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On the algebraic structure of isotropic generalized elasticity theories

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Abstract

In this paper the algebraic structure of the isotropic n th-order gradient elasticity is investigated. In the classical isotropic elasticity it is well-known that the constitutive relation can be broken down into two uncoupled relations between elementary part of the strain and the stress tensors (deviatoric and spherical). In this paper we demonstrate that this result can not be generalized and since 2nd-order isotropic elasticity there exist couplings between elementary parts of higher-order strain and stress tensors. Therefore, and in certain way, n th-order isotropic elasticity have the same kind of algebraic structure as anisotropic classical elasticity. This structure is investigated in the case of 2nd-order isotropic elasticity, and moduli characterizing the behavior are provided.

Dedicated to Prof. Antonio Di Carlo in recognition of his academic activity

1 Introduction

In the last years it has been widely recognized that classical continuum mechanics was unable to describe a variety of important mechanical and physical phenomena. In particular, the size effects and non-local behaviors due to the discrete nature of matter at a sufficiently small scale, the presence of microstructural defects or the existence of internal constraints cannot be captured by classical continuum mechanics. Higher gradient models are also needed when continuum models are introduced for describing systems in which strong inhomogeneities of physical properties are present at eventually different length scales (see e.g. [1, 35, 33, 40, 42, 43, 44]), in continuum systems in which some *microscopical* degrees of freedom can *capture* a relevant amount of deformation energy (see e.g. [10, 9]), in fracture (see e.g. [13, 22]), and so on. In [28, 14] second strain-gradient elasticity, was investigated by Mindlin who shows that a second gradient of strain is needed to describe, in a continuous manner, capillarity and cohesion effects in elastic continuum.

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Furthermore, it has recently been proved, that some microstructure can be specifically designed to render any higher-grade effects predominant [2]. Aside from strain-gradient theories, gradient effects have recently been considered both in the context of plate theory [23, 24], and in the context of stress-gradient [15].

In the present contribution n th-order strain-gradient isotropic (SO(3)-invariance) elasticity is considered. By n th-grade elastic materials we mean those materials whose mechanical response depends on the present value of the first n deformation-gradients [11, 34]: for $n = 1$ classical elasticity tensor is retrieved, while for case $n = 2$ Mindlin's strain-gradient elasticity [27, 26, 12, 29, 5] is obtained. At the present time, few results (see e.g.[11, 31, 32]) concerning a general theory of n th-gradient elastic material are known. Therefore a general picture of this extended theory has to be drawn. The aim of the present paper is to provide a general result concerning the algebraic structure of isotropic n th-order elasticity tensors. More specifically we will show that, contrary to classical elasticity, for n th-order isotropic elasticity the harmonic parts¹ of higher-order strain and stress tensors are always coupled. In classical elasticity these kinds of coupling appear for anisotropic classes greater or equal to transverse isotropy [41]. Therefore, as already observed in the case of strain-gradient elasticity [29], the algebraic structure of n th-order elasticity tensors is very similar to anisotropic classical elasticity tensors.

The paper is organized as follows In a first section the general context of our study will be presented. After introducing the framework of n th-gradient elastic materials, the harmonic decomposition, which is a central tool in our study, is presented together with a method to easily compute its structure. In a second time, the Walpole and Kelvin tensor decompositions are defined and their links investigated. **This point is important since these two decompositions seem to be often merged.** The section is concluded by stating (without proofs at this stage) our main results. In the next section (§.3) two specific situations are studied: the classical and the second-order elasticity. For the second-order elasticity, and following [41, 29], an intrinsic representation, based on a kinematic interpretation of strain-gradient tensor is provided. This interpretation give rises to 5 moduli associated with *elementary* mechanisms. The last section is devoted to the proofs of our results. In appendices some details and explicit constructions concerning second-order elasticity tensors are provided.

Notations Let us define some notations that will be used throughout the paper, other less important one will be introduced in the core of the text. The following matrix groups will be considered in the present paper :

- GL(3): the group of all the invertible transformations of \mathbb{R}^3 , i.e. if $F \in \text{GL}(3)$ then $\det(F) \neq 0$;
- O(3) : the orthogonal group, i.e. the group of all isometries of \mathbb{R}^3 i.e. if $Q \in \text{O}(3)$ $\det(Q) \pm 1$ and $Q^{-1} = Q^T$, where the superscript T denotes the transposition.
- SO(3) : the special orthogonal group, i.e. the subgroup of O(3) of elements satisfying $\det(Q) = 1$. This is the group of 3D rotations;
- \mathcal{O} : the octahedral group, i.e. the group of rotations that let a cube invariant.

¹As detailed latter on, the harmonic decomposition is the correct generalization of the deviatoric and spheric decomposition of a second-order symmetric tensor.

Vector spaces will be denoted using blackboard fonts, and their tensorial order indicated by using formal indices. Generic tensor spaces will be denoted \mathbb{T} , and space of harmonic tensors k -th order harmonic tensor \mathbb{H}^k . Harmonic tensor spaces are $\text{SO}(3)$ -invariant, their elements are *completely symmetric* and *traceless* k -th order tensors [6, 21, 16]. In 3D we have $\dim(\mathbb{H}^k) = 2k + 1$, **algorithms to compute the harmonic decomposition can be found in the following references**[37, 20, 3, 4, 38]. The precise meaning of these spaces will be detailed in the text. When needed, index symmetries of both space and their element are expressed as follows: (...) indicates invariance under permutation of the indices in parentheses, and $\underline{\quad}$ indicates invariance with respect to permutations of the underlined blocks.

In the following, the tensorial product of \mathbb{R}^3 is denoted \otimes , while \otimes^k indicates the k -order tensor product and S^k its completely symmetrized version. Classically the spatial derivation will be denoted by ∇ .

2 Setting of the problem and main results

2.1 n th-gradient elastic materials

Primary and dual quantities

In n th grade elasticity, the mechanical response is supposed to depend on the first n deformation-gradients. Those quantities are defined recursively

$$\begin{cases} \varepsilon^{(1)} = \varepsilon \\ \varepsilon^{(k+1)} = \varepsilon^{(k)} \otimes \nabla \end{cases}$$

where ε and $\varepsilon^{(k)}$ denote, respectively, the classical symmetric strain tensor and the k th-order strain tensor. Due to the fact that ε is a second-order symmetric tensor, $\varepsilon^{(k)}$ is a $(k + 1)$ th-order tensor that belongs to the space $\mathbb{T}^{(k)}$ defined as follows:

$$\mathbb{T}^{(k)} = S^2(\mathbb{R}^3) \times S^{k-1}(\mathbb{R}^3) \subseteq \otimes^{k+1}\mathbb{R}^3 \quad (1)$$

Elements of $\mathbb{T}^{(k)}$ are $(k + 1)$ th-order tensors symmetric with respect to their 2 first indices and their $k - 1$ last. Using the convention for index symmetries defined in the introduction, elements of $\mathbb{T}^{(k)}$ have the following shape:

$$\mathbb{T}_{(i_1 i_2)(i_3 \dots i_{k-1})}$$

By duality $\sigma^{(k)}$ is defined as the energetic conjugate of $\varepsilon^{(k)}$, and for $k = 1$ the symmetric Cauchy stress tensor is retrieved. It is worth noting that the $\sigma^{(k)}$ is not the gradient of $\sigma^{(k-1)}$.

Constitutive law

The object of our study concerns the following n th-order elasticity relation:

$$\sigma^{(n)} = \mathbb{C}^{(n)} \varepsilon^{(n)}$$

where $\mathbb{C}^{(n)}$ is a $2(n+1)$ th-order tensor. It is worth noting that this tensorial law is just a part of the complete n th-order gradient elasticity law which also contains coupling terms with the other deformation gradients [27, 28]. Therefore, in this paper, *nth-order elasticity relation* only refers to this part of the complete n th-order gradient elasticity law. For $n = 1$ the classical Hooke's law is obtained, and for $n = 2$ we obtained the second-order elasticity studied, for example, in [12, 5]. The vector space of $\mathbb{C}^{(n)}$ is defined as the space of symmetric endomorphism of $\mathbb{T}^{(n)}$. In terms of tensorial products, this space is constructed as

$$\mathbb{C}^{(n)} = S^2(\mathbb{T}^{(n)}) \subseteq \otimes^{2(n+1)} \mathbb{R}^3$$

Hence elements of $\mathbb{C}^{(n)}$ inherit the minor symmetries of $\sigma^{(n)}$ and $\varepsilon^{(n)}$, completed by the major one. In other terms, elements of $\mathbb{C}^{(n)}$ have the following shape:

$$\mathbb{C}_{\underline{(i_1 i_2)(j_1 \dots j_{n-1})}} \underline{(k_1 k_2)(l_1 \dots l_{n-1})}$$

For our study, we need to break tensor spaces into elementary building blocks. The definition of these elementary spaces depends on the considered group action. In the present situation, tensor spaces will be decomposed into $\text{SO}(3)$ -invariant spaces. This decomposition is the (correct) generalization of the decomposition of second-order symmetric tensors into deviatoric and spheric parts.

2.2 The harmonic decomposition

The harmonic decomposition has been widely used in the mechanical community for studying anisotropic elasticity [6, 8, 16, 17]. In order to present our results in self-contained way, basic definitions concerning this decomposition are summed up here. A more general and rigorous presentation can be found in [21, 38], **and some historical considerations concerning its uses in mechanics in [16]**. In \mathbb{R}^3 , under $\text{SO}(3)$ -action any tensor space \mathbb{V} can be decomposed orthogonally into a sum of harmonic tensor spaces of different orders²:

$$\mathbb{V} = \bigoplus_{i=0}^p \alpha_i \mathbb{H}^i$$

where p indicates the tensorial order of \mathbb{V} , \mathbb{H}^i is the vector space of i th-order harmonic tensors and α_i indicates the number of copies of \mathbb{H}^i in the decomposition. Elements of \mathbb{H}^i are i th-order *completely symmetric traceless* tensors, the dimension of their vector space is $2i + 1$. The denomination harmonic is related to a classical isomorphism in \mathbb{R}^3 between the space of harmonic polynomials (i.e. polynomials with null Laplacian) of degree i and the space of i th-order completely symmetric traceless tensors. If needed, an explicit isomorphism, denoted h in the following, can be computed using, for example, an algorithm proposed by Spencer [37, 20]. The following property is important [19]

Property 2.1. *The isomorphism h that realizes the harmonic decomposition is uniquely defined iff $\alpha_i \leq 1$ for $i \in [0, p]$.*

According to the problem under investigation, the explicit knowledge of h may be required or not. This point is important because the explicit computation of an isomorphism is a task which complexity increases very quickly with the tensorial order. At

²In the harmonic decomposition of a tensor space, the equality sign means that there exists an isomorphism h between the right- and the left-hand side of the decomposition. In order to avoid the use of too many notations we do not use a specific sign to indicate this isomorphism.

the opposite, the determination of the structure of the decomposition, i.e. the number of spaces together with their multiplicities is almost straightforward using the Clebsch-Gordan product.

2.3 Clebsch-Gordan product

For our need, in order to determine the algebraic structure of n th-order elasticity tensors, we only need to know the structure of the harmonic decomposition. To compute it we use the tensorial product of group representations. More details can be found in [21, 3]. The computation rule is simple. Consider two harmonic tensor spaces \mathbb{H}^i and \mathbb{H}^j , whose product space is noted $\mathbb{T}^{i+k} := \mathbb{H}^i \otimes \mathbb{H}^j$. This space admits the following SO(3)-invariant decomposition:

$$\mathbb{G}^{i+j} = \mathbb{H}^i \otimes \mathbb{H}^j = \bigoplus_{k=|i-j|}^{i+j} \mathbb{H}^k \quad (2)$$

In the following this rule will be referred to as the Clebsch-Gordan product. For example, consider \mathbb{H}_a^1 and \mathbb{H}_b^1 two different first-order harmonic spaces. Elements of such spaces are vectors. According to formula (2), the SO(3)-invariant decomposition of \mathbb{T}^2 is:

$$\mathbb{G}^2 = \mathbb{H}_a^1 \otimes \mathbb{H}_b^1 = \mathbb{H}^2 \oplus \mathbb{H}^1 \oplus \mathbb{H}^0$$

As an example, the tensorial product of two vector spaces generates a second-order tensor space. The resulting structure is composed of a scalar (\mathbb{H}^0), a vector (\mathbb{H}^1) and a deviator (\mathbb{H}^2). This computation rule has to be completed by the following property [21]

Property 2.2. *The decomposition of an even-order (resp. odd-order) completely symmetric tensor, i.e. invariant under any index permutation, only contains even-order (resp. odd-order) harmonic spaces.*

Applying these rules, the harmonic structure of $\mathbb{T}^{(n)}$ (the space of n th-order stress and strain tensors), is easily determined. Results for low order spaces are provided in the following table:

Vector space	Elements	Harmonic decomposition	Dimension	Unicity
\mathbb{T}^1	$\varepsilon_{(ij)}$	$\mathbb{H}^2 \oplus \mathbb{H}^0$	6	Y
\mathbb{T}^2	$\varepsilon_{(ij),k}$	$\mathbb{H}^3 \oplus \mathbb{H}^2 \oplus 2\mathbb{H}^1$	18	N
\mathbb{T}^3	$\varepsilon_{(ij),(kl)}$	$\mathbb{H}^4 \oplus \mathbb{H}^3 \oplus 3\mathbb{H}^2 \oplus \mathbb{H}^1 \oplus 2\mathbb{H}^0$	36	N

It can be observed, on these examples, that, according to the property 2.1, \mathbb{T}^1 is the only space having an uniquely defined harmonic decomposition. This fact will be important for the forthcoming discussion.

2.4 Tensor decompositions

In the following³ two different tensor decompositions will be discussed, the *Spectral or Kelvin's decomposition* and the *Harmonic-induced or Walpole's decomposition*. To that aim, some orthonormal bases have to be introduced. On \mathbb{T} three different bases will be considered:

³In this subsection all (n) superscripts will be dropped down to enlighten notations.

- \mathcal{S} : the spatial-basis, constructed by tensor products of the canonical one (see Appendix A for an explicit example);
- $\mathcal{E}(C)$: the eigen-basis, constructed from the eigenvectors of an element $C \in \mathbb{T} \otimes^S \mathbb{T}$, the notation keeps track of that dependence;
- $\mathcal{H}(\mathbb{T})$: the harmonic-basis, constructed from an $\text{SO}(3)$ -irreducible spaces decomposition of \mathbb{T} , the notation keeps track of that dependence.

Similarly on $\mathbb{T} \otimes^S \mathbb{T}$ the following bases will be considered:

- $\mathcal{S} \otimes^S \mathcal{S}$: in this basis the components of C are denoted $C_{\mathcal{S}}$. This corresponds to the classical matrix representation of a constitutive tensor [25, 5];
- $\mathcal{E}(C) \otimes^S \mathcal{E}(C)$: in this basis the components of C are denoted $C_{\mathcal{E}}$. In this basis the matrix of C is diagonal. This representation correspond to the *Spectral or Kelvin's decomposition*, which has been widely used in the context of classical elasticity (see e.g. [36, 25, 7]);
- $\mathcal{H}(\mathbb{T}) \otimes^S \mathcal{H}(\mathbb{T})$: in this basis the components of C are denoted $C_{\mathcal{H}}$. The associated matrix representation has been less investigated in the literature, but it can be associated with some constructions of Walpole [41, 29, 30]. Hence this decomposition will be referred to as the *Harmonic-induced or Walpole's decomposition*⁴.

The links that relate the different bases are indicated in the following diagram:

$$\begin{array}{ccccccc}
\forall C \in \mathbb{T} \otimes^S \mathbb{T}, & \mathcal{S} \otimes^S \mathcal{S} & \xrightarrow{d} & \mathcal{E}(C) \otimes^S \mathcal{E}(C) & \rightsquigarrow & \mathcal{E}(C) & \\
& \searrow^{\phi(h)} & & \uparrow^{d^*} & & \uparrow^{\psi(d^*)} & \\
& & & \mathcal{H}(\mathbb{T}) \otimes^S \mathcal{H}(\mathbb{T}) & \leftarrow \rightsquigarrow & \mathcal{H}(\mathbb{T}) & \xleftarrow{h} \mathcal{S}
\end{array}$$

where d and d^* indicate diagonalization processes, $\psi(d)$ and $\psi(d^*)$ the related changes of basis, and $\phi(h)$ the change of basis induced by the harmonic decomposition h of \mathbb{T} . In this diagram the wavy arrows indicate that the change of basis in the destination space is induced by an operation in the original space. As a consequence the different components of C are related in the following way:

$$\begin{array}{ccc}
& C_{\mathcal{E}} & \\
d \nearrow & & \uparrow^{d^*} \\
C_{\mathcal{S}} & \xrightarrow{\phi(h)} & C_{\mathcal{H}}
\end{array}$$

It has to be observed that the former diagram commutes, that is:

$$d = d^* \circ \phi(h)$$

Therefore $C_{\mathcal{H}}$ maybe considered as an intermediary step in the transformation from $C_{\mathcal{S}}$ to $C_{\mathcal{E}}$.

⁴To be more exact, this decomposition corresponds to the isotropic Walpole decomposition, since Walpole decomposition can be associated with any G -invariant decomposition of \mathbb{T} , in which G is a subgroup of $\text{SO}(3)$.

The basis $\mathcal{H}(\mathbb{T})$ is determined by the $\text{SO}(3)$ -structure of \mathbb{T} , while $\mathcal{E}(C)$ is specific to each element of $\mathbb{T} \otimes^S \mathbb{T}$. As a consequence the isotropic Walpole's decomposition is determined by the $\text{SO}(3)$ -structure⁵ of \mathbb{T} meanwhile the Kelvin's decomposition is specific to each element of $\mathbb{T} \otimes^S \mathbb{T}$.

Therefore there is an intrinsic dissymmetry between these two decompositions. But as in some specific situations they coincide, these decompositions are often merged and the Walpole's one is sometimes referred to as spectral, which is false. For classical elasticity, these decompositions are identical only for both the isotropic and the cubic systems. Furthermore, and as it will be demonstrated, the Walpole representation for higher-order isotropic elasticity is no more diagonal.

2.5 Main results

In §.4 the following theorem is proved:

Theorem 2.3 (Theorem 1). *If the harmonic decomposition of \mathbb{T} is unique, and if C is isotropic then its matrix representation $C_{\mathcal{H}}$ is diagonal. In such a case d^* reduces to a permutation and the Walpole and the Kelvin decompositions are equivalent.*

This theorem can be reformulated in another way. To that aim let $[\mathbb{T} \otimes^S \mathbb{T}]^{\text{SO}(3)}$ denotes the subspace of isotropic elements of $\mathbb{T} \otimes^S \mathbb{T}$, and $\sharp^{\text{SO}(3)}(\mathbb{T})$ the number of harmonic components in the decomposition of \mathbb{T} . We have the alternative formulation to Theorem 2.3:

Theorem 2.4 (Theorem 1'). *Let consider $C \in [\mathbb{T} \otimes^S \mathbb{T}]^{\text{SO}(3)}$, the Walpole and the Kelvin representation of C are equivalent iff*

$$\dim([\mathbb{T} \otimes^S \mathbb{T}]^{\text{SO}(3)}) = \sharp^{\text{SO}(3)}(\mathbb{T})$$

Even if we did not investigate in that direction in the present paper, we may conjecture that the Theorem 1' can be extended to anisotropic situations, i.e.

Conjecture 2.5. *Let consider $C \in [\mathbb{T} \otimes^S \mathbb{T}]^G$, the space of G -invariant symmetric endomorphisms of \mathbb{T} and $\sharp^G(\mathbb{T})$ be the number of G -irreducible components in the decomposition of \mathbb{T} . The Walpole and the Kelvin representation of C are equivalent iff*

$$\dim([\mathbb{T} \otimes^S \mathbb{T}]^G) = \sharp^G(\mathbb{T})$$

For example this situation occurs for cubic classical elasticity in which the 3 different coefficients correspond to the number of \mathcal{O} -irreducible spaces of \mathbb{T}^1 . As well-known [41] in this situation the Walpole and the Kelvin's decomposition coincide.

Now we have to determine in which situations the condition of the former theorems is verified. The answer is given by the following theorem

⁵Or more generally a G -invariant decomposition for anisotropic tensors, in which G is a subgroup of $\text{SO}(3)$.

Theorem 2.6 (Theorem 2). *Let \mathbb{T} be a n th-order tensor space only endowed with minor symmetries, then the harmonic decomposition of \mathbb{T} is unique or, equivalently:*

$$\dim([\mathbb{T} \otimes^S \mathbb{T}]^{\text{SO}(3)}) = \#^{\text{SO}(3)}(\mathbb{T})$$

if $\mathbb{T} = S^n(\mathbb{R}^3)$, $n \geq 0$ or $\mathbb{T} = \otimes^2(\mathbb{R}^3)$.

A direct application of Theorem 2.4 shows that only \mathbb{T}^1 meets this requirement. Therefore for n th-gradient elasticity, Theorem 2.3 is true only for first order elasticity, and since $n = 2$ there exist couplings between harmonic components of higher order stresses and strains. In classical elasticity this kind of coupling appears for transverse isotropy and lower symmetry classes (except the cubic one). In other terms, n th-order isotropic elasticity have the same kind of algebraic structure as anisotropic classical elasticity.

3 Illustrations : First and second-order elasticity

In this section some practical illustrations of our results will be given. For second-order elasticity this explicit construction gives insights into the physic encoded by the second-order constitute law. As a result, we obtain the explicit expressions of five moduli, four of them related to *elementary physical mechanisms* while the last one is a coupling modulus between these mechanisms. In order not to lengthen too much the section, some explicit constructions for second-order elasticity, such as the 18×18 passage matrix, are postponed to the appendix.

3.1 $n = 1$: Classical elasticity

In classical elasticity, the constitutive relation reads⁶

$$\sigma = \mathbb{C} : \varepsilon, \quad \sigma, \varepsilon \in \mathbb{T}^1 = \mathbb{T}_{(ij)}, \quad \mathbb{C} \in S^2(\mathbb{T}^{(1)}) = \mathbb{T}_{(ij)(kl)} \quad (3)$$

As well-known the isotropic elasticity tensor is defined by 2 coefficients, and $\mathbb{T}_{(ij)}$ contains two harmonic spaces. Hence Theorem 1' applies and the Walpole and the Kelvin's decomposition coincide. It is known that, for isotropic behavior, \mathbb{C} can be expressed in the following operational form:

$$\mathbb{C} = 3K\mathbb{P}^{\mathbb{H}^0} + 2G\mathbb{P}^{\mathbb{H}^2} \quad (4)$$

where K and G are respectively known as the bulk and the shear modulus, while $\mathbb{P}^{\mathbb{H}^0}$ and $\mathbb{P}^{\mathbb{H}^2}$ are the hydrostatic and the deviatoric projectors. Therefore $\mathbb{P}^{\mathbb{H}^0}$ and $\mathbb{P}^{\mathbb{H}^2}$ are intrinsic basis vectors for isotropic classical elasticity tensors. These basis vectors satisfy the following properties:

- Partition of the identity⁷:

$$\text{Id}_{\mathbb{T}^1} = \mathbb{P}^{\mathbb{H}^2} + \mathbb{P}^{\mathbb{H}^0}$$

- Multiplication table:

\times	$\mathbb{P}^{\mathbb{H}^2}$	$\mathbb{P}^{\mathbb{H}^0}$
$\mathbb{P}^{\mathbb{H}^2}$	$\mathbb{P}^{\mathbb{H}^2}$	0
$\mathbb{P}^{\mathbb{H}^0}$	0	$\mathbb{P}^{\mathbb{H}^0}$

⁶In this subsection, to avoid cumbersome notations, superscripts that indicate the order of relations are dropped down.

⁷In this expression $\text{Id}_{\mathbb{T}^1}$ denotes the identity operator for the space \mathbb{T}^1 .

According to these properties the constitutive law (3) can be broken down into two uncoupled relations

$$\begin{cases} \sigma^{\mathbb{H}^2} = 2G\varepsilon^{\mathbb{H}^2} \\ \sigma^{\mathbb{H}^0} = 3K\varepsilon^{\mathbb{H}^0} \end{cases} \quad (5)$$

This decomposition of classical elasticity is well-known and has been established by numerous methods in the literature (see e.g. [36, 41, 25, 7]). If we consider shear and spherical strains (resp. stresses) as elementary strains (resp. stresses), this decomposition means that for isotropic system these mechanisms are not coupled. From a computational point of view, this representation has many advantages since the inversion of the constitutive matrix is direct [41, 29].

The following observation is important: *the system (5) means that the constitutive equation (3) expressed in the basis induced by the harmonic decomposition of \mathbb{T}^1 is diagonal.* Therefore in this case the Walpole and Kelvin decompositions coincide. This fact was indeed clear from the Theorem 1'.

3.2 $n = 2$: Second-order elasticity

It has been observed that for classical isotropic elasticity elementary parts of stress and strain are not coupled. Let consider now the linear relation of second order elasticity:

$$\tau = A \cdot \eta, \quad \tau, \eta \in \mathbb{T}^2 = \mathbb{T}_{(ij)k}, \quad A \in S^2(\mathbb{T}^{(2)}) = \mathbb{T}_{(ij)k} \underline{(lm)n}$$

where τ, A, η stand respectively for $\sigma^{(2)}, C^{(2)}, \varepsilon^{(2)}$. Therefore both τ and η are third-order tensors, while A is a sixth-order one.

\mathbb{T}^2 harmonic decomposition

As we are interested in the Walpole decomposition of A , the first point is to analyze the harmonic decomposition of both η and τ . As calculated in the table (2.3), \mathbb{T}^2 admits the following harmonic decomposition:

$$\mathbb{T}^2 \cong \mathbb{H}^3 \oplus \mathbb{H}^2 \oplus \mathbb{H}^{1,a} \oplus \mathbb{H}^{1,b} \quad (6)$$

It can be observed that any element of this space contains 2 vectorial parts, therefore the isomorphism associated with the decomposition is not unique. From Theorem 1 we know that the Walpole decomposition will not be diagonal and, contrary to classical elasticity, isotropic elementary couplings between harmonic spaces will occur. This coupling only concerns the vector parts since both \mathbb{H}^3 and \mathbb{H}^2 are uniquely defined. Therefore many constructions are possible, but among them some are more natural since they give a physical interpretation to the harmonic decomposition. In the following we will only consider a kinematic interpretation of the decomposition (see [4] for another interpretations). The approach consists in splitting $T_{(ij)k}$ first into a complete symmetric part and a remainder one before proceeding to the harmonic decomposition. This approach is summed-up by

the following diagram:

$$\begin{array}{ccc}
& \mathbb{T}_{(ij)k} & \\
\text{Sym} \swarrow & & \searrow \text{Asym=Id-Sym} \\
\mathbb{S}_{(ijk)} & & \mathbb{R}_{ij} \\
\downarrow \mathcal{H} & & \downarrow \mathcal{H} \\
\mathbb{H}^3 \oplus \mathbb{H}_a^1 & & \mathbb{H}^{\#2} \oplus \mathbb{H}_b^1
\end{array}$$

where Sym, Asym and \mathcal{H} respectively stand for the symmetrization, anti-symmetrization and the harmonic decomposition processes. $T_{(ij)k}$ is first split into a full symmetric tensor and an asymmetric one:

$$T_{(ij)k} = S_{ijk} + \frac{1}{3} (\epsilon_{jkl} R_{li} + \epsilon_{ikl} R_{lj})$$

where ϵ_{ijk} denotes the Levi-Civita symbol in 3D. The space of full symmetric third-order tensors is 10-dimensional meanwhile the space of the remaining one is 8 dimensional, those spaces are in direct sum. In the strain-gradient literature [27] the complete symmetric part $S_{(ijk)}$, defined:

$$S_{(ijk)} = \frac{1}{3} (T_{(ij)k} + T_{(ki)j} + T_{(jk)i})$$

is related to the *stretch-gradient* part of $T_{(ij)k}$. The remaining traceless non-symmetric part R_{ij} :

$$R_{ij} = \epsilon_{ipq} T_{(jp)q}$$

is the *rotation-gradient* part of $T_{(ij)k}$. At this stage, this decomposition coincides with the type III formulation of Mindlin strain gradient elasticity [26]. According to its words, *this third form of the theory is the most convenient one for reduction to the theory in which the potential energy-density is a function of the strain and the gradient of the rotation*. The terms $V^{\nabla \text{rot}}$ is the components of the strain gradient tensor that give rise to couple-stress [27].

This first decomposition, which is sometimes referred to as the Schur decomposition, is $\text{GL}(3)$ -invariant, meaning that each component is $\text{GL}(3)$ -irreducible. In other terms, this decomposition of the strain-gradient tensors into two *mechanisms* (stretch-gradient and rotation-gradient) is preserved under any invertible transformation. Under $\text{SO}(3)$ -action each part can further be decomposed into harmonic components by removing their different traces:

- $\mathbb{S}_{(ijk)}$ splits into a 3rd-order deviator ($\dim \mathbb{H}^3 = 7$) and a vector ($\dim \mathbb{H}_a^1 = 3$);
- \mathbb{R}_{ij} splits into a (pseudo-)deviator ($\dim \mathbb{H}^2 = 5$) and a vector ($\dim \mathbb{H}_b^1 = 3$).

Stretch-gradient tensors:

The space $\mathbb{S}_{(ijk)}$ is isomorphic to $\mathbb{H}^3 \oplus \mathbb{H}_{\nabla \text{str}}^1$, hence the isomorphism that realizes the decomposition is unique. Doing some algebra we obtain

$$S_{(ijk)} = H_{(ijk)} + \frac{1}{5} (V_i^{\nabla \text{str}} \delta_{(jk)} + V_j^{\nabla \text{str}} \delta_{(ik)} + V_k^{\nabla \text{str}} \delta_{(ij)})$$

with

$$V_i^{\nabla \text{str}} = S_{(pp)i} = \frac{1}{3} (T_{ppi} + 2T_{ipp}) ; H_{(ijk)} = S_{(ijk)} - \frac{1}{5} (V_i^{\nabla \text{str}} \delta_{(jk)} + V_j^{\nabla \text{str}} \delta_{(ik)} + V_k^{\nabla \text{str}} \delta_{(ij)})$$

In this interpretation $V^{\nabla str}$ is the vector part of *the stretch-gradient tensor*.

Rotation-gradient tensors:

The space \mathbb{R}_{ij} is isomorphic to $\mathbb{H}^2 \oplus \mathbb{H}_{\nabla rot}^1$, hence the isomorphism that realizes the decomposition is unique. Doing some algebra we obtain

$$R_{ij} = H_{(ij)} + \epsilon_{ijp} V_p^{\nabla rot}$$

with

$$V_i^{\nabla rot} = \frac{1}{2} \epsilon_{ipq} (R_{pq} - R_{qp}) = \frac{1}{2} (T_{ppi} - T_{ipp}) ; H_{(ij)} = R_{ij} - \frac{1}{2} \epsilon_{ijp} V_p^{\nabla rot} = \frac{1}{2} (R_{pq} + R_{qp})$$

In this formulation $V^{\nabla rot}$ is the vector part of *the rotation-gradient tensor*, and is embedded in the third-order tensor in the following way:

$$T(V^{\nabla rot})_{ijk} = \frac{1}{3} (-V_i^{\nabla rot} \delta_{(jk)} - V_j^{\nabla rot} \delta_{(ik)} + 2V_k^{\nabla str} \delta_{(ij)})$$

Therefore a physical meaning can be given to the harmonic decomposition of $\mathbb{T}^{(2)}$, since the strain-gradient tensor encodes two orthogonal effects: stretch-gradient and rotation-gradient. These effects are canonically defined and preserved under invertible changes of variables. The harmonic decomposition of these elementary effects correspond to their decomposition in spherical harmonics [21].

Walpole decomposition of second-order elasticity tensor

Using the expressions of $H_{(ijk)}$, $V_i^{\nabla str}$, R_{ij} , $V_i^{\nabla rot}$ one can easily build a transformation matrix P from $\mathcal{S}(\mathbb{T}^2)$, the spatial basis of \mathbb{T}^2 , to $\mathcal{H}(\mathbb{T}^2)$, a basis related to its harmonic decomposition (see the Appendix for the explicit expressions of the chosen basis vectors). As the decomposition is not unique the result depends on the physical interpretation chosen for the vector parts. In this new basis, A has the following expression:

$$A_{\mathcal{H}(\mathbb{T}^2)} = \begin{pmatrix} m_{s_3} \text{Id}_7 & 0 & 0 & 0 \\ 0 & m_{s_1} \text{Id}_3 & 0 & m_{c_1} \text{Id}_3 \\ 0 & 0 & m_{r_2} \text{Id}_5 & 0 \\ 0 & m_{c_1} \text{Id}_3 & 0 & m_{r_1} \text{Id}_3 \end{pmatrix}_{\mathcal{H}(\mathbb{T}^2)}$$

where Id_k indicates the k th-order identity matrix. In this particular basis, it clearly appears that the isotropic behavior is coupled. Hence contrary to classical elasticity (both in 2D and 3D), in the isotropic system, the matrix of A is not diagonal. Therefore, in the generic situation, a coupling always exists between $\mathbb{H}_{\nabla str}^1$ and $\mathbb{H}_{\nabla rot}^1$. A physical consequence is that, for example, even for isotropic material, pure stretch-gradient generates couple-stress [27, 18]. This coupling will obviously disappear if the coupling modulus m_{c_1} is equal to 0.

Using this matrix representation, and as usually done in classical elasticity (c.f. relation (4)), A can be rewritten in an operational intrinsic form. To that aim let us consider $\{h_i\}$ the orthonormal basis of $\mathcal{H}(\mathbb{T}^2)$, this basis is defined by the concatenation of the

orthonormal bases of, respectively, \mathbb{H}^3 , $\mathbb{H}^{1,\nabla str}$, \mathbb{H}^2 and $\mathbb{H}^{1,\nabla rot}$. From this basis let define the three following sets:

$$\{H_i\} = \{h_i \otimes h_i\}, \quad \{K_i\} = \{h_{i+7} \otimes h_{i+15}\}, \quad \{K_i^T\} = \{h_{i+15} \otimes h_{i+7}\}$$

From the first set we can define the following projectors:

$$P^{\mathbb{H}^3} := \sum_{i=1}^7 H_i, \quad P^{\mathbb{H}^{1,\nabla str}} := \sum_{i=8}^{10} H_i, \quad P^{\mathbb{H}^2} := \sum_{i=11}^{15} H_i, \quad P^{\mathbb{H}^{1,\nabla rot}} := \sum_{i=16}^{18} H_i$$

These projectors satisfy the following property:

$$\text{Id}_{\mathbb{T}^2} = P^{\mathbb{H}^3} + P^{\mathbb{H}^{1,\nabla str}} + P^{\mathbb{H}^2} + P^{\mathbb{H}^{1,\nabla rot}}$$

To form a basis for isotropic elements of $\mathbb{C}^{(2)}$, the projectors have to be completed by the two following coupling operators⁸:

$$Q^{\nabla str/\nabla rot} := \sum_{i=1}^3 K_i, \quad Q^{\nabla rot/\nabla str} := \sum_{i=1}^3 K_i^T$$

Hence, any second order elasticity tensor can be rewritten in terms of these operators:

$$A = m_{s_3} P^{\mathbb{H}^3} + m_{s_1} P^{\mathbb{H}^{1,\nabla str}} + m_{c_1} (Q^{\nabla str/\nabla rot} + Q^{\nabla rot/\nabla str}) + m_{r_2} P^{\mathbb{H}^2} + m_{r_1} P^{\mathbb{H}^{1,\nabla rot}}$$

This relation is analogous to the relation (4) in classical elasticity. Now, we can compute the multiplication table of the basis ($P^{\mathbb{H}^3}$, $P^{\mathbb{H}^2}$, $P^{\mathbb{H}^{1,\nabla str}}$, $Q^{\nabla str/\nabla rot}$, $Q^{\nabla rot/\nabla str}$, $P^{\mathbb{H}^{1,\nabla rot}}$):

\times	$P^{\mathbb{H}^3}$	$P^{\mathbb{H}^2}$	$P^{\mathbb{H}^{1,\nabla str}}$	$Q^{\nabla str/\nabla rot}$	$Q^{\nabla rot/\nabla str}$	$P^{\mathbb{H}^{1,\nabla rot}}$
$P^{\mathbb{H}^3}$	$P^{\mathbb{H}^3}$	0	0	0	0	0
$P^{\mathbb{H}^2}$	0	$P^{\mathbb{H}^2}$	0	0	0	0
$P^{\mathbb{H}^{1,\nabla str}}$	0	0	$P^{\mathbb{H}^{1,\nabla str}}$	$Q^{\nabla str/\nabla rot}$	0	0
$Q^{\nabla str/\nabla rot}$	0	0	0	0	$P^{\mathbb{H}^{1,\nabla str}}$	$Q^{\nabla str/\nabla rot}$
$Q^{\nabla rot/\nabla str}$	0	0	$P^{\mathbb{H}^{1,\nabla rot}}$	$Q^{\nabla rot/\nabla str}$	0	0
$P^{\mathbb{H}^{1,\nabla rot}}$	0	0	0	0	$P^{\mathbb{H}^{1,\nabla rot}}$	$Q^{\nabla rot/\nabla str}$

This multiplication table corresponds to the irreducible algebra of second-order elasticity tensor as identified by Monchiet and Bonnet in [29]. As a consequence the constitutive law (3.2) can be broken down into four relations

$$\begin{cases} \tau^{\mathbb{H}^3} = m_{s_3} \eta^{\mathbb{H}^3} \\ \tau^{\mathbb{H}^2} = m_{r_2} \eta^{\mathbb{H}^2} \\ \tau^{\mathbb{H}^{1,\nabla str}} = m_{s_1} \eta^{\mathbb{H}^{1,\nabla str}} + m_{c_1} \eta^{\mathbb{H}^{1,\nabla rot}} \\ \tau^{\mathbb{H}^{1,\nabla rot}} = m_{c_1} \eta^{\mathbb{H}^{1,\nabla str}} + m_{r_1} \eta^{\mathbb{H}^{1,\nabla rot}} \end{cases}$$

Following Walpole the algebra associated with the basis vectors of our representation can be decomposed into 3 irreducible algebras⁹:

⁸In this notation $Q^{A/B}$ denotes the coupling operator from the the source space B to the destination space A .

⁹The algebra we obtain is similar to the algebra obtained in classical elasticity for transverse isotropy [41].

- \mathcal{A}^3 , which is associated with the projection on \mathbb{H}^3 . Since there is an unique \mathbb{H}^3 in the decomposition, \mathcal{A}^3 is unary;
- \mathcal{A}^2 , which is associated with the projection on \mathbb{H}^2 . Since there is an unique \mathbb{H}^2 in the decomposition, \mathcal{A}^2 is unary;
- \mathcal{A}^1 , which is associated with the projection on $\mathbb{H}^{1,\nabla str} \oplus \mathbb{H}^{1,\nabla rot}$. Since there is 2 \mathbb{H}^1 in the decomposition, \mathcal{A}^1 is isomorphic to the matrix algebra of 2×2 matrices.

Therefore our construction based on the harmonic decomposition of \mathbb{T}^2 allows to construct the Walpole decomposition of tensor $C^{(2)} = A$. Our result is agreement with the previous results given by Monchiet and Bonnet in [29] on the same topic. However, contrary to the aforementioned paper, our construction is based on a physical interpretation of the harmonic decomposition of the strain gradient tensor. Therefore the moduli $(m_{s_3}, m_{s_1}, m_{r_2}, m_{r_1}, m_{c_1})$ are, in our representation, related to kinematic mechanisms.

In the context of the kinetic interpretation of the harmonic decomposition of \mathbb{T}^2 , the moduli have the following expressions:

Mechanism	Type	Expressions
Stretch-gradient	\mathbb{H}^3	$m_{s_3} = A_{111111} - A_{111221} - 4A_{122133} - 2A_{122331}$
	\mathbb{H}^1	$m_{s_1} = A_{111111} + \frac{2}{3}(A_{111221} + 4A_{122133} + 2A_{122331})$
Rotation-gradient	\mathbb{H}^2	$m_{r_2} = -\frac{A_{111111}}{2} - A_{111221} + 2A_{122133} + 4A_{122331} + \frac{3A_{221221}}{2}$
	\mathbb{H}^1	$m_{r_1} = \frac{1}{6}(-3A_{111111} + 2A_{111221} + 20A_{122133} - 8A_{122331} + 9A_{221221})$
Coupling	\mathbb{H}^1	$m_{c_1} = \frac{\sqrt{5}}{3}(-2A_{111221} + 4A_{122133} + 2A_{122331})$

The second-order elasticity tensors will become singular in the following situations:

$$\{m_{s_3} = 0; m_{r_2} = 0; m_{s_1}m_{r_1} - m_{c_1}^2 = 0\}$$

The first two conditions are obvious since m_{s_3} and m_{r_2} are also eigenvalues of A , the third one is equivalent to the vanishing of the determinant $\begin{vmatrix} m_{s_1} & m_{c_1} \\ m_{c_1} & m_{r_1} \end{vmatrix}$ of the algebra generated by the vector components of the harmonic decomposition of \mathbb{T}^2 . As already noted if the coupling modulus m_{c_1} is set to zero, the Walpole and the Kelvin's decomposition become equivalent.

This representation of the second order elasticity tensor gives an insight into the physics encoded in the operator, since the physical meaning of the moduli $\{m_i\}$ are rather clear:

- m_{s_3} and m_{s_1} are moduli related to the stretch-gradient part of $\eta_{(ij)k}$, m_{s_3} is linked with the septor part of the stretch-gradient while m_{s_1} concerns the vector part. Furthermore m_{s_3} is also an eigenvalue of the operator, hence m_{s_3} can not vanish without making A singular. At the opposite, the vector part m_{s_1} can be 0.
- m_{r_2} and m_{r_1} are moduli related to the rotation-gradient part of $\eta_{(ij)k}$, m_{r_2} is linked with the septor part of the stretch-gradient while m_{r_1} concerns the vector part. As before, m_{r_2} is also an eigenvalue of the operator, hence m_{r_2} can not vanish without making A singular. At the opposite, the vector part m_{r_1} can be 0.

- m_{c_1} is the coupling modulus between the rotation-gradient and the vector part of the stretch gradient.

These quantities are interesting for, at least, two reasons. First the computation of the different ratio of these quantities are necessary to quantify the relative importance of the different mechanisms and, at the end, to justify to neglect some of them. In this view the following quantity, which measures the relative effect of rotation-gradient vs stretch-gradient can be defined :

$$r/s = \frac{\sqrt{m_{r_2}^2 + m_{r_1}^2}}{\sqrt{m_{s_3}^2 + m_{s_1}^2}}$$

The second and closely related interest is that it makes easy to properly impose kinematic constraints. For example, if the material is only sensitive to the rotation gradient, the moduli associated with the other mechanisms should be 0, in other terms

$$m_{s_3} = m_{s_1} = m_{c_1} = 0$$

4 Construction of the proofs

The proofs of our theorems are rather direct, and make use of the Schur's lemma. Therefore before constructing our proof, this technical lemma has to be introduced.

4.1 The Schur's lemma

Let us define the notion of an intertwining operator.

Definition 4.1. *Let (E_1, ρ_1) and (E_2, ρ_2) be two G representations. A linear operator $T : E_1 \rightarrow E_2$ is said to be an intertwining operator between ρ_1 and ρ_2 if the following graph commutes for all $g \in G$:*

$$\begin{array}{ccc} E_1 & \xrightarrow{T} & E_2 \\ \downarrow \rho_1 & & \downarrow \rho_2 \\ E_1 & \xrightarrow{T} & E_2 \end{array}$$

i.e. if

$$\forall g \in G, \rho_2(g) \circ T = T \circ \rho_1(g)$$

Theorem 4.2 (Schur's lemma). *Let T be an intertwining operator between irreducible G -representations (E_1, ρ_1) and (E_2, ρ_2) .*

- *If ρ_1 and ρ_2 are inequivalent, then $T = 0$;*
- *If $E_1 = E_2 = E$ and $\rho_1 = \rho_2 = \rho$, then $T = \lambda \text{Id}_E$.*

Now let consider an harmonic space \mathbb{H}^k , and A_k an isotropic self-adjoint endomorphism of \mathbb{H}^k we have the following lemma:

Lemma 4.3. *If A_k is an isotropic self-adjoint endomorphism of \mathbb{H}^k then $A_k = \lambda \text{Id}_{\mathbb{H}^k}$.*

Proof. To demonstrate the lemma we need to show that A_k is an intertwining operator between the irreducible the $\text{SO}(3)$ -representation (\mathbb{H}^k, ρ_k) and itself. Therefore we need to show that the following diagram commutes

$$\begin{array}{ccc} \mathbb{H}^k & \xrightarrow{A_k} & \mathbb{H}^k \\ \downarrow \rho_k & & \downarrow \rho_k \\ \mathbb{H}^k & \xrightarrow{A_k} & \mathbb{H}^k \end{array}$$

i.e. if

$$\forall g \in G, \rho_k(g) \circ A_k = A_k \circ \rho_k(g)$$

The former condition is equivalent to:

$$A_k = \rho_k(g) A_k \rho_k^{-1}(g)$$

This condition is satisfied for any element of the symmetry group of A_k . As A_k is supposed to be isotropic the relation is verified for any $g \in \text{SO}(3)$. Therefore A_k is an intertwining operator between $\text{SO}(3)$ -representations. Since \mathbb{H}^k is irreducible a direct application of Schur's lemma terminates the proof. \square

4.2 Main proofs

Theorem 4.4 (Theorem 1). *If the harmonic decomposition of \mathbb{T} is unique, and if \mathbb{C} is isotropic then its matrix representation $C_{\mathcal{H}}$ is diagonal. In such a case d^* reduces to a permutation and the Walpole and the Kelvin decompositions are equivalent.*

Proof. Let consider an $\text{SO}(3)$ -irreducible decomposition of \mathbb{T} , this decomposition is supposed to be constructed from an isomorphism h :

$$\mathbb{T} = \underset{h}{\bigoplus}_{k=1}^p \mathbb{H}^{\alpha_k}; \quad \alpha_p = n$$

where $\{\alpha_k\}$ is an increasing series that contains p terms. If $\sharp(\mathbb{T})$ denotes the number of different irreducible spaces involved in the decomposition, we have $\sharp(\mathbb{T}) = p$.

Now we have to compute the Clebsch-Gordan decomposition of $\mathbb{T} \otimes^S \mathbb{T}$. Since $\mathbb{T} = \bigoplus_{k=1}^p \mathbb{H}^{\alpha_k}$, the symmetric tensor product $\mathbb{T} \otimes^S \mathbb{T}$ can be decomposed into a direct sum as:

$$\mathbb{T} \otimes^S \mathbb{T} = \left(\bigoplus_{k=1}^p \mathbb{H}^{\alpha_k} \right) \otimes^S \left(\bigoplus_{k=1}^p \mathbb{H}^{\alpha_k} \right) = \underbrace{\left(\bigoplus_{k=1}^p \mathbb{H}^{\alpha_k} \otimes^S \mathbb{H}^{\alpha_k} \right)}_S \oplus \underbrace{\left(\bigoplus_{1 \leq i < j \leq p} \mathbb{H}^{\alpha_i} \otimes \mathbb{H}^{\alpha_j} \right)}_C \quad (7)$$

This expression shows that any symmetric endomorphism of \mathbb{T} can be decomposed into a direct sum of:

1. S-term: p self-adjoint endomorphisms between \mathbb{H}^{α_k} ;
2. C-term: $\frac{p(p-1)}{2}$ coupling morphisms between the p irreducible spaces.

Each of these terms can be decomposed into $\text{SO}(3)$ -irreducible spaces, this is done using the following Clebsch-Gordan rules:

- for the classical product: $\mathbb{H}^p \otimes \mathbb{H}^q = \bigoplus_{i=|p-q|}^{p+q} \mathbb{H}^i$
- for the symmetric product: $\mathbb{H}^k \otimes^S \mathbb{H}^k = \bigoplus_{i=0}^k \mathbb{H}^{2i}$

An isotropic space $[\mathbb{T} \otimes^S \mathbb{T}]^{\text{SO}(3)}$ can only contain scalars, i.e. harmonic spaces of order 0. Looking at the Clebsch-Gordan products, it appears that only same order products can generate scalar terms. In the relation (7) it can be seen that same order products can either be produced by self-products (S-term) or cross-products between same order terms (C-term).

$$[\mathbb{T} \otimes^S \mathbb{T}]^{\text{SO}(3)} = \bigoplus_{k=1}^q \mathbb{H}^{0,k} = \underbrace{\left(\bigoplus_{i=1}^p \mathbb{H}^{0,i} \right)}_S \oplus \underbrace{\left(\bigoplus_{j=1}^{q-p} \mathbb{H}^{0,j} \right)}_C$$

Let us consider now the hypothesis that the decomposition of \mathbb{T} is unique. It is known that it is the case iff the order of spaces in the harmonic decomposition of \mathbb{T} are all different. This hypothesis means:

$$\forall k \in [1, p-1], \alpha_{k+1} > \alpha_k$$

As a consequence in the relation (7) there is no cross-product between same order terms and, hence no isotropic coupling between harmonic spaces. Under this hypothesis we therefore have

$$[\mathbb{T} \otimes^S \mathbb{T}]^{\text{SO}(3)} = \underbrace{\left(\bigoplus_{k=1}^p \mathbb{H}^{\alpha_k} \otimes^S \mathbb{H}^{\alpha_k} \right)}_S = \underbrace{\left(\bigoplus_{i=1}^p \mathbb{H}^{0,i} \right)}_S \quad (8)$$

Therefore any $C \in [\mathbb{T} \otimes^S \mathbb{T}]^{\text{SO}(3)}$ is defined by a collection $\{C_k\}$ of p self-adjoint isotropic endomorphisms. A direct application of the lemma 4.3, shows that for all k , $C_k = \lambda \text{Id}_{\mathbb{H}^k}$. Therefore under the condition of the theorem the matrix of C expressed in $\mathcal{H}(\mathbb{T}) \otimes^S \mathcal{H}(\mathbb{T})$ is diagonal. Since the matrix of C expressed in $\mathcal{E}(C) \otimes^S \mathcal{E}(C)$ is also diagonal, they may differ by a permutation. Therefore, in such a case, d^* reduces to a permutation, and the Kelvin and Walpole decompositions are equivalent. \square

Corollary 4.5. *If the harmonic decomposition of \mathbb{T} is unique, then $\dim([\mathbb{T} \otimes^S \mathbb{T}]^{\text{SO}(3)}) = \sharp(\mathbb{T})$.*

Proof. It is a direct consequence of the previous theorem. Getting back at the relation (8), it shows that if \mathbb{T} is unique there is not coupling between harmonic spaces. Therefore the isotropic components \mathbb{H}^0 are generated by self products of the elements of the harmonic decomposition of \mathbb{T} . Hence an element $C \in [\mathbb{T} \otimes^S \mathbb{T}]^{\text{SO}(3)}$ has as many components as the number of harmonic components in \mathbb{T} . \square

Theorem 4.6 (Theorem 2). *Let \mathbb{T} be a n th-order tensor space only endowed with minor symmetries, then the harmonic decomposition of \mathbb{T} is unique or, equivalently:*

$$\dim([\mathbb{T} \otimes^S \mathbb{T}]^{\text{SO}(3)}) = \sharp^{\text{SO}(3)}(\mathbb{T})$$

if $\mathbb{T} = S^n(\mathbb{R}^3)$, $n \geq 0$ or $\mathbb{T} = \otimes^2(\mathbb{R}^3)$.

Proof. Let \mathbb{T} be a subset of $G^n = \otimes^n(\mathbb{R}^3)$ the space of n th-order tensor. If this subspace is only defined in terms of index symmetries¹⁰ we have:

$$S^n(\mathbb{R}^3) \subseteq \mathbb{T} \subseteq \otimes^n(\mathbb{R}^3)$$

With the following important property [21]:

$$S^n(\mathbb{R}^3) = \begin{cases} \bigoplus_{k=0}^{\frac{n}{2}} \mathbb{H}^{2k} & \text{if } n \text{ even} \\ \bigoplus_{k=0}^{\frac{n-1}{2}} \mathbb{H}^{2k+1} & \text{if } n \text{ odd} \end{cases}$$

Therefore if $\mathbb{T} = S^n(\mathbb{R}^3)$ its harmonic decomposition of \mathbb{T} is unique. At the opposite if $\mathbb{T} = \otimes^n(\mathbb{R}^3)$, the harmonic decomposition of \mathbb{T} is unique provided $n < 3$. This fact is easy to verify, lets compute some decompositions using the Clebsch-Gordan product:

$$\otimes^1(\mathbb{R}^3) = \mathbb{H}^1 \quad ; \quad \otimes^2(\mathbb{R}^3) = \mathbb{H}^0 \oplus \mathbb{H}^1 \oplus \mathbb{H}^2 \quad ; \quad \otimes^3(\mathbb{R}^3) = 2\mathbb{H}^0 \oplus 3\mathbb{H}^1 \oplus 2\mathbb{H}^2 \oplus \mathbb{H}^3$$

Therefore the harmonic decomposition of $\otimes^3(\mathbb{R}^3)$ is not unique, and since for $n > 3$, $\otimes^n(\mathbb{R}^3) = (\otimes^{n-3}(\mathbb{R}^3)) \otimes (\otimes^3(\mathbb{R}^3))$, the harmonic decomposition of $\otimes^{n>3}(\mathbb{R}^3)$ is not unique. \square

5 Conclusion

In this paper the algebraic structure of the isotropic n th-order gradient elasticity has been investigated. Studying the harmonic tensor decomposition of the n -th order strain gradient tensor, it has been shown that, and contrary to classical elasticity, higher-order isotropic constitutive tensors always describe coupled mechanisms. As demonstrated, this fact indeed occurs each time the number of isotropic coefficients exceeds the number of elementary spaces in the harmonic decomposition of the n -th order strain-gradient tensor. Besides this general result, an explicit construction of this phenomenon has been provided for Mindlin strain-gradient elasticity. This construction is important for at least two reasons:

- its principle is general and can be applied in many other situations;
- for strain-gradient elasticity, it provides a physical interpretation to higher-order moduli. Their explicit knowledge is both interesting to analyze the second-order kinematic of a microstructured media and also to impose (if needed) kinematic constrains on the behavior.

It has to be noticed that, as the harmonic decomposition of strain-gradient tensors is not unique, many other interpretations of this decomposition can be proposed. This situation has to be compared with the multiple choices one can made for the classical isotropic moduli. The use of these *physically-based* moduli may be a powerful tool to study the degeneracy that occurs in higher-order continua.

¹⁰This hypothesis excludes, for example, traceless subspace.

A Matrix representations of strain-gradient elasticity: main results

Any second order elasticity tensor A can be represented by a matrix relative to an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. This question has been fully investigated in [5]. Here we sum-up the main results. As expressed by the relation (3.2), the second-order elasticity tensor A is a self-adjoint endomorphism of $\mathbb{T}_{(ij)k}$. In order to express the strain gradient η or the hyperstress tensor τ as a 18-dimensional vector and write A as a symmetric matrix, we introduce the following orthonormal basis vectors:

$$\hat{\mathbf{e}}_\alpha = \left(\frac{1 - \delta_{ij}}{\sqrt{2}} + \frac{\delta_{ij}}{2} \right) (\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i) \otimes \mathbf{e}_k, \quad 1 \leq \alpha \leq 18 \quad (9)$$

where the Einstein summation convention does not apply. Then, the aforementioned tensors can be expressed as

$$\eta = \sum_{\alpha=1}^{18} \hat{\eta}_\alpha \hat{\mathbf{e}}_\alpha, \quad \tau = \sum_{\alpha=1}^{18} \hat{\tau}_\alpha \hat{\mathbf{e}}_\alpha, \quad A = \sum_{\alpha, \beta=1}^{18} \hat{A}_{\alpha\beta} \hat{\mathbf{e}}_\alpha \otimes \hat{\mathbf{e}}_\beta, \quad (10)$$

so that the relation (3.2) can be conveniently written in the matrix form

$$\hat{\tau}_\alpha = \hat{A}_{\alpha\beta} \hat{\eta}_\beta. \quad (11)$$

Using the orthonormal basis (9), the relationship between the matrix components $\hat{\eta}_\alpha$ and η_{ijk}

$$\hat{\eta}_\alpha = \begin{cases} \eta_{ijk} & \text{if } i = j, \\ \sqrt{2}\eta_{ijk} & \text{if } i \neq j; \end{cases} \quad (12)$$

and, obviously the same relation between $\hat{\tau}_\alpha$ and τ_{ijk} hold. For the constitutive tensor we have the following correspondence.

$$\hat{A}_{\alpha\beta} = \begin{cases} A_{ijklmn} & \text{if } i = j \text{ and } l = m, \\ \sqrt{2}A_{ijklmn} & \text{if } i \neq j \text{ and } l = m \text{ or } i = j \text{ and } l \neq m, \\ 2A_{ijklmn} & \text{if } i \neq j \text{ and } l \neq m. \end{cases} \quad (13)$$

It remains to choose an appropriate three-to-one subscript correspondence between ijk and α . The correspondence specified in Table 1 was chosen in order to make the 6th order tensor block diagonal for diehdral classes [5]. This constitute the so-called spatial basis \mathcal{S} discussed in the main part of the paper. The knowledge of the harmonic decomposition of $\mathbb{T}_{(ij)k}$, studied in section 3.2, allows to construct a transformation matrix from \mathcal{S} to $\mathcal{H}(\mathbb{T}_{(ij)k})$. This matrix is decomposed here into four elementary matrices, each of them

α	1	2	3	4	5	
ijk	111	221	122	331	133	Privileged direction: 1
α	6	7	8	9	10	
ijk	222	112	121	332	233	Privileged direction: 2
α	11	12	13	14	15	
ijk	333	113	131	223	232	Privileged direction: 3
α	16	17	18			
ijk	123	132	231			No privileged direction

Table 1: The three-to-one subscript correspondence for 3D strain-gradient elasticity

associated with an harmonic space of $\mathbb{T}_{(ij)k}$.

$$[P_{\mathbb{H}^3}] = \begin{pmatrix} \sqrt{\frac{2}{5}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{10}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{2}{5}} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{10}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{2}{5}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{10}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad [P_{\mathbb{H}^1, \nabla str}] = \begin{pmatrix} \sqrt{\frac{3}{5}} & 0 & 0 \\ \frac{1}{\sqrt{15}} & 0 & 0 \\ \sqrt{\frac{2}{15}} & 0 & 0 \\ \frac{1}{\sqrt{15}} & 0 & 0 \\ \sqrt{\frac{2}{15}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{5}} & 0 \\ 0 & \frac{1}{\sqrt{15}} & 0 \\ 0 & \sqrt{\frac{2}{15}} & 0 \\ 0 & \frac{1}{\sqrt{15}} & 0 \\ 0 & \sqrt{\frac{2}{15}} & 0 \\ 0 & 0 & \sqrt{\frac{3}{5}} \\ 0 & 0 & \frac{1}{\sqrt{15}} \\ 0 & 0 & \sqrt{\frac{2}{15}} \\ 0 & 0 & \frac{1}{\sqrt{15}} \\ 0 & 0 & \sqrt{\frac{2}{15}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[\mathbf{P}_{\mathbb{H}^2}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix}, \quad [\mathbf{P}_{\mathbb{H}^1, \nabla rot}] = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & 0 \\ \frac{1}{\sqrt{6}} & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & 0 \\ \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & -\frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{6}} \\ 0 & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{6}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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