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SOFIC MEASURES AND DENSITIES OF LEVEL SETS

ALAIN THOMAS

ABSTRACT. The Bernoulli convolution associated to the real $\beta > 1$ and the integer $d \geq \beta$ is a probability measure $\eta_{\beta,d}$ on \mathbb{R} , solution of the self-similarity relation $\eta = \sum_{k=0}^{d-1} p_k \cdot \eta \circ S_k$ where (p_0, \dots, p_{d-1}) is a probability vector and $S_k(x) = \frac{x+k}{\beta}$. If β is an integer or a Pisot algebraic number, the study of this measure is close to the study of the order of growth of the number of representations in base β with digits in $\{0, 1, \dots, d-1\}$. In the case $\beta = 2$ and $d = 3$ it is also related to the continued fractions.

0. INTRODUCTION

The different sections of this paper are relatively independent. A sofic probability measure on a space $\{0, 1, \dots, b-1\}^{\mathbb{N}}$, i.e. the image of a Markov probability measure by a shift-commuting continuous map, is representable by products of matrices as explained in Theorem 5. Now the measures defined by Bernoulli convolution [12], i.e.

$$\eta_{\beta,d} := \bigstar_{n=1}^{\infty} \left(\sum_{a=0}^{d-1} \delta_{\frac{a}{\beta^n}} \right)$$

where $\beta \in \mathbb{R}$, $d \in \mathbb{N}$ and $d \geq \beta > 1$, are an example of sofic measures if β is an integer or a Pisot number, i.e., an algebraic number whose conjugates belong to the open unit disk. A transductor of normalization and a Markov chain are naturally associated to β and d . The matrices associated to the measure $\eta_{\beta,d}$ are easy to define when β is a integer, let b , and have been studied by different authors; for instance they are the transposes of the ones of Protasov [13]. In this case the measure $\eta_{b,d}$ is related to the number of representations of the integer n in base b with d digits, let $\mathcal{N}(n)$, and the Hausdorff dimensions of the level sets of $\eta_{b,d}$ are related with the exponential densities of some sets of integers on which $\mathcal{N}(n)$ has a given order of growth (Proposition 13).

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1. SOFIC SUBSHIFTS

By subshifts we mean closed subsets of $\{0, 1, \dots, b-1\}^{\mathbb{N}}$ invariant by the shift $\sigma : (\omega_n)_{n \in \mathbb{N}} \mapsto (\omega_{n+1})_{n \in \mathbb{N}}$. Let us give two equivalent definitions of the sofic subshifts.

Definition 1. *A sofic subshift is a subshift recognizable by a finite automaton [1].*

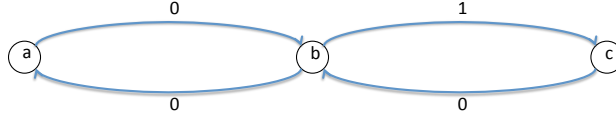
Definition 2. *A subshift is a sofic iff it is the image of a topological Markov subshift by a letter-to-letter morphism.*

In this definition, "letter-to-letter morphism" can be replaced by "continuous morphism": indeed a continuous morphism has the form

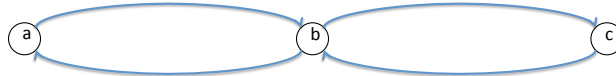
$$\varphi((\omega_n)_{n \in \mathbb{N}}) = (\psi(\omega_{n-s}\omega_{n-s+1} \dots \omega_{n+s}))_{n \in \mathbb{N}}$$

where ψ is defined on $\{0, 1, \dots, s\}^{2s+1}$, being understood that $\omega_n = 0$ for $n \leq 0$.

Proof of the equivalence. Let us see – on a representative example – why a subshift recognizable by a finite automaton is the image of a topological Markov subshift by a letter-to-letter morphism.

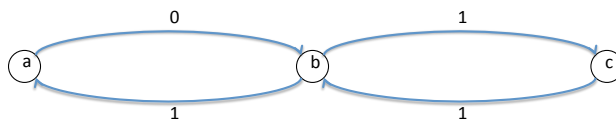


The sofic subshift we consider, is constituted by the labels of the infinite paths, namely the sequences of 0 and 1 without factors $10^{2i}1$, $i \in \mathbb{N} \cup \{0\}$. It excludes an infinity of words, it is not of finite type. Clearly it is the image, by a letter-to-letter morphism, of the Markov subshift of alphabet $\{(a, 0), (b, 0), (b, 1), (c, 1)\}$ that excludes some words of length two: it excludes the words that are not in the list $((a, 0), (b, 0)), ((a, 0), (b, 1)), ((b, 0), (a, 0)), ((b, 1), (c, 0)), ((c, 0), (b, 0)), ((c, 0), (b, 1))$. Conversely let us consider for instance the Markov subshift S defined by the following graph:



and the letter-to-letter morphism $\varphi : S \rightarrow \{0, 1\}^{\mathbb{N}}$ defined from the map $\psi : \{a, b, c\} \rightarrow \{0, 1\}$, $\psi(a) = 0$, $\psi(b) = 1$ and $\psi(c) = 1$. According

to the second definition, $\varphi(S)$ is a sofic subshift. It excludes the words $01^{2i}0$, $i \in \mathbb{N} \cup \{0\}$. It is also a sofic subshift in the sense of the first definition, because it is recognizable by the automaton whose arrows from each state x have label $\psi(x)$:



□

2. MARKOV, SOFIC AND LINEARLY REPRESENTABLE MEASURES

Definition 3. (i) A (homogeneous) Markov probability measure is a measure μ on the product set $\{0, 1, \dots, b - 1\}^{\mathbb{N}}$ defined by setting for any cylinder set $[\omega_1 \dots \omega_n]$

$$(1) \quad \mu[\omega_1 \dots \omega_n] = p_{\omega_1} p_{\omega_1 \omega_2} \dots p_{\omega_{n-1} \omega_n}$$

where $p = (p_0 \dots p_{b-1})$ is a positive probability vector and $P =$

$$\begin{pmatrix} p_{00} & \dots & p_{0(b-1)} \\ \vdots & \ddots & \vdots \\ p_{(b-1)0} & \dots & p_{(b-1)(b-1)} \end{pmatrix} \text{ a nonnegative stochastic matrix.}$$

Clearly the support of μ is a Markov subshift, and μ is invariant (or stationary) iff p is a left eigenvector of P .

(ii) A probability measure on $\{0, 1, \dots, b - 1\}^{\mathbb{N}}$, let ν , is called sofic if it is the image of a Markov probability measure by a continuous morphism. This morphism can be chosen letter-to-letter.

(iii) Let $\{R_0, \dots, R_{b-1}\}$ be a set of nonnegative r -dimensional row vectors, $\mathcal{M} = \{M_0, \dots, M_{b-1}\}$ a set of $r \times r$ nonnegative matrices and C a positive r -dimensional column vector, such that

$$\left(\sum_i R_i \right) C = 1 \quad \text{and} \quad \left(\sum_i M_i \right) C = C.$$

One can define a probability measure on $\{0, 1, \dots, b - 1\}^{\mathbb{N}}$, let η , by setting

$$(2) \quad \eta[\omega_1 \dots \omega_n] = R_{\omega_1} M_{\omega_2} \dots M_{\omega_n} C.$$

We call it a \mathcal{M} -measure.

Remark 4. If $\sum_i M_i$ is irreducible and if $R_i = R M_i$ for any i , where R is the positive left eigenvector of $\sum_i M_i$, the \mathcal{M} -measure is called a "linearly representable measure" for instance in [2] or in

<http://www.unc.edu/math/Faculty/petersen/Papers/06Jan2010.pdf>

and is σ -invariant. In this case $\eta[\omega_1 \dots \omega_n] = RM_{\omega_1} \dots M_{\omega_n} C$.

Theorem 5. *A probability measure on $\{0, 1, \dots, b-1\}^{\mathbb{N}}$ is sofic if and only if it is a \mathcal{M} -measure.*

Proof. We first see that the Markov measures are \mathcal{M} -measures. Indeed the product in (1) can be seen as a product of submatrices of p and P :

$$(3) \quad \mu[\omega_1 \dots \omega_n] = \pi_{\omega_1} P_{\omega_2} \dots P_{\omega_n} \mathbb{U}$$

where, for any i and j ,

$$(4) \quad \begin{aligned} \pi_i &= (0 \ \dots \ 0 \ p_i \ 0 \ \dots \ 0) \\ P_j &= \begin{pmatrix} 0 & \dots & 0 & p_{1j} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & p_{(b-1)j} & 0 & \dots & 0 \end{pmatrix} \text{ and } \mathbb{U} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \end{aligned}$$

Now any sofic measure ν defined from a Markov measure μ and a map

$$\psi : \{0, 1, \dots, b-1\} \rightarrow \{0, 1, \dots, b'-1\}$$

is a \mathcal{M} -measure, because (3) implies

$$(5) \quad \nu[\omega_1 \dots \omega_n] = \left(\sum_{\psi(i)=\omega_1} \pi_i \right) \left(\sum_{\psi(j)=\omega_2} P_j \right) \dots \left(\sum_{\psi(k)=\omega_n} P_k \right) \mathbb{U}.$$

Conversely let η be a \mathcal{M} -measure. In the formula (2) we may suppose that the entries c_i of the column vector C are 1: indeed let

$$\Delta = \begin{pmatrix} c_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & c_{r-1} \end{pmatrix},$$

the formula remains true if we replace C by

$\Delta^{-1}C = \mathbb{U}$, the matrices M_i by $\Delta^{-1}M_i\Delta$ and R_i by $R_i\Delta$. According to the hypotheses of Definition 3(iii), the following matrices are a probability vector and a stochastic matrix respectively:

$$p = (R_0 \ R_1 \ \dots \ R_{b-1}) \text{ and } P = \begin{pmatrix} M_0 & M_1 & \dots & M_{b-1} \\ \vdots & \vdots & \ddots & \vdots \\ M_0 & M_1 & \dots & M_{b-1} \end{pmatrix}.$$

We define from p and P the submatrices π_i and P_j , as in (4). Clearly, the formula (2) is equivalent to (5) with $\psi(i) = \lfloor \frac{i}{r} \rfloor$, $i \in \{0, 1, \dots, rb-1\}$. Now (5) implies that η is the image of the Markov measure defined by p and P , so η is sofic. \square

3. LEVEL SETS ASSOCIATED TO A MAP $f : \mathbb{N} \rightarrow \mathbb{R}_+^*$

Let us first define the exponential densities associated to f . In the examples we give later, f has a polynomial rate of growth in the sense that there exists $0 \leq \alpha_1 \leq \alpha_2$ such that

$$(6) \quad n^{\alpha_1} \leq f(n) \leq n^{\alpha_2} \quad (n \text{ large enough}).$$

We are interested to find for certain values of $\alpha \geq 0$, a subset $\mathcal{E}(\alpha) \subset \mathbb{N}$ as large as possible such that $\log(f(n)) \sim \log(n^\alpha)$ for $\mathcal{E}(\alpha) \ni n \rightarrow \infty$. Let us construct a subset $\mathcal{E}(\alpha)$ that satisfies this condition and has maximal lower and upper exponential densities. The lower and upper exponential densities of a subset $S \subset \mathbb{N}$ are defined by

$$\begin{aligned} d_{\text{exp}}^-(S) &:= \liminf_{N \rightarrow \infty} \frac{\log(\#S \cap [0, N])}{\log N} \\ d_{\text{exp}}^+(S) &:= \limsup_{N \rightarrow \infty} \frac{\log(\#S \cap [0, N])}{\log N}. \end{aligned}$$

Definition and theorem 6. *Given $f : \mathbb{N} \rightarrow \mathbb{R}_+^*$, one associate to any $\alpha \geq 0$ and $\varepsilon > 0$ the level set*

$$(7) \quad \mathcal{E}(\alpha, \varepsilon) := \left\{ n \in \mathbb{N} : \alpha - \varepsilon \leq \frac{\log f(n)}{\log n} \leq \alpha + \varepsilon \right\}.$$

and to any $\alpha \geq 0$ the limit-densities – that are also lower bounds

$$\begin{aligned} \delta^-(\alpha) &:= \lim_{\varepsilon \rightarrow 0} d_{\text{exp}}^-(\mathcal{E}(\alpha, \varepsilon)) \\ \delta^+(\alpha) &:= \lim_{\varepsilon \rightarrow 0} d_{\text{exp}}^+(\mathcal{E}(\alpha, \varepsilon)). \end{aligned}$$

If f satisfies the condition (6), it is clear from the compactity of $[\alpha_1, \alpha_2]$ that there exists α such that $\mathcal{E}(\alpha, \varepsilon)$ is infinite for any $\varepsilon > 0$. In this case we define $\mathcal{E}(\alpha)$ as follows: there exists a set of integers of the form

$$(8) \quad \mathcal{E}(\alpha) := \bigcup_k \mathcal{E}(\alpha, 1/k) \cap [N_{k-1}, N_k[$$

where the increasing sequence (N_k) is chosen in order that $\mathcal{E}(\alpha)$ has lower exponential density $\delta^-(\alpha)$, upper exponential density $\delta^+(\alpha)$, and

$$\lim_{\substack{n \rightarrow \infty \\ n \in \mathcal{E}(\alpha)}} \frac{\log f(n)}{\log n} = \alpha.$$

Proof. For any subset $S \subset \mathbb{N}$ the quantity $\frac{\log(\#S \cap [0, N])}{\log N}$ oscillate indefinitely, approximately between its lower bound and its upper bound. So for any $k \in \mathbb{N}$ we can find infinitely many $N_k \in \mathbb{N}$ and $N'_k \leq N_k$ such

that

$$\frac{\log(\#\mathcal{E}(\alpha, 1/k) \cap [0, N_k])}{\log N_k} \geq \delta^+(\alpha) - \frac{1}{k}$$

$$\frac{\log(\#\mathcal{E}(\alpha, 1/k) \cap [0, N'_k])}{\log N'_k} \leq \delta^-(\alpha) + \frac{1}{k}.$$

Choosing successively each N_k large enough with respect to N_{k-1} and N'_{k-1} , $\mathcal{E}(\alpha)$ defined in (8) has lower exponential density $\delta^-(\alpha)$ and upper exponential density $\delta^+(\alpha)$. \square

Example 7. Let $f(n) = n^{1+\sin n}$ then, given $s \in [0, 2]$ and $\varepsilon > 0$, the set of the integers n such that $1 + \sin n \in [s - \varepsilon, s + \varepsilon]$ has a positive usual density, hence the graph of δ^\pm is



4. LEVEL SETS ASSOCIATED TO A SET OF MATRICES

There exist a lot of partial results (for instance [4] and [5]) about the level sets of the \mathcal{M} -measures defined in (2). Given a \mathcal{M} -measure η , the level sets can be defined by

$$E(\alpha) := \left\{ (\omega_n)_{n \in \mathbb{N}} : \lim_{n \rightarrow \infty} \frac{\log \eta[\omega_1 \dots \omega_n]}{\log(1/b^n)} = \alpha \right\}$$

and we denote by $d(\alpha) := \text{H-dim}(E(\alpha))$ their Hausdorff dimensions.

On the other side it is natural to associate to the \mathcal{M} -measure η , the function $f : \mathbb{N} \rightarrow \mathbb{R}$ defined for any $n = \varepsilon_1 b^{k-1} + \dots + \varepsilon_k b^0$ by

$$f(n) = R_{\varepsilon_1} M_{\varepsilon_2} \dots M_{\varepsilon_k} C$$

and to specify the relation between the singularity spectrum $\alpha \mapsto d(\alpha)$ associated to η and the density spectrum $\alpha \mapsto \delta^\pm(\alpha)$ associated to f . In the second case α cannot exceed $\frac{\log \rho}{\log b}$, where ρ is the joint spectral radius of the matrices M_ε .

Remark 8. *When f is the partition functions in integral base or in base $c\beta^k + o(1)$, β a Pisot number and $c \in \mathbb{Q}(\beta)$ (see [6]), there exists a real $\alpha_0 > 0$ such that $\frac{\log f(n)}{\log n}$ tends to α_0 when n tends to infinity along a subset $S_{\alpha_0} \subset \mathbb{N}$ of natural density 1. This imply that $\mathcal{E}(\alpha_0, \varepsilon)$ also has density 1 and consequently $\delta^-(\alpha_0) = \delta^+(\alpha_0) = 1$. The problem is to find some other values of α such that $\delta^+(\alpha) \neq 0$; or better, to prove that that the graphs of d and δ^\pm are concave on a nontrivial interval.*

5. BERNOULLI CONVOLUTION AND NUMBER OF REPRESENTATIONS IN INTEGRAL BASE

Given two integers $b \leq d$ we consider the bi-infinite matrix

$$(9) \quad M_\infty := \begin{pmatrix} \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots \\ \dots & 0 & p_{d-1} & \dots & p_{d-b-1} & \dots & p_0 & \dots & 0 & 0 & \dots \\ \dots & 0 & 0 & \dots & p_{d-1} & \dots & p_b & \dots & p_0 & 0 & \dots \\ \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots \end{pmatrix},$$

each row containing the same probability vector $p = (p_0 \dots p_{d-1})$, shifted b times to the right from the previous row.

5.1. This matrix is related to the Bernoulli convolution in integral base. Let us define more generally the Bernoulli convolution in real base $\beta > 1$ with d digits, $d \geq \beta$: it is the probability measure $\eta = \eta_{\beta,d}$ defined by setting, for any interval $I \subset \mathbb{R}$

$$(10) \quad \eta(I) := P_p \left(\left\{ (\omega_k)_{k \in \mathbb{N}} : \sum_k \frac{\omega_k}{\beta^k} \in I \right\} \right)$$

where $\omega_k \in \{0, 1, \dots, d-1\}$ and P_p the product probability defined on $\{0, \dots, d-1\}^{\mathbb{N}}$ from the probability vector p , that we suppose positive. Now we suppose that $\beta = b \in \mathbb{N}$ and we prove that the measure $\eta = \eta_{b,d}$ is in a sense the \mathcal{M} -measure associated to the following submatrices of

the infinite matrix M_∞ , of size $a = \lceil \frac{d-1}{b-1} \rceil$:

$$(11) \quad \begin{pmatrix} p_0 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} p_1 & p_0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \dots, \begin{pmatrix} p_{b-1} & p_{b-2} & \dots & p_0 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{pmatrix}$$

that we call M_0, M_1, \dots, M_{b-1} .

Proposition 9. *The measure of the translated b -adic interval $\varepsilon_0 + I_{\varepsilon_1 \dots \varepsilon_k}$ where $\varepsilon_0 \in \{0, 1, \dots, a-1\}$, $\varepsilon_i \in \{0, 1, \dots, b-1\}$ for $i \geq 1$ and $I_{\varepsilon_1 \dots \varepsilon_k} := \left[\sum_{i=1}^k \frac{\varepsilon_i}{b^i}, \sum_{i=1}^k \frac{\varepsilon_i}{b^i} + \frac{1}{b^k} \right)$, is given by the formula*

$$(12) \quad \eta(\varepsilon_0 + I_{\varepsilon_1 \dots \varepsilon_k}) = E_{\varepsilon_0} M_{\varepsilon_1} \dots M_{\varepsilon_k} C$$

where E_0, E_1, \dots, E_{a-1} are the canonical basis row vectors and C the unique positive eigenvalue of $\sum_{\varepsilon} M_{\varepsilon}$. Consequently the measure $\eta_{\varepsilon_0, \text{symb}}$ defined as follows is a \mathcal{M} -measure on $\{0, 1, \dots, b-1\}^{\mathbb{N}}$:

$$(13) \quad \eta_{\varepsilon_0, \text{symb}}[\varepsilon_1 \dots \varepsilon_k] := \frac{\eta(\varepsilon_0 + I_{\varepsilon_1 \dots \varepsilon_k})}{\eta(\varepsilon_0 + [0, 1))}.$$

Proof. We denote $I = I_{\varepsilon_1 \dots \varepsilon_n}$ and $I' = I_{\varepsilon_2 \dots \varepsilon_n}$, and we put $p_\omega = 0$ if $\omega \notin \{0, 1, \dots, d-1\}$.

$$\begin{aligned} \eta(\varepsilon_0 + I) &= \sum_{\omega=0}^{d-1} P \left(\left\{ \omega_1 = \omega \text{ and } \sum_k \frac{\omega_{k+1}}{b^k} \in \varepsilon_0 b + \varepsilon_1 - \omega + I' \right\} \right) \\ &= \sum_{\omega \in \mathbb{Z}} p_\omega \eta(\varepsilon_0 b + \varepsilon_1 - \omega + I') \\ &= \sum_{q \in \mathbb{Z}} p_{\varepsilon_0 b + \varepsilon_1 - q} \eta(q + I'). \end{aligned}$$

In fact q belongs to $\{0, 1, \dots, a-1\}$, otherwise $\eta(q + I')$ is null. Since the coefficients $p_{\varepsilon_0 b + \varepsilon_1 - q}$ are the entries of the matrix M_{ε_1} we deduce

$$(14) \quad \begin{pmatrix} \eta(I) \\ \eta(I+1) \\ \vdots \\ \eta(I+a-1) \end{pmatrix} = M_{\varepsilon_1} \begin{pmatrix} \eta(I') \\ \eta(I'+1) \\ \vdots \\ \eta(I'+a-1) \end{pmatrix} \\ = M_{\varepsilon_1} \dots M_{\varepsilon_k} \begin{pmatrix} \eta([0, 1)) \\ \eta([1, 2)) \\ \vdots \\ \eta([a-1, a)) \end{pmatrix}.$$

In the particular case $k = 1$, we make the sum or the l.h.s. of (14)

for $\varepsilon_1 = 0, 1, \dots, b - 1$ and deduce that $\begin{pmatrix} \eta([0, 1)) \\ \eta([1, 2)) \\ \vdots \\ \eta([a - 1, a)) \end{pmatrix}$ is a nonneg-

ative eigenvector of $\sum_{\varepsilon} M_{\varepsilon}$. The sum of its entries is 1 because $[0, a]$ contains the support of η . Since $\sum_{\varepsilon} M_{\varepsilon}$ is obviously irreducible, by Perron-Frobenius its nonnegative eigenvector is unique and the formula (12) follows. \square

5.2. The bi-infinite matrix (9) is also related – in the case $p_k = \frac{1}{d}$ – to the normalization [9] in integral base. The normalization of a infinite sequence $(\omega_i)_{i \in \mathbb{N}}$ with terms in $\{0, \dots, d - 1\}$, is defined by

$$\begin{aligned} \mathbf{n}((\omega_i)_{i \geq 1}) &:= (\varepsilon_i)_{i \geq -(h-1)} \\ \varepsilon_i &\in \{0, 1, \dots, b - 1\}, \varepsilon_{-(h-1)} \neq 0 \\ \sum_{i=1}^{\infty} \frac{\omega_i}{b^i} &= \sum_{i=-(h-1)}^{\infty} \frac{\varepsilon_i}{b^i} \end{aligned}$$

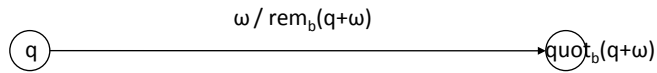
and the ε_i are not eventually $b - 1$. The normalization of a finite sequence with terms in $\{0, \dots, d - 1\}$ is a special case:

$$\begin{aligned} \mathbf{n}(\omega_1, \dots, \omega_k) &:= (\varepsilon_{-(h-1)}, \dots, \varepsilon_k) \in \{0, 1, \dots, b - 1\}^{k+h}, \varepsilon_{-(h-1)} \neq 0, \\ \sum_{i=1}^k \frac{\omega_i}{b^i} &= \sum_{i=-(h-1)}^k \frac{\varepsilon_i}{b^i} \end{aligned}$$

or equivalently

$$(15) \quad \omega_1 b^{k-1} + \dots + \omega_k b^0 = \varepsilon_{-(h-1)} b^{k+h-1} + \dots + \varepsilon_k b^0.$$

Proposition 10. *One obtains the canonical digits ε_i by entering successively, from the initial state 0, the digits $\omega_k, \omega_{k-1}, \dots, \omega_1, 0, 0, \dots$ in the following transducer with set of states $\{0, 1, \dots, a - 1\}$, $a = \lceil \frac{d-1}{b-1} \rceil$. The arrows of initial state $q \in \{0, 1, \dots, a - 1\}$ and input label ω have final state and output label the quotient and the remainder of the Euclidean division of $q + \omega$ by b respectively:*



Proof. We put $\omega_i = 0$ (resp. $\varepsilon_i = 0$) for any $i \in \mathbb{Z} \setminus \{1, \dots, k\}$ (resp. for any $i \in \mathbb{Z} \setminus \{-(h-1), \dots, k\}$). We call the " i^{th} carry", the integer

$$q_i = \sum_{j>i} \frac{\omega_j - \varepsilon_j}{b^{j-i}} = \sum_{j \leq i} (\varepsilon_j - \omega_j) b^{i-j}.$$

From the first equality we have $-1 < q_i < \frac{d-1}{b-1}$, or equivalently $0 \leq q_i \leq a-1$. From the second we have $q_i = bq_{i-1} + \varepsilon_i - \omega_i$, showing that q_{i-1} and ε_i are obtained by Euclidean division of $q_i + \omega_i$ by b .

Let us prove that q_{i-1} belong to the set of states, that is, $0 \leq q_{i-1} < a$. Indeed $0 \leq q_{i-1} = \lfloor \frac{q_i + \omega_i}{b} \rfloor \leq \frac{a+d-2}{b}$; now the inequality $\frac{a+d-2}{b} < a$ is equivalent to $a > \frac{d-2}{b-1}$, and $a \geq \frac{d-1}{b-1} > \frac{d-2}{b-1}$ is true from the definition of a . \square

5.3. Number of b -representations with digits in $\{0, 1, \dots, d-1\}$.

In order to compute the number of representations of an integer, let $n = \varepsilon_1 b^{k-1} + \dots + \varepsilon_k b^0$, we consider the subgraph obtained by fixing the output : the output label is a given integer $\varepsilon \in \{0, 1, \dots, b-1\}$. Then the input label, from a state q to a state q' , is $\omega = bq' + \varepsilon - q$ if this integer belongs to $\{0, 1, \dots, d-1\}$. The incidence matrix of this subgraph is the transpose of the matrix dM_ε defined in (11), where we replace each p_i by $\frac{1}{d}$, proving that:

Proposition 11. *The number of representations of n with digits in $\{0, 1, \dots, d-1\}$, let $\mathcal{N}(n)$, is the number of paths from the state 0 to the state 0 with successive outputs $\varepsilon_k, \dots, \varepsilon_1$:*

$$\mathcal{N}(n) = d^k \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} M_{\varepsilon_1} \dots M_{\varepsilon_k} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Remark 12. *Let $q \in \{0, 1, \dots, a-1\}$. Since $\mathcal{N}(n-q)$ is the number of paths from the state q to to the state 0 with successive outputs $\varepsilon_k, \dots, \varepsilon_1$, it is also the $(q+1)^{\text{th}}$ entry of $d^k \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} M_{\varepsilon_1} \dots M_{\varepsilon_k}$. So, from (12),*

$$(16) \quad d^k \eta(I_{\varepsilon_1 \dots \varepsilon_k}) = \begin{pmatrix} \mathcal{N}(n) & \mathcal{N}(n-1) & \dots & \mathcal{N}(n-a+1) \end{pmatrix} C =: f(n).$$

5.4. The relation between the singularity spectrum and the density spectrum. The singularity spectrum d_η of the measure $\eta = \eta_{b,d}$ is related to the density spectrum δ^\pm of the number of representations in base b with d digits. Suppose that $d_\eta(\alpha) = \text{H-dim}(E(\alpha))$ exists, and let us cover $E(\alpha)$ by some b -adic intervals $I_{\varepsilon_1 \dots \varepsilon_k}$. By the definition

of $E(\alpha)$, for k large enough we must consider only the intervals such that

$$(17) \quad \alpha - \varepsilon \leq \frac{\log \eta(I_{\varepsilon_1 \dots \varepsilon_k})}{\log(1/b^k)} \leq \alpha + \varepsilon.$$

Let $n = \varepsilon_1 b^{k-1} + \dots + \varepsilon_k b^0$ and $\gamma := \frac{\log d}{\log b} - \alpha$. From (16),

$$(b^k)^{\gamma - \varepsilon} \leq f(n) \leq (b^k)^{\gamma + \varepsilon}.$$

We recognize the set $\mathcal{E}_f(\gamma, \varepsilon)$ of (7). By definition of the Hausdorff dimension, given $s_1 < d(\alpha) < s_2$, $(b^{-k})^{s_1} \# \mathcal{E}_f(\gamma, \varepsilon)$ tends to infinity and $(b^{-k})^{s_2} \# \mathcal{E}_f(\gamma, \varepsilon)$ tends to 0 when $k \rightarrow \infty$. Consequently for k large enough

$$(b^k)^{s_1} \leq \# \mathcal{E}_f(\gamma, \varepsilon) \leq (b^k)^{s_2},$$

which proves that

$$(18) \quad \delta_f^-(\gamma) = \delta_f^+(\gamma) = d_\eta(\alpha).$$

Let us deduce the following

Proposition 13. *The density spectrum $\delta_{\mathcal{N}}^\pm$ of the number of representations in base b with d digits satisfies*

$$\delta_{\mathcal{N}}^-(\gamma) = \delta_{\mathcal{N}}^+(\gamma) = d_\eta(\alpha)$$

with $\gamma := \frac{\log d}{\log b} - \alpha$, for any α such that $d_\eta(\alpha)$ exists.

Proof. The function f defined in (16) has the same order of growth as the function \mathcal{N} because – from [6, Lemma 9 (ii)] – for any $n \in \mathbb{N}$

$$\frac{1}{C \log n} \leq \frac{\mathcal{N}(n)}{\mathcal{N}(n-1)} \leq C \log n.$$

Hence $\delta_f^\pm = \delta_{\mathcal{N}}^\pm$ and the proposition follows from (18). \square

6. THE GENERAL FRAMEWORK IN PISOT BASE

Let us call \mathbf{n} the normalization map [9] in the integral or Pisot base $\beta > 1$; it associates to each sequence $(\omega_i)_{i \in \mathbb{N}} \in \{0, 1, \dots, d-1\}^{\mathbb{N}}$ the sequence $(\varepsilon_i)_{i \in \mathbb{Z}} \in \{0, 1, \dots, \lceil \beta \rceil - 1\}^{\mathbb{Z}}$ satisfying both conditions:

$$(19) \quad \begin{aligned} \sum_{i \in \mathbb{N}} \frac{\omega_i}{\beta^i} &= \sum_{i \in \mathbb{Z}} \frac{\varepsilon_i}{\beta^i} \\ \forall i \in \mathbb{Z}, \sum_{j>i} \frac{\varepsilon_j}{\beta^j} &< \frac{1}{\beta^i} \text{ (Parry admissibility condition [11]).} \end{aligned}$$

Clearly $\sum_{i \in \mathbb{Z}} \frac{\varepsilon_i}{\beta^i}$ is not a bi-infinite sum but a sum of the form $\sum_{i > -h} \frac{\varepsilon_i}{\beta^i}$ with positive first term $\varepsilon_{-(h-1)}\beta^{h-1}$. It is the Parry expansion [11] of the real $x = \sum_{i \in \mathbb{N}} \frac{\omega_i}{\beta^i}$, and $h \in \mathbb{Z}$ is the integer such that $\beta^{h-1} \leq x < \beta^h$.

6.1. The transducer associated to β and d . The states of the transducer are the carries of the normalization of the sequences $(\omega_i)_{i \in \mathbb{N}} \in \{0, 1, \dots, d-1\}^{\mathbb{N}}$. More precisely given $i \in \mathbb{Z}$, the sum $\sum_{j > i} \frac{\omega_j}{\beta^j}$ is not equal to $\sum_{j > i} \frac{\varepsilon_j}{\beta^j}$ but to $\sum_{j > i} \frac{\varepsilon_j}{\beta^j}$ plus a real number that we denote by $\frac{q_i}{\beta^i}$. So we call "the i^{th} carry", the real q_i defined by the formulas

$$(20) \quad \frac{q_i}{\beta^i} = \sum_{j > i} \frac{\omega_j - \varepsilon_j}{\beta^j} = \sum_{j \leq i} \frac{\varepsilon_j - \omega_j}{\beta^j}$$

understanding that $\omega_j = 0$ for $j \leq 0$. They imply that

$$(21) \quad q_i + \omega_i = \beta q_{i-1} + \varepsilon_i \text{ and } q_i \in (-1, \alpha], \alpha := \frac{d-1}{\beta-1}.$$

This allows to define the transducer of normalization:

- The set of states is

$$S := \left\{ \sum_{j=0}^{i-1} \alpha_j \beta^j : i \in \mathbb{N}, \alpha_j \in \mathbb{Z}, -d < \alpha_j < \lceil \beta \rceil \right\} \cap (-1, \alpha].$$

Clearly the carries defined in (20) belong to S . From Garsia's separation lemma [10] S is finite, let

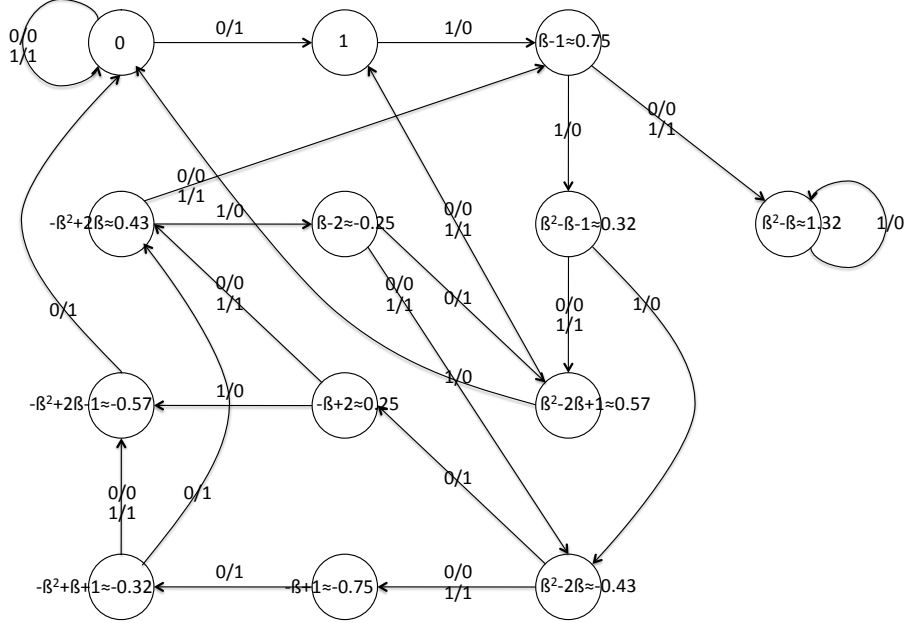
$$\{\mathbf{i}_0 = 0, \mathbf{i}_1, \dots, \mathbf{i}_{a-1}\} := S.$$

- The arrows have the form $q \xrightarrow{\omega / \varepsilon} \beta q - \omega + \varepsilon$.

Notice that the arrows are in the opposite direction of the ones considered in integral base because, except in this case, this is more natural: as seen in Example 14, the Euclidean division used in Proposition 10 do not have analogue in non-integral base. Using (21), the couple $((\omega_i)_{i \in \mathbb{Z}}, (\varepsilon_i)_{i \in \mathbb{Z}})$ formed by any sequence and the normalized sequence is represented by a path.

We construct at the same time the set S and the transducer: this set contains 0 and, for any $q \in S$, it contains $\beta q - \omega + \varepsilon$ for any $\omega \in \{0, 1, \dots, d-1\}$ and $\varepsilon \in \{0, 1, \dots, \lceil \beta \rceil - 1\}$ such that $\beta q - \omega + \varepsilon \in (-1, \alpha]$.

Example 14. If β is the positive root of $X^3 - 2X^2 + X - 1$ and $d = 2$, the set S has thirteen elements and the transducer is the following:



6.2. How to normalize a sequence $(\omega_i)_{i \in \mathbb{N}}$? We first notice that in a non-integral base there do not exist a literal transducer of normalization. For instance in base $\beta = \frac{1+\sqrt{5}}{2}$, if we normalize from the left to the right a sequence of the form $(01)^n x$ without knowing if the digit x is 0 or 1, we obtain $(01)^n 0$ if $x = 0$ and $1(00)^n$ if $x = 1$. So all the terms of the normalized sequence depend on the value of x , while the successive carries cannot depend on x because x is at the right. As well if we normalize from the right to the left a sequence of the form $x(11)^n$ without knowing if the digit x is 0 or 1, we obtain $(10)^n 0$ if $x = 0$ and $(10)^n 01$ if $x = 1$.

Proposition 15. (i) Let $(\omega_i)_{i \in \mathbb{N}} \in \{0, 1, \dots, d-1\}^{\mathbb{N}}$. The normalized sequence defined in (19) is the unique admissible sequence which is the output label of a bi-infinite path of input label $\dots 000\omega_1\omega_2\omega_3\dots$.

(ii) Let $\omega_1 \dots \omega_k \in \{0, 1, \dots, d-1\}^k$. If there exist an admissible sequence $\varepsilon_{-(h-1)} \dots \varepsilon_k$ such that $\varepsilon_{-(h-1)} \neq 0$ and

$$\omega_1 \beta^{k-1} + \dots + \omega_k \beta^0 = \varepsilon_{-(h-1)} \beta^{k+h-1} + \dots + \varepsilon_k \beta^0,$$

then $\varepsilon_{-(h-1)} \dots \varepsilon_k$ is the unique admissible output label of a path of input label $0 \dots 0\omega_1 \dots \omega_k$, with initial and final state 0.

Proof. We consider a bi-infinite path with input label $\dots 000\omega_1\omega_2\omega_3\dots$. From the definition of the transducer, the states and the output label

satisfy for any $i < i'$ in \mathbb{Z}

$$(22) \quad \frac{q_i}{\beta^i} = \sum_{i < j \leq i'} \frac{\omega_j - \varepsilon_j}{\beta^j} + \frac{q_{i'}}{\beta^{i'}}.$$

In the case of the item (i), since $\frac{q_{i'}}{\beta^{i'}}$ has a null limit when $i' \rightarrow +\infty$, as well as $\frac{q_i}{\beta^i}$ when $i \rightarrow -\infty$ because $q_{i-1} < q_i$ when $\omega_i = 0$, we deduce from (22) that $\sum_{i \in \mathbb{Z}} \frac{\varepsilon_i}{\beta^i}$ is the Parry expansion of $\sum_{i \in \mathbb{N}} \frac{\omega_i}{\beta^i}$.

In the case of (ii), we replace i' by k in (22) so $q_{i'}$ is null, as well as q_i for i less than or equal to some integer $-h$. Then $\sum_{i=-(h-1)}^k \frac{\varepsilon_i}{\beta^i}$ is the

Parry expansion of $\sum_{i=0}^k \frac{\omega_i}{\beta^i}$. □

6.3. The number of redundant representations. For any finite sequence $\varepsilon_1 \dots \varepsilon_k \in \{0, 1, \dots, d-1\}^k$ we denote by $\mathcal{N}(\varepsilon_1 \dots \varepsilon_k)$ the number of $\omega_1 \dots \omega_k \in \{0, 1, \dots, d-1\}^k$ such that

$$(23) \quad \omega_1 \beta^{k-1} + \dots + \omega_k \beta^0 = \varepsilon_1 \beta^{k-1} + \dots + \varepsilon_k \beta^0.$$

Notice that, this time, the carries $q_i = \sum_{j>i} \frac{\omega_j - \varepsilon_j}{\beta^{j-i}}$ do not belong to $(-1, \alpha]$ but to $(-\alpha, \alpha)$. So the set of states we consider is

$$S' := \left\{ \sum_{j=0}^{i-1} \alpha_j \beta^j : i \in \mathbb{N}, \alpha_j \in \mathbb{Z}, -d < \alpha_j < d \right\} \cap (-\alpha, \alpha).$$

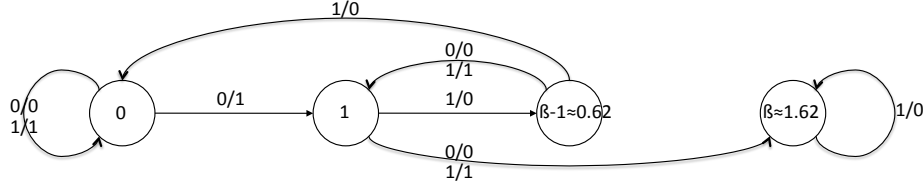
We denote by $\mathbf{i}'_0 = 0, \mathbf{i}'_1, \dots, \mathbf{i}'_{a'-1}$ the elements of S' . Now we count the number of $\omega_1 \dots \omega_k \in \{0, 1, \dots, d-1\}^k$ that satisfy (23), by using the $(0, 1)$ -matrices $N_\varepsilon = (n_{ij})_{\substack{0 \leq i \leq a'-1 \\ 0 \leq j \leq a'-1}}$, $\varepsilon \in \{0, 1, \dots, d-1\}$, defined by

$$n_{ij} = 1 \Leftrightarrow \beta \mathbf{i}'_i - \mathbf{i}'_j + \varepsilon \in \{0, 1, \dots, d-1\}.$$

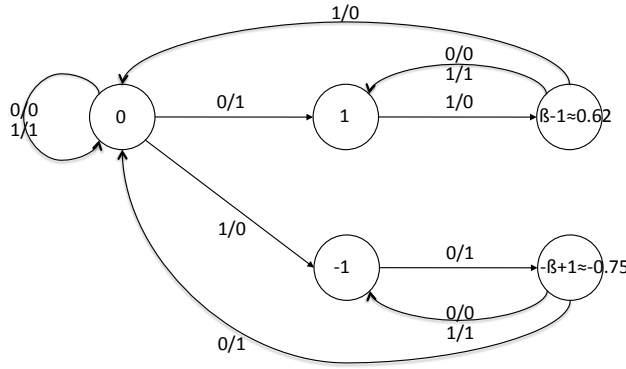
We obtain the following

Proposition 16. $\mathcal{N}(\varepsilon_1 \dots \varepsilon_k) = (1 \ 0 \ \dots \ 0) N_{\varepsilon_1} \dots N_{\varepsilon_k} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$

Example 17. If $\beta = \frac{1+\sqrt{5}}{2}$, the set S has four elements and elements and S' has five elements. The transducer with set of states S' is



and the transducer with set of states S'



6.4. The Markov chain associated to a Bernoulli convolution.

Unfortunately the normalization map is not continuous: for instance in base $\beta = \frac{1+\sqrt{5}}{2}$, $\lim_{k \rightarrow \infty} \mathbf{n}(1^k 0^\infty) \neq \mathbf{n}(1^\infty)$. So, to prove that the Bernoulli convolution defined in (10) is sofic, it is not possible to apply directly the methods of §2. Nevertheless it is possible to prove it by means of the following completed transducer, which is in fact a topological Markov chain. The states of the completed transducer are the couples

$$(\varepsilon, \mathbf{i}_i) \in \mathcal{S} := \{0, \dots, \lceil \beta \rceil - 1\} \times \{\mathbf{i}_0, \dots, \mathbf{i}_{a-1}\}.$$

Let us define some transition probabilities between the states of the completed transducer: when we have, in the initial transducer, two successive arrows $\xrightarrow{\omega / \varepsilon} q \xrightarrow{\omega' / \varepsilon'} q'$ we put

$$(24) \quad p_{(\varepsilon, q), (\varepsilon', q')} := p_{\omega'} = p_{\beta q - q' + \varepsilon'} \quad (\text{independent on } \varepsilon)$$

assuming that $p_\omega = 0$ for any $\omega \in \mathbb{R} \setminus \{0, 1, \dots, d - 1\}$.

Example 18. In Example 14 the graph of the completed transducer is

and $\begin{pmatrix} v_0 \\ \vdots \\ v_{13} \end{pmatrix} = \begin{pmatrix} v_{14} \\ \vdots \\ v_{26} \end{pmatrix} = \begin{pmatrix} v_{27} \\ \vdots \\ v_{39} \end{pmatrix}$ is a positive eigenvector of $M_0 + M_1 M_0 + M_1^2 M_0^2$.

Let now β be a Pisot number, $d \geq \beta$ an integer, and η the Bernoulli convolution defined in (10). Generalizing (13), we consider the restriction of η to a interval $[q, q+1)$, where $q \in S$ is one of the states of the transducer defined in Subsection 6.1. Especially we want to compute by means of matrices the measure η_q defined on the symbolic space $\{0, 1, \dots, [\beta] - 1\}^{\mathbb{N}}$ by

$$(25) \quad \eta_q[\varepsilon_1 \dots \varepsilon_n] := \frac{\eta(q + I_{\varepsilon_1 \dots \varepsilon_n})}{\eta([q, q+1))}$$

where $I_{\varepsilon_1 \dots \varepsilon_n}$ is the set of elements of $[0, 1)$ whose Parry expansion begins by $\varepsilon_1 \dots \varepsilon_n$.

- On the following diagram the map s_q associates, to any sequence $(\omega_i)_{i \in \mathbb{N}}$ in the set

$$\Omega_q := \left\{ (\omega_i)_{i \in \mathbb{N}} : 0 \leq \omega_i \leq d-1, \sum \frac{\omega_i}{\beta^i} \in [q, q+1) \right\},$$

the Parry-admissible sequence $(\varepsilon_i)_{i \in \mathbb{N}}$ such that

$$\sum_{i \in \mathbb{N}} \frac{\omega_i}{\beta^i} = q + \sum_{i \in \mathbb{N}} \frac{\varepsilon_i}{\beta^i}.$$

The one-to-one map $S_q : \Omega_q \rightarrow \mathcal{S}^{\mathbb{N}}$ is defined by

$$S_q((\omega_i)_{i \in \mathbb{N}}) := ((\varepsilon_i, q_i))_{i \in \mathbb{N}}$$

where $q_0 = q$ and $\forall i \in \mathbb{N}$, $q_i = \beta q_{i-1} - \omega_i + \varepsilon_i$. And $\pi : \mathcal{S}^{\mathbb{N}} \rightarrow \{0, \dots, [\beta] - 1\}^{\mathbb{N}}$ is the projection. The diagram is

$$\begin{array}{ccc} (\omega_i)_{i \in \mathbb{N}} \in \Omega_q & \xrightarrow{S_q} & ((\varepsilon_i, q_i))_{i \in \mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \\ & \searrow s_q & \downarrow \pi \\ & & (\varepsilon_i)_{i \in \mathbb{N}} \in \mathcal{Adm} \end{array}$$

The probability measure P_q on Ω_q is simply $\frac{P(\cdot)}{P(\Omega_q)}$, and its image m_q by the one-to-one map S_q satisfies

$$(26) \quad m_q[(\varepsilon_1, q_1) \dots (\varepsilon_k, q_k)] = \frac{1}{P(\Omega_q)} p_{\omega_1} \dots p_{\omega_k} \pi_k$$

where $\omega_i = \beta q_{i-1} - q_i + \varepsilon_i$ for $1 \leq i \leq k$ and

$$(27) \quad \pi_k = P \left(\sum_{i \geq 1} \frac{\omega_{k+i}}{\beta^i} \in [0, M_k) \right),$$

M_k being the positive real such that $I_{\varepsilon_1 \dots \varepsilon_n} = [0, \frac{M_k}{\beta^k})$.

So m_q is not necessarily Markov but it has in a sense some transition probabilities, denoted by $p_{(\varepsilon, q), (\varepsilon', q')}$ in (24).

– Let us define now the Markov measure, assuming from now that the expansion of the unity is periodic, in order to simplify the formula. The expansion of the unity [11] is a sequence $(\alpha_i)_{i \in \mathbb{N}}$ such that

$$1 = \sum_{i \in \mathbb{N}} \frac{\alpha_i}{\beta^i} \quad \text{and} \quad \forall i \geq 0, \quad 0 < \sum_{j > i} \frac{\alpha_j}{\beta^j} \leq \frac{1}{\beta^i}$$

We suppose that this sequence has a period, let T . Then the β -shift, i.e. the set of the sequences of nonnegative integers $(\varepsilon_i)_{i \in \mathbb{N}}$ such that

$$\sum_{j > i} \frac{\varepsilon_j}{\beta^j} \leq \frac{1}{\beta^i} \text{ for any } i \geq 0, \text{ is coded by the words } w \text{ of the form}$$

$$w = \alpha_1 \dots \alpha_{i-1} \alpha'_i \text{ with } 0 \leq i \leq T \text{ and } 0 \leq \alpha'_i < \alpha_i$$

and the word $w_{\max} = \alpha_1 \dots \alpha_T$. Let $\mathcal{W} = \{v_1, \dots, v_{|\mathcal{W}|}\}$ be this set of words.

Proposition 19. *When $\varepsilon_1 \dots \varepsilon_k$ is a concatenation of words of \mathcal{W} ,*

$$(28) \quad m_q[(\varepsilon_1, q_1) \dots (\varepsilon_k, q_k)] = \frac{P(\Omega_{q_k})}{P(\Omega_q)} p_{(0, q), (\varepsilon_1, q_1)} \dots p_{(\varepsilon_{k-1}, q_{k-1}), (\varepsilon_k, q_k)}.$$

Proof. If $\varepsilon_1 \dots \varepsilon_k$ is a concatenation of words of \mathcal{W} it is clear, from the definition of the Parry-admissible sequences, that the interval $I_{\varepsilon_1 \dots \varepsilon_k}$ is simply $[0, \frac{1}{\beta^k})$ when $\varepsilon_1 \dots \varepsilon_k$. So in (27) we have $M_k = 1$, and in this case (26) is equivalent to (28). \square

Theorem 20. *The measure η_q defined in (25) is sofic. It is the projection of the Markov measure of transition matrix*

$$M := \Delta^{-1} \begin{pmatrix} M_{v_1} & \dots & M_{v_{|\mathcal{W}|}} \\ \vdots & \ddots & \vdots \\ M_{v_1} & \dots & M_{v_{|\mathcal{W}|}} \end{pmatrix} \Delta$$

where

$$w = \prod_i \varepsilon_i \Rightarrow M_w := \prod_i \begin{pmatrix} P_{(0, i_0), (\varepsilon_i, i_0)} & \dots & P_{(0, i_0), (\varepsilon_i, i_{a-1})} \\ \vdots & \ddots & \vdots \\ P_{(0, i_{a-1}), (\varepsilon_i, i_0)} & \dots & P_{(0, i_{a-1}), (\varepsilon_i, i_{a-1})} \end{pmatrix}$$

and Δ is the diagonal matrix with diagonal entries v_0, \dots, v_{a-1} ($\lceil \beta \rceil$ times), the v_i being the entries of the positive eigenvector of $\sum_{w \in \mathcal{W}} M_w$. The initial probabilities are the entries of the $(i+1)^{\text{th}}$ row of ΔM , where i the integer such that $q = \mathbf{i}_i$. Moreover, for any $w_1, \dots, w_\ell \in \mathcal{W}$

$$(29) \quad \eta_q[w_1 \dots w_\ell] = \underset{\text{((i+1)th position)}}{\begin{pmatrix} 0 & \dots & 1 & \dots & 0 \end{pmatrix}} M_{w_1} \dots M_{w_\ell} \begin{pmatrix} v_0 \\ \dots \\ v_{a-1} \end{pmatrix}.$$

Proof. Let $\varepsilon_1 \dots \varepsilon_k = w_1 \dots w_\ell$, and let us sum the r.h.s. of (28) for any $q_1, \dots, q_{|w_1|-1}$, then for any $q_{|w_1|+1}, \dots, q_{|w_1|+|w_2|-1}$, and so one. We obtain – with the notation $q'_j = q_{|w_1|+\dots+|w_j|}$

$$(30) \quad \sum m_q[(\varepsilon_1, q_1) \dots (\varepsilon_k, q_k)] = \frac{P(\Omega_{q_k})}{P(\Omega_r)} p'_{(0,r),(w_1,q'_1)} \dots p'_{(w_{\ell-1},q'_{\ell-1}),(w_\ell,q'_\ell)}$$

where $p'_{(w_{j-1},r'_{j-1}),(w_j,r'_j)}$ is the (i, i') -entry of the matrix M_{w_j} if r'_{j-1} is the i^{th} state and r'_j the i'^{th} state. This sum represents the probability that the canonical digits $\varepsilon_1, \dots, \varepsilon_k$ and the carries q'_1, \dots, q'_ℓ have some given values. Considering the case $k = 1$ we deduce that

$$\sum_{(\varepsilon_1, q_1)} \frac{P(\Omega_{q_1})}{P(\Omega_q)} p'_{(0,q),(\varepsilon'_1, q'_1)} = 1,$$

because m_q is a probability. So the $P(\Omega_q)$ for $q \in S$ are the entries of a right eigenvector of $\sum_{w \in \mathcal{W}} M_w$ and, according to the notations of the theorem, we can put $v_i = P(\Omega_{\mathbf{i}_i})$ and deduce the first assertion of the theorem. We obtain (29) by summing (30) for any $q'_1, \dots, q'_\ell \in S$. \square

7. WHAT ABOUT THE NUMBER OF REPRESENTATIONS IN A POLYNOMIAL BASE?

Let $\mathcal{N}_{\text{polynomial}}(n)$ be the number of representations $n = \sum \omega_k G_k$ with $\omega_k \leq d-1$, in the polynomial base $G_k = a_s k^s + \dots + a_0 k^0$. The expected result:

$$\mathcal{N}_{\text{polynomial}}(n) \sim \frac{e^{\alpha n^{\frac{1}{s+1}}}}{\beta n^{\frac{2s+1}{2s+2}}}$$

with α, β constants does not allow to consider some level sets. This formula was proved in the case $d = 2, G_k = k$ by Hardy and Ramanujan (1918) and generalized by several authors. For instance Roth and Szekeres (1954) have a formula, assuming only that G_k has the order of k^s , but it is less precise in the way that the exponent $\alpha n^{\frac{1}{s+1}}$ must be replaced by $n^{\frac{1}{s+1} + \varepsilon_n}$ with $\lim_{n \rightarrow \infty} \varepsilon_n = 0$, and the exponent $\beta n^{\frac{2s+1}{2s+2}}$ by $n^{\frac{2s+1}{2s+2} + \varepsilon'_n}$ with $\lim_{n \rightarrow \infty} \varepsilon'_n = 0$.

8. REPRESENTATIONS IN BASE $b = 2$ WITH THREE DIGITS

The number of representations of the integer

$$n = {}_2 1^{a_s} 0^{a_{s-1}} \dots 0^{a_1} 1^{a_0},$$

with digits in $\{0, 1, 2\}$, let $\mathcal{N}(n)$, is – according to [3, Proposition 6.11] – the denominator q_s of the continued fraction $[0; a_1, \dots, a_s] :=$

$$\frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_s}}}$$

such that

$$(31) \quad \lim_{\substack{n \rightarrow \infty \\ n \in S}} \frac{\log \mathcal{N}(n)}{\log n} = \alpha_0.$$

From the classical formula giving the denominator of the continued fraction in terms of a_1, \dots, a_s ,

$$\alpha_0 = \lim_{k \rightarrow \infty} \frac{1}{2^k \log(2^k)} \sum_{s, a_1, \dots, a_s} \log \left(\sum_{h, i_0, \dots, i_{h+1}} \prod_{j=0}^h a_{i_j, i_{j+1}} \right),$$

summing for $s, a_1, \dots, a_s \in \mathbb{N}$ such that $\sum a_i \leq k$ and for $h \in \mathbb{N} \cup \{0\}$, $(i_0, \dots, i_{h+1}) \in \{0\} \times \mathbb{N}^h \times \{s+1\}$, where the notation $a_{i', i}$ means a_i if $i - i'$ is an odd positive integer and 0 otherwise. And by the following computation $\alpha_0 \simeq 0.56$:

*f := proc(n); if n = 2 * floor((1/2) * n) then f((1/2) * n) + f((1/2) * n - 1) else f((1/2) * n - 1/2) end if end proc;*

f(0) := 1;

s := seq(ln(f(n)), n = 0..16383);

*alpha_0 := evalf(sum(s[n], n = 8192..16383)/(8192 * ln(8192)));*

Proposition 21. *The constant α_0 defined in (31) is also the Lebesgue-a.e. value of $\lim_{s \rightarrow \infty} \frac{\log q_s(x(t))}{s \log 4}$, where $x(t) = [0; a_1(t), a_2(t), \dots]$ for $t = {}_2 0.1^{a_1(t)} 0^{a_2(t)} 1^{a_3(t)} \dots$ in the interval $[\frac{1}{2}, 1)$.*

Proof. From [6, Remark 21], $\alpha_0 \log 2 = \gamma$, defined in [8, Theorem 1.1]. Now by [6, Corollary 1.2], $\frac{\log 3}{\log 2} - \alpha_0$ is the almost sure value of the local dimension of the Bernoulli convolution $\eta = \eta_{2,3}$ in base 2 with three digits. Let $\varepsilon_1 \varepsilon_2 \dots$ be the canonical 2-expansion of $t \in [\frac{1}{2}, 1)$, and let

$$n = n(t, k) = \varepsilon_1 2^{k-1} + \dots + \varepsilon_k 2^0.$$

By (16) one has

$$(32) \quad \mathcal{N}(n) + \mathcal{N}(n-1) = 2 \cdot 3^k \eta(I_{\varepsilon_1 \dots \varepsilon_k}).$$

Now for $k = k(s) = a_1(t) + \dots + a_s(t)$ one has $\mathcal{N}(n(t, k)) = q_s(x(t))$. Using the inequality $\frac{1}{C \log n} \leq \frac{\mathcal{N}(n)}{\mathcal{N}(n-1)} \leq C \log n$ proved in [6, Lemma 9 (ii)], the relation (32) implies that

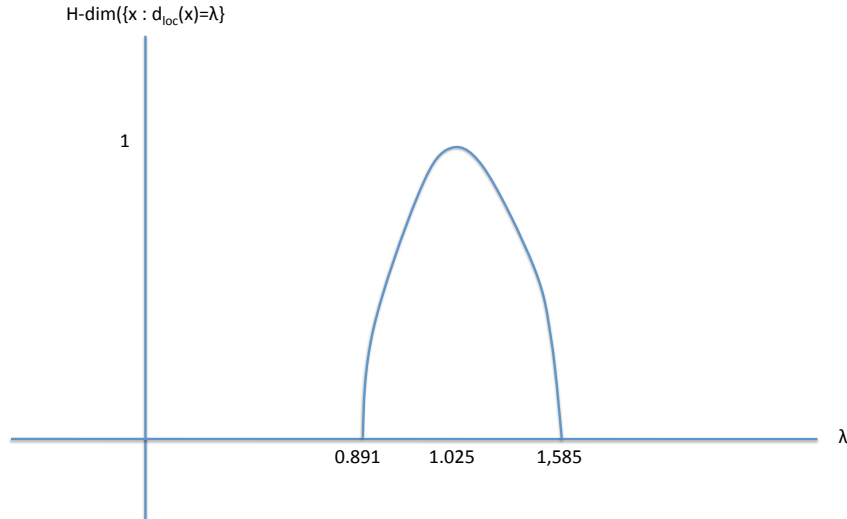
$$(33) \quad \lim_{s \rightarrow \infty} \frac{\log q_s(x(t))}{k(s)} = \lim_{k \rightarrow \infty} \frac{\log (2 \cdot 3^k \eta(I_{\varepsilon_1 \dots \varepsilon_k}))}{k}$$

and, $k(s)$ being Lebesgue-almost surely equivalent to $2s$, the local dimension of η at t , let $d_{\text{loc}}(t)$, satisfies

$$\lim_{s \rightarrow \infty} \frac{\log q_s(x(t))}{s \log 4} = \frac{\log 3}{\log 2} - d_{\text{loc}}(t) = \alpha_0 \text{ for Lebesgue-a.e. } t.$$

□

The Levy constant $\lim_{s \rightarrow \infty} \frac{\log q_s(x)}{s} = \frac{\pi^2}{12 \log 2} \simeq 1.18657$ for Lebesgue-a.e. x , is larger than the value of $\lim_{s \rightarrow \infty} \frac{\log q_s(x(t))}{s} = \alpha_0 \log 4 \simeq 0.78$ for Lebesgue-a.e. t . Now it is well known that the partial quotients $a_i(x)$ of Lebesgue-a.e. real x satisfy $\lim_{s \rightarrow \infty} \frac{a_1(x) + \dots + a_s(x)}{s} = \infty$. For any real x that satisfies the Levy condition and $\lim_{s \rightarrow \infty} \frac{a_1(x) + \dots + a_s(x)}{s} = \infty$, and for the real t such that $x(t) = x$, the l.h.s. in (33) tends to 0 hence $d_{\text{loc}}(t) = \frac{\log 3}{\log 2} \simeq 1.585$ (the largest local dimension of η , according to [7, Theorem 2.5]). Since $\frac{\log 3}{\log 2} - \alpha_0 \simeq 1.025$ and $\frac{\log 3}{\log 2} - \frac{\log \frac{1+\sqrt{5}}{2}}{\log 2} \simeq 0.891$, the graph of the singularity spectrum of $\eta_{2,3}$ is



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