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# Some structural characterizations of linear descriptor systems

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**Abstract:** Different properties of general linear descriptor systems are reviewed (existence of solution, consistency of initial condition, impulse controllability and controllability) and structurally characterized. The invariants are associated to a known feedback canonical form of descriptor systems. The aim is to sort the systems by inclusion properties depending on these characterizations.

*Keywords:* Control, descriptor systems, structure, invariants.

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## 1. INTRODUCTION

Descriptor systems provide a convenient and natural description of dynamical systems (see Duan (2010) for examples). However, even in the restricted case of linear time-invariant descriptor systems, that is considered here, they may present problems like solvability, initial condition consistency, impulsive response and different characteristics of controllability. The descriptor representation of the considered systems has the following form

$$E\dot{x} = Ax + Bu \quad (1)$$

where:  $E : \mathcal{X}_d \rightarrow \mathcal{X}_{eq}$ ,  $A : \mathcal{X}_d \rightarrow \mathcal{X}_{eq}$  and  $B : \mathcal{U} \rightarrow \mathcal{X}_{eq}$  are linear maps. The linear spaces  $\mathcal{X}_d \approx \mathbb{R}^{n_d}$ ,  $\mathcal{X}_{eq} \approx \mathbb{R}^{n_{eq}}$ ,  $\mathcal{U} \approx \mathbb{R}^m$ , are called the descriptor, the equation, and the input spaces, respectively. No special assumptions are made on the matrix pencil  $sE - A$ , in the most general case it can be rectangular ( $n_d \neq n_{eq}$ ). This type of systems has attracted much attention during the last two decades (Geerts (1993), Ishihara and Terra (2001), Hou and Muller (1999), Hou (2004), Zhang (2006), Duan and Chen (2007), ...). Most of the properties that will be considered here have been described in the previous cited references. The aim here is to give structural characterizations of these properties as well as a classification of the systems in terms of the invariants of the feedback canonical form introduced in Loiseau et al. (1991). Section 2 states the properties that are considered and recalls their classical known characterizations. Section 3 brings back the definition of the feedback canonical form and its associated lists of invariants. Section 4 gives the structural characterizations in terms of these invariants. Section 5 shows how easily they can be applied on the analysis of the control problem of a constrained manipulator Mills and Goldenberg (1989). Section 6 is devoted to the conclusion.

## 2. SOME PROPERTIES OF SINGULAR SYSTEMS

Following Hautus and Silverman (1983), Geerts (1993) proposed a distributional framework to describe the solu-

tion of (1) (the same framework was adopted in Özçaldıran and Haliloğlu (1993) and Przyluski and Sosnowski (1994)). The considered set of *impulsive-smooth* distributions, denoted  $\mathcal{C}_{\text{imp}}^{n_d}$  for the state ( $\mathcal{C}_{\text{imp}}^m$  for the input) is decomposed as  $\mathcal{C}_{\text{imp}}^{n_d} = \mathcal{C}_{\text{p-imp}}^{n_d} \oplus \mathcal{C}_{\text{sm}}^{n_d}$  where  $\mathcal{C}_{\text{p-imp}}^{n_d}$  and  $\mathcal{C}_{\text{sm}}^{n_d}$  denote the set of pure impulses and smooth distributions respectively. The distributional version of (1) is also presented:  $pEx = Ax + Bu + Ex_0$  where  $x_0 \in \mathcal{X}_d$  is the initial condition,  $Ex_0$  stands for  $E x_0 \delta$  ( $\delta$  denotes the Dirac delta function),  $pEx$  stands for  $\delta^{(1)} * Ex$  (where  $*$  denotes the convolution and  $\delta$ ) and such that when  $pEx$  is smooth,  $pEx = E\dot{x} + Ex_0^+$  where  $E\dot{x}$  stands for the distribution that can be identified with ordinary derivative. For a given initial condition and a given input, the solution set of the state trajectories is

$$S_C(x_0, u) := \{x \in \mathcal{C}_{\text{imp}}^{n_d} \mid [pE - A]x = Bu + Ex_0\} \quad (2)$$

It is well known that for the general systems (1), this set may be empty (existence of solution) or may have several solutions (non uniqueness). Moreover, the solutions may be impulsive or smooth and even in this case with jumps at the origin.

### 2.1 Solvability, consistent initial conditions

Geerts (1993) proposed the following definitions for the solvability of (1)

*Definition 1.* The system (1) is

- *C-solvable* if  $\forall x_0 \in \mathcal{X}_d, \exists u \in \mathcal{C}_{\text{imp}}^m : S_C(x_0, u) \neq \emptyset$ ,
- *C-solvable in the function sense* if  $\forall x_0 \in \mathcal{X}_d, \exists u \in \mathcal{C}_{\text{sm}}^m : S_C(x_0, u) \cap \mathcal{C}_{\text{sm}}^{n_d} \neq \emptyset$ .

Let us note that *C-solvability* is concerned with distributional (or impulsive) solutions whereas *C-solvable in the function sense* is concerned with ordinary (or smooth) solutions. But, even in the latter case, for some initial conditions there may exist jumps at the origin. Geerts

(1993) introduced the following definition to distinguish these two notions of consistent of initial conditions

*Definition 2.* A point  $x_0 \in \mathcal{X}_d$  is called

- *C-consistent* if  
 $\exists u \in \mathcal{C}_{sm}^m \exists x \in S_C(x_0, u) \cap \mathcal{C}_{sm}^{n_d} : x(0^+) = x_0$ .  
The set of *C-consistent* points is denoted  $I_c$ .
- *weakly C-consistent* if  
 $\exists u \in \mathcal{C}_{sm}^m : S_C(x_0, u) \cap \mathcal{C}_{sm}^{n_d} \neq \emptyset$ .  
The set of *weakly C-consistent* points is denoted  $I_c^w$ .

*C-consistency* avoids jumps at the origin for at least one smooth solution. *Weak C-consistency* enables jumps at the origin among the smooth (piece-wise continuous) solutions. Note that  $I_c^w = \mathcal{X}_d$  if and only if the system (1) is *C-solvable in the function sense* (Geerts (1993), proposition 4.2). Also note that Hou (2004) introduced the notion of admissible initial condition for which there exists an  $u \in \mathcal{C}_{imp}^m$  such that there exists  $x \in S_C(x_0, u)$ ; Clearly an arbitrary initial condition is admissible in and only if the system is *C-solvable*.

In his seminal paper Geerts (1993) also gave characterizations of these properties.

*Theorem 3.* (Th. 3.5 of Geerts (1993)). The system (1) is *C-solvable* if and only if

$$\forall \eta(s) \in M^{1 \times n_{eq}}(s) : \eta(s) [sE - A \ B] = 0 \Leftrightarrow \eta(s) [E \ A \ B] = 0 \quad (3)$$

where  $M^{1 \times n_{eq}}(s)$  is the set of  $1 \times n_{eq}$  matrices with elements in the field of rational function  $\mathbb{R}(s)$ .

*Corollary 4.* (Ishihara and Terra (2001), Hou (2004)). The system (1) is *C-solvable* if and only if

$$\text{rank} [sE - A \ B] = \text{rank} [E \ A \ B] \quad (4)$$

*Corollary 5.* (Corollary 3.6 of Geerts (1993)). If  $[E \ A \ B]$  is full row rank, the system (1) is *C-solvable* if and only if  $[sE - A \ B]$  is right invertible as a rational matrix

*Theorem 6.* (Prop. 4.2 and Th. 4.5 of Geerts (1993)). Assume that in (1)  $[E \ A \ B]$  is full row rank. Then, the three following assertions are equivalent

$$\begin{aligned} &\text{The system (1) is } C\text{-solvable in the function sense} \\ &I_c^w = \mathcal{X}_d \\ &ImE + AKerE + ImB = \mathcal{X}_{eq} \end{aligned}$$

Note that Corollary 5 and Theorem 6 are the exact results of Geerts (1993) under the assumption “ $[E \ A \ B]$  is full row rank”. This condition is nothing else but the lack of redundant equations in (1). So there is no loss of generality with this assumption. However, it is possible to state general results: Theorem 3 and Corollary 4 come from Geerts (1993) and Ishihara and Terra (2001) for *C-solvability*. The following one gives a general condition for the *C-solvability in the function sense*.

*Corollary 7.* The system (1) is *C-solvable in the function sense*, or is such that  $I_c^w = \mathcal{X}_d$  if and only if

$$ImA \subset ImE + AKerE + ImB \quad (5)$$

**Proof.** Suppose that  $[E \ A \ B]$  is full row rank. Then  $ImA \subset ImE + AKerE + ImB \Leftrightarrow ImE + AKerE + ImB = \mathcal{X}_{eq}$ . Suppose that  $[E \ A \ B]$  is not full row rank. There exist  $\bar{n}_{eq}$ ,  $Y$ ,  $\bar{E}$ ,  $\bar{A}$  and  $\bar{B}$  such that

$$[E \ A \ B] = \begin{bmatrix} I_{\bar{n}_{eq}} \\ Y \end{bmatrix} [\bar{E} \ \bar{A} \ \bar{B}] \quad (6)$$

with  $[\bar{E} \ \bar{A} \ \bar{B}]$  full row rank.

The new system (1) defined with  $(\bar{E}, \bar{A}, \bar{B})$  (denote it  $(\bar{\Sigma})$ ) is obtained from (1) with  $(E, A, B)$  (denote it  $(\Sigma)$ ) by removing the redundant equations. Clearly,  $I_{c(\bar{\Sigma})}^w = I_{c(\Sigma)}^w$ .

Also  $ImA \subset ImE + AKerE + ImB \Leftrightarrow Im\bar{A} \subset Im\bar{E} + \bar{A}Ker\bar{E} + Im\bar{B}$ . Now, since for  $(\bar{\Sigma})$ ,  $[\bar{E} \ \bar{A} \ \bar{B}]$  is full row rank and  $Im\bar{A} \subset Im\bar{E} + \bar{A}Ker\bar{E} + Im\bar{B}$  then  $I_{c(\bar{\Sigma})}^w = \mathcal{X}_d$  (see theorem 6) from which one deduces that  $ImA \subset ImE + AKerE + ImB \Rightarrow I_{c(\Sigma)}^w = \mathcal{X}_d$ . For the reverse part, if  $I_{c(\Sigma)}^w = \mathcal{X}_d$  then  $I_{c(\bar{\Sigma})}^w = \mathcal{X}_d$  which implies  $Im\bar{A} \subset Im\bar{E} + \bar{A}Ker\bar{E} + Im\bar{B}$  and then  $ImA \subset ImE + AKerE + ImB$ .

*Theorem 8.* (Th. 4.5 of Geerts (1993)). Assume that in (1)  $[E \ A \ B]$  is full row rank. Then,  $I_c = \mathcal{X}_d$  if and only if

$$ImE + ImB = \mathcal{X}_{eq} \quad (7)$$

*Corollary 9.* For the system (1),  $I_c = \mathcal{X}_d$  if and only if

$$ImA \subset ImE + ImB \quad (8)$$

The proof is very similar to the proof of the Corollary 7.

Note that the condition (8) is also the condition which in Frankowska (1990) characterizes *strict* singular systems which are such that for all  $x_0 \in \mathcal{X}_d$  there exists a trajectory,  $x(t) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathcal{X}_d)$  (infinitely differentiable functions from  $\mathbb{R}^+$  to  $\mathcal{X}_d$ ) satisfying  $x(0) = x_0$ .

## 2.2 Controllability

If  $E = I$  (identity) (1) becomes a classical linear system for which only one clearly identified notion of controllability exists. If  $sE - A$  is square and invertible (regular continuous descriptor systems) things become more complicated. A system can be said controllable in the sense of Verghese et al. (1981) or Controllable in the sense of Cobb (1984), Yip and Sincovec (1981) or Rosenbrock (1974). In fact, Cobb (1984) showed that these last three notions are equivalent. Our aim, in this section, is not to review all these notions defined for regular descriptor systems, but to recall the definition of different kinds of controllability that have been introduced for general rectangular systems (impulse controllability (Geerts (1993), Ishihara and Terra (2001), Hou (2004)), controllability (Frankowska (1990), Özçaldıran and Haliloğlu (1993)) and what will be called here strong controllability (Özçaldıran and Haliloğlu (1993)) and their known characterizations.

Originally, the idea and name of *impulse controllability* can be found in Cobb (1983), for regular system and in Geerts (1993) for more general systems, but the now established general definition appeared in Ishihara and Terra (2001).

*Definition 10.* (Ishihara and Terra (2001)). The system (1) is called *impulse controllable* if for every initial condition there exists a smooth (impulse-free) control  $u(t)$  and a smooth state trajectory  $x(t)$  solution of (1).

In fact, this definition is nothing else but the definition 1 of *C-solvable in the function sense*. So the following corollary is an immediate consequence of Corollary 7,

*Corollary 11.* The system (1) is *impulse controllable* or *C-solvable in the function sense*, or such that  $I_c^w = \mathcal{X}_d$  iff

$$ImA \subset ImE + AKerE + ImB \quad (9)$$

In the case of general rectangular descriptor systems one finds the following definition of controllability.

*Definition 12.* (Frankowska (1990)). The system (1) is said to be *controllable* if for every pair of states  $x_1, x_2 \in \mathcal{X}_d$  and every  $T > 0$  there exists a trajectory of (1) such that  $x(0) = x_1$  and  $x(T) = x_2$ .

Note that in Frankowska (1990), the solution was supposed to be an absolutely continuous function; in Aubin and Frankowska (1991) the following theorem is stated with smooth trajectories (infinitely differentiable function).

*Theorem 13.* (Aubin and Frankowska (1991)). The system (1) is *controllable* if and only if the reachable subspace (defined below) is equal to the descriptor space

$$\mathcal{R}_{\mathcal{X}_d}^* = \mathcal{X}_d \quad (10)$$

In this case, for every pair of states  $x_1, x_2 \in \mathcal{X}_d$  and every  $T > 0$  there exists a trajectory  $x(t) \in C^\infty(\mathbb{R}^+, \mathcal{X}_d)$  of (1) such that  $x(0) = x_1$  and  $x(T) = x_2$ .

$\mathcal{R}_{\mathcal{X}_d}^* = \mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{S}_{\mathcal{X}_d}^*$  where  $\mathcal{V}_{\mathcal{X}_d}^*$  and  $\mathcal{S}_{\mathcal{X}_d}^*$  are respectively the limits of the following algorithms,

$$\begin{cases} \mathcal{V}_{\mathcal{X}_d}^0 = \mathcal{X}_d \\ \mathcal{V}_{\mathcal{X}_d}^\mu = A^{-1}(E\mathcal{V}_{\mathcal{X}_d}^{\mu-1} + ImB) \end{cases} \longrightarrow \mathcal{V}_{\mathcal{X}_d}^* \quad (11)$$

$$\begin{cases} \mathcal{S}_{\mathcal{X}_d}^0 = KerE \\ \mathcal{S}_{\mathcal{X}_d}^\mu = E^{-1}(A\mathcal{S}_{\mathcal{X}_d}^{\mu-1} + ImB) \end{cases} \longrightarrow \mathcal{S}_{\mathcal{X}_d}^* \quad (12)$$

*Remark 14.* If  $\mathcal{R}_{\mathcal{X}_d}^* = \mathcal{X}_d$  then  $\mathcal{V}_{\mathcal{X}_d}^* = \mathcal{X}_d$  so  $ImA \subset ImE + ImB$ : naturally, the system is *strict* (Frankowska (1990)) or the set of *C-consistent* points is equal to the descriptor space (i.e.  $I_c = \mathcal{X}_d$ ) (Geerts (1993)).

Note that the same definition and characterization can be found in Özçaldıran and Haliloğlu (1993) but with the denomination *complete controllability*. In fact, Özçaldıran and Haliloğlu (1993) contains a deeper study of controllability type properties: controllability (the ability to reach zero from any point) or reachability (ability to reach any point from zero) is considered, and the term *complete* in Özçaldıran and Haliloğlu (1993) is associated to the idea that the trajectory, to reach zero from any point or a point from zero, is a smooth trajectory without any jump. Theorems 2.7 and 2.8 of Özçaldıran and Haliloğlu (1993) state that complete controllability of (1) is equivalent to complete reachability of (1) and is nothing else but the controllability property of (1) defined by Frankowska (1990) and characterized by  $\mathcal{R}_{\mathcal{X}_d}^* = \mathcal{X}_d$ .

Also note that Theorem 2.5 of Özçaldıran and Haliloğlu (1993) contains the following characterization

*Theorem 15.* (Özçaldıran and Haliloğlu (1993)). For the system (1), whatever are  $x_0 \in \mathcal{X}_d$  and  $T > 0$  there exists a piece-wise continuous trajectory  $x(t)$  (with possible jumps) such that  $x(0) = x_0$  and  $x(T) = 0$  if and only if

$$\mathcal{R}_{\mathcal{X}_d}^* + KerE = \mathcal{X}_d \quad (13)$$

Note that, in this case, controllability and reachability are no longer equivalent. Because this property is said (in Özçaldıran and Haliloğlu (1993)) to be equivalent to the strong controllability of Verghese et al. (1981) of the regular case (this will become clearer in section 4), this property will also be called *strong controllability* here, although in Özçaldıran and Haliloğlu (1993) the term strong was associated to an other property ...

*Remark 16.* If  $\mathcal{R}_{\mathcal{X}_d}^* + KerE = \mathcal{X}_d$  then  $E\mathcal{R}_{\mathcal{X}_d}^* + AKerE + ImB = ImE + ImA + ImB$  so  $ImA \subset ImE + AKerE + ImB$ : as expected, the set of *weakly C-consistent* points is equal to the descriptor space (i.e.  $I_c^w = \mathcal{X}_d$ ).

Obviously, the characterizations (4) of *C-solvability*, (5) or (9) of *C-solvability in the function sense*, *weak C-consistency* or *impulse controllability*, (8) of *C-consistency*, (10) of controllability and (13) of *strong controllability* are all invariant under the action of static state feedbacks. As a consequence, it is possible to give structural equivalent characterizations (section 4) using the invariants of the feedback canonical form of  $(E, A, B)$  triples introduced in Loiseau et al. (1991) (section 3).

### 3. THE FEEDBACK CANONICAL FORM OF $(E, A, B)$ TRIPLES (Loiseau et al. (1991))

#### 3.1 The group of transformation

The feedback canonical form of  $(E, A, B)$  triples is the canonical form under the action of the group  $\mathcal{T}$ :

$$\mathcal{T} = \{(W, V, G, F) \text{ of appropriate dimensions / } W, V, G \text{ are invertible}\}$$

where

- $V$  is a change of basis of the descriptor space  $(\mathcal{X}_d)$ ,
- $W$  is a change of basis of the equation space  $(\mathcal{X}_{eq})$ ,
- $G$  is a change of basis of the input space  $(\mathcal{U})$ ,
- $F$  is a proportional state feedback:  $\mathcal{X}_d \rightarrow \mathcal{U}$

#### 3.2 The canonical form and the list of invariants

For a given triple  $(E, A, B)$  there exists  $(V, W, G, F) \in \mathcal{T}$  such that

$$(W^{-1}EV, W^{-1}(A + BF)V, W^{-1}BG) = (E_c, A_c, B_c) \quad (14)$$

where  $(E_c, A_c, B_c)$  is the canonical form under the action of the group  $\mathcal{T}$ . This is the 6-block diagonal form described in figure 1. The blocks are characterized by indices which are invariant under the action of the group  $\mathcal{T}$ . They can be obtained in a geometric way with the help of the algorithms (11), (12) and the following one

$$\begin{cases} \mathcal{T}_{\mathcal{X}_d}^0 = \{0\} \\ \mathcal{T}_{\mathcal{X}_d}^\mu = E^{-1}(A\mathcal{T}_{\mathcal{X}_d}^{\mu-1} + ImB) \end{cases} \longrightarrow \mathcal{T}_{\mathcal{X}_d}^* = \mathcal{S}_{\mathcal{X}_d}^* \quad (15)$$

introduced in Özçaldıran (1985) and Malabre (1987). One can find the following characterizations in Loiseau et al. (1991) (or in Lebret and Loiseau (1994), generalization of this canonical form to  $(E, A, B, C)$  quadruples : system (1) with an output equation  $y = Cx$ ),

*Theorem 17.* The invariants of the feedback canonical form are characterized by

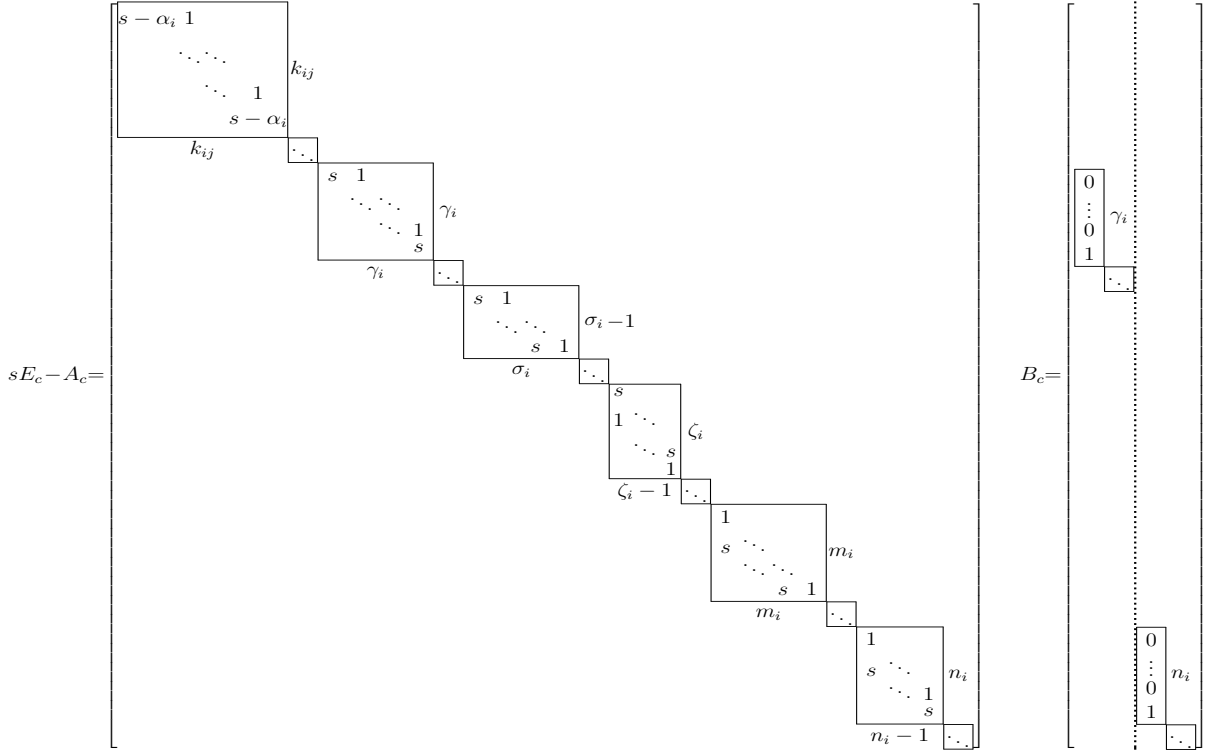


Fig. 1. Feedback canonical form of  $(E, A, B)$  triple.

- $\{s - \alpha_i\}^{k_{ij}}$  is the list of invariant factor of  $(E\mathcal{V}_{\mathcal{X}_d}^* / E\mathcal{R}_{\mathcal{X}_d}^* \| A + BF \| \mathcal{V}_{\mathcal{X}_d}^* / \mathcal{R}_{\mathcal{X}_d}^*)$ , the map induced in the quotient spaces  $\mathcal{V}_{\mathcal{X}_d}^* / \mathcal{R}_{\mathcal{X}_d}^*$  and  $E\mathcal{V}_{\mathcal{X}_d}^* / E\mathcal{R}_{\mathcal{X}_d}^*$  where  $F$  is such that  $(A + BF)\mathcal{V}_{\mathcal{X}_d}^* \subset E\mathcal{V}_{\mathcal{X}_d}^*$

$$\begin{array}{ccc}
 \mathcal{V}_{\mathcal{X}_d}^* & \xrightarrow{A+BF} & E\mathcal{V}_{\mathcal{X}_d}^* \\
 \downarrow & & \downarrow \\
 \frac{\mathcal{V}_{\mathcal{X}_d}^*}{\mathcal{R}_{\mathcal{X}_d}^*} & \xrightarrow{\frac{E\mathcal{V}_{\mathcal{X}_d}^*}{E\mathcal{R}_{\mathcal{X}_d}^*} \| A+BF \| \frac{\mathcal{V}_{\mathcal{X}_d}^*}{\mathcal{R}_{\mathcal{X}_d}^*}} & \frac{E\mathcal{V}_{\mathcal{X}_d}^*}{E\mathcal{R}_{\mathcal{X}_d}^*}
 \end{array}$$

- $^1 \text{card}\{i/\gamma_i \geq \mu\} = \dim \left( \frac{\mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{T}_{\mathcal{X}_d}^\mu}{\mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{S}_{\mathcal{X}_d}^{\mu-1}} \right) \quad \forall \mu \geq 1$
- $\text{card}\{i/\sigma_i \geq \mu\} = \dim \left( \frac{\mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{S}_{\mathcal{X}_d}^{\mu-1}}{\mathcal{V}_{\mathcal{X}_d}^* \cap \mathcal{T}_{\mathcal{X}_d}^{\mu-1}} \right) \quad \forall \mu \geq 1$
- $\text{card}\{i/\zeta_i \geq \mu\} = \dim \left( \frac{\mathcal{S}_{\mathcal{X}_d}^* \cap \mathcal{V}_{\mathcal{X}_d}^{\mu-2}}{\mathcal{S}_{\mathcal{X}_d}^* \cap \mathcal{V}_{\mathcal{X}_d}^{\mu-1}} \right) \quad \forall \mu \geq 2$
- $\text{card}\{i/\zeta_i \geq 1\} = \dim \left( \frac{\mathcal{X}_d}{A\mathcal{S}_{\mathcal{X}_d}^* + \text{Im}B + \text{Im}E} \right)$
- $\text{card}\{i/m_i = \mu\} = \dim \left( \frac{\text{Ker}E \cap \mathcal{V}_{\mathcal{X}_d}^{\mu-1}}{\text{Ker}E \cap \mathcal{V}_{\mathcal{X}_d}^\mu} \right) \quad \forall \mu \geq 1$
- $\text{card}\{i/n_i = \mu\} = \dim \left( \frac{\text{Ker}E + E^{-1}(\text{Im}B) \cap \mathcal{V}_{\mathcal{X}_d}^{\mu-2}}{\text{Ker}E + E^{-1}(\text{Im}B) \cap \mathcal{V}_{\mathcal{X}_d}^{\mu-1}} \right) \quad \forall \mu \geq 2$
- $\text{card}\{i/n_i = 1\} = \dim \left( \frac{\text{Im}B}{\text{Im}B \cap \text{Im}E} \right)$

<sup>1</sup>  $\text{card}\{\cdot\}$  denote the “number of element of the set”.

Note that these lists of invariants completely characterize the feedback canonical form. So to obtain it, one just has to compute the different steps of the algorithms (11), (12), (15) and then the above lists. One does not have to find the changes of bases and the state feedback that would lead to the canonical form.

#### 4. STRUCTURAL CHARACTERIZATION OF THE PROPERTIES

*Corollary 18.* The system (1) is *C-solvable* if and only if

$$\text{card}\{i/\zeta_i \geq 2\} = 0 \quad (16)$$

**Proof.** Note that the invariants  $\zeta_i = 1$  are associated to blocks with one row without column. This corresponds to an empty row of the pencil  $sE_c - A_c$  and of the matrix  $B_c$ . This is the characteristic of a redundant equation for (1). Note that  $\text{card}\{i/\zeta_i = 1\} = \dim(\mathcal{X}_{eq}) - \text{rank}[E \ A \ B]$ . If  $[E \ A \ B]$  is full row rank,  $\text{card}\{i/\zeta_i = 1\} = 0$ , and one can easily check with each of the six types of blocks of the feedback canonical form that  $\text{rank}[sE - A \ B] = \underline{n}_{eq}$  or equivalently  $[sE - A \ B]$  is right invertible if and only if  $\text{card}\{i/\zeta_i \geq 2\} = \text{card}\{i/\zeta_i \geq 1\} = 0$  (see Corollary (5)). In the most general case, the equivalent characterization of Corollary (4) is (16).

*Corollary 19.* The system (1) is

- *C-solvable in the function sense*
- or with  $I_c^w = \mathcal{X}_d$  (weak-C-consistent initial condition)
- or *impulse controllable*

if and only if

$$\begin{aligned}
 \text{card}\{i/\zeta_i \geq 2\} &= 0 \\
 \text{card}\{i/m_i \geq 2\} &= 0 \\
 \text{card}\{i/n_i \geq 2\} &= 0
 \end{aligned} \quad (17)$$

**Proof.** Once again, it is easy to check with each of the six block of the feedback canonical form that (5) is fulfilled if and only if the three conditions of (17) are.

To obtain an equivalent characterization of Theorem (6) one just has to add  $\text{card}\{i/\zeta_i = 1\} = 0$ .

The presence of possible jumps at the origin in the smooth trajectory  $x(t)$  solution of (1) corresponds here to the possible existence of invariant  $m_i = 1$ ; this will be clearer with the characterization of  $I_c = \mathcal{X}_d$  (C-consistent initial condition) for which  $\text{card}\{i/m_i \geq 1\} = 0$  (see Corollary 21).

*Corollary 20.* The system (1) is *strong controllable* ( $I_c^w = \mathcal{X}_d$  and “there exist an input such that the solution converges to zero in finite time with possible jump”) if and only if

$$\begin{aligned} \text{The list of invariant factor is empty} \\ \text{card}\{i/\zeta_i \geq 2\} &= 0 \\ \text{card}\{i/m_i \geq 2\} &= 0 \\ \text{card}\{i/n_i \geq 2\} &= 0 \end{aligned} \quad (18)$$

**Proof.** The geometric characterization of *strong controllability* (see Theorem (15)) is  $\mathcal{R}_{\mathcal{X}_d}^* + \text{Ker}E = \mathcal{X}_d$ . Since  $\mathcal{R}_{\mathcal{X}_d}^*$ , is the domain of the blocks  $(sI - \alpha_j)^{k_{ij}}$ ,  $\gamma_i$  and  $\sigma_i$  (see Leuret and Loiseau (1994)) and since it is easy to identify  $\text{Ker}E$  on the feedback canonical form, the structural characterization is easy to obtain.

Once again, the acceptance of jumps explains the presence of the invariants  $m_i = 1$ . For other reasons (strong equivalence), these  $1 \times 1$  blocks are also not excluded by Verghese et al. (1981) in their study of controllability of the case of regular systems. This explains that in accordance with their terminology, it was proposed in subsection 2.2 to call this controllability property *strong controllability*.

*Corollary 21.* For the system (1),  $I_c = \mathcal{X}_d$  (C-Consistent initial condition) if and only if

$$\begin{aligned} \text{card}\{i/\zeta_i \geq 2\} &= 0 \\ \text{card}\{i/m_i \geq 1\} &= 0 \\ \text{card}\{i/n_i \geq 2\} &= 0 \end{aligned} \quad (19)$$

Remember that this is also the condition for a system to be strict (Frankowska (1990)). The difference ( $AKerE$ ) between (5) and (8) explains the disappearance of the invariants  $m_i = 1$ . This is consistent with the property that for *C-consistency initial conditions* jump at the origin are not any more accepted in the trajectory solutions.

*Corollary 22.* The system (1) is controllable (definition 12) if and only if

$$\begin{aligned} \text{The list of invariant factor is empty} \\ \text{card}\{i/\zeta_i \geq 2\} &= 0 \\ \text{card}\{i/m_i \geq 1\} &= 0 \\ \text{card}\{i/n_i \geq 2\} &= 0 \end{aligned} \quad (20)$$

The difference between Corollary (22) and Corollary (21) is the same as between Corollary (20) and Corollary (19). Strong controllability and controllability differ by the  $1 \times 1$  blocks  $m_i = 1$ .

One can verify that some of the above given characterizations appeared in Korotka et al. (2011).

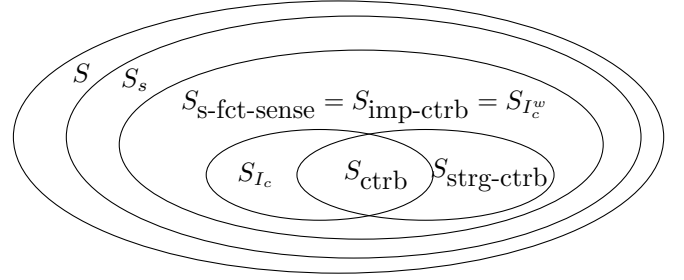


Fig. 2. Inclusions based on the properties of descriptor systems.

Figure 2 gives a synthetic view of the inclusion subsets of descriptor systems depending on the above properties

- $S$  is the set of all possible systems (1),
- $S_s$  is the set of *C-solvable* systems,
- $S_{s-fct-sense}$  is the set of *C-solvable in the function sense* systems,
- $S_{imp-ctrb}$  is the set of *impulse controllable* systems,
- $S_{I_c^w}$  is the set of systems such that  $I_c^w = \mathcal{X}_d$ ,
- $S_{I_c}$  is the set of systems such that  $I_c = \mathcal{X}_d$ ,
- $S_{strg-ctrb}$  is the set of *strongly controllable* systems,
- $S_{ctrb}$  is the set of *controllable* systems,

## 5. AN EXAMPLE

The following “force and position control of manipulators during constrained motion task” is fully described in Mills and Goldenberg (1989) (also referenced in Loiseau and Zagalak (2009) (p. 1190)). In Duan (2010) (p. 12), one can find a three-link planar version (three bodies, three actuators) in the (x,y) frame. The robot has to clean a surface defined by  $x = l$ , the end effector has to be perpendicular to this surface. In the very general case, one has to model the motion of the free robot (classical equations are here in the joint coordinates)

$$M_\theta(\ddot{\theta}) + C_\theta(\theta, \dot{\theta}) + G_\theta(\theta) = u_\theta + \partial\psi_\theta/\partial\theta.\mu$$

and adds the equation of the constraint function

$$\psi_\theta = 0$$

In the three-link planar version  $\theta \in \mathbb{R}^3$  and the Lagrangian multiplier which defined the generalized constraint force is  $\mu \in \mathbb{R}^2$ . Note that in Cartesian coordinates defined by the position and orientation of the end effector ( $z = [x, y, \phi]$ ), the equations would have the same structure (1.25, 1.26 of Duan (2010)). One can linearize the model with the following working point

$$\begin{aligned} z_{wp} &= [x_{wp}=l \quad y_{wp}=cste_1 \quad \phi_{wp}=0]^T \\ \dot{z}_{wp} &= [\dot{x}_{wp}=0 \quad \dot{y}_{wp}=cste_2 \quad \dot{\phi}_{wp}=0]^T \\ \ddot{z}_{wp} &= [\ddot{x}_{wp}=0 \quad \ddot{y}_{wp}=0 \quad \ddot{\phi}_{wp}=0]^T \\ \mu_{wp} &= [\mu_{wp1}=cste_3 \quad \mu_{wp2}=0]^T \end{aligned}$$

With the following state vector  $[\delta z^T \quad \delta \dot{z}^T \quad \delta \mu^T]^T$  where  $\delta z = z - z_{wp}$ ,  $\delta \mu = \mu - \mu_{wp}$ , the linearized model is a descriptor system (1) with

$$E = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & M_{wp} & 0 \\ 0 & 0 & 0_2 \end{bmatrix} \quad A = \begin{bmatrix} 0 & I & 0 \\ -K_{wp} & -D_{wp} & F_{wp}^T \\ F_{wp} & 0 & 0_2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ S_{wp} \\ 0 \end{bmatrix}$$

where  $M_{wp}$ ,  $S_{wp}$  are invertible  $3 \times 3$  matrices and  $F_{wp} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

The particular structure of this mechanical system makes easy the computation of the feedback canonical form. With

$$W = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & M_{wp} & 0 \\ 0 & 0 & I_2 \end{bmatrix}, \quad V = I_5, \quad F^T = \begin{bmatrix} K_{wp}^T \\ D_{wp}^T \\ -F_{wp} \end{bmatrix}^T, \quad G = S_{wp}^{-1} M_{wp}$$

one can easily found that it has just 5 invariants,  $\gamma_1 = 2$ ,  $\gamma_2 = 2$ ,  $\sigma_1 = 0$ ,  $\sigma_2 = 0$ ,  $n_1 = 3$ ,  $n_2 = 3$ .

Since  $\text{card}\{i/\zeta_i \geq 2\} = 0$ , the system is  $C$ -solvable. But since  $\text{card}\{i/n_i \geq 2\} = 2$ , the system is not  $C$ -solvable in the function sense or equivalently it is not impulse controllable (and consequently it is not strongly controllable and not controllable). This means that there exist some initial conditions for which there does not exist smooth input ( $u \in \mathcal{C}_{sm}^m$ ) such that the state solution is smooth. In other words, for some initial conditions impulsive behavior is unavoidable. This statement confirms the result of Mills and Goldenberg (1989) (see (61) which says that "to exhibit no impulsive behavior, initial conditions should belong to a particular subset"; more development, here, would have shown that  $\delta x(0)=0$  and  $\delta \phi(0)=0$  should be strictly respected. This statement also confirms the statement of Loiseau and Zagalak (2009): "the impulse behavior of this system cannot be removed by state feedback".

## 6. CONCLUSION

Some properties of solvability, consistency of initial condition and controllability have been listed for continuous linear time invariant descriptor systems. For each of them the known definition and known characterizations in terms of a matrix pencil, space inclusions or geometric algorithms has been given. The new point here is the unified simple characterization which have been given in terms of the invariants of the feedback canonical form of  $(E, A, B)$  triple. An example taken from the literature illustrates the idea that for mechanical systems with constraints these characterizations can be obtained easily without numerical code to compute the canonical form or its invariants.

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