_{[Z,X,Y]}^* := \inf\{\mathcal{S} \subset \mathcal{V} \mid \mathcal{S} = Z^{-1}(X\mathcal{S} + \text{Im } Y)\}$ .

We distinguish two cases.

- For the square brackets  $[\mathcal{V} : X, Z, 0]$  and  $[Z, X, 0]$ , we write:  $\mathcal{V}_{[X,Z]}^*$ ,  $\mathcal{V}_{[X,Z]}^\mu$ ,  $\mathcal{S}_{[Z,X]}^*$  and  $\mathcal{S}_{[Z,X]}^\mu$ , instead of:  $\mathcal{V}_{[\mathcal{V}:X,Z,0]}^*$ ,  $\mathcal{V}_{[\mathcal{V}:X,Z,0]}^\mu$ ,  $\mathcal{S}_{[Z,X,0]}^*$  and  $\mathcal{S}_{[Z,X,0]}^\mu$ , respectively, where  $\mu \in \mathbb{Z}^+$ .
- For the square bracket  $[\mathcal{K}_{\overline{C}} : \overline{A}, I, Y]$ , we write:  $\overline{\mathcal{V}}_Y^*$  and  $\overline{\mathcal{V}}_Y^\mu$ , instead of:  $\mathcal{V}_{[\mathcal{K}_{\overline{C}}:\overline{A},I,Y]}^*$  and  $\mathcal{V}_{[\mathcal{K}_{\overline{C}}:\overline{A},I,Y]}^\mu$ , where  $\mu \in \mathbb{Z}^+$ .

**Subspaces.** Note that in the particular case  $X = A : \mathcal{X} \rightarrow \mathcal{X}$ ,  $Y = B : \mathcal{U} \rightarrow \mathcal{X}$ , and  $Z = I$ , the equalities  $\mathcal{V}_{[\mathcal{X}:A,I,B]}^* = \mathcal{X}$  and  $\mathcal{S}_{[I,A,B]}^* = \langle A \mid \mathcal{B} \rangle$  hold true.

Given the linear transformations  $X = A : \mathcal{X}_d \rightarrow \mathcal{X}_{eq}$ ,  $Y = B : \mathcal{U} \rightarrow \mathcal{X}_{eq}$ , and  $Z = E : \mathcal{X}_d \rightarrow \mathcal{X}_{eq}$ , it is observed that

- the *supremal*  $(A, E, B)$  *invariant subspace* contained in  $\mathcal{X}_d$  and the *infimal*  $(E, A, B)$  *invariant subspace related to*  $\mathcal{B}$ ,  $\mathcal{V}_{[\mathcal{X}_d:A,E,B]}^*$  and  $\mathcal{S}_{[E,A,B]}^*$ , are

identified by  $\mathcal{V}_{\mathcal{X}_d}^*$  and  $\mathcal{S}_{\mathcal{X}_d}^*$ , respectively, and the respective subspaces of their algorithms (ALG–V) and (ALG–S) are identified by  $\mathcal{V}_{\mathcal{X}_d}^\mu$  and  $\mathcal{S}_{\mathcal{X}_d}^\mu$  ( $\mu \in \mathbb{Z}^+$ ), respectively;

- the supremal  $(A, E, B)$  invariant subspace contained in  $\mathcal{K}_C$ ,  $\mathcal{V}_{[\mathcal{K}_C: A, E, B]}^*$ , is identified by  $\mathcal{V}^*$ , and the respective subspaces of its algorithm (ALG–V) are identified by  $\mathcal{V}^\mu$  ( $\mu \in \mathbb{Z}^+$ );
- the unobservable space  $\mathcal{V}_{[\mathcal{K}_C: A, E, 0]}^*$  is identified by  $\mathcal{N}$ ; and the closed loop unobservable space  $\mathcal{V}_{[\mathcal{K}_C: A_{F_p}, E_{F_d}, 0]}^*$  is identified by  $\mathcal{N}_{(F_p, F_d)}$ .

Let us note that:

- (i)  $\mathcal{V}_{[\mathcal{K}: A, E, B]}^* = \mathcal{V}_{[\mathcal{K}: A_{F_p}, E_{F_d}, B]}^*$ ,
  - (ii)  $\mathcal{S}_{[\mathcal{K}: E, A, B]}^* = \mathcal{S}_{[\mathcal{K}: E_{F_d}, A_{F_p}, B]}^*$ , and
  - (iii) for any  $F_d$ , there exists  $F_p$  such that:  $A_{F_p} \mathcal{V}_{[\mathcal{K}: A_{F_p}, E_{F_d}, B]}^* \subset E_{F_d} \mathcal{V}_{[\mathcal{K}: A_{F_p}, E_{F_d}, B]}^*$ .
- The set of such pairs  $(F_p, F_d)$  is identified by  $\mathbf{F}(\mathcal{V}_{[\mathcal{K}: A, E, B]}^*)$ .

## 1. INTRODUCTION

One of the most studied concepts in System Theory is the one of *reachability*. This concept is normally associated with the set of vectors that can be reached from the origin in a finite time, following trajectories solutions of the system, generated by the input system. Here, the term *input system* refers to an exogenous signal which is available for controlling the output system.

### 1.1. State Space Representations

For the case of state space representations  $\mathfrak{R}^{ss}(A, B)$ ,

$$dx/dt = Ax + Bu, \quad (1.1)$$

9 where  $u \in \mathcal{U} \approx \mathbb{R}^m$  is the input variable,  $x \in \mathcal{X} \approx \mathbb{R}^n$  is the state variable, and  
 10 with the usual assumption  $\text{Ker } B = \{0\}$ , Kalman (1960, 1963) introduced his  
 11 famous reachability matrix:  $\mathcal{R}_{[A, B]} = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$ . He showed that  
 12 given any  $x_0, x_1 \in \mathcal{X}$ , there exists a control law<sup>1</sup>  $u(\cdot) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{U})$ , generating  
 13 a trajectory  $x(\cdot) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{X})$  solution of (1.1), starting from the given initial  
 14 condition  $x(0) = x_0 \in \mathcal{X}$ , and reaching the desired final state  $x(t_1) = x_1 \in \mathcal{X}$ ,  
 15 in a finite time  $t_1 \in \mathbb{R}^+$ , iff,  $\text{rank}(\mathcal{R}_{[A, B]}) = n$ ; in this case, the representation  
 16 (1.1) is called *reachable*. This concept is known as *state reachability*<sup>2</sup>, and  
 17 when the pair  $(A, B)$  satisfies such a rank condition, we identify it as a *state*  
 18 *reachable pair*.

19 A. *State reachability*. Brunovsky (1970) showed that for a given *reachable*  
 20 state space representation (1.1), there exist a linear map  $F_B: \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  
 21 isomorphisms  $T_B: \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $G_B: \mathbb{R}^m \rightarrow \mathbb{R}^m$ , such that the pair  $(A_B, B_B)$ ,  
 22 where  $A_B = T_B^{-1}(A + BF_B)T_B$  and  $B_B = T_B^{-1}BG_B$ , is expressed in the Brunovsky

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<sup>1</sup> In this paper, we restrain our discussion to infinitely differentiable functions. This is not restrictive since  $\mathcal{C}^\infty(\mathbb{R}^+, \mathcal{V})$  is dense in  $\mathcal{L}_1^{\text{loc}}(\mathbb{R}^+, \mathcal{V})$  (see Polderman & Willems, 1998, Corollary 2.4.12).

<sup>2</sup> In many text books, this property is called *state controllability*, or simply *controllability*. Let us note that *controllability* only characterizes the system's property of reaching the origin  $x_1 = 0$ , from any state  $x_0 \neq 0$ , in a finite time  $t_1$ . Since in the continuous time-invariant linear systems case both properties, *reachability* and *controllability*, are mutually implied, they are often treated indistinguishably, but in the general case of the implicit representations, this is no longer the case; for example the implicit representation,  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} dx/dt = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$ , is trivially controllable but not reachable.

canonical form, namely:

$$\begin{aligned} A_{\mathcal{B}} &= \text{BDM} \{A_{\mathcal{B}_1}, \dots, A_{\mathcal{B}_m}\}, \quad B_{\mathcal{B}} = \text{BDM} \{b_{\mathcal{B}_1}, \dots, b_{\mathcal{B}_m}\}, \\ [A_{\mathcal{B}_i} | b_{\mathcal{B}_i}] &= [\text{Tu} \{(\underline{e}_{\kappa_i}^2)^T\} | \underline{e}_{\kappa_i}^{\kappa_i}], \quad i \in \{1, \dots, m\}, \end{aligned} \quad (1.2)$$

where the set

$$S_{\kappa} = \left\{ \{\kappa_1, \kappa_2, \dots, \kappa_m\} \subset \mathbb{Z} \mid \kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_m \geq 1 \text{ \& } \sum_{i=1}^m \kappa_i = n \right\} \quad (1.3)$$

is known as the set of *reachability indices*. They are also geometrically characterized as follows

$$\text{card} \{\kappa_i \geq 1\} = \dim(\mathcal{B}) \quad \text{and} \quad \text{card} \{\kappa_i \geq \mu\} = \dim \left( \frac{\sum_{i=0}^{\mu-1} A^i \mathcal{B}}{\sum_{i=0}^{\mu-2} A^i \mathcal{B}} \right), \quad \forall \mu \geq 2.$$

Another important success was the introduction of the reachable space  $\langle A | \mathcal{B} \rangle$ . In (Wonham, 1985) is showed that a pair  $(A, B)$  is reachable iff:

$$\langle A | \mathcal{B} \rangle = \mathcal{X}. \quad (1.4)$$

Note that  $\langle A | \mathcal{B} \rangle = \mathcal{X}$  iff  $\text{rank}(\mathcal{R}_{[A, B]}) = n$ . Wonham (1985) showed that the reachability Gramian  $W_{t_1} = \int_0^{t_1} \exp(\tau A) B B^T \exp(\tau A^T) d\tau$ , with  $t \in [0, t_1]$ , is non-singular iff (1.4) is satisfied. Thus, with the control law

$$u(t) = B^T \exp((t_1 - t)A^T) W_{t_1}^{-1} (x_1 - \exp(t_1 A) x_0), \quad (1.5)$$

we get a trajectory  $x(\cdot) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{X})$  solution of (1.1), such that  $x(0) = x_0 \in \mathcal{X}$  and  $x(t_1) = x_1 \in \mathcal{X}$ .

Another well known result concerning state reachability is the one related with pole assignment. Indeed, *the pair  $(A, B)$  is reachable iff for every symmetric (with respect to the real line) set of complex numbers  $\Lambda$ , of cardinality  $n$ , there exists a proportional state feedback  $u = Fx$  such that the spectrum of  $(\lambda I - A_F)$  is  $\Lambda$*  (see for example Theorems 2.1 and 9.3.1 of Wonham (1985) and Polderman & Willems (1998), respectively).

40 *B. Behavioral reachability.* Willems (1983, 1991) defined an *input/state sys-*  
 41 *tem* as the triple  $\Sigma_{i/s} = (\mathbb{R}^+, \mathcal{U} \times \mathcal{X}, \mathfrak{B}_{[A,B]})$ , with behavior<sup>3</sup>

$$\mathfrak{B}_{[A,B]} = \left\{ (u, x) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{U} \times \mathcal{X}) \mid \begin{bmatrix} (I \frac{d}{dt} - A) & -B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0 \right\}. \quad (1.6)$$

42 In the behavioral framework of Willems (1983, 1991), the system  $\Sigma_{i/s} =$   
 43  $(\mathbb{R}^+, \mathcal{U} \times \mathcal{X}, \mathfrak{B}_{[A,B]})$  is called<sup>4</sup> *reachable* if *for any given*  $(u_0, x_0), (u_1, x_1) \in$   
 44  $\mathcal{U} \times \mathcal{X}$  *and*  $t_1 > 0$ , *it is possible to find a trajectory*  $(u, x) \in \mathfrak{B}_{[A,B]}$ , *such that*  
 45  $(u(0), x(0)) = (u_0, x_0)$  *and*  $(u(t_1), x(t_1)) = (u_1, x_1)$  (c.f. Polderman & Willems,  
 46 1998, Definition 5.2.2). In the following, this *reachability* concept is called  
 47 *behavioral reachability*.

48 In (Polderman & Willems, 1998, Theorem 5.2.27) is proved that for the  
 49 case of state space representations  $\mathfrak{R}^{ss}(A, B)$ , *state reachability* is equivalent to  
 50 *behavioral reachability*. Although the *behavioral reachability* is well charac-  
 51 terized, it could be interesting to find an explicit control law  $u(\cdot) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{U})$ ,  
 52 similar to (1.5), which ensures  $x(t_1) = x_1$  and  $u(t_1) = u_1$ . This will be done in  
 53 Section 2.

## 54 1.2. Implicit Representations

55 As a generalization of proper linear systems, Rosenbrock (1970) intro-  
 56 duced the *implicit representations*  $\mathfrak{R}^{imp}(E, A, B)$ , which are a set of differential

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<sup>3</sup> The original definition given by Willems (1983, 1991) is  $\mathfrak{B}_{[A,B]} = \left\{ (u, x) \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}^+, \mathcal{U} \times \mathcal{X}) \mid \exists x_0 \in \mathcal{X} \text{ s.t. } x(t) = \exp(At)x_0 + \int_0^t \exp(A(t-\tau))Bu(\tau)d\tau \right\}$ . But since we restrict our attention to infinitely differentiable functions (see footnote <sup>1</sup>), weak and strong solutions coincide (see Polderman & Willems, 1998, Theorem 2.3.11).

<sup>4</sup> For consistency of this paper, we say *reachable* instead of *controllable*, as is stated in (Polderman & Willems, 1998) (see also footnote <sup>2</sup>).



57 and algebraic equations (Brenan *et al*, 1996) of the following form (see also  
58 Lewis, 1992)

$$E dx/dt = Ax + Bu, \quad (1.7)$$

59 where  $E : \mathcal{X}_d \rightarrow \mathcal{X}_{eq}$ ,  $A : \mathcal{X}_d \rightarrow \mathcal{X}_{eq}$  and  $B : \mathcal{U} \rightarrow \mathcal{X}_{eq}$  are linear maps. The lin-  
60 ear spaces  $\mathcal{X}_d \approx \mathbb{R}^{n_d}$ ,  $\mathcal{X}_{eq} \approx \mathbb{R}^{n_{eq}}$ , and  $\mathcal{U} \approx \mathbb{R}^m$  are called the descriptor, the  
61 equation, and the input spaces, respectively. In order to avoid redundant  
62 components in the input variable  $u$ , and linear dependence on the descriptor  
63 equations (1.7), as usually, we assume throughout the paper that the following  
64 hypotheses are verified:

65 [H1]  $\text{Ker } B = 0$ , and

66 [H2]  $\text{Im } E + \text{Im } A + \mathcal{B} = \mathcal{X}_{eq}$ .

67 For the case of regular implicit representations, *i.e.* representations where  
68 the linear transformations  $E$  and  $A$  are square and the pencil  $[\lambda E - A]$  is in-  
69 vertible (Gantmacher, 1977), the *reachability* was studied by Verghese, Lévy  
70 and Kailath (1981) from a transfer function point of view, Yip and Sincovec  
71 (1981) in the time domain, Cobb (1984) from a distributional point of view,  
72 and by Özçaldıran (1985) from a geometric point of view.

73 In the case of implicit representations, where the linear transformations  
74  $E$  and  $A$  are square and the pencil  $[\lambda E - A]$  is not necessarily invertible,  
75 Özçaldıran (1986) extended his *reachability* geometric characterization for  
76 the case of regular implicit representations (Özçaldıran, 1985), by means of  
77 the supremal  $(A, E, B)$  reachability subspace contained in  $\mathcal{X}_d$ , defined as

$$\mathcal{R}_{\mathcal{X}_d}^* = \mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{S}_{\mathcal{X}_d}^*. \quad (1.8)$$

78 This is a nice generalization of the classical case,  $\mathfrak{R}^{ss}(A, B) = \mathfrak{R}^{imp}(I, A, B)$ ,  
79 where the reachable space  $\mathcal{R}_{\mathcal{X}_d}^*$  is equal to  $\langle A \mid \mathcal{B} \rangle$ , namely equal to  $\mathcal{V}_{[\mathcal{X}: A, I, B]}^* \cap$

80  $\mathcal{S}_{[I,A,B]}^*$ . Thus, for representations  $\mathfrak{R}^{imp}(E, A, B)$ , with  $E$  and  $A$  not necessarily  
 81 square, it was natural to associate its reachability with  $\mathcal{R}_{\mathcal{X}_d}^*$ .

82 [Frankowska \(1990\)](#) firmly established the pertinence of this reachability  
 83 concept, using differential inclusions to relate it with behavioral properties.

84 One major difficulty when studying reachability for implicit systems (1.7)  
 85 is that their solution set does not only depend on the initial conditions  $x(0)$   
 86 and on the external control input  $u$ , but also depends on a possible internal  
 87 free variable (degree of freedom), which is completely unknown.

### 88 1.3. Outline

89 In this paper, we study the reachability notion in the sense of [Frankowska](#)  
 90 (1990), showing some connections with the important works of [Willems](#)  
 91 (1991) and [Geerts \(1993\)](#), and we consider the relationships between the  
 92 reachability property and the complete pole assignment ability.

93 The paper is organized as follows: In Section 2, we consider the behav-  
 94 ioral reachability problem for state space representations, namely the ability  
 95 of reaching the input-state pair  $(u(\cdot), x(\cdot))$ . In Section 3, we formalize the  
 96 notion of *implicit systems*, following the behavioral point of view, and we  
 97 also study the equivalences between the notions of existence of solution and  
 98 impulse controllability. In Section 4, we study the reachability notion of  
 99 [Frankowska \(1990\)](#) for implicit systems. In Section 5, we consider the exis-  
 100 tent relationships between the reachability property and the complete pole  
 101 assignment ability, and in Section 6, we conclude the paper.

## 102 2. BEHAVIORAL REACHABILITY PROBLEM

103 We consider the following problem.

104 **Problem 1.** Let us consider an *input/state system*  $\Sigma_{i/s} = (\mathbb{R}^+, \mathcal{U} \times \mathcal{X}, \mathfrak{B}_{[A,B]})$   
 105 represented by (1.1), and with the behavior (1.6). Given  $(u_0, x_0), (u_1, x_1) \in$   
 106  $B^{-1} \langle A \mid \mathcal{B} \rangle \times \langle A \mid \mathcal{B} \rangle$  and  $t_1 > 0$ , find a trajectory  $(u, x) \in \mathfrak{B}_{[A,B]}$ , such that  
 107  $(u(0), x(0)) = (u_0, x_0)$  and  $(u(t_1), x(t_1)) = (u_1, x_1)$ .

108 This is the *behavioral reachability* problem, and in (Polderman & Willems,  
 109 1998, Theorem 5.2.27) is proved that for the case of *state space representa-*  
 110 *tions*, *state reachability* is equivalent to *behavioral reachability*. So, condition  
 111 (1.4) guarantees the existence of a solution for Problem 1.

112 One could think that the control law (1.5), proposed by Wonham (1985),  
 113 solves Problem 1, but this proposition only guarantees the *reachability* of the  
 114 state variable,  $x(0) = x_0$  and  $x(t_1) = x_1$ , and nothing about the input variable  $u$ ,  
 115 which is let completely free at the end points  $u(0)$  and  $u(t_1)$ . An intermediary  
 116 step towards the solution of Problem 1 is given by the next result proved in  
 117 Appendix A.

118 **Lemma 1.** *Let the state space representation (1.1) be reachable, with the*  
 119 *reachability indices set (1.3). Let the linear map  $F_{\mathcal{B}}: \mathbb{R}^n \rightarrow \mathbb{R}^m$  and the iso-*  
 120 *morphisms  $T_{\mathcal{B}}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $G_{\mathcal{B}}: \mathbb{R}^m \rightarrow \mathbb{R}^m$  be such that the pair  $(A_{\mathcal{B}}, B_{\mathcal{B}})$ ,*  
 121 *where  $A_{\mathcal{B}} = T_{\mathcal{B}}^{-1}(A + BF_{\mathcal{B}})T_{\mathcal{B}}$  and  $B_{\mathcal{B}} = T_{\mathcal{B}}^{-1}BG_{\mathcal{B}}$ , is expressed in the Brunovsky*  
 122 *canonical form (1.2). Let the reachability matrices,  $\mathcal{R}_{[A_{\mathcal{B}}, B_{\mathcal{B}}]}$  and  $\mathcal{R}_{[A_{\mathcal{B}_i}, b_{\mathcal{B}_i}]}$ ,*  
 123  *$i \in \{1, \dots, m\}$ , of the pair  $(A_{\mathcal{B}}, B_{\mathcal{B}})$  and the pairs  $(A_{\mathcal{B}_i}, b_{\mathcal{B}_i})$ ,  $i \in \{1, \dots, m\}$ , respec-*  
 124 *tively, be defined as follows:*

$$\begin{aligned} \mathcal{R}_{[A_{\mathcal{B}}, B_{\mathcal{B}}]} &= \text{BDM} \left\{ \mathcal{R}_{[A_{\mathcal{B}_1}, b_{\mathcal{B}_1}]}, \dots, \mathcal{R}_{[A_{\mathcal{B}_m}, b_{\mathcal{B}_m}]} \right\}, \\ \mathcal{R}_{[A_{\mathcal{B}_i}, b_{\mathcal{B}_i}]} &= \begin{bmatrix} b_{\mathcal{B}_i} & A_{\mathcal{B}_i} b_{\mathcal{B}_i} & \cdots & A_{\mathcal{B}_i}^{\kappa_i-1} b_{\mathcal{B}_i} \end{bmatrix}. \end{aligned} \quad (2.1)$$

125 *Let us assume that we have found trajectories  $f_i \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^1)$ , satisfying:*

126 (i) for  $j = 0, 1$

$$\mathfrak{D}(\frac{d}{dt})f(t_j) = G_{\mathcal{B}}^{-1}(u_j - F_{\mathcal{B}}x_j), \quad (2.2)$$

127 where  $\mathfrak{D}(d/dt) = \text{DM}\{d^{\kappa_1}/dt^{\kappa_1}, \dots, d^{\kappa_m}/dt^{\kappa_m}\}$ ,  $f(t) = \begin{bmatrix} f_1(t) & \dots & f_m(t) \end{bmatrix}^T$

128 and  $t_0 = 0$ .

129 (ii) If  $\bar{w}_i(t) = \begin{bmatrix} \frac{d^{\kappa_i-1}f_i(t)}{dt^{\kappa_i-1}} & \dots & \frac{df_i(t)}{dt} & f_i(t) \end{bmatrix}^T$ ,  $1 \leq i \leq m$ , and  $\bar{w}(t) = \begin{bmatrix} \bar{w}_1^T(t) & \dots & \bar{w}_m^T(t) \end{bmatrix}^T$  then, for  $j = 0, 1$ ,

$$\bar{w}(t_j) = \mathcal{R}_{[A_{\mathcal{B}}, B_{\mathcal{B}}]}^{-1} T_{\mathcal{B}}^{-1} x_j. \quad (2.3)$$

131 Then, applying the control law,

$$u(t) = F_{\mathcal{B}}x(t) + G_{\mathcal{B}}\mathfrak{D}(d/dt)f(t), \quad (2.4)$$

132 to the system represented by (1.1), we get:

$$x(t) = T_{\mathcal{B}}\mathcal{R}_{[A_{\mathcal{B}}, B_{\mathcal{B}}]}\bar{w}(t), \quad (2.5)$$

133 with

$$(u(0), x(0)) = (u_0, x_0) \quad \text{and} \quad (u(t_1), x(t_1)) = (u_1, x_1). \quad (2.6)$$

134 Let us now propose the trajectories:<sup>5</sup>

$$\begin{aligned} f_i(t) &= \begin{bmatrix} t^{2\kappa_i+1} & \dots & t^{\kappa_i+1} \end{bmatrix} \mathbf{a}_{i,1} + \begin{bmatrix} t^{\kappa_i} & \dots & 1 \end{bmatrix} \mathbf{a}_{i,0}, \\ \mathbf{a}_{i,1} &= \begin{bmatrix} a_{i,2\kappa_i+1} & \dots & a_{i,\kappa_i+1} \end{bmatrix}^T \in \mathbb{R}^{\kappa_i+1}, \text{ and } \mathbf{a}_{i,0} = \begin{bmatrix} a_{i,\kappa_i} & \dots & a_{i,0} \end{bmatrix}^T \in \mathbb{R}^{\kappa_i+1}, \end{aligned} \quad (2.7)$$

135 with  $i \in \{1, \dots, m\}$ , and let us define the following auxiliary matrices:

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<sup>5</sup> Lewis (1986) did a similar proposition when he introduced a “fast” input in his reachability consideration.

$$X_{(i,0)}(t) = \begin{bmatrix} \kappa_i!/0! & 0 & \dots & 0 & 0 \\ (\kappa_i!/1!)t & (\kappa_i-1)!/0! & 0 \dots & 0 & 0 \\ \vdots & \vdots & \dots & . & . \\ (\kappa_i!/ \kappa_i!)t^{\kappa_i} & ((\kappa_i-1)!/(\kappa_i-1)!)t^{\kappa_i-1} & \dots & (1!/1!)t & 0!/0! \end{bmatrix}, \quad (2.8)$$

136

$$X_{(i,1)}(t) = \begin{bmatrix} ((2\kappa_i+1)!/(\kappa_i+1)!)t^{\kappa_i+1} & \dots & ((\kappa_i+1)!/1!)t \\ \vdots & \dots & \vdots \\ ((2\kappa_i+1)!/(2\kappa_i+1)!)t^{2\kappa_i+1} & \dots & ((\kappa_i+1)!/(\kappa_i+1)!)t^{\kappa_i+1} \end{bmatrix}. \quad (2.9)$$

137 The following Lemma gives a selection of the coefficient vectors  $\mathbf{a}_{i,0}$  and  $\mathbf{a}_{i,1}$ ,  
 138  $i \in \{1, \dots, m\}$ , for  $f$  in (2.7) to satisfy assumptions (2.2) and (2.3) of Lemma  
 139 1 (see Appendix B for the proof).

140 **Lemma 2.** For  $i \in \{1, \dots, m\}$ , the determinants of the auxiliary matrices (2.8)  
 141 and (2.9) satisfy

$$\det(X_{(i,0)}(t)) = \prod_{\ell=0}^{\kappa_i} \ell! \quad \text{and} \quad \det(X_{(i,1)}(t)) = t^{(\kappa_i+1)^2} \prod_{\ell=0}^{\kappa_i} \ell! \quad (2.10)$$

142 Moreover, if we select the coefficient vectors  $\mathbf{a}_{i,0}$  and  $\mathbf{a}_{i,1}$ , as follows:

$$\begin{aligned} \mathbf{a}_{i,0} &= X_{(i,0)}^{-1}(0)v_0, \quad \mathbf{a}_{i,1} = X_{(i,1)}^{-1}(t_1)(v_1 - X_{(i,0)}(t_1)\mathbf{a}_{i,0}), \\ v_j &= \begin{bmatrix} ((\underline{e}_m^i)^T G_{\mathcal{B}}^{-1}(u_j - F_{\mathcal{B}}x_j))^T & \left( \mathcal{R}_{[A_{\mathcal{B}_i}, B_{\mathcal{B}_i}]}^{-1} P_i T_{\mathcal{B}}^{-1} x_j \right)^T \end{bmatrix}^T, \quad j \in \{0, 1\}, \end{aligned} \quad (2.11)$$

143 where:

$$P_i = \begin{bmatrix} \underline{e}_n^{\hat{n}_i+1} & \dots & \underline{e}_n^{\hat{n}_i+\kappa_i} \end{bmatrix}^T, \quad \hat{n}_1 = 0 \quad \text{and} \quad \hat{n}_{i \geq 2} = \sum_{j=1}^{i-1} \kappa_j, \quad (2.12)$$

144 then the function  $f$  defined by (2.7) fulfills assumptions (2.2) and (2.3) of  
 145 Lemma 1.

Let us note from Lemma 2 that the proposed solutions only depend on the set of reachability indices  $S_\kappa$ , and on the fixed final time  $t_1$ . Hence, once  $S_\kappa$  and  $t_1$  are given, the matrices  $X_{i,0}(0)$ ,  $X_{i,0}(t_1)$  and  $X_{i,1}(t_1)$  are uniquely determined. And thus, the values of  $\mathfrak{a}_{i,0}$  and  $\mathfrak{a}_{i,1}$  only depend on the boundary points,  $(u_0, x_0)$  and  $(u_1, x_1)$ , of the trajectory  $(u, x) \in \mathfrak{B}_{[A,B]}$ .

From the above observation, it is possible to track a given trajectory  $(\bar{u}, \bar{x}) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^{m+n})$ , with a delayed time  $t_1$ . Indeed, we only need to fix a sampling time  $t_1 \in \mathbb{R}^{+*}$ , and to apply iteratively Lemma 1 with the settings  $(u_0, x_0) = (u(kt_1), x(kt_1))$  and  $(u_1, x_1) = (\bar{u}(kt_1), \bar{x}(kt_1))$ .

Otherwise written, in each sampling interval  $[kt_1, (k+1)t_1)$ , we find a trajectory  $(u, x) \in \mathfrak{B}_{[A,B]} \cap \mathcal{C}^\infty(\mathbb{R}^+ \cap [kt_1, (k+1)t_1), \mathcal{U} \times \mathcal{X})$ , such that  $(u(kt_1), x(kt_1)) = (u_0, x_0)$  and  $\lim_{\sigma \rightarrow t_1} (u(kt_1 + \sigma), x(kt_1 + \sigma)) = (u_1, x_1)$ .

We have proved in this way the following Theorem.

**Theorem 1.** *Let us consider an input/state system  $\Sigma_{i/s} = (\mathbb{R}^+, \mathcal{U} \times \mathcal{X}, \mathfrak{B}_{[A,B]})$ , represented by (1.1). If (1.4) is satisfied, then for any sequence  $(\bar{u}_k, \bar{x}_k) \in \mathbb{R}^{m+n}$ ,  $k \in \mathbb{Z}^+$ , and a given sampling time  $t_1 \in \mathbb{R}^{+*}$ , there exists a control law  $u \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^m)$ , such that  $(u(kt_1), x(kt_1)) = (\bar{u}_{k-1}, \bar{x}_{k-1})$ .*

### 3. IMPLICIT SYSTEMS

In this Section, we formalize the notion of *implicit system* following the behavioral point of view. For this, let us first state the following definition:

**Definition 1.** An implicit representation  $\mathfrak{R}^{imp}(E, A, B)$  is called an *input/descriptor system*, when for all initial condition  $x_0 \in \mathcal{X}_d$ , there exists at least one solution  $(u, x) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{U} \times \mathcal{X}_d)$ , such that  $x(0) = x_0$ . The *input/descriptor*

169 *system* is defined by the triple<sup>6</sup>  $\Sigma_{i/d} = (\mathbb{R}^+, \mathcal{U} \times \mathcal{X}_d, \mathfrak{B}_{[E,A,B]})$ , with behavior:

$$\mathfrak{B}_{[E,A,B]} = \left\{ (u, x) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{U} \times \mathcal{X}) \mid \begin{bmatrix} (E \frac{d}{dt} - A) & -B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0 \right\} \quad (3.1)$$

170 At this point, it is important to clarify what exactly means the sentence  
 171 “*there exists at least one solution*”. For this, we are going to recall hereafter  
 172 the notions of *existence of solution* introduced by Geerts (1993) and Aubin  
 173 & Frankowska (1991).

### 174 3.1. Existence of solution for every initial condition

175 Following (Hautus, 1976) and (Hautus & Silverman, 1983), Geerts (1993)  
 176 generalized the solvability results of (Geerts & Mehrmann, 1990). One advan-  
 177 tage of this generalization is that the solvability is introduced in a very nat-  
 178 ural way, passing from the distributional framework (Schwartz, 1978) to the  
 179 usual time domain with ordinary differential equations; this is precisely the  
 180 starting point of the so called behavioral approach (Polderman & Willems,  
 181 1998), chosen in this paper.

182 Geerts (1993) considered the linear combinations of impulsive and smooth  
 183 distributions, with  $\mu$  coordinates, denoted by  $\mathcal{C}_{\text{imp}}^\mu$ , as the signal sets. The  
 184 set  $\mathcal{C}_{\text{imp}}^\mu$  is a subalgebra and is also decomposed as  $\mathcal{C}_{\text{p-imp}}^\mu \oplus \mathcal{C}_{\text{sm}}^\mu$ , where  $\mathcal{C}_{\text{p-imp}}^\mu$   
 185 and  $\mathcal{C}_{\text{sm}}^\mu$  denote the subalgebras of pure impulses<sup>7</sup> and smooth distributions<sup>8</sup>,

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<sup>6</sup> See also Polderman & Willems (1998) and Kuijper (1992b).

<sup>7</sup> The unit element of this subalgebra is the Dirac delta distribution  $\delta$ . Any linear combination of  $\delta$  and its distributional derivatives  $\delta^{(\ell)}$ ,  $\ell > 1$ , is called impulsive.

<sup>8</sup> The set of regular distributions are distributions that are functions; namely piecewise continuous integrable, or measurable functions. In those papers, they assume that the regular distributions  $u(t)$  are smooth on  $[0, \infty)$ , *i.e.* that a function  $v : [0, \infty) \rightarrow \mathbb{R}$  exists, ar-

186 respectively (Schwartz, 1978). He introduced the following definitions for  
 187 the distributional version of the *implicit representation* (1.7)  $\mathfrak{R}_{dist}^{imp}(E, A, B)$ :<sup>9</sup>  
 188  $pEx = Ax + Bu + Ex_0$  (c.f. Definitions 3.1 and 4.1, Geerts, 1993)<sup>10</sup>.

189 **Definition 2.** (Geerts, 1993) Given the *solution set*  $S_C(x_0, u) := \{x \in \mathcal{C}_{imp}^{n_d} \mid$   
 190  $[pE - A]x = Bu + Ex_0\}$ , the *implicit representation*  $\mathfrak{R}_{dist}^{imp}(E, A, B)$  is:

- 191 • *C-solvable* if  $\forall x_0 \in \mathcal{X}_d \exists u \in \mathcal{C}_{imp}^m : S_C(x_0, u) \neq \emptyset$ ,
- 192 • *C-solvable in the function sense* if  $\forall x_0 \in \mathcal{X}_d \exists u \in \mathcal{C}_{sm}^m : S_C(x_0, u) \cap \mathcal{C}_{sm}^n \neq \emptyset$ .

193 Given the “consistent initial conditions set”  $\mathcal{I}_C := \{z_0 \in \mathcal{X}_d \mid \exists u \in \mathcal{C}_{sm}^m \exists x \in$   
 194  $S_C(z_0, u) \cap \mathcal{C}_{sm}^{n_d} : x(0^+) = z_0\}$ , and the “weakly consistent initial conditions set”  
 195  $\mathcal{I}_C^w := \{z_0 \in \mathcal{X}_d \mid \exists u \in \mathcal{C}_{sm}^m \exists x \in S_C(z_0, u) \cap \mathcal{C}_{sm}^{n_d}\}$ , a point  $x_0 \in \mathcal{X}_d$  is called *C-*  
 196 *consistent* if  $x_0 \in \mathcal{I}_C$ , and *weakly C-consistent* if  $x_0 \in \mathcal{I}_C^w$ .

197 Let us note that:

- 198 (i) *C-solvability* is concerned with distributional solutions,
- 199 (ii) *C-solvability in the function sense* is concerned with solutions only com-  
 200 posed by ordinary functions arbitrarily often differentiable,
- 201 (iii) the two notions of consistency, *C-consistent* and *weakly C-consistent*,  
 202 lead to *smooth solutions*, namely with no impulses, but

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bitrarily often differentiable including at  $t = 0$ , such that  $u(t) = 0$  for  $t < 0$  and  $u(t) = v(t)$ , for  
 $t \geq 0$  (Hautus & Silverman, 1983). These distributions are identified as ordinary functions  
 with support on  $\mathbb{R}^+$ .

<sup>9</sup>  $Ex_0$  stands for  $Ex_0\delta$ ,  $x_0 \in \mathcal{X}_d$  being the initial condition, and  $pEx$  stands for  
 $\delta^{(1)} * Ex$  ( $*$  denotes convolution); if  $pEx$  is smooth and  $E\dot{x}$  stands for the distribution  
 that can be identified with the ordinary derivative  $E dx/dt$ , then  $pEx = E\dot{x} + Ex_{0+}$ .

<sup>10</sup> He also considered the *B-free* case  $\mathfrak{R}_{dist}^{imp}(E, A, f)$ :  $pEx = Ax + f + Ex_0$ .



203 (iv) *C-consistency* avoids jumps at the origin, namely the *smooth solutions*  
 204 are continuous on the left, and  
 205 (iv) *weakly C-consistent* enables jumps at the origin, but they are piece-wise  
 206 continuous *smooth solutions*.

207 [Geerts \(1993\)](#) characterized the existence of solutions for every initial  
 208 condition in his Corollary 3.6, Proposition 4.2 and Theorem 4.5. Hereafter  
 209 we summarize these results with their geometric equivalences.

210 **Theorem 2.** ([Geerts, 1993](#)) If **[H2]** is fulfilled, then

- 211 •  $\mathfrak{R}_{dist}^{imp}(E, A, B)$  is C-solvable if and only if  $[(\lambda E - A) \quad -B]$  is right invert-
- 212 ible as a rational matrix, i.e. if and only if<sup>11</sup>

$$E\mathcal{V}_{\mathcal{X}_d}^* + A\mathcal{S}_{\mathcal{X}_d}^* + \mathcal{B} = \mathcal{X}_{eq} . \quad (3.2)$$

- 213 •  $\mathfrak{R}_{dist}^{imp}(E, A, B)$  is C-solvable in the function sense if and only if  $\mathcal{I}_C^w = \mathcal{X}_d$ ,
- 214 namely, if and only if  $\text{Im } E + A\mathcal{K}_E + \mathcal{B} = \mathcal{X}_{eq}$ , i.e. if and only if<sup>12</sup>

$$E\mathcal{V}_{\mathcal{X}_d}^* = \text{Im } E . \quad (3.3)$$

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<sup>11</sup>  $[\lambda[E \quad 0] - [A \quad B]]$  is right invertible iff (see [Loiseau \(1985\)](#) and [Armentano \(1986\)](#))  $\mathcal{X}_{eq} = [E \quad 0]\mathcal{V}_{[[A \quad B], [E \quad 0]]}^* + [A \quad B]\mathcal{S}_{[[E \quad 0], [A \quad B]]}^*$ , namely iff  $E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} + A\mathcal{S}_{\mathcal{X}_d}^* = \mathcal{X}_{eq}$  (from **(ALG-V)** and **(ALG-S)** we get  $\mathcal{V}_{[[A \quad B], [E \quad 0]]}^* = \mathcal{V}_{\mathcal{X}_d}^* \oplus \mathcal{U}$  and  $\mathcal{S}_{[[E \quad 0], [A \quad B]]}^* = \mathcal{S}_{\mathcal{X}_d}^* \oplus \mathcal{U}$ ).

<sup>12</sup> From **(ALG-V)** and **[H2]**, one obtains the following sequence of implications:  
 $\text{Im } E + \mathcal{B} + A\mathcal{K}_E = \mathcal{X}_{eq} \Rightarrow \mathcal{V}_{\mathcal{X}_d}^1 + \mathcal{K}_E = \mathcal{X}_d \Rightarrow E\mathcal{V}_{\mathcal{X}_d}^1 = \text{Im } E \Rightarrow E\mathcal{V}_{\mathcal{X}_d}^* = \text{Im } E$   
 $\Rightarrow \mathcal{X}_d = \mathcal{V}_{\mathcal{X}_d}^* + \mathcal{K}_E = A^{-1}(E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) + \mathcal{K}_E \Rightarrow \text{Im } A = \text{Im } A \cap (E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) + A\mathcal{K}_E \Rightarrow$   
 $\mathcal{X}_{eq} = \text{Im } E + \mathcal{B} + A\mathcal{K}_E.$

215 •  $\mathcal{I}_C = \mathcal{X}_d$  if and only if  $\text{Im } E + \mathcal{B} = \mathcal{X}_{eq}$  i.e. if and only if <sup>13</sup>

$$E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} = \mathcal{X}_{eq} . \quad (3.4)$$

### 216 3.2. Existence of a viable solution

217 In order to study the reachability for implicit systems, Frankowska (1990)  
 218 introduced the set-valued map (the set of all admissible velocities)  $\mathbf{F} : \mathcal{X}_d \rightsquigarrow$   
 219  $\mathcal{X}_d$ ,  $\mathbf{F}(x) = E^{-1}(Ax + \mathcal{B}) = \{v \in \mathcal{X} \mid Ev \in Ax + \mathcal{B}\}$ , and the differential inclusion

$$dx/dt \in \mathbf{F}(x), \quad \text{where } x(0) = x_0, \quad (3.5)$$

220 Frankowska (1990) showed that the solutions of (1.7) and the ones of (3.5)  
 221 are the same. She also clarified the meaning of a viable solution and she  
 222 characterized the largest subspace of such viable solutions.

223 **Definition 3.** (Frankowska, 1990, Aubin & Frankowska, 1991)

- 224 • An absolutely continuous function  $x : \mathbb{R}^+ \rightarrow \mathcal{X}_d$  is called a *trajectory* of  
 225 (3.5), if  $x(0) = x_0$  and  $dx/dt \in \mathbf{F}(x)$  for almost every  $t \in \mathbb{R}^+$ , that is to say,  
 226 if there exists a measurable function  $u : \mathbb{R}^+ \rightarrow \mathcal{U}$  such that  $x(0) = x_0$  and  
 227  $Edx/dt = Ax + Bu$ , for almost every  $t \in \mathbb{R}^+$ .
- 228 • Let  $\mathcal{K}$  be a subspace<sup>14</sup> of  $\mathcal{X}_d$ . A trajectory  $x$  of (3.5) is called *viable*  
 229 *in  $\mathcal{K}$* , if  $x(t) \in \mathcal{K}$  for all  $t \geq 0$ . The set of such trajectories is called

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<sup>13</sup> Directly follows from (ALG-V) and [H2].

<sup>14</sup> We restrict our discussion to subspaces of finite dimensional vector spaces. In (Frankowska, 1990) and in (Aubin & Frankowska, 1991) these definitions are stated in the more general framework of closed sets of normed vector spaces.

230 the set of viable solutions in  $\mathcal{X}$ . The subspace  $\mathcal{X}$  is called a viability  
 231 domain of  $\mathbf{F}$ , if for all  $x \in \mathcal{X} : \mathbf{F}(x) \cap \mathcal{X} \neq \emptyset$ . The subspace  $\mathcal{X}$  is called  
 232 the viability kernel of (3.5) when it is the largest viability domain of  $\mathbf{F}$ .

233 **Theorem 3.** (*Aubin & Frankowska, 1991*) The supremal  $(A, E, B)$ -invariant  
 234 subspace contained in  $\mathcal{X}_d, \mathcal{V}_{\mathcal{X}_d}^*$ , is the viability kernel of  $\mathcal{X}_d$  for the set-valued  
 235 map  $\mathbf{F} : \mathcal{X}_d \rightsquigarrow \mathcal{X}_d, \mathbf{F}(x) = E^{-1}(Ax + \mathcal{B})$ . Moreover, for all  $x_0 \in \mathcal{V}_{\mathcal{X}_d}^*$  there exists  
 236 a trajectory  $x \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{V}_{\mathcal{X}_d}^*)$  solution of (1.7) satisfying  $x(0) = x_0$ .

237 Frankowska (1990) called a singular system “strict” when the viability  
 238 kernel coincides with the whole descriptor space  $\mathcal{X}_d$ , namely

$$\mathcal{V}_{\mathcal{X}_d}^* = \mathcal{X}_d. \quad (3.6)$$

239 In order to clarify ideas, let us extract from (Bonilla & Malabre, 1997,  
 240 Section 2.1) the following result:

241 **Result 1.** There exists a subspace  $\mathcal{X}_1$  such that:

$$\mathcal{X}_d = \mathcal{V}_{\mathcal{X}_d}^* \oplus \mathcal{X}_1, \quad \mathcal{X}_{eq} = (E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) \oplus A\mathcal{X}_1, \quad \text{and} \quad \mathcal{X}_1 \approx A\mathcal{X}_1. \quad (3.7)$$

242 Moreover, when projecting on  $\mathcal{X}_1$  any trajectory  $x \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{V}_{\mathcal{X}_d}^*)$  solution  
 243 of (1.7), we always get a null trajectory.

244 Furthermore, for all  $x_0 \in \mathcal{V}_{\mathcal{X}_d}^*$  there exists at least one trajectory  $(u, x_\rho) \in$   
 245  $\mathcal{C}^\infty(\mathbb{R}^+, \mathcal{U} \times \mathcal{V}_{\mathcal{X}_d}^*)$  solution of (1.7), satisfying  $x_\rho(0) = x_0$ .

246 **PROOF OF RESULT 1.** From the algorithm shown in (Fig. 1, Bonilla & Mal-  
 247 abre, 1997) and from [H2], we get the geometric decompositions (3.7), and  
 248 under these decompositions, (1.7) takes the following form:

$$\left[ \begin{array}{c|c} E_\rho & * \\ \hline 0 & \bar{X}_{\rho-1} \end{array} \right] \frac{d}{dt} \left[ \begin{array}{c} x_\rho \\ \bar{x}_{\rho-1} \end{array} \right] = \left[ \begin{array}{c|c} A_\rho & 0 \\ \hline 0 & I_1 \end{array} \right] \left[ \begin{array}{c} x_\rho \\ \bar{x}_{\rho-1} \end{array} \right] + \left[ \begin{array}{c} B_\rho \\ 0 \end{array} \right] u, \quad (3.8)$$

249 where  $x_\rho \in \mathcal{V}_{\mathcal{X}_d}^*$ ,  $\bar{x}_{\rho-1} \in \mathcal{X}_1$ ,  $I_1 : \mathcal{X}_1 \leftrightarrow A\mathcal{X}_1$  is an isomorphism,  $\bar{X}_{\rho-1}$  is a nilpo-  
 250 tent matrix (an upper triangular matrix with zeros in its diagonal). Then  
 251  $\bar{x}_{\rho-1} \equiv 0$ .

252 If we now apply the following geometric decompositions:

$$E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} = E\mathcal{V}_{\mathcal{X}_d}^* \oplus \mathcal{B}_C, \quad \mathcal{B} = (\mathcal{B} \cap E\mathcal{V}_{\mathcal{X}_d}^*) \oplus \mathcal{B}_C, \quad \mathcal{U} = B^{-1}E\mathcal{V}_{\mathcal{X}_d}^* \oplus B^{-1}\mathcal{B}_C, \quad (3.9)$$

253 where  $\mathcal{B}_C$  is some complementary subspace of  $\mathcal{B} \cap E\mathcal{V}_{\mathcal{X}_d}^*$ , we get for  $\Re^{imp}(E_\rho, A_\rho,$   
 254  $B_\rho)$  (recall (3.8)):

$$\begin{bmatrix} \bar{E}_\rho \\ 0 \end{bmatrix} \frac{d}{dt} x_\rho = \begin{bmatrix} \bar{A}_\rho \\ \hat{A}_\rho \end{bmatrix} x_\rho + \begin{bmatrix} \bar{B}_\rho & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \quad (3.10)$$

255 Since  $\text{Im } \bar{E}_\rho = E\mathcal{V}_{\mathcal{X}_d}^*$ , there exists  $\bar{E}_\rho^r : E\mathcal{V}_{\mathcal{X}_d}^* \rightarrow \mathcal{V}_{\mathcal{X}_d}^*$  such that  $\bar{E}_\rho \bar{E}_\rho^r = I$ . Then,  
 256 one solution of (3.10) is given by

$$\begin{aligned} x_\rho(t) &= \exp(\bar{E}_\rho^r \bar{A}_\rho t) x_0 + \int_0^t \exp(\bar{E}_\rho^r \bar{A}_\rho(t-\tau)) \bar{E}_\rho^r \bar{B}_\rho u_1(\tau) d\tau, \\ u_2(t) &= -\hat{A}_\rho x_\rho(t). \end{aligned} \quad \square$$

257 Thus, the subspaces  $E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} \subset \mathcal{X}_{eq}$  and  $\mathcal{V}_{\mathcal{X}_d}^* \subset \mathcal{X}_d$  characterize the set of  
 258 all possible trajectories of (1.7) which are not identically zero for any input  
 259  $u$ . The projection of any trajectory solution of (1.7) on the quotient space  
 260  $\mathcal{X}_d/\mathcal{V}_{\mathcal{X}_d}^*$ , in correspondence with the projection on  $\mathcal{X}_{eq}/(E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B})$  for the  
 261 equation space, results in an identically null function (see Bonilla & Malabre,  
 262 1995, Corollary 2.1). Let us note that when Assumption [H2] holds, the  
 263 geometric conditions  $E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} = \mathcal{X}_{eq}$  and  $\mathcal{V}_{\mathcal{X}_d}^* = \mathcal{X}_d$  are equivalent<sup>15</sup>.

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<sup>15</sup> From (ALG-V) and [H2]:  $E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} = \mathcal{X}_{eq} \Rightarrow \mathcal{V}_{\mathcal{X}_d}^* = A^{-1}(E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) = \mathcal{X}_d$ ;  $\mathcal{V}_{\mathcal{X}_d}^* = \mathcal{X}_d$   
 $\Rightarrow \text{Im } E = E\mathcal{V}_{\mathcal{X}_d}^*$  &  $\text{Im } A = \text{Im } A \cap (E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) \Rightarrow \mathcal{X}_{eq} = \text{Im } E + \text{Im } A + \mathcal{B} = E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}$ .

### 264 3.3. Discussion about existence of solution

265 An important contribution of Geerts (1993) is to give conditions under  
 266 which the distributional and time-domain frameworks lead to the same con-  
 267 clusions with respect to the shape of the resulting system's solution trajec-  
 268 tories (*c.f.* (3.4) and (3.3)), namely the resulting distributions are identified as  
 269 ordinary functions, with support on  $\mathbb{R}^+$ , and the generalized derivatives can  
 270 be identified with ordinary derivatives. Also, it is well connected with the  
 271 viability discussion of Frankowska (1990) and Aubin & Frankowska (1991);  
 272 indeed, a singular system is strict if and only if the consistent initial condi-  
 273 tion set  $\mathcal{I}_C$  coincides with the whole descriptor variable space  $\mathcal{X}_d$  (*c.f.* (3.6)  
 274 and (3.4), and recall Assumption [H2]).

275 Regarding the set of weakly consistent initial conditions Geerts (1993)  
 276 notes, in his abstract and conclusion, that the condition that this set equals  
 277 to the whole state space (under the Assumption [H2]) is equivalent to the  
 278 impulse controllability for regular systems (Cobb, 1984) (or controllability of  
 279 the infinite part in the sense of Verghese *et al* (1981)). This correspondence  
 280 has been generalized to non regular systems and one can note that the nowa-  
 281 days most commonly adopted definition for impulse controllability is the one  
 282 cited by Ishihara & Terra (2001)<sup>16</sup>: *a general singular system is impulse*  
 283 *controllable if for every initial condition there exists a smooth (impulse-free)*  
 284 *control  $u(t)$ , and a smooth (impulse-free, but with possible jumps, especially*  
 285 *at the origin) variable descriptor trajectory solution of the system.*

286 More generally, one can verify that the paper of Geerts (1993) is the

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<sup>16</sup>Notice that in this paper is stated that the definition comes from Geerts (1993).

287 main reference on solvability properties, consistency of initial conditions, the  
 288 ability to find control such that no impulsive phenomenon appears (see for  
 289 examples [Hou & Müller \(1999\)](#), [Ishihara & Terra \(2001\)](#), [Hou \(2004\)](#) and  
 290 [Zhang \(2006\)](#)).

291 However, one should also cite [Özçaldıran & Haliloğlu \(1993\)](#) who proved  
 292 that there exists a pair of smooth distributions (without jumps), satisfying  
 293  $\mathfrak{R}_{dist}^{imp}(E, A, B)$  if and only if  $x(0_*) \in \mathcal{V}_{\mathcal{X}_d^*}$ , namely  $\mathcal{V}_{\mathcal{X}_d^*} = \mathcal{X}_d$  (see their Proposi-  
 294 tion 1.3), and [Przyłuski & Sosnowski \(1994\)](#) who proved that the subspace  
 295  $\mathcal{V}_{\mathcal{X}_d^*} + \mathcal{K}_E$  characterizes the set of initial conditions, for which there exists a  
 296 pair of smooth distributions (with possible jumps) satisfying  $\mathfrak{R}_{dist}^{imp}(E, A, B)$ ,  
 297 namely  $E\mathcal{V}_{\mathcal{X}_d^*} = \text{Im } E$  (see their Proposition 1).

298 In Figure 1, we summarize all the above discussion.

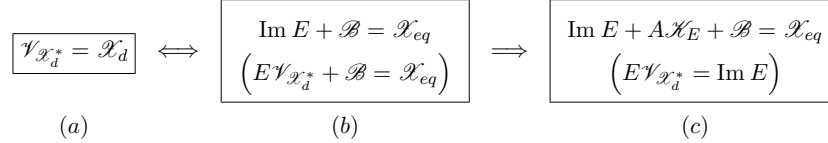


Figure 1: Connexions between the notions of existence of solution and impulse controllability (under the Assumption **[H2]**).

299 Fig. 1(a) is the condition of viable solution of [Aubin & Frankowska \(1991\)](#)  
 300 or smooth solution (without any jump) of [Özçaldıran & Haliloğlu \(1993\)](#). Fig.  
 301 1(b) is the condition that the set of consistent initial condition equals the  
 302 whole space of [Geerts \(1993\)](#). Fig. 1(c) is the condition of *C-solvability in the*  
 303 *function sense* of [Geerts \(1993\)](#) or the condition of [Przyłuski & Sosnowski](#)  
 304 [\(1994\)](#) that the set of initial conditions of smooth solutions (with possible  
 305 jumps) equals the whole space, or the impulse controllability condition of  
 306 [Ishihara & Terra \(2001\)](#), or the impulse-mode controllability with arbitrary

307 initial conditions of Hou (2004).

308 Finally let us note that if the notion of weakly consistent initial conditions  
309 as defined by Geerts (1993) is associated to the notion of impulse controlla-  
310 bility, the notion of consistent initial conditions as defined by Geerts (1993)  
311 is associated to the notion of reachability of Frankowska (1990) (in the more  
312 general non regular case) since the system must be strict to be reachable.  
313 See also the controllability discussion found in Korotka *et al* (2011).

#### 314 4. REACHABILITY FOR IMPLICIT SYSTEMS

315 For the case of implicit systems, Frankowska (1990) extended the classical  
316 reachability definition as follows.

317 **Definition 4.** (Frankowska, 1990) The implicit representation (1.7) is called  
318 *reachable* if for any pair of vectors  $x_0, x_1 \in \mathcal{X}_d$  and for any pair of real numbers  
319  $t_1 > t_0 \geq 0$ , there exists a trajectory  $x(\cdot)$  solution of (1.7), such that  $x(t_0) = x_0$   
320 and  $x(t_1) = x_1$ .

321 Frankowska (1990) has established in her Theorem 4.4 that  $\mathcal{R}_{\mathcal{X}_d}^*$  (see (1.8))  
322 is the reachable space of implicit systems like (1.7), with  $E$  and  $A$  not nec-  
323 essarily square. Hereafter, we recall Corollary 2.4 of Aubin and Frankowska  
324 (1991) which is *ad hoc* for our paper.

325 **Theorem 4.** (Aubin & Frankowska, 1991) For any  $t_1 > 0$  and for a system  
326 like (1.7), with  $E$  and  $A$  not necessarily square, the reachable space of (1.7)  
327 at time  $t_1$  from the initial descriptor variable  $x(0)$  is equal to  $\mathcal{R}_{\mathcal{X}_d}^*$ . Moreover,  
328  $\mathcal{R}_{\mathcal{X}_d}^*$  is the supremal subspace such that for all  $x_0, x_1 \in \mathcal{R}_{\mathcal{X}_d}^*$  and  $t_1 > 0$ , there

329 exists a trajectory  $x \in C^\infty(\mathbb{R}^+ \mathcal{R}_{\mathcal{X}_d}^*)$  solution of (1.7) satisfying  $x(0) = x_0$  and  
 330  $x(t_1) = x_1$ .

331 In this Section we are interested in generalizing and solving Problem 1  
 332 in the case of an *input/descriptor system*  $\Sigma_{i/d} = (\mathbb{R}^+, \mathcal{U} \times \mathcal{X}_d, \mathfrak{B}_{[E,A,B]})$ , with  
 333 behavior (3.1).

334 **Problem 2.** Let us consider a *input/descriptor system*  $\Sigma_{i/d} = (\mathbb{R}^+, \mathcal{U} \times \mathcal{X}_d,$   
 335  $\mathfrak{B}_{[E,A,B]})$  represented by (1.7), and with the behavior (3.1). Given  $(u_0, x_0),$   
 336  $(u_1, x_1) \in B^{-1}E\mathcal{R}_{\mathcal{X}_d}^* \times \mathcal{R}_{\mathcal{X}_d}^*$  and  $t_1 > 0$ , find a trajectory  $(u, x) \in \mathfrak{B}_{[E,A,B]}$ , such  
 337 that  $(u(0), x(0)) = (u_0, x_0)$  and  $(u(t_1), x(t_1)) = (u_1, x_1)$ .

338 For answering this question, we proceed as follows.

- 339 (i) We first apply some geometric decompositions to the subspaces  $\mathcal{X}_d$  and  
 340  $\mathcal{X}_{eq}$ , inspired by Proposition 2.2 of Aubin and Frankowska (1991); the aim  
 341 of these decompositions is to point out a part of the implicit representation,  
 342 more or less explicit, which is expressed as a state space representation.
- 343 (ii) We next show that such a state space representation is reachable in the  
 344 classical sense.
- 345 (iii) Finally, based on Section 2, we answer Problem 2.

#### 346 4.1. State reachability

347 The following Lemma is proved in Appendix C.

348 **Lemma 3.** When  $\mathcal{R}_{\mathcal{X}_d}^* = \mathcal{X}_d$ , the implicit representation (1.7) can be re-  
 349 stricted to  $\mathcal{R}_{\mathcal{X}_d}^*$  in the domain, and to  $A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}$  in the codomain.

350 Moreover, the spaces  $\mathcal{R}_{\mathcal{X}_d}^*$ ,  $\mathcal{B}$ ,  $A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}$  and  $\mathcal{U}$  can be decomposed as fol-  
 351 lows:  $\mathcal{R}_{\mathcal{X}_d}^* = \mathcal{R}_C \oplus (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E)$ ,  $\mathcal{B} = (\mathcal{B} \cap E\mathcal{R}_{\mathcal{X}_d}) \oplus \mathcal{B}_C$ ,  $A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B} = E\mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{B}_C$



352 and  $\mathcal{U} = B^{-1}E\mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{U}_C$ , where  $\mathcal{R}_C$  and  $\mathcal{U}_C$  are complementary subspaces such  
 353 that  $\mathcal{R}_C \approx E\mathcal{R}_{\mathcal{X}_d}^*$  and  $\mathcal{U}_C = B^{-1}\mathcal{B}_C \approx \mathcal{B}_C$ . Under these decompositions, the im-  
 354 plicit representation (1.7), restricted to  $\mathcal{R}_{\mathcal{X}_d}^*$  in the domain and to  $A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}$   
 355 in the codomain, takes the following form:

$$\begin{aligned} \bar{E}dx/dt &= \bar{A}x + \bar{B}u, \\ \bar{E} &= \begin{bmatrix} I_C & 0 \\ 0 & 0 \end{bmatrix}, \bar{A} = \begin{bmatrix} \bar{A}_{1,1} & \bar{A}_{1,2} \\ \bar{A}_{2,1} & \bar{A}_{2,2} \end{bmatrix}, \bar{B} = \begin{bmatrix} \bar{B}_1 & 0 \\ 0 & I_{\mathcal{U}_C} \end{bmatrix}, \end{aligned} \quad (4.1)$$

356 where  $I_C : \mathcal{R}_C \leftrightarrow E\mathcal{R}_{\mathcal{X}_d}^*$ , and  $I_{\mathcal{U}_C} : \mathcal{U}_C \leftrightarrow \mathcal{B}_C$  are isomorphisms.

357 In order to locate the state reachability part of (4.1), let us first define  
 358 the natural projections:

$$\begin{aligned} P_C : \mathcal{R}_{\mathcal{X}_d}^* &\rightarrow \mathcal{R}_C / (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E), \quad P_\ell : \mathcal{R}_{\mathcal{X}_d}^* \rightarrow (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) / \mathcal{R}_C, \\ Q_1 : \mathcal{U} &\rightarrow B^{-1}E\mathcal{R}_{\mathcal{X}_d}^* / B^{-1}\mathcal{B}_C, \quad Q_2 : \mathcal{U} \rightarrow B^{-1}\mathcal{B}_C / B^{-1}E\mathcal{R}_{\mathcal{X}_d}^*. \end{aligned}$$

359 Let us next apply to  $\mathfrak{R}^{imp}(\bar{E}, \bar{A}, \bar{B})$  the reachability algorithm of [Özçaldıran](#)  
 360 (1985),  $\bar{\mathcal{R}}^0 = \{0\}$ ,  $\bar{\mathcal{R}}^{\mu+1} = \bar{E}^{-1} \left( \bar{A}\bar{\mathcal{R}}^\mu + (\bar{\mathcal{B}}_1 \oplus \mathcal{B}_C) \right)$ , whose limit is  $\mathcal{R}_{\mathcal{X}_d}^*$ ; namely:  
 361  $\bar{\mathcal{R}}^1 = I_C^{-1}\bar{\mathcal{B}}_1 \oplus (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E)$  and  $\bar{\mathcal{R}}^{\mu+1} = I_C^{-1} \left( \bar{A}_{1,1}P_C\bar{\mathcal{R}}^\mu + \text{Im} [\bar{A}_{1,2} \bar{B}_1] \right) \oplus (\mathcal{R}_{\mathcal{X}_d}^* \cap$   
 362  $\mathcal{K}_E)$ , for  $\mu \geq 1$ . We thus obtain  $I_C P_C \bar{\mathcal{R}}^{\mu+1} = \bar{A}_{1,1}^\mu \text{Im} \bar{B}_1 + \sum_{i=0}^{\mu-1} \bar{A}_{1,1}^i \text{Im} [\bar{A}_{1,2} \bar{B}_1]$ ,  
 363 which implies:

$$E\mathcal{R}_{\mathcal{X}_d}^* = \langle \bar{A}_{1,1} \mid \text{Im} [\bar{A}_{1,2} \bar{B}_1] \rangle. \quad (4.2)$$

364 Thus,  $(\bar{A}_{1,1}, [\bar{A}_{1,2} \bar{B}_1])$  is a *state reachable pair*.

#### 365 4.2. Behavioral reachability

366 Given any initial condition  $x_0 \in \mathcal{R}_{\mathcal{X}_d}^*$ , the solution set of (4.1) is charac-  
 367 terized by the following behavior

$$\mathfrak{B}_{[\overline{E}, \overline{A}, \overline{B}]} = \left\{ (u, x) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{U} \times \mathcal{R}_{\mathcal{X}_d}^*) \mid \exists x_0 \in \mathcal{R}_{\mathcal{X}_d}^* \text{ s.t. } P_{\mathcal{C}}x(t) = \right. \\ \left. \exp(I_{\mathcal{C}}^{-1}\overline{A}_{1,1}t) P_{\mathcal{C}}x_0 + \int_0^t \exp(I_{\mathcal{C}}^{-1}\overline{A}_{1,1}(t-\tau)) I_{\mathcal{C}}^{-1} (\overline{A}_{1,2}P_{\ell}x(\tau) + \overline{B}_1Q_1u(\tau)) d\tau, \right. \\ \left. Q_2u(t) = -I_{\mathcal{Q}_{\mathcal{C}}}^{-1} (\overline{A}_{2,1}P_{\mathcal{C}}x(t) + \overline{A}_{2,2}P_{\ell}x(t)) \right\}, \quad (4.3)$$

368 which behavioral equations are

$$\begin{aligned} \frac{d}{dt} I_{\mathcal{C}} P_{\mathcal{C}}x &= \overline{A}_{1,1} P_{\mathcal{C}}x + \begin{bmatrix} \overline{A}_{1,2} & \overline{B}_1 \end{bmatrix} \begin{bmatrix} P_{\ell}x \\ Q_1u \end{bmatrix}, \\ 0 &= \overline{A}_{2,1} P_{\mathcal{C}}x + \overline{A}_{2,2} P_{\ell}x + I_{\mathcal{B}_{\mathcal{C}}} Q_2u. \end{aligned} \quad (4.4)$$

369 Let us note that

370 (i) the component  $P_{\mathcal{C}}x$  is the part of the descriptor variable which needs a  
371 control law to reach the desired goal.

372 (ii) The component  $P_{\ell}x$  is the free part of the descriptor variable which acts  
373 as some kind of internal input variable, together with the component  $Q_1u$   
374 which is the effective external control input variable.

375 (iii) The component  $Q_2u$  of the external control variable must be equal to  
376 a component of the descriptor variable. This is because we have chosen a  
377 purely integral description. This part of the input corresponds to algebraic  
378 relationships linked with purely derivative actions.

379 From Lemmas 1 and 2, we get the following theorem which gives a solution  
380 to Problem 2.

381 **Theorem 5.** Consider the reachable part (4.4) of the implicit representation  
382 (1.7). Denote:  $n = \dim(E\mathcal{R}_{\mathcal{X}_d}^*)$  and  $m = \dim(\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) + \dim(B^{-1}E\mathcal{R}_{\mathcal{X}_d}^*)$ .  
383 Let  $\{\kappa_1, \kappa_2, \dots, \kappa_m\} \subset \mathbb{Z}^+$  be the reachability indices of the pair  $(\overline{A}_{1,1}, [\overline{A}_{1,2} \ \overline{B}_1])$ ,  
384 with  $\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_m \geq 1$  and  $\kappa_1 + \kappa_2 + \dots + \kappa_m = n$ .

385 Let the linear map  $F: \mathcal{R}_C \rightarrow (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) \times (B^{-1}E\mathcal{R}_{\mathcal{X}_d}^*)$  and the isomor-  
 386 phisms  $T: \mathcal{R}_C \leftrightarrow \mathcal{R}_C$  and  $G: (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) \times (B^{-1}E\mathcal{R}_{\mathcal{X}_d}^*) \leftrightarrow (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) \times (B^{-1}$   
 387  $E\mathcal{R}_{\mathcal{X}_d}^*)$  be such that the pair  $(\bar{A}_B, \bar{\Gamma}_B)$ , where  $\bar{A}_B = (I_C T)^{-1}(\bar{A}_{1,1} + [\bar{A}_{1,2} \ \bar{B}_1] F)$   
 388  $(I_C T)$  and  $\bar{\Gamma}_B = (I_C T)^{-1} [\bar{A}_{1,2} \ \bar{B}_1] G$ , is expressed in the Brunovsky canonical  
 389 form (1.2). The reachability matrix  $\mathcal{R}_{[\bar{A}_B, \bar{\Gamma}_B]}$  is expressed in terms of the  
 390 reachability matrices  $\mathcal{R}_{[\bar{A}_{B_i}, \bar{\Gamma}_{B_i}]}$  as in (2.1).

391 Let  $x_0, x_1 \in \mathcal{R}_{\mathcal{X}_d}^*$ ,  $Q_1 u_0, Q_1 u_1 \in B^{-1}E\mathcal{R}_{\mathcal{X}_d}^*$ , and  $t_1 > 0$  be given. If we apply

$$\begin{bmatrix} P_t x(t) \\ Q_1 u(t) \end{bmatrix} = F P_C x(t) + G \mathfrak{D}(d/dt) f(t), \quad (4.5)$$

392 where  $f(t) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^m)$  and  $\mathfrak{D}(d/dt)$  are defined as in Lemmas 1 and 2, we  
 393 get

$$P_C x(t) = T \mathcal{R}_{[\bar{A}_B, \bar{\Gamma}_B]} \bar{w}(t), \quad (4.6)$$

394 and

$$(u(t_i), x(t_i)) = \left( \begin{bmatrix} Q_1 u_i \\ -I_{\mathcal{U}_C}^{-1} \begin{bmatrix} \bar{A}_{2,1} & \bar{A}_{2,2} \end{bmatrix} x_i \end{bmatrix}, x_i \right), \quad i \in \{0, 1\}, t_0 = 0. \quad (4.7)$$

#### 395 4.3. Comments on the reachability

396 For the general case of implicit systems, represented by (1.7) with  $E$  and  
 397  $A$  not necessarily square, Frankowska (1990) has been the first to give a  
 398 functional interpretation of reachability. For this, she has used the Viability  
 399 Theory. More precisely, she has shown that reachability is equivalent to  
 400 finding a trajectory  $x \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{X}_d)$  solution of (1.7), starting from the initial  
 401 condition  $x_0$  and reaching the desired  $x_1$  in a given finite time  $t_1$ , namely  
 402  $x(0) = x_0$  and  $x(t_1) = x_1$  (see Theorem 4). Moreover, Frankowska (1990) has  
 403 shown that reachability is geometrically characterized by the well known

404 reachable space  $\mathcal{R}_{\mathcal{X}_d}^*$ . Of course,  $\mathcal{R}_{\mathcal{X}_d}^*$  is contained in the viability kernel  $\mathcal{V}_{\mathcal{X}_d}^*$ .  
 405 This guarantees the existence of at least one trajectory solution of (1.7),  
 406 leaving from  $x_0$ . This is also clear from  $\mathcal{R}_{\mathcal{X}_d}^* = \mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{S}_{\mathcal{X}_d}^*$ .

407 One interesting thing found in the proof of (Aubin & Frankowska, 1991,  
 408 Proposition 2.2) was to put forward the importance of the state space rep-  
 409 resentation (4.4) of the implicit equation(1.7). This fact has enabled us to  
 410 apply systematically the results of the classical State Space Control The-  
 411 ory. More precisely, thanks to the reachability of the pair  $(\bar{A}_{1,1}, [\bar{A}_{1,2} \ \bar{B}_1])$   
 412 (see (4.2)), it is possible to find trajectories  $f_i \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^1)$  (see (2.7), (2.11),  
 413 (2.12), (2.8), and (2.9)) for synthesizing the control law (4.5) (see also (2.4))  
 414 which guarantees (4.7) (see Lemmas 1 and 2 and Theorem 5).

415 The aim of Theorem 5 was not to prove once more the sufficiency of The-  
 416 orem 4, but to interpret the reachability of (1.7) in the classical state space  
 417 framework. This interpretation allows us to have a better understanding of  
 418 the existing mechanisms in the linear implicit systems reachability. Indeed,  
 419 there exist two control actions. The first one is due to the free variable  $P_\ell x$ ,  
 420 and another one is due to the control input  $Q_1 u$  (see (4.3)). The control input  
 421  $Q_2 u$  is algebraically linked to the descriptor variable components, the state  
 422 variable  $P_C x$  and the free variable  $P_\ell x$ , by means of the algebraic restriction  
 423 (4.4.b) (when it exists).

424 For systems composed by *infinite elementary divisors*<sup>17</sup>, the matrix  $Q_1$

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<sup>17</sup> Kronecker showed that any pencil  $[\lambda E - A]$ ,  $\lambda \in \mathbb{C}$ , is strictly equivalent to a canon-  
 ical matrix, composed by four kind of blocks: (i) *finite elementary divisors* (integral ac-  
 tions), e.g.  $\begin{bmatrix} (\lambda - \alpha) & 1 \\ 0 & (\lambda - \alpha) \end{bmatrix}$ , (ii) *infinite elementary divisors* (derivative actions), e.g.

425 is null and the square matrix  $Q_2$  is invertible. In this case, the equations  
 426 (4.5) and (4.6) describe the behavior of a system fed-back by the control law  
 427 (4.4b). Indeed, from (4.5) and (4.6), we get:

$$x(t) = \begin{bmatrix} I \\ F \end{bmatrix} T\mathcal{R}_{[\bar{A}_B, \bar{\Gamma}_B]} \bar{w}(t) + \begin{bmatrix} 0 \\ G \end{bmatrix} \mathfrak{D}(\mathrm{d}/\mathrm{d}t)f(t).$$

428 And from (4.4b), (4.5) and (4.6), we have:

$$Q_2 u(t) = -I_{\mathcal{U}_C}^{-1} \left( (\bar{A}_{2,1} + \bar{A}_{2,2}F) T\mathcal{R}_{[\bar{A}_B, \bar{\Gamma}_B]} \bar{w}(t) + \bar{A}_{2,2}G\mathfrak{D}(\mathrm{d}/\mathrm{d}t)f(t) \right).$$

429 It is remarkable that in the systems represented by *column minimal in-*  
 430 *dices*, it is possible to have reachable systems without any control. This  
 431 phenomenon is possible because of the existence of the free variable  $P_\ell x$ ,  
 432 which acts as an internal control signal.

## 433 5. POLE ASSIGNMENT

434 One of the most important features of the reachability of a state space  
 435 representation (1.1) is the complete assignability of the closed loop spectrum  
 436 by means of a state feedback. This equivalence is no longer the case when  
 437 dealing with implicit representations (1.7). For the implicit description case,  
 438 a geometric condition has to be added in order to guarantee such a *pole*  
 439 *assignment ability*. In the sequel we give geometric conditions, which enable

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$\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$ , (iii) *column minimal indices* (internal variable structure), *e.g.*  $\begin{bmatrix} \lambda & 1 \end{bmatrix}$ , and  
 (iv) *row minimal indices* (internal behavioral restrictions), *e.g.*  $\begin{bmatrix} \lambda \\ 1 \end{bmatrix}$ ; see [Gantmacher \(1977\)](#).

us to assign the closed loop spectrum of: (i) a reachable implicit description  
 (1.7), and (ii) a reachable and observable implicit description with output  
 equation,  $\mathfrak{R}^{imp}(E, A, B, C)$ :

$$E dx/dt = Ax + Bu, \quad y = Cx, \quad (5.8)$$

where  $C : \mathcal{X}_d \rightarrow \mathcal{Y}$  is a linear map, and the linear space  $\mathcal{Y}$  is the output space.

At this point, it is useful to clarify what we mean by spectrum and observable part.

*A. Spectrum.* We distinguish between the finite spectrum,  $\sigma_f(A, E) = \{\lambda \in \mathbb{C} \mid \exists v \neq 0 \text{ s.t. } Av = \lambda Ev\}$ , and the infinite spectrum,  $\sigma_\infty(E, A) = \{\mu \in \mathbb{C} \mid \exists w \neq 0 \text{ s.t. } Ew = \mu Aw\}$  (c.f. [Gantmacher \(1977\)](#), [Wong \(1974\)](#), [Armentano \(1986\)](#)); the elements of  $\sigma_f(A, E)$  are called poles, and the elements of  $\sigma_\infty(E, A)$  are called poles at infinity. Note that for the four kind of blocks of the Kronecker canonical form<sup>18</sup>: (i)  $\sigma_f(A, E) = \emptyset$  and  $\sigma_\infty(E, A) = \emptyset$  for its *row minimal indices* blocks, (ii)  $\sigma_f(A, E) = \emptyset$  for its *infinite elementary divisors* blocks, (iii)  $\text{card}\{\sigma_f(A, E)\} = \infty$  and  $\text{card}\{\sigma_\infty(E, A)\} = \infty$  for its *column minimal indices* blocks.

*B. Observable part.* With respect to the observable part, let us recall that it was shown in ([Bonilla & Malabre, 1995](#)) that the third condition of [Kuijper \(1992a\)](#) –  $\begin{bmatrix} sE - A \\ C \end{bmatrix}$  has full column rank for all  $s \in \mathbb{C}$  – for getting a minimal implicit representation (among all externally equivalent<sup>19</sup> representations

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<sup>18</sup> See footnote 17.

<sup>19</sup> Two representations are called externally equivalent if the corresponding sets of all possible trajectories for the external variables, expressed in an *input/output partition*  $(u, y)$ , are the same ([Willems, 1983](#), [Polderman & Willems, 1998](#)).

of the same type), is equivalent to have a null unobservable space, namely:  
 $\mathcal{N} = \{0\}$ . Indeed, if we decompose the descriptor and equation spaces as:  
 $\mathcal{X}_d = \mathcal{X}_{ob} \oplus \mathcal{N}$  and  $\mathcal{X}_{eq} = \mathcal{W}_{ob} \oplus E\mathcal{N}$ , where  $\mathcal{X}_{ob}$  and  $\mathcal{W}_{ob}$  are some complementary subspaces, (5.8) takes the following form:

$$\begin{aligned} \begin{bmatrix} E_{ob} & 0 \\ Z & E_{\mathcal{N}} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x_{ob} \\ x_{\mathcal{N}} \end{bmatrix} &= \begin{bmatrix} A_{ob} & 0 \\ X & A_{\mathcal{N}} \end{bmatrix} \begin{bmatrix} x_{ob} \\ x_{\mathcal{N}} \end{bmatrix} + \begin{bmatrix} B_{ob} \\ Y \end{bmatrix} u \\ y &= \begin{bmatrix} C_{ob} & 0 \end{bmatrix} \begin{bmatrix} x_{ob} \\ x_{\mathcal{N}} \end{bmatrix} \end{aligned} \quad (5.9)$$

And the implicit descriptions  $\mathfrak{R}^{imp}(E, A, B, C)$  and  $\mathfrak{R}^{imp}(E_{ob}, A_{ob}, B_{ob}, C_{ob})$  are externally equivalents (c.f. [Bonilla & Malabre, 1995](#), Theorem 2.1). The point we want to enlighten here is that, since  $E_{\mathcal{N}}$  is epic, there then exists  $E_{\mathcal{N}}^r$  such that  $E_{\mathcal{N}} E_{\mathcal{N}}^r = I$ , which implies that all the homogeneous trajectories of (5.9), beginning at any initial condition  $\begin{bmatrix} 0 \\ x_0 \end{bmatrix} \in \mathcal{N}$ ,  $x_{\mathcal{N}}(t) = \exp(E_{\mathcal{N}}^r A_{\mathcal{N}} t) x_0$ , always remain inside  $\mathcal{N} \subset \mathcal{X}_C$ . Thus, like in the classical state representations, they are called unobservable trajectories; and since  $\mathcal{N}$  is the supremal  $(A, E)$  invariant subspace contained in  $\mathcal{X}_C$  with this property,  $\mathfrak{R}^{imp}(E_{ob}, A_{ob}, B_{ob}, C_{ob})$  is called the *observable part* of  $\mathfrak{R}^{imp}(E, A, B, C)$ .

### 5.1. Pole Assignment for a Reachable Implicit Description

**Theorem 6.** ([Bonilla & Malabre, 1993](#)) *Given an implicit system represented by (1.7), for every finite symmetric (with respect to the real line) set of complex numbers  $\Lambda$  of cardinality  $\dim(\mathcal{R}_{\mathcal{X}_d}^*)$ , there exists a proportional and derivative descriptor feedback  $u = F_p x + F_d dx/dt$ , such that  $\sigma_f(A_{F_p}, E_{F_d}) = \Lambda$ , if and only if*

$$\mathcal{R}_{\mathcal{X}_d}^* = \mathcal{X}_d, \quad (5.10)$$

$$\dim(E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) \geq \dim(\mathcal{V}_{\mathcal{X}_d}^*). \quad (5.11)$$

479 [Bonilla & Malabre \(1993\)](#) named this property *external reachability*. In  
480 that paper, condition (5.11) is expressed in its equivalent form:

$$\dim(\mathcal{B}/(\mathcal{B} \cap E\mathcal{V}_{\mathcal{X}_d}^*)) \geq \dim(\mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{K}_E). \quad (5.12)$$

481 Let us note that the geometric condition (5.10) is the *reachability* con-  
482 dition of [Frankowska \(1990\)](#) (c.f. Theorem 4) and the geometric condi-  
483 tion (5.11) is the *descriptor variable uniqueness* condition of [Lebret \(1991\)](#),  
484 namely the closed loop left invertibility property, which enables us to assign  
485 the poles by means of a proportional and derivative feedback.

486 **Lemma 4.** ([Lebret, 1991](#)) *There exists a proportional and derivative descrip-*  
487 *tor feedback  $u = F_p x + F_d dx/dt + v$ , such that the fed-back implicit represen-*  
488 *tation  $\mathfrak{R}^{imp}(E_{F_d}, A_{F_p}, B)$  satisfies  $\text{Ker}(\lambda E_{F_d} - A_{F_p}) = \{0\}$  iff (5.11) is satisfied.*

489 Let us also note that in the case of a strict singular system, the ge-  
490 ometric condition (5.11) is translated to (c.f. (3.6), (3.4) and Fig. 1):  
491  $\dim(\mathcal{X}_{eq}) \geq \dim(\mathcal{X}_d)$ . In other words, it is not possible to assign all the spec-  
492 trum of an implicit system having one degree of freedom, as for example the  
493 ones considered in ([Bonilla & Malabre, 2003](#)).

494 We have the following Corollary of Theorem 6, proved in [Appendix D](#).

495 **Corollary 1.** *Let the implicit representation (1.7) satisfy the geometric con-*  
496 *ditions (5.10) and (5.11). Then:*

- 497 1. *If  $\mathcal{U}_c = \{0\}$ , the implicit representation (4.1) reduces to the following*  
498 *reachable state space representation ( $\overline{\mathcal{B}}_1 = \text{Im } \overline{B}_1$ ):*

$$dx/dt = \overline{A}_{1,1}x + \overline{B}_1u \quad \text{with} \quad \langle \overline{A}_{1,1} \mid \overline{\mathcal{B}}_1 \rangle = \mathcal{X}_d. \quad (5.13)$$



499 2. If  $\mathcal{U}_C \neq \{0\}$ , there exists a map  $\bar{V}_\ell : \mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E \rightarrow \mathcal{B}_C$  such that  $\text{Ker } \bar{V}_\ell =$   
500  $\{0\}$ . Then, applying the proportional feedback

$$u = \begin{bmatrix} 0 & 0 \\ -I_{\mathcal{U}_C}^{-1} \bar{A}_{2,1} & -I_{\mathcal{U}_C}^{-1} (\bar{A}_{2,2} + \bar{V}_\ell) \end{bmatrix} x + v, \quad (5.14)$$

501 we get

$$\begin{bmatrix} I_C & 0 \\ 0 & 0 \end{bmatrix} dx/dt = \begin{bmatrix} \bar{A}_{1,1} & 0 \\ 0 & -I \end{bmatrix} x + \begin{bmatrix} \bar{B}_1 & \bar{A}_{1,2} \bar{V}_\ell^g I_{\mathcal{U}_C} \\ 0 & \bar{V}_\ell^g I_{\mathcal{U}_C} \end{bmatrix} v, \quad (5.15)$$

502 where  $\bar{V}_\ell^g : \mathcal{B}_C \rightarrow \mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E$  is some left inverse of  $\bar{V}_\ell$ , and

$$I_C \mathcal{R}_C = E \mathcal{R}_{\mathcal{X}_d}^* = \langle \bar{A}_{1,1} \mid \bar{\mathcal{B}}_1 + \bar{A}_{1,2} (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) \rangle \quad \text{and} \quad \mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E = \bar{V}_\ell^g I_{\mathcal{U}_C} \mathcal{U}_C. \quad (5.16)$$

503 Furthermore, applying the proportional and derivative feedback

$$u = \begin{bmatrix} 0 & 0 \\ -I_{\mathcal{U}_C}^{-1} \bar{A}_{2,1} & -I_{\mathcal{U}_C}^{-1} (\bar{A}_{2,2} + \bar{V}_\ell) \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & -I_{\mathcal{U}_C}^{-1} \bar{V}_\ell \end{bmatrix} dx/dt + v, \quad (5.17)$$

504 we get

$$dx/dt = \begin{bmatrix} \bar{A}_{1,1} & \bar{A}_{1,2} \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} \bar{B}_1 & 0 \\ 0 & \bar{V}_\ell^g I_{\mathcal{U}_C} \end{bmatrix} v, \quad (5.18)$$

505 with

$$\left\langle \begin{bmatrix} \bar{A}_{1,1} & \bar{A}_{1,2} \\ 0 & 0 \end{bmatrix} \middle| \text{Im} \begin{bmatrix} \bar{B}_1 & 0 \\ 0 & \bar{V}_\ell^g I_{\mathcal{U}_C} \end{bmatrix} \right\rangle = \langle \bar{A}_{1,1} \mid \bar{\mathcal{B}}_1 + \bar{A}_{1,2} (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) \rangle \oplus \mathcal{U}_C = \mathcal{X}_d. \quad (5.19)$$

506 From this Corollary, we realize that with a proportional feedback, we  
507 can only modify the finite spectrum of  $\bar{A}_{1,1} = R \bar{A} |_{(\mathcal{R}_{\mathcal{X}_d}^* / \mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E)}$ , where  
508  $R : A \mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B} \rightarrow E \mathcal{R}_{\mathcal{X}_d}^* / \mathcal{B}_C$  is the natural projection. To assign all the finite  
509 spectrum of  $\bar{A}$ , we need a proportional and derivative feedback.

## 5.2. Pole Assignment for a Reachable and Observable Implicit Description

In this section, we are going to consider the reachability of the observable part after feedback, of the implicit representation (5.8). For this, let us recall that the supremal  $(A, E, B)$ -invariant subspace contained in  $\text{Ker } C$ ,  $\mathcal{V}^*$   $= \sup\{\mathcal{V} \subset \mathcal{X}_C \mid A\mathcal{V} \subset E\mathcal{V} + \text{Im } B\}$ , that characterizes the biggest part of a given implicit representation  $\mathfrak{R}^{imp}(E, A, B, C)$ , can be made unobservable by means of a suitable proportional and derivative descriptor feedback (*c.f.* the early Geometric Algorithms Section).

Given a proportional and derivative descriptor feedback  $u = F_p^*x + F_d^*dx/dt$ , where  $(F_p^*, F_d^*) \in \mathbf{F}(\mathcal{V}^*)$ , let us consider the *quotient implicit representation*  $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$ , where the linear applications  $E_*$ ,  $A_*$ ,  $B_*$  and  $C_*$  are the induced maps uniquely defined by

$$E_*\Phi = \Pi E_{F_d^*}, \quad A_*\Phi = \Pi A_{F_p^*}, \quad B_* = \Pi B, \quad \text{and} \quad C = C_*\Phi, \quad (5.20)$$

where  $\Phi: \mathcal{X}_d \rightarrow \mathcal{X}_d/\mathcal{V}^*$  and  $\Pi: E\mathcal{X}_d \rightarrow E\mathcal{X}_d/E_{F_d^*}\mathcal{V}^*$  are the canonical projections. In Appendix E, we prove the following Theorem.<sup>20</sup>

**Theorem 7.** *Given an implicit system represented by (5.8), for every symmetric (with respect to the real line) set of complex numbers  $\Lambda$  of cardinality  $\dim((\mathcal{X}_{\mathcal{X}_d}^* + \mathcal{V}^*)/\mathcal{V}^*)$ , there exists a proportional and derivative descriptor feedback  $u = F_p^*x + F_d^*dx/dt + v$ , with  $(F_p^*, F_d^*) \in \mathbf{F}(\mathcal{V}^*)$ , such that  $\sigma_f(A_*, E_*) = \Lambda$ , where  $E_*$  and  $A_*$  are the induced maps (5.20), if and only if:*

$$(\mathcal{X}_{\mathcal{X}_d}^* + \mathcal{V}^*)/\mathcal{V}^* = \mathcal{X}_d/\mathcal{V}^*, \quad (5.21)$$

$$\dim\left((E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B})/(E\mathcal{V}^* + \mathcal{B})\right) + \dim(\mathcal{B}) \geq \dim\left(\mathcal{V}_{\mathcal{X}_d}^*/\mathcal{V}^*\right). \quad (5.22)$$

<sup>20</sup>For a related result for regular systems see Schumacher (1980).

530 Let us note that (5.22) is equivalent to:<sup>21</sup>

$$\dim \left( \mathcal{B} / (\mathcal{B} \cap E\mathcal{V}_{\mathcal{X}_d}^*) \right) \geq \dim \left( \mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{K}_E \right) - \dim \left( \mathcal{V}^* \cap E^{-1}\mathcal{B} \right). \quad (5.23)$$

531 For the implicit representations (5.8), satisfying Theorem 7, we will say that  
 532 they have the *externally reachable output dynamics property*<sup>22</sup>. Theorem 7  
 533 is important because it enables us to tackle systems having an internal vari-  
 534 able structure (see for example Bonilla & Malabre (1991), Bonilla & Malabre  
 535 (2003), and Bonilla & Malabre (2008)). Let us also note that the geomet-  
 536 ric condition (5.22) is the *descriptor variable uniqueness property* notion of  
 537 Lebret (1991), namely the closed loop left invertibility property of the ob-  
 538 servable part of the system.

539 **Lemma 5.** (Lebret, 1991) *There exists a proportional and derivative de-*  
 540 *scriptor feedback  $u = F_p x + F_d dx/dt + v$ , such that the fed-back implicit rep-*  
 541 *resentation  $\mathfrak{R}^{imp}(E_{F_d}, A_{F_p}, B)$  satisfies  $\text{Ker}(\lambda E_{F_d} - A_{F_p}) \subset \mathcal{N}_{(F_p, F_d)}$  iff (5.22) is*  
 542 *satisfied.*

543 Let us finally note that, when comparing (5.22) with (5.11), we realize  
 544 that Theorem 7 is indeed establishing the external reachability of the observ-  
 545 able part after feedback. Also note that in the case  $\mathcal{V}^* = \{0\}$ , (5.22) and

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<sup>21</sup> This equivalence follows from the equivalence between (5.11) and (5.12), and from the fact that  $\mathcal{B} \cap E\mathcal{V}^* = E(\mathcal{V}^* \cap E^{-1}\mathcal{B})$  implies that  $\dim(\mathcal{V}^* \cap E^{-1}\mathcal{B}) = \dim(\mathcal{V}^*) + \dim(\mathcal{B} \cap \text{Im } E) - \dim(E\mathcal{V}^* + \mathcal{B} \cap \text{Im } E)$ .

<sup>22</sup> The *externally reachable output dynamics* notion is a simplification of the one of *reachable with output dynamics assignment* (see Bonilla et al, 1994, Definition 6).

(5.11) are the same; and in the case  $\mathcal{V}^* = \mathcal{V}_{\mathcal{X}_d}^*$ , we get the trivial condition  $\dim(\mathcal{B}) \geq 0$ .

Let us finish this Section with an academic example.

*Academic Example.* Let us consider a perturbed linear system represented by the state space representation,  $\mathfrak{R}^{ss}(\bar{A}, [\bar{B} \ \bar{S}], \bar{C})$ :

$$d\bar{x}/dt = \bar{A}\bar{x} + \begin{bmatrix} \bar{B} & \bar{S} \end{bmatrix} \begin{bmatrix} u \\ q \end{bmatrix} \quad \text{and} \quad y = \bar{C}\bar{x}, \quad (5.24)$$

where  $q \in \mathcal{Q} \approx \mathbb{R}^\eta$ ,  $u \in \mathcal{U} \approx \mathbb{R}^m$ ,  $y \in \mathcal{Y} \approx \mathbb{R}^p$  and  $\bar{x} \in \bar{\mathcal{X}} \approx \mathbb{R}^{\bar{n}}$ , are the disturbance, the input, the output, and the state variables, respectively. We assume that the three following assumptions hold true:

**[H1]**  $\text{Ker } \bar{B} = \{0\}$  and  $\text{Ker } \bar{S} = \{0\}$ ,

**[H2]**  $q(\cdot) \in \mathcal{C}^m(\mathbb{R}^+, \mathcal{Q})$ ,  $q(t)$ ,  $dq(t)/dt$ ,  $\dots$ ,  $d^m q(t)/dt^m \in \mathcal{L}^\infty$ ,  $\forall t \geq 0$ ,

**[H3]**  $q$  is a measured disturbance.

We want to solve the Disturbance Decoupling Problem with a PD Feedback (DDP-PDF).

**Problem 3 (DDP-PDF).** Under which conditions does there exist a proportional and derivative feedback  $u = (\bar{F}_{p1} + \bar{F}_{d1}d/dt)\bar{x} + (\bar{F}_{p2} + \bar{F}_{d2}d/dt)q + v$ , such that the closed-loop transfer function matrix between  $q$  and  $y$  is identically zero, and the finite spectrum of the observable part of the closed loop system is assigned at will.

For solving this problem, let us rewrite (5.24) in the descriptor form (5.8) with

$$E = \begin{bmatrix} I_{\bar{n}} & | & 0 \end{bmatrix}, \quad A = \begin{bmatrix} \bar{A} & | & \bar{S} \end{bmatrix}, \quad B = \begin{bmatrix} \bar{B} \end{bmatrix}, \quad C = \begin{bmatrix} \bar{C} & | & 0 \end{bmatrix}, \quad (5.25)$$

566 where  $x = \begin{bmatrix} \bar{x}^T & q^T \end{bmatrix}^T \in \mathcal{X}_d = \overline{\mathcal{X}} \oplus \mathcal{Q} \approx \mathbb{R}^{\bar{n}+\eta}$  and  $\mathcal{X}_{eq} = \overline{\mathcal{X}} \approx \mathbb{R}^{\bar{n}}$ . In this im-  
 567 plicit representation, the perturbation  $q$  is acting as the free part of the de-  
 568 scriptor variable  $x$ . Then from Theorem 7, the **DDP-PDF** is solvable if and  
 569 only if the implicit representation (5.8) and (5.25) satisfies (5.21) and (5.22),  
 570 namely if and only if both following conditions hold true (see Appendix F):

$$\langle \bar{A} \mid \text{Im} [\bar{B} \bar{S}] \rangle + \overline{\mathcal{V}}_{[\bar{B} \bar{S}]}^* = \overline{\mathcal{X}}, \quad (5.26)$$

571

$$\dim \left( \overline{\mathcal{V}}_{[\bar{B} \bar{S}]}^* \cap \overline{\mathcal{B}} \right) \geq \dim \left( \frac{\text{Im } \bar{S}}{\text{Im } \bar{S} \cap (\overline{\mathcal{V}}_{[\bar{B} \bar{S}]}^* + \overline{\mathcal{B}})} \right). \quad (5.27)$$

Let us consider for example:  $\bar{A} = T_u \{e_3^2\}$ ,  $\bar{S} = ae_3^1 + be_3^2$ , with  $|a| + |b| \neq 0$ ,  $\bar{B} = e_3^3$   
 and  $\bar{C} = (e_3^1)^T$ . We have for this case  $\text{Im } \bar{S} = \text{span} \{ae_3^1 + be_3^2\}$ ,  $\overline{\mathcal{B}} = \text{span} \{e_3^3\}$ ,  
 and  $\text{Im} [\bar{B} \bar{S}] = \text{span} \{ae_3^1 + be_3^2, e_3^3\}$ , then  $\langle \bar{A} \mid \text{Im} [\bar{B} \bar{S}] \rangle = \text{span} \{e_3^1, e_3^2, e_3^3\} = \overline{\mathcal{X}}$ ,  
 $\overline{\mathcal{V}}_{[\bar{B} \bar{S}]}^* = \text{span} \{ae_3^2, e_3^3\}$ ,  $\overline{\mathcal{V}}_{[\bar{B} \bar{S}]}^* \cap \overline{\mathcal{B}} = \text{span} \{e_3^3\}$ , and  $\text{Im } \bar{S} \cap (\overline{\mathcal{V}}_{[\bar{B} \bar{S}]}^* + \overline{\mathcal{B}}) = \{0\}$ .  
 Therefore (5.26) and (5.27) are satisfied, and the DDP-PDF has solution.  
 Indeed, applying to (5.24) and (5.25) the PD feedback

$$u = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} d\bar{x}/dt + \begin{bmatrix} -1/\tau & 0 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 1/\tau \end{bmatrix} v,$$

we obtain the closed loop system described by:

$$\tau dy/dt + y = v, \quad \bar{x}_1 = y, \quad \bar{x}_2 = dy/dt - aq, \quad \text{and} \quad \bar{x}_3 = d^2y/dt^2 - adq/dt - bq.$$

572 Let us note that  $\overline{\mathcal{V}}_{\bar{B}}^* = \{0\}$ , and that  $\text{Im } \bar{S} \cap (\overline{\mathcal{V}}_{\bar{B}}^* + \overline{\mathcal{B}}) = \{0\}$ , so there is no  
 573 purely proportional solutions (see for example Wonham (1985)).

## 574 6. CONCLUDING REMARKS

575 The notion of reachability introduced by Frankowska (1990) generalizes  
 576 the property introduced by Yip & Sincovec (1981) in the regular case. Fur-  
 577 thermore, Cobb (1984) indicates that this last property is consistent with

578 that of [Rosenbrock \(1974\)](#) introduced in a purely structural framework. In  
 579 the same paper, [Cobb \(1984\)](#) enlightens with time domain characterizations  
 580 the difference between the reachability in the sense of [Rosenbrock \(1974\)](#)  
 581 and the reachability in the sense of [Verghese \*et al\* \(1981\)](#) based, once again,  
 582 on pure structural tools (Kronecker canonical forms and/or Smith canon-  
 583 ical forms). In the regular case, for which the system can be decomposed  
 584 into two parts, a finite or slow subsystem, and an infinite or fast subsys-  
 585 tem, [Cobb \(1984\)](#) showed that [Rosenbrock \(1974\)](#) reachability is equivalent  
 586 to the reachability of the finite part and controllability of the infinite part.  
 587 He also showed that [Verghese \*et al\* \(1981\)](#) reachability is equivalent to the  
 588 reachability of the finite part associated to the impulse controllability of the  
 589 infinite part. The impulse controllability as defined by [Cobb \(1984\)](#), or the  
 590 controllability of the infinite part in the sense of [Verghese \*et al\* \(1981\)](#) is not  
 591 any more defined by the idea to reach a desired descriptor variable but by the  
 592 ability of the system to generate a maximal class of impulses using piecewise  
 593 smooth, non impulsive controls.

594 One can deduce from this analysis that if a regular system is reachable  
 595 (reachability of the finite and controllability of the infinite part) in the sense  
 596 of [Cobb \(1984\)](#), [Yip & Sincovec \(1981\)](#), [Rosenbrock \(1974\)](#) or [Frankowska](#)  
 597 [\(1990\)](#) (the four notion are equivalent in this case) then any vector is a consis-  
 598 tent initial condition in the sense of [Geerts \(1993\)](#). The converse implication  
 599 is not true. In general, reachability is not a consequence of the fact that ev-  
 600 ery vector of the descriptor space defines a consistent initial condition. The  
 601 condition is necessary but not sufficient for reachability.

602 In this paper we have given a geometric interpretation of the implicit sys-

603 tems reachability Theorem of Frankowska (1990) and we have also found some  
604 interesting connections between the works (Frankowska, 1990) and (Geerts,  
605 1993). The geometric interpretation has enabled us to have a better under-  
606 standing of the existing mechanisms in the linear implicit systems reachabil-  
607 ity. For this, we have first interpreted the *viability* notion from a geometric  
608 point of view. We have next solved Problem 2, with Theorem 5, which is a  
609 generalization of Problem 1, solved with Theorem 1.

610 We have also studied the existing relationships, between the reachabil-  
611 ity property and the capability of the complete pole assignment ability. In  
612 Theorem 6, we have considered the pole assignment problem of a reachable  
613 implicit description,  $\mathfrak{R}^{imp}(E, A, B)$ ; we have also shown in Corollary 1, that  
614 with a proportional feedback, we can only modify the spectrum of the re-  
615 striction to  $\mathcal{R}_{\mathcal{X}_d}^*/\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E$  in the domain and  $E\mathcal{R}_{\mathcal{X}_d}^*$  in the co-domain; to  
616 assign all the spectrum, we need a proportional and derivative feedback. In  
617 Theorem 7, we have considered the pole assignment problem of a reachable  
618 and observable implicit description with output equation,  $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$ .

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## 738 Appendix A. Proof of Lemma 1

739 For the existence of such  $F_{\mathcal{B}}$ ,  $T_{\mathcal{B}}$  and  $G_{\mathcal{B}}$ , see for example Theorems 5.9 and  
 740 5.10 and Corollary 5.3 of Wonham (1985). Doing the change of state variable:  
 741  $T_{\mathcal{B}}^{-1}x = \xi = \begin{bmatrix} \xi_1^T & \dots & \xi_m^T \end{bmatrix}^T$ , we obtain the following set of closed loop state  
 742 space representations (see (1.1), (1.2), and (2.4)):  $d\xi_i/dt = A_{\mathcal{B},i}\xi_i + b_{\mathcal{B},i}d^{\kappa_i}f_i/dt^{\kappa_i}$ ,  
 743  $i \in \{1, \dots, m\}$ , which solutions are (integrate by parts  $n_i$  times each solution):

$$\begin{aligned}
 \xi_i(t) &= \exp(A_{\mathcal{B},i}t) \xi_i(0) + \int_0^t \exp(A_{\mathcal{B},i}(t-\tau)) b_{\mathcal{B},i} \frac{d^{\kappa_i} f_i(\tau)}{d\tau^{\kappa_i}} d\tau \\
 &= \exp(A_{\mathcal{B},i}t) \left( \xi_i(0) - \sum_{j=0}^{\kappa_i-1} A_{\mathcal{B},i}^j b_{\mathcal{B},i} \frac{d^{\kappa_i-(j+1)} f_i(0)}{dt^{\kappa_i-(j+1)}} \right) + \sum_{j=0}^{\kappa_i-1} A_{\mathcal{B},i}^j b_{\mathcal{B},i} \frac{d^{\kappa_i-(j+1)} f_i(t)}{dt^{\kappa_i-(j+1)}} \\
 &= \exp(A_{\mathcal{B},i}t) \left( \xi_i(0) - \mathcal{R}_{[A_{\mathcal{B},i}, b_{\mathcal{B},i}]} \bar{w}_i(0) \right) + \mathcal{R}_{[A_{\mathcal{B},i}, b_{\mathcal{B},i}]} \bar{w}_i(t), \quad i \in \{1, \dots, m\}, \\
 \xi(t) &= \exp(A_{\mathcal{B}}t) \left( \xi(0) - \mathcal{R}_{[A_{\mathcal{B}}, B_{\mathcal{B}}]} \bar{w}(0) \right) + \mathcal{R}_{[A_{\mathcal{B}}, B_{\mathcal{B}}]} \bar{w}(t), \\
 x(t) &= \exp((A + BF_{\mathcal{B}})t) \left( x(0) - T_{\mathcal{B}} \mathcal{R}_{[A_{\mathcal{B}}, B_{\mathcal{B}}]} \bar{w}(0) \right) + T_{\mathcal{B}} \mathcal{R}_{[A_{\mathcal{B}}, B_{\mathcal{B}}]} \bar{w}(t).
 \end{aligned} \tag{A1}$$

744 Therefore, (A1), (2.2) and (2.3) imply (2.5) and (2.6).  $\square$

## 745 Appendix B. Proof of Lemma 2

746 Let us first compute  $\det(X_{(i,1)}(t))$ , for  $\kappa_i \geq 2$ . For this, we first do the decom-  
 747 position  $X_{(i,1)}(t) = D_{i,\ell}(t) \tilde{X}_{(i,\kappa_i+1)} D_{i,r}(t)$ ,  $i \in \{1, \dots, m\}$ , where  $D_{i,\ell}(t) =$   
 748  $\text{DM}\left\{\frac{t}{(\kappa_i+1)!}, \dots, t^{\kappa_i+1}/(2\kappa_i+1)!\right\}$ ,  $D_{i,r}(t) = \text{DM}\left\{(2\kappa_i+1)!t^{\kappa_i}, \dots, (\kappa_i+1)!\right\}$  and

$$\tilde{X}_{(i,\kappa_i+1)} = \begin{bmatrix} (\kappa_i+1)!/(\kappa_i+1)! & \dots & (\kappa_i+1)!/1! \\ \vdots & \dots & \vdots \\ (2\kappa_i+1)!/(2\kappa_i+1)! & \dots & (2\kappa_i+1)!/(\kappa_i+1)! \end{bmatrix}. \tag{B1}$$

749 Defining the following column elementary matrices:

$$\begin{aligned}
750 \quad T_{i,1} &= \left[ \begin{array}{c|c|c|c|c|c|c} e_{\kappa_i+1}^1 & (e_{\kappa_i+1}^2 - e_{\kappa_i+1}^1) & (e_{\kappa_i+1}^3 - \kappa_i e_{\kappa_i+1}^2) & \cdots & (e_{\kappa_i+1}^{\kappa_i+1} - 2e_{\kappa_i+1}^{\kappa_i}) & & \\ \hline e_{\kappa_i+1}^1 & e_{\kappa_i+1}^2 & (e_{\kappa_i+1}^3 - (\kappa_i + 2)e_{\kappa_i+1}^2) & (e_{\kappa_i+1}^4 - (\kappa_i + 1)e_{\kappa_i+1}^3) & \cdots & (e_{\kappa_i+1}^{\kappa_i+1} - 4e_{\kappa_i+1}^{\kappa_i}) & \\ \hline \end{array} \right], \quad T_{i,2} = \left[ \right. \\
751 \quad & \left. \begin{array}{c|c|c|c|c|c|c} e_{\kappa_i+1}^1 & e_{\kappa_i+1}^2 & (e_{\kappa_i+1}^3 - (\kappa_i + 2)e_{\kappa_i+1}^2) & (e_{\kappa_i+1}^4 - (\kappa_i + 1)e_{\kappa_i+1}^3) & \cdots & (e_{\kappa_i+1}^{\kappa_i+1} - 4e_{\kappa_i+1}^{\kappa_i}) & \\ \hline \end{array} \right. \\
752 \quad & \left. \right], \quad \dots, \quad T_{i,\kappa_i-1} = \left[ \begin{array}{c|c|c|c|c|c|c} e_{\kappa_i+1}^1 & \cdots & e_{\kappa_i+1}^{\kappa_i-1} & (e_{\kappa_i+1}^{\kappa_i} - (2\kappa_i - 1)e_{\kappa_i+1}^{\kappa_i-1}) & (e_{\kappa_i+1}^{\kappa_i+1} - (2\kappa_i - \\
753 \quad & 2)e_{\kappa_i+1}^{\kappa_i}) & \cdots & e_{\kappa_i+1}^1 & \cdots & e_{\kappa_i+1}^{\kappa_i} & (e_{\kappa_i+1}^{\kappa_i+1} - (2\kappa_i)e_{\kappa_i+1}^{\kappa_i}) \end{array} \right], \text{ we then get:}
\end{aligned}$$

$$\tilde{X}_{(i,\kappa_i+1)} \prod_{j=1}^{\kappa_i} T_{i,j} = \begin{bmatrix} 0! & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1! & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 1 & \prod_{\ell=\kappa_i}^{\kappa_i} \ell & \prod_{\ell=\kappa_i-1}^{\kappa_i} \ell & \cdots & \prod_{\ell=3}^{\kappa_i} \ell & \prod_{\ell=2}^{\kappa_i} \ell & \kappa_i! \end{bmatrix}. \quad (\text{B2})$$

754 which implies (2.10.b).

755 For the second statement, let us first note that (2.7)-(2.9), (2.2) and (2.3),  
756 imply:

$$X_{(i,1)}(t)\mathbf{a}_{i,1} + X_{(i,0)}(t)\mathbf{a}_{i,0} = \begin{bmatrix} d^{\kappa_i} f_i(t)/dt^{\kappa_i} \\ \bar{w}_i(t) \end{bmatrix}, \quad (\text{B3})$$

757 with  $i \in \{1, \dots, m\}$ . And let us next note that (2.2) and (2.3) are equivalent  
758 to:

$$\frac{d^{\kappa_i} f(t_j)}{dt^{\kappa_i}} = (e_m^i)^T G_B^{-1}(u(t_j) - F_B x(t_j)) \quad \text{and} \quad \bar{w}_i(t_j) = \mathcal{R}_{[A_B, B_B]}^{-1} P_i T_B^{-1} x(t_j), \quad (\text{B4})$$

759 with  $i \in \{1, \dots, m\}$  and  $j \in \{0, 1\}$ , and where  $t_0 = 0$ ,  $u(t_0) = u_0$ ,  $u(t_1) = u_1$ ,  $x(t_0) =$   
760  $x_0$ , and  $x(t_1) = x_1$ . Therefore, (2.8)-(2.10), (B3) and (2.11) imply (B4).  $\square$

## 761 Appendix C. Proof of Lemma 3

762 Let us first prove that the spaces  $\mathcal{X}_d$ ,  $\mathcal{X}_{eq}$  and  $\mathcal{U}$  can be decomposed as  
763 follows:

$$\begin{aligned}
\mathcal{X}_d &= \mathcal{R}_C \oplus (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) \oplus \mathcal{X}_2 \oplus \mathcal{X}_1, \quad \mathcal{X}_{eq} = E\mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{B}_C \oplus E\mathcal{X}_2 \oplus A\mathcal{X}_1, \\
\mathcal{U} &= B^{-1}E\mathcal{R}_{\mathcal{X}_d}^* \oplus B^{-1}\mathcal{B}_C,
\end{aligned} \quad (\text{C1})$$

764 where:

$$\begin{aligned}\mathcal{X}_d &= \mathcal{V}_{\mathcal{X}_d}^* \oplus \mathcal{X}_1, \quad \mathcal{V}_{\mathcal{X}_d}^* = \mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{X}_2, \quad \mathcal{R}_{\mathcal{X}_d}^* = \mathcal{R}_C \oplus (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E), \\ \mathcal{X}_{eq} &= (E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) \oplus A\mathcal{X}_1, \quad E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} = (A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \oplus E\mathcal{X}_2, \\ A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B} &= E\mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{B}_C, \quad \mathcal{B} = (\mathcal{B} \cap E\mathcal{R}_{\mathcal{X}_d}^*) \oplus \mathcal{B}_C.\end{aligned}\tag{C2}$$

765 And also:

$$\mathcal{R}_C \approx E\mathcal{R}_{\mathcal{X}_d}^*, \quad \mathcal{X}_2 \approx E\mathcal{X}_2, \quad \mathcal{X}_1 \approx A\mathcal{X}_1, \quad \mathcal{B}_C \approx B^{-1}\mathcal{B}_C = \mathcal{U}_C,\tag{C3}$$

$$\mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{K}_E = \mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E, \quad \mathcal{B} \cap E\mathcal{V}_{\mathcal{X}_d}^* = \mathcal{B} \cap E\mathcal{R}_{\mathcal{X}_d}^*.\tag{C4}$$

766 **1.** From (1.8), (ALG-S) and (ALG-V), we get:

$$\mathcal{V}_{\mathcal{X}_d}^* = A^{-1}(E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) \quad \text{and} \quad \mathcal{R}_{\mathcal{X}_d}^* = \mathcal{V}_{\mathcal{X}_d}^* \cap E^{-1}(A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}).\tag{C5}$$

767 Indeed:  $\mathcal{V}_{\mathcal{X}_d}^* \cap E^{-1}(A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) = \mathcal{V}_{\mathcal{X}_d}^* \cap E^{-1}(A((A^{-1}(E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B})) \cap \mathcal{S}_{\mathcal{X}_d}^*) + \mathcal{B})$   
 768  $= \mathcal{V}_{\mathcal{X}_d}^* \cap E^{-1}((E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) \cap A\mathcal{S}_{\mathcal{X}_d}^* + \mathcal{B}) = \mathcal{V}_{\mathcal{X}_d}^* \cap E^{-1}((E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) \cap (A\mathcal{S}_{\mathcal{X}_d}^* + \mathcal{B}))$   
 769  $= \mathcal{V}_{\mathcal{X}_d}^* \cap (\mathcal{V}_{\mathcal{X}_d}^* + E^{-1}\mathcal{B}) \cap \mathcal{S}_{\mathcal{X}_d}^* = \mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{S}_{\mathcal{X}_d}^* = \mathcal{R}_{\mathcal{X}_d}^*$  (see also [Özçaldıran, 1985](#),  
 770 [Malabre, 1987](#)). From (C5) and Result 1, we get:

$$\begin{aligned}\mathcal{X}_d &= \mathcal{V}_{\mathcal{X}_d}^* \oplus \mathcal{X}_1, \quad \mathcal{X}_{eq} = (E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) \oplus A\mathcal{X}_1, \\ \mathcal{V}_{\mathcal{X}_d}^* &= \mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{X}_2, \quad \mathcal{R}_{\mathcal{X}_d}^* = \mathcal{R}_C \oplus (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E).\end{aligned}\tag{C6}$$

$$E\mathcal{R}_{\mathcal{X}_d}^* = E\mathcal{V}_{\mathcal{X}_d}^* \cap (A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \quad \text{and} \quad A\mathcal{R}_{\mathcal{X}_d}^* \subset A\mathcal{V}_{\mathcal{X}_d}^* \subset E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}.\tag{C7}$$

772 **2.** From (C5.b), we get (C4.a), which implies together with (C6.c):

$$E\mathcal{V}_{\mathcal{X}_d}^* = E\mathcal{R}_{\mathcal{X}_d}^* \oplus E\mathcal{X}_2.\tag{C8}$$

773 Indeed, the direct sum comes from the fact that  $\mathcal{X}_2 \cap \mathcal{K}_E \subset \mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{K}_E =$   
 774  $\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E$  implies that  $(\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{X}_2) \cap \mathcal{K}_E = (\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{X}_2) \cap (\mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) = (\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{X}_2)$   
 775  $\cap (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) = \mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E = \mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E + \mathcal{X}_2 \cap \mathcal{K}_E.$

776 Moreover, since:  $\mathcal{X}_2 \cap \mathcal{K}_E = (\mathcal{X}_2 \cap \mathcal{V}_{\mathcal{X}_d}^*) \cap \mathcal{K}_E = \mathcal{X}_2 \cap (\mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) = \mathcal{X}_2$   
 777  $\cap (\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E) = (\mathcal{X}_2 \cap \mathcal{R}_{\mathcal{X}_d}^*) \cap \mathcal{K}_E = \{0\}$ , we get:  $\dim(E\mathcal{X}_2) = \dim(\mathcal{X}_2)$ , thus  
 778 (C3.b) follows.

779 **3.** From (C8) and (C7), we get:

$$E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} = (E\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \oplus E\mathcal{X}_2. \quad (\text{C9})$$

780 Indeed, since:  $\{0\} = (E\mathcal{R}_{\mathcal{X}_d}^*) \cap (E\mathcal{X}_2) = E\mathcal{V}_{\mathcal{X}_d}^* \cap (A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \cap (E\mathcal{X}_2) =$

781  $(A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \cap (E\mathcal{X}_2)$ , we get:  $E\mathcal{X}_2 \cap (E\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \subset E\mathcal{X}_2 \cap (A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) = \{0\}$ .

782 Moreover, (C9), (C7) and (C8) imply:

$$\begin{aligned} E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} &= (E\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \oplus E\mathcal{X}_2 = (E\mathcal{V}_{\mathcal{X}_d}^* \cap (A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) + \mathcal{B}) \oplus E\mathcal{X}_2 \\ &= ((E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B}) \cap (A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B})) \oplus E\mathcal{X}_2 = (A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \oplus E\mathcal{X}_2. \end{aligned} \quad (\text{C10})$$

783 **4.** From (C7.a) and (3.9), there exist subspaces,  $\mathcal{W}_C$  and  $\mathcal{B}_C$ , such that:

$$A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B} = E\mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{W}_C, \quad \mathcal{B} = ((E\mathcal{R}_{\mathcal{X}_d}^*) \cap \mathcal{B}) \oplus \mathcal{B}_C, \quad \mathcal{W}_C \supset \mathcal{B}_C \quad (\text{C11})$$

784 From (C8), (C10), and (C11), we get:  $E\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{B} = (E\mathcal{R}_{\mathcal{X}_d}^* \oplus E\mathcal{X}_2) + \mathcal{B} =$

785  $E\mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{B}_C \oplus E\mathcal{X}_2 = (A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \oplus E\mathcal{X}_2$ , that is to say:  $E\mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{W}_C = A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}$

786  $\approx E\mathcal{R}_{\mathcal{X}_d}^* \oplus \mathcal{B}_C$ . Hence:

$$\mathcal{W}_C = \mathcal{B}_C \quad (\text{C12})$$

787 **5.** From the geometric decompositions (C6), (C10), (C11), and (C12), the

788 subspaces  $\mathcal{X}_d$ ,  $\mathcal{X}_{eq}$ , and  $\mathcal{U}$  take the form (C1)-(C2).

789 **6.** From (C2.c,a) and since:  $\text{Ker } A \subset \mathcal{V}_{\mathcal{X}_d}^*$  and  $\text{Ker } B = \{0\}$ , we get (C3.a,c,d).

790 **7.** To prove (C4.b), note first that (C8) and (C9) imply  $\mathcal{B} \cap E\mathcal{V}_{\mathcal{X}_d}^* =$

791  $\mathcal{B} \cap (E\mathcal{R}_{\mathcal{X}_d}^* + E\mathcal{X}_2)$  and  $(E\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \cap E\mathcal{X}_2 = \{0\}$ . Let  $x \in \mathcal{B} \cap (E\mathcal{R}_{\mathcal{X}_d}^* + E\mathcal{X}_2)$ ,

792 there then exist  $z \in E\mathcal{R}_{\mathcal{X}_d}^*$ ,  $y \in E\mathcal{X}_2$ , and  $b \in \mathcal{B}$  such that  $x = z + y = b$ ,



793 which implies  $y = b - z \in (E\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \cap E\mathcal{X}_2 = \{0\}$ , *i.e.*  $x \in \mathcal{B} \cap E\mathcal{R}_{\mathcal{X}_d}^*$ . There-  
 794 fore:  $\mathcal{B} \cap E\mathcal{V}_{\mathcal{X}_d}^* = \mathcal{B} \cap (E\mathcal{R}_{\mathcal{X}_d}^* + E\mathcal{X}_2) \subset \mathcal{B} \cap E\mathcal{R}_{\mathcal{X}_d}^* \subset \mathcal{B} \cap E\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B} \cap E\mathcal{X}_2 \subset$   
 795  $\mathcal{B} \cap (E\mathcal{R}_{\mathcal{X}_d}^* + E\mathcal{X}_2) = \mathcal{B} \cap E\mathcal{V}_{\mathcal{X}_d}^*$ .

796 Let us next note that under the geometric decompositions, (C1)-(C3), the  
 797 implicit representation (1.7) takes the following form (recall (3.7) and (3.8)):

$$\begin{bmatrix} \bar{E} & 0 & * \\ 0 & I_2 & * \\ 0 & 0 & \bar{X}_{\rho-1} \end{bmatrix} \frac{d}{dt}x = \begin{bmatrix} \bar{A} & \hat{A} & 0 \\ 0 & \hat{A}_3 & 0 \\ 0 & 0 & I_1 \end{bmatrix} x + \begin{bmatrix} \bar{B} \\ 0 \\ 0 \end{bmatrix} u, \quad (\text{C13})$$

798 where  $I_2 : \mathcal{X}_2 \leftrightarrow E\mathcal{X}_2$  is an isomorphism, and the matrices  $\bar{E}$ ,  $\bar{A}$  and  $\bar{B}$ , are  
 799 the ones shown in (4.1). Then, when  $\mathcal{R}_{\mathcal{X}_d} = \mathcal{X}_d$ , we get (4.1).  $\square$

## 800 Appendix D. Proof of Corollary 1

801 Let us first note that (5.10) implies that the implicit representation (C13)  
 802 is only composed by the linear transformations (4.1).

803 Let us next note that Lemma 3 and (5.11) imply that (see (C2)-(C4)):

$$\mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{K}_E = \mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E \quad \text{and} \quad \mathcal{B}/(\mathcal{B} \cap E\mathcal{V}_{\mathcal{X}_d}^*) = \mathcal{B}/(\mathcal{B} \cap E\mathcal{R}_{\mathcal{X}_d}^*) \approx \mathcal{B}_C \approx \mathcal{U}_C, \quad (\text{D1})$$

$$\dim(A\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{B}) \geq \dim(\mathcal{R}_{\mathcal{X}_d}^*). \quad (\text{D2})$$

805 **Case 1.** If  $\mathcal{U}_C = \{0\}$ , then (5.12) and (D1) imply:  $\mathcal{R}_{\mathcal{X}_d}^* \cap \mathcal{K}_E = \{0\}$ . Thus, the  
 806 blocks  $\bar{A}_{1,2}$ ,  $\bar{A}_{2,1}$ ,  $\bar{A}_{2,2}$ , and  $I_{\mathcal{U}_C}$  actually disappear from (4.1), corresponding  
 807 to 0 row and 0 column. Moreover  $\bar{B}_1 \neq 0$ , because the pair  $(\bar{A}_{1,1}, [\bar{A}_{1,2} \ \bar{B}_1])$  is  
 808 reachable (see (4.2)). Namely, we get (5.13).

809 **Case 2.** The existence of  $\bar{V}_\ell$  is implied by (D2). From (5.14) and (4.1), we  
 810 get (5.15). From (5.15) and (4.2), we get (5.16). From (5.17) and (4.1), we  
 811 get (5.18). From (5.18) and (4.2), we get (5.19).  $\square$

## 812 Appendix E. Proof of Theorem 7

813 Let us first propose a PD descriptor feedback  $u = F_p^*x + F_d^*dx/dt + v$ ,  
 814 where the pair of linear transformations  $(F_p, F_d)$  is chosen such that:

$$(F_p^*, F_d^*) \in \mathbf{F}(\mathcal{V}^*) \text{ and } \mathcal{B} \cap E_{F_d^*}\mathcal{V}^* = \{0\}. \quad (\text{E1})$$

815 Let us next, consider the *quotient implicit representation*  $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$   
 816 defined by (5.20). Let us note that  $\text{Ker } B_* = B^{-1}E_{F_d^*}\mathcal{V}^* \approx \mathcal{B} \cap E_{F_d^*}\mathcal{V}^*$  implies  
 817  $\text{Ker } B_* = \{0\}$ , and that  $\Phi\mathcal{N}_{(F_p^*, F_d^*)} = \Phi \sup\{\mathcal{V} \subset \text{Ker } C \mid A_{F_p^*}\mathcal{V} \subset E_{F_d^*}\mathcal{V}\} = \Phi\mathcal{V}^* =$   
 818  $\{0\}$  implies the observability of the *quotient implicit representation*  
 819  $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$ . The proof of Theorem 7 is done in 4 steps:

820 *i)*  $\mathfrak{R}^{imp}(E_{F_d^*}, A_{F_p^*}, B, C)$  is externally equivalent to  $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$ . This  
 821 fact follows from (Theorem 2.1, Bonilla & Malabre, 1995), which states,  
 822 among others, the *external equivalency* between  $\mathfrak{R}^{imp}(E_{F_d^*}, A_{F_p^*}, B, C)$  and  
 823  $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$  (see also Kuijper & Schumacher, 1991).

824 *ii)*  $\mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^* = \Phi\mathcal{V}_{\mathcal{X}_d}^*$  and  $\mathcal{S}_{\mathcal{X}_d/\mathcal{V}^*}^* = \Phi\mathcal{S}_{\mathcal{X}_d}^*$ . For the case of the *quotient im-*  
 825 *PLICIT representation*  $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$  the corresponding algorithms (ALG–  
 826 V) and (ALG–S), for computing  $\mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^*$  and  $\mathcal{S}_{\mathcal{X}_d/\mathcal{V}^*}^*$ , take the following form:

$$\begin{aligned} \mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^0 &= \mathcal{X}_d/\mathcal{V}_d^*, \quad \mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^{\mu+1} = A_*^{-1}(E_*\mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^\mu + \mathcal{B}_*), \\ \mathcal{S}_{\mathcal{X}_d/\mathcal{V}^*}^0 &= \{0\}, \quad \mathcal{S}_{\mathcal{X}_d/\mathcal{V}^*}^{\mu+1} = E_*^{-1}(A_*\mathcal{S}_{\mathcal{X}_d/\mathcal{V}^*}^\mu + \mathcal{B}_*). \end{aligned} \quad (\text{E2})$$

827 It is clear that:  $\mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^0 = \Phi \mathcal{V}_{\mathcal{X}_d}^0$  and  $\mathcal{S}_{\mathcal{X}_d/\mathcal{V}^*}^0 = \Phi \mathcal{S}_{\mathcal{X}_d}^0$ . Let us assume that:  
828  $\mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^\mu = \Phi \mathcal{V}_{\mathcal{X}_d}^\mu$  and  $\mathcal{S}_{\mathcal{X}_d/\mathcal{V}^*}^\mu = \Phi \mathcal{S}_{\mathcal{X}_d}^\mu$ , then from (E2) and from (5.20), we  
829 get:  $\mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^{\mu+1} = (\Phi \mathcal{X}_d) \cap A_*^{-1} (E_* \Phi \mathcal{V}_{\mathcal{X}_d}^\mu + \Pi \mathcal{B}) = \Phi \Phi^{-1} A_*^{-1} \Pi (E_{F_d^*} \mathcal{V}_{\mathcal{X}_d}^\mu + \mathcal{B}) = \Phi A_{F_d^*}^{-1}$   
830  $\Pi^{-1} \Pi (E_{F_d^*} \mathcal{V}_{\mathcal{X}_d}^\mu + \mathcal{B}) = \Phi A_{F_d^*}^{-1} (E_{F_d^*} \mathcal{V}_{\mathcal{X}_d}^\mu + \mathcal{B} + E_{F_d^*} \mathcal{V}^*) = \Phi (\mathcal{X}_d \cap A_{F_d^*}^{-1} (E_{F_d^*} \mathcal{V}_{\mathcal{X}_d}^\mu + \mathcal{B}))$   
831  $= \Phi \mathcal{V}_{\mathcal{X}_d}^{\mu+1}$ , and  $\mathcal{S}_{\mathcal{X}_d/\mathcal{V}^*}^{\mu+1} = (\Phi \mathcal{X}_d) \cap E_*^{-1} (A_* ((\Phi \mathcal{X}_d) \cap (\Phi \mathcal{S}_{\mathcal{X}_d}^\mu)) + \Pi \mathcal{B}) = \Phi \Phi^{-1} E_*^{-1}$   
832  $\Pi (A_{F_d^*} (\mathcal{X}_d \cap \mathcal{S}_{\mathcal{X}_d}^\mu) + \mathcal{B}) = \Phi E_{F_d^*}^{-1} \Pi^{-1} \Pi (A_{F_d^*} (\mathcal{X}_d \cap \mathcal{S}_{\mathcal{X}_d}^\mu) + \mathcal{B}) = \Phi E_{F_d^*}^{-1} (A_{F_d^*}$   
833  $(\mathcal{X}_d \cap \mathcal{S}_{\mathcal{X}_d}^\mu) + \mathcal{B} + E_{F_d^*} \mathcal{V}^*) = \Phi E_{F_d^*}^{-1} (A_{F_d^*} (\mathcal{X}_d \cap \mathcal{S}_{\mathcal{X}_d}^\mu) + \mathcal{B}) + \Phi \text{Ker } E_{F_d^*} =$   
834  $\Phi (\mathcal{X}_d \cap E_{F_d^*}^{-1} (A_{F_d^*} (\mathcal{X}_d \cap \mathcal{S}_{\mathcal{X}_d}^\mu) + \mathcal{B})) = \Phi \mathcal{S}_{\mathcal{X}_d}^{\mu+1}.$

835 *iii) If (5.21) and (5.23) are satisfied, then  $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$  satisfies The-*  
836 *orem 6.* Since:  $(\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{S}_{\mathcal{X}_d}^*) \cap \text{Ker } \Phi = (\mathcal{V}_{\mathcal{X}_d}^* + \mathcal{S}_{\mathcal{X}_d}^*) \cap \mathcal{V}^* = \mathcal{V}^* = \mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{V}^* +$   
837  $\mathcal{S}_{\mathcal{X}_d}^* \cap \mathcal{V}^*$ , we get from (5.21):  $\mathcal{R}_{\mathcal{X}_d/\mathcal{V}^*}^* = \mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^* \cap \mathcal{S}_{\mathcal{X}_d/\mathcal{V}^*}^* = \Phi \mathcal{V}_{\mathcal{X}_d}^* \cap \Phi \mathcal{S}_{\mathcal{X}_d}^* =$   
838  $\Phi (\mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{S}_{\mathcal{X}_d}^*) = \Phi \mathcal{R}_{\mathcal{X}_d}^* = \Phi (\mathcal{R}_{\mathcal{X}_d}^* + \mathcal{V}^*) = \Phi \mathcal{X}_d = \mathcal{X}_d/\mathcal{V}^*$ , which is the first con-  
839 dition of Theorem 6. On the other hand, since for any  $F_d^* : \mathcal{X}_d \rightarrow \mathcal{U}$ ,  $E^{-1} \mathcal{B} =$   
840  $E_{F_d^*}^{-1} \mathcal{B}$ , we have:  $\dim(\mathcal{K}_E) + \dim(\text{Im } E \cap \mathcal{B}) = \dim(E_{F_d^*}^{-1} \mathcal{B})$ , which together with  
841 (5.23) imply:<sup>23</sup>

$$842 \quad \dim(\mathcal{B}) \geq \dim(E_{F_d^*}^{-1} \mathcal{B} / (\mathcal{V}^* \cap E_{F_d^*}^{-1} \mathcal{B})) = \dim(\Phi E_{F_d^*}^{-1} \mathcal{B}) = \dim(E_*^{-1} \mathcal{B}_*),$$

843 then:  $\dim(\mathcal{B}_*) = \dim(\Pi \mathcal{B}) \geq \dim(E_*^{-1} \mathcal{B}_*) - \dim(\mathcal{B} \cap \text{Ker } \Pi) = \dim(E_*^{-1} \mathcal{B}_*)$   
844  $- \dim(B \text{Ker } B_*) = \dim(E_*^{-1} \mathcal{B}_*) - \dim(\text{Ker } B_*) = \dim(E_*^{-1} \mathcal{B}_*)$ , that is to say:  
845  $\dim(\mathcal{B}_*/(\mathcal{B}_* \cap \text{Im } E_*)) \geq \dim(\mathcal{K}_{E_*})$ , which is the second condition<sup>24</sup> of Theorem  
846 6.

847 *iv) If  $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$  satisfies Theorem 6, then (5.21) and (5.23) are sat-*  
848 *isfied.* From the first condition of Theorem 6, we have:  $\mathcal{X}_d/\mathcal{V}^* = \mathcal{R}_{\mathcal{X}_d/\mathcal{V}^*}^* =$

<sup>23</sup> Note that:  $\mathcal{X}_d = \mathcal{R}_{\mathcal{X}_d}^* + \mathcal{V}^* \subset \mathcal{V}_{\mathcal{X}_d}^* \subset \mathcal{X}_d$ , and recall (5.20).

<sup>24</sup> Note that:  $\mathcal{X}_d/\mathcal{V}^* = \mathcal{R}_{\mathcal{X}_d/\mathcal{V}^*}^* \subset \mathcal{V}_{\mathcal{X}_d/\mathcal{V}^*}^* \subset \mathcal{X}_d/\mathcal{V}^*$ .

849  $(\Phi \mathcal{V}_{\mathcal{X}_d}^*) \cap (\Phi \mathcal{S}_{\mathcal{X}_d}^*)$ , which implies:  $\mathcal{X}_d = \mathcal{V}_{\mathcal{X}_d}^* \cap (\mathcal{S}_{\mathcal{X}_d}^* + \mathcal{V}^*) = \mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{S}_{\mathcal{X}_d}^* + \mathcal{V}^*$   
 850  $= \mathcal{R}_{\mathcal{X}_d}^* + \mathcal{V}^*$ , which is the first condition of Theorem 7. From the sec-  
 851 ond condition of Theorem 6, we have:<sup>24</sup>  $\dim(\Pi \mathcal{B}) = \dim(\mathcal{B}_*) \geq \dim(\mathcal{K}_{E_*}) +$   
 852  $\dim(\mathcal{B}_* \cap \text{Im } E_*) = \dim(E_*^{-1} \mathcal{B}_*) = \dim(E_*^{-1} \Pi \mathcal{B}) = \dim(\Phi E_{F_d}^{-1} \mathcal{B})$ . Then (recall  
 853 (E1)):  $\dim(\mathcal{B}) \geq \dim(\Phi E_{F_d}^{-1} \mathcal{B}) + \dim(\mathcal{B} \cap \text{Ker } \Pi) = \dim(\Phi E^{-1} \mathcal{B}) + \dim(\mathcal{B} \cap$   
 854  $E_{F_d}^* \mathcal{V}^*) = \dim(E^{-1} \mathcal{B}) - \dim(\mathcal{V}^* \cap E^{-1} \mathcal{B}) = \dim(\mathcal{K}_E) + \dim(\mathcal{B} \cap \text{Im } E) - \dim(\mathcal{V}^* \cap$   
 855  $E^{-1} \mathcal{B})$ , which is the second condition<sup>23</sup> of Theorem 7.  $\square$

## 856 Appendix F. Geometric Inequalities (5.26) and (5.27)

857 From (ALG-V), (ALG-S) and (5.25), we obtain:  $\mathcal{V}_{\mathcal{X}_d}^* = \overline{\mathcal{X}} \oplus \mathcal{Q}$  and  $\mathcal{S}_{\mathcal{X}_d}^*$   
 858  $= \langle \overline{A} \mid \text{Im} [\overline{B} \ \overline{S}] \rangle \oplus \mathcal{Q}$ , which imply:  $E \mathcal{V}_{\mathcal{X}_d}^* = \overline{\mathcal{X}}$  and  $\mathcal{R}_{\mathcal{X}_d}^* = \langle \overline{A} \mid \text{Im} [\overline{B} \ \overline{S}] \rangle \oplus \mathcal{Q}$ .

859 From (ALG-V) and (5.25), we get:  $\mathcal{V}^0 = E^{-1} \text{Im } E = E^{-1} \overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^0$  and  $\mathcal{V}^1$   
 860  $= E^{-1} \mathcal{K}_{\overline{C}} = E^{-1} \overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^1$ , then:  $E \mathcal{V}^0 = \overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^0$  and  $E \mathcal{V}^1 = \overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^1$ . Let us as-  
 861 sume that:  $E \mathcal{V}^\mu = \overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^\mu$ , then:  $\mathcal{V}^{\mu+1} = (E^{-1} \mathcal{K}_{\overline{C}}) \cap [\overline{A} \ \overline{S}]^{-1} (\overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^\mu + \overline{\mathcal{B}})$ ,  
 862 which implies:  $E \mathcal{V}^{\mu+1} = \mathcal{K}_{\overline{C}} \cap E [\overline{A} \ \overline{S}]^{-1} (\overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^\mu + \overline{\mathcal{B}}) = \mathcal{K}_{\overline{C}} \cap \overline{A}^{-1} (\overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^\mu$   
 863  $+ \text{Im} [\overline{B} \ \overline{S}]) = \overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^{\mu+1}$ . Thus:  $E \mathcal{V}^* = \overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^*$ .

864 From the previous paragraphs we have the following equivalences:  $\mathcal{R}_{\mathcal{X}_d}^*$   
 865  $+ \mathcal{V}^* = \mathcal{X}_d \Leftrightarrow \langle \overline{A} \mid \text{Im} [\overline{B} \ \overline{S}] \rangle \oplus \mathcal{Q} + \mathcal{V}^* = \overline{\mathcal{X}} \oplus \mathcal{Q} \Leftrightarrow E^{-1} \langle \overline{A} \mid \text{Im} [\overline{B} \ \overline{S}] \rangle + \mathcal{V}^* =$   
 866  $\overline{\mathcal{X}} \oplus \mathcal{Q} \Rightarrow \text{Im } E \cap \langle \overline{A} \mid \text{Im} [\overline{B} \ \overline{S}] \rangle + E \mathcal{V}^* = \overline{\mathcal{X}} \Rightarrow \langle \overline{A} \mid \text{Im} [\overline{B} \ \overline{S}] \rangle + \overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^* = \overline{\mathcal{X}}$   
 867  $\Rightarrow E^{-1} (\langle \overline{A} \mid \text{Im} [\overline{B} \ \overline{S}] \rangle + E \mathcal{V}^*) = \overline{\mathcal{X}} \oplus \mathcal{Q} \Rightarrow E^{-1} \langle \overline{A} \mid \text{Im} [\overline{B} \ \overline{S}] \rangle + \mathcal{V}^* = \overline{\mathcal{X}} \oplus \mathcal{Q};$   
 868 which imply (5.26).

869 From the two first paragraphs, (5.22) takes the form:

$$\dim \left( \frac{\overline{\mathcal{X}}}{\overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^* + \overline{\mathcal{B}}} \right) + \dim(\overline{\mathcal{B}}) \geq \dim \left( \frac{\overline{\mathcal{X}}}{\overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^*} \right) + \dim \left( \frac{\{0\} \oplus \mathcal{Q}}{\mathcal{V}^* \cap \mathcal{K}_E} \right) \quad (\text{F1})$$

870 From (ALG-V), (5.25) and the second paragraph, we obtain:

$$\begin{aligned}
\mathcal{K}_E \cap \mathcal{V}^* &= \mathcal{K}_E \cap (E^{-1} \mathcal{K}_{\overline{C}}) \cap \begin{bmatrix} \overline{A} & \overline{S} \end{bmatrix}^{-1} \left( \overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^* + \overline{\mathcal{B}} \right) \\
&= \mathcal{K}_E \cap \begin{bmatrix} \overline{A} & \overline{S} \end{bmatrix}^{-1} \left( \overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^* + \overline{\mathcal{B}} \right) = \{0\} \oplus \overline{S}^{-1} \left( \overline{\mathcal{V}}_{[\overline{B} \ \overline{S}]}^* + \overline{\mathcal{B}} \right)
\end{aligned}
\tag{F2}$$

871 From (F1) and (F2) we get (5.27) (recall that  $\text{Ker } \overline{S} = \{0\}$ ). □