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# Semilinear fractional elliptic equations with gradient nonlinearity involving measures

Huyuan Chen<sup>1</sup>

Departamento de Ingeniería Matemática  
Universidad de Chile, Chile

Laurent Véron<sup>2</sup>

Laboratoire de Mathématiques et Physique Théorique  
Université François Rabelais, Tours, France

## Abstract

We study the existence of solutions to the fractional elliptic equation (E1)  $(-\Delta)^\alpha u + \epsilon g(|\nabla u|) = \nu$  in a bounded regular domain  $\Omega$  of  $\mathbb{R}^N$  ( $N \geq 2$ ), subject to the condition (E2)  $u = 0$  in  $\Omega^c$ , where  $\epsilon = 1$  or  $-1$ ,  $(-\Delta)^\alpha$  denotes the fractional Laplacian with  $\alpha \in (1/2, 1)$ ,  $\nu$  is a Radon measure and  $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is a continuous function. We prove the existence of weak solutions for problem (E1)-(E2) when  $g$  is subcritical. Furthermore, the asymptotic behavior and uniqueness of solutions are described when  $\nu$  is Dirac mass,  $g(s) = s^p$ ,  $p \geq 1$  and  $\epsilon = 1$ .

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**Key words:** Fractional Laplacian, Radon measure, Green kernel, Dirac mass.

**MSC2010:** 35R11, 35J61, 35R06

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<sup>1</sup>chenhuyuan@yeah.net

<sup>2</sup>Laurent.Veron@lmpt.univ-tours.fr

# 1 Introduction

Let  $\Omega \subset \mathbb{R}^N (N \geq 2)$  be an bounded  $C^2$  domain and  $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$  be a continuous function. The purpose of this paper is to study the existence of weak solutions to the semilinear fractional elliptic problem with  $\alpha \in (1/2, 1)$ ,

$$\begin{aligned} (-\Delta)^\alpha u + \epsilon g(|\nabla u|) &= \nu & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c, \end{aligned} \quad (1.1)$$

where  $\epsilon = 1$  or  $-1$  and  $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$  with  $\beta \in [0, 2\alpha - 1]$ . Here  $\rho(x) = \text{dist}(x, \Omega^c)$  and  $\mathfrak{M}(\Omega, \rho^\beta)$  is the space of Radon measures in  $\Omega$  satisfying

$$\int_{\Omega} \rho^\beta d|\nu| < +\infty. \quad (1.2)$$

In particular, we denote  $\mathfrak{M}^b(\Omega) = \mathfrak{M}(\Omega, \rho^0)$ . The associated positive cones are respectively  $\mathfrak{M}_+(\Omega, \rho^\beta)$  and  $\mathfrak{M}_+^b(\Omega)$ . According to the value of  $\epsilon$ , we speak of an absorbing nonlinearity the case  $\epsilon = 1$  and a source nonlinearity the case  $\epsilon = -1$ . The operator  $(-\Delta)^\alpha$  is the fractional Laplacian defined as

$$(-\Delta)^\alpha u(x) = \lim_{\epsilon \rightarrow 0^+} (-\Delta)_\epsilon^\alpha u(x),$$

where for  $\epsilon > 0$ ,

$$(-\Delta)_\epsilon^\alpha u(x) = - \int_{\mathbb{R}^N} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} \chi_\epsilon(|x - z|) dz \quad (1.3)$$

and

$$\chi_\epsilon(t) = \begin{cases} 0, & \text{if } t \in [0, \epsilon], \\ 1, & \text{if } t > \epsilon. \end{cases}$$

In a pioneering work, Benilan and Brezis [7], [1] studied the existence and uniqueness of the solution to the semilinear Dirichlet elliptic problem

$$\begin{aligned} -\Delta u + h(u) &= \nu & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{aligned} \quad (1.4)$$

where  $\nu$  is a bounded measure in  $\Omega$  and the function  $h$  is nondecreasing, positive on  $(0, +\infty)$  and satisfies that

$$\int_1^{+\infty} (h(s) - h(-s)) s^{-2\frac{N-1}{N-2}} ds < +\infty.$$

Later on, Véron [29] improved this result in replacing the Laplacian by more general uniformly elliptic second order differential operator, where  $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$  with  $\beta \in [0, 1]$  and  $h$  is a nondecreasing function satisfying

$$\int_1^{+\infty} (h(s) - h(-s)) s^{-2\frac{N+\beta-1}{N+\beta-2}} ds < +\infty.$$

The general semilinear elliptic problems involving measures such as the equations involving boundary measures have been intensively studied; it was initiated by Gmira and Véron [16] and then this subject has been extended in various ways, see [4, 6, 18, 19, 20, 21] for details and [22] for a general panorama. In a recent work, Nguyen-Phuoc and Véron [24] obtained the existence of solutions to the viscous Hamilton-Jacobi equation

$$\begin{aligned} -\Delta u + h(|\nabla u|) &= \nu & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{aligned} \tag{1.5}$$

when  $\nu \in \mathfrak{M}^b(\Omega)$ ,  $h$  is a continuous nondecreasing function vanishing at 0 which satisfies

$$\int_1^{+\infty} h(s) s^{-\frac{2N-1}{N-1}} ds < +\infty.$$

During the last years there has also been a renewed and increasing interest in the study of linear and nonlinear integro-differential operators, especially, the fractional Laplacian, motivated by great applications in physics and by important links on the theory of Levy processes. Many estimates of its Green kernel and generation formula can be found in the references [8, 11, 12, 13, 10, 14, 26, 28, 27]. Recently, Chen and Véron [13] studied the semilinear fractional elliptic equation

$$\begin{aligned} (-\Delta)^\alpha u + h(u) &= \nu & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c, \end{aligned} \tag{1.6}$$

where  $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$  with  $\beta \in [0, \alpha]$ . They proved the existence and uniqueness of the solution to (1.6) when the function  $h$  is nondecreasing and satisfies

$$\int_1^{+\infty} (h(s) - h(-s)) s^{-1-k_{\alpha,\beta}} ds < +\infty,$$

where

$$k_{\alpha,\beta} = \begin{cases} \frac{N}{N-2\alpha}, & \text{if } \beta \in [0, \frac{N-2\alpha}{N}\alpha], \\ \frac{N+\alpha}{N-2\alpha+\beta}, & \text{if } \beta \in (\frac{N-2\alpha}{N}\alpha, \alpha]. \end{cases} \tag{1.7}$$

Our interest in this article is to investigate the existence of weak solutions to fractional equations involving nonlinearity in the gradient term and with Radon measure. In order the fractional Laplacian be the dominant operator in terms of order of differentiation, it is natural to assume that  $\alpha \in (1/2, 1)$ .

**Definition 1.1** *We say that  $u$  is a weak solution of (1.1), if  $u \in L^1(\Omega)$ ,  $\nabla u \in L^1_{loc}(\Omega)$ ,  $g(|\nabla u|) \in L^1(\Omega, \rho^\alpha dx)$  and*

$$\int_{\Omega} [u(-\Delta)^\alpha \xi + \epsilon g(|\nabla u|)\xi] dx = \int_{\Omega} \xi d\nu, \quad \forall \xi \in \mathbb{X}_\alpha, \tag{1.8}$$

where  $\mathbb{X}_\alpha \subset C(\mathbb{R}^N)$  is the space of functions  $\xi$  satisfying:

(i)  $\text{supp}(\xi) \subset \bar{\Omega}$ ,

(ii)  $(-\Delta)^\alpha \xi(x)$  exists for all  $x \in \Omega$  and  $|(-\Delta)^\alpha \xi(x)| \leq C$  for some  $C > 0$ ,

(iii) there exist  $\varphi \in L^1(\Omega, \rho^\alpha dx)$  and  $\varepsilon_0 > 0$  such that  $|(-\Delta)_\varepsilon^\alpha \xi| \leq \varphi$  a.e. in  $\Omega$ , for all  $\varepsilon \in (0, \varepsilon_0]$ .

We denote by  $G_\alpha$  the Green kernel of  $(-\Delta)^\alpha$  in  $\Omega$  and by  $\mathbb{G}_\alpha[\cdot]$  the associated Green operator defined by

$$\mathbb{G}_\alpha[\nu](x) = \int_{\Omega} G_\alpha(x, y) d\nu(y), \quad \forall \nu \in \mathfrak{M}(\Omega, \rho^\alpha). \quad (1.9)$$

Using bounds on  $\mathbb{G}_\alpha[\nu]$ , we obtain in section 2 some crucial estimates which will play an important role in our construction of weak solutions. Our main result in the case  $\epsilon = 1$  is the following.

**Theorem 1.1** *Assume  $\epsilon = 1$  and let  $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$  be a continuous function verifying  $g(0) = 0$  and*

$$\int_1^{+\infty} g(s) s^{-1-p_\alpha^*} ds < +\infty, \quad (1.10)$$

where

$$p_\alpha^* = \frac{N}{N - 2\alpha + 1}. \quad (1.11)$$

Then for any  $\nu \in \mathfrak{M}_+(\Omega, \rho^{2\alpha-1})$ , problem (1.1) admits a nonnegative weak solution  $u_\nu$  which satisfies

$$u_\nu \leq \mathbb{G}_\alpha[\nu]. \quad (1.12)$$

As in the case  $\alpha = 1$ , uniqueness remains an open question. We note that the critical value  $p_\alpha^*$  is independent of  $\beta$ . A similar fact was first observed when dealing with problem (1.6) where the critical value  $k_{\alpha, \beta}$  defined by (1.7) does not depend on  $\beta \in [0, \frac{N-2\alpha}{N}\alpha]$ .

When  $\epsilon = -1$ , we have to consider the critical value  $p_{\alpha, \beta}^*$  which depends truly on  $\beta$  and is expressed by

$$p_{\alpha, \beta}^* = \frac{N}{N - 2\alpha + 1 + \beta}. \quad (1.13)$$

We observe that  $p_{\alpha, 0}^* = p_\alpha^*$  and  $p_{\alpha, \beta}^* < p_\alpha^*$  when  $\beta > 0$ . In the source case, the assumptions on  $g$  are of a different nature than in the absorption case, namely

(G)  $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is a continuous function which satisfies

$$g(s) \leq c_1 s^p + \sigma_0, \quad \forall s \geq 0, \quad (1.14)$$

for some  $p < p_{\alpha, \beta}^*$  where  $c_1 > 0$  and  $\sigma_0 > 0$ .

Our main result concerning the source case is the following.

**Theorem 1.2** *Assume  $\epsilon = -1$ ,  $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$  with  $\beta \in [0, 2\alpha - 1)$  is non-negative,  $g$  satisfies (G) and*

*(i)  $p \in (0, 1)$ , or*

*(ii)  $p = 1$  and  $c_1$  is small enough, or*

*(iii)  $p \in (1, p_{\alpha, \beta}^*)$ ,  $\sigma_0$  and  $\|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}$  are small enough.*

*Then problem (1.1) admits a weak nonnegative solution  $u_\nu$  which satisfies*

$$u_\nu \geq \mathbb{G}_\alpha[\nu]. \quad (1.15)$$

Recently, Bidaut-Véron, García-Huidobro and Véron in [5] obtained the existence of a renormalized solution of

$$-\Delta_p u = |\nabla u|^q + \nu \quad \text{in } \Omega,$$

when  $\nu \in \mathfrak{M}^b(\Omega)$ . We make use of some idea in [5] in the proof of Theorem 1.2 and extend some results in [5] to elliptic equations involving  $(-\Delta)^\alpha$  with  $\alpha \in (1/2, 1)$  and  $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$  with  $\beta \in [0, 2\alpha - 1)$ .

In the last section we assume that  $\Omega$  contains 0 and give pointwise estimates of the positive solutions

$$\begin{aligned} (-\Delta)^\alpha u + |\nabla u|^p &= \delta_0 & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c, \end{aligned} \quad (1.16)$$

when  $0 < p < p_\alpha^*$ . Combining properties of the Riesz kernel with a bootstrap argument, we prove that any weak solution of (1.16) is regular outside 0 and is actually a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u + |\nabla u|^p &= 0 & \text{in } \Omega \setminus \{0\}, \\ u &= 0 & \text{in } \Omega^c. \end{aligned} \quad (1.17)$$

These pointwise estimates are quite easy to establish in the case  $\alpha = 1$ , but much more delicate when the diffusion operator is non-local. We give sharp asymptotics of the behaviour of  $u$  near 0 and prove that the solution of (1.16) is unique in the class of positive solutions.

The paper is organized as follows. In Section 2, we study the Green operator and prove the key estimate

$$\|\nabla \mathbb{G}_\alpha[\nu]\|_{M^{p_\alpha^*}(\Omega, \rho^\alpha dx)} \leq c_2 \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}.$$

Section 3 is devoted to prove Theorem 1.1 and Theorem 1.2. In Section 4, we consider the case where  $\epsilon = 1$  in (1.1) and  $\nu$  is a Dirac measure. We obtain precise asymptotic estimate and derive uniqueness.

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## 2 Preliminaries

### 2.1 Marcinkiewicz type estimates

In this subsection, we recall some definitions and properties of Marcinkiewicz spaces.

**Definition 2.1** Let  $\Theta \subset \mathbb{R}^N$  be a domain and  $\mu$  be a positive Borel measure in  $\Theta$ . For  $\kappa > 1$ ,  $\kappa' = \kappa/(\kappa - 1)$  and  $u \in L^1_{loc}(\Theta, d\mu)$ , we set

$$\|u\|_{M^\kappa(\Theta, d\mu)} = \inf \left\{ c \in [0, \infty] : \int_E |u| d\mu \leq c \left( \int_E d\mu \right)^{\frac{1}{\kappa'}}, \forall E \subset \Theta, E \text{ Borel} \right\} \quad (2.1)$$

and

$$M^\kappa(\Theta, d\mu) = \{u \in L^1_{loc}(\Theta, d\mu) : \|u\|_{M^\kappa(\Theta, d\mu)} < \infty\}. \quad (2.2)$$

$M^\kappa(\Theta, d\mu)$  is called the Marcinkiewicz space of exponent  $\kappa$ , or weak  $L^\kappa$ -space and  $\|\cdot\|_{M^\kappa(\Theta, d\mu)}$  is a quasi-norm.

**Proposition 2.1** [2, 9] Assume  $1 \leq q < \kappa < \infty$  and  $u \in L^1_{loc}(\Theta, d\mu)$ . Then there exists  $c_3 > 0$  dependent of  $q, \kappa$  such that

$$\int_E |u|^q d\mu \leq c_3 \|u\|_{M^\kappa(\Theta, d\mu)} \left( \int_E d\mu \right)^{1-q/\kappa},$$

for any Borel set  $E$  of  $\Theta$ .

The next estimate is the key-stone of the proof of Theorem 1.1.

**Proposition 2.2** Let  $\Omega \subset \mathbb{R}^N$  ( $N \geq 2$ ) be a bounded  $C^2$  domain and  $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$  with  $\beta \in [0, 2\alpha - 1]$ . Then

$$\|\nabla \mathbb{G}_\alpha[|\nu|]\|_{M^{p_\alpha^*}(\Omega, \rho^\alpha dx)} \leq c_2 \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}, \quad (2.3)$$

where  $\nabla \mathbb{G}_\alpha[|\nu|](x) = \int_\Omega \nabla_x G_\alpha(x, y) d|\nu|(y)$  and  $p_\alpha^*$  is given by (1.11).

*Proof.* For  $\lambda > 0$  and  $y \in \Omega$ , we set

$$\omega_\lambda(y) = \left\{ x \in \Omega \setminus \{y\} : |\nabla_x G_\alpha(x, y)| \rho^\alpha(x) > \lambda \rho^\beta(y) \right\}, \quad m_\lambda(y) = \int_{\omega_\lambda(y)} dx.$$

From [11], there exists  $c_4 > 0$  such that for any  $(x, y) \in \Omega \times \Omega$  with  $x \neq y$ ,

$$G_\alpha(x, y) \leq c_4 \min \left\{ \frac{1}{|x - y|^{N-2\alpha}}, \frac{\rho^\alpha(x)}{|x - y|^{N-\alpha}}, \frac{\rho^\alpha(y)}{|x - y|^{N-\alpha}} \right\}, \quad (2.4)$$

$$G_\alpha(x, y) \leq c_4 \frac{\rho^\alpha(y)}{\rho^\alpha(x)|x-y|^{N-2\alpha}}, \quad (2.5)$$

and by Corollary 3.3 in [3], we have

$$|\nabla_x G_\alpha(x, y)| \leq N G_\alpha(x, y) \max \left\{ \frac{1}{|x-y|}, \frac{1}{\rho(x)} \right\}. \quad (2.6)$$

This implies that for any  $s \in [0, 2\alpha - 1]$  there exists  $c_5 > 0$  such that

$$|\nabla_x G_\alpha(x, y)| \leq \max \left\{ \frac{c_5 \rho^\alpha(y)}{\rho^\alpha(x)|x-y|^{N-2\alpha+1}}, \frac{c_5 \rho^{\alpha-1-s}(x) \rho^s(y)}{|x-y|^{N-\alpha}} \right\}. \quad (2.7)$$

Therefore, for  $2\alpha - 1 - s \geq 0$ , we derive

$$\begin{aligned} |\nabla_x G_\alpha(x, y)| \rho^\alpha(x) &\leq \max \left\{ \frac{c_5 \rho^\alpha(y)}{|x-y|^{N-2\alpha+1}}, \frac{c_5 \rho^{2\alpha-1-s}(x) \rho^s(y)}{|x-y|^{N-\alpha}} \right\} \\ &\leq \max \left\{ \frac{c_5 \rho^\alpha(y)}{|x-y|^{N-2\alpha+1}}, \frac{c_5 \rho_\Omega^{2\alpha-1-s} \rho^s(y)}{|x-y|^{N-\alpha}} \right\}, \end{aligned}$$

where  $\rho_\Omega = \sup_{z \in \Omega} \rho(z)$ . Since  $N - 2\alpha + 1 > N - \alpha$  and  $2\alpha - 1 - s \geq 0$ , there exists some  $c_6 > 0$  such that

$$\omega_\lambda(y) \subset \left\{ x \in \Omega : |x-y| \leq c_6 \max \left\{ \rho^{\frac{\alpha-\beta}{N-2\alpha+1}}(y) \lambda^{-\frac{1}{N-2\alpha+1}}, \rho^{\frac{s-\beta}{N-\alpha}}(y) \lambda^{-\frac{1}{N-\alpha}} \right\} \right\}.$$

Choosing  $s = \beta \in [0, 2\alpha - 1]$  we deduce that for any  $\lambda > 1$  there holds

$$\omega_\lambda(y) \subset \left\{ x \in \Omega : |x-y| \leq c_6 \rho_\Omega^{\frac{\alpha-\beta}{N-2\alpha+1}} \lambda^{-\frac{1}{N-2\alpha+1}} \right\}. \quad (2.8)$$

As a consequence,

$$m_\lambda(y) \leq c_7 \lambda^{-p_\alpha^*},$$

where  $c_7 > 0$  independent of  $y$  and  $\lambda$ .

Let  $E \subset \Omega$  be a Borel set and  $\lambda > 1$ , then

$$\int_E |\nabla_x G_\alpha(x, y)| \rho^\alpha(x) dx \leq \int_{\omega_\lambda(y)} |\nabla_x G_\alpha(x, y)| \rho^\alpha(x) dx + \lambda \rho^\beta(y) \int_E dx.$$

Noting that

$$\begin{aligned} \int_{\omega_\lambda(y)} |\nabla_x G_\alpha(x, y)| \rho^\alpha(x) dx &= - \int_{\lambda \rho^\beta(y)}^\infty s dm_s(y) \\ &= \lambda \rho^\beta(y) m_\lambda(y) + \int_{\lambda \rho^\beta(y)}^\infty m_s(y) ds \\ &\leq c_8 \rho^\beta(y) \lambda^{1-p_\alpha^*}, \end{aligned}$$



for some  $c_8 > 0$ , we derive

$$\int_E |\nabla_x G_\alpha(x, y)| \rho^\alpha(x) dx \leq \rho^\beta(y) \left( c_8 \lambda^{1-p_\alpha^*} + \lambda \int_E dx \right).$$

Choosing  $\lambda = (\int_E dx)^{-\frac{1}{p_\alpha^*}}$  yields

$$\int_E |\nabla_x G_\alpha(x, y)| \rho^\alpha(x) dx \leq c_8 \rho^\beta(y) \left( \int_E dx \right)^{\frac{p_\alpha^*-1}{p_\alpha^*}} \quad \forall y \in \Omega.$$

Therefore,

$$\begin{aligned} \int_E |\nabla \mathbb{G}_\alpha[|\nu|](x)| \rho^\alpha(x) dx &= \int_\Omega \int_E |\nabla_x G_\alpha(x, y)| \rho^\alpha(x) dx d|\nu(y)| \\ &\leq \int_\Omega \rho^\beta(y) d|\nu(y)| \sup_{x \in K} \rho^{-\beta}(y) \int_E |\nabla_x G_\alpha(x, y)| \rho^\alpha(x) dx \\ &\leq c_8 \int_\Omega \rho^\beta(y) d|\nu(y)| \left( \int_E dx \right)^{\frac{p_\alpha^*-1}{p_\alpha^*}} \\ &\leq c_8 \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)} \left( \int_E dx \right)^{\frac{p_\alpha^*-1}{p_\alpha^*}}. \end{aligned} \tag{2.9}$$

As a consequence,

$$\|\nabla \mathbb{G}_\alpha[|\nu|]\|_{MP_\alpha^*(\Omega, \rho^\alpha dx)} \leq c_8 \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)},$$

which ends the proof.  $\square$

**Proposition 2.3** [13] *Assume that  $\nu \in L^1(\Omega, \rho^\beta dx)$  with  $0 \leq \beta \leq \alpha$ . Then for  $r \in (1, \frac{N}{N-2\alpha+\beta})$  there exists  $c_9 > 0$  such that for any  $\nu \in L^1(\Omega, \rho^\beta dx)$*

$$\|\mathbb{G}_\alpha[\nu]\|_{W^{2\alpha-\gamma, r}(\Omega)} \leq c_9 \|\nu\|_{L^1(\Omega, \rho^\beta dx)}, \tag{2.10}$$

where  $r' = \frac{r}{r-1}$ ,  $\gamma = \beta + \frac{N}{r'}$  if  $\beta > 0$  and  $\gamma > \frac{N}{r'}$  if  $\beta = 0$ .

**Proposition 2.4** *If  $0 \leq \beta < 2\alpha - 1$  the mapping  $\nu \mapsto |\nabla \mathbb{G}_\alpha[\nu]|$  is compact from  $L^1(\Omega, \rho^\beta dx)$  into  $L^q(\Omega)$  for any  $q \in [1, p_{\alpha, \beta}^*)$  and there exists  $c_{10} > 0$  such that*

$$\left( \int_\Omega |\nabla \mathbb{G}_\alpha[\nu]|^q dx \right)^{\frac{1}{q}} \leq c_{10} \int_\Omega |\nu| \rho^\beta(x) dx. \tag{2.11}$$

*Proof.* For  $\nu \in L^1(\Omega, \rho^\beta dx)$  with  $0 \leq \beta < 2\alpha - 1$  and  $p \in (1, \frac{N}{N-2\alpha+\beta})$ , we obtain from Proposition 2.3 that

$$\mathbb{G}_\alpha[\nu] \in W^{2\alpha-\gamma, p}(\Omega),$$

where  $2\alpha - \gamma > 1$ . Therefore,  $|\nabla \mathbb{G}_\alpha[\nu]| \in W^{2\alpha-\gamma-1,p}(\Omega)$  and

$$\|\nabla \mathbb{G}_\alpha[\nu]\|_{W^{2\alpha-\gamma-1,p}(\Omega)} \leq c_9 \|\nu\|_{L^1(\Omega, \rho^\beta(x)dx)}. \quad (2.12)$$

By [23, Th 6.5], the embedding of  $W^{2\alpha-\gamma-1,p}(\Omega)$  into  $L^q(\Omega)$  is compact for  $q \in [1, \frac{Np}{N-(2\alpha-\gamma-1)p})$ . When  $\beta > 0$ ,

$$\begin{aligned} \frac{Np}{N-(2\alpha-\gamma-1)p} &= \frac{Np}{N-(2\alpha-\beta-N\frac{p-1}{p}-1)p} \\ &= \frac{N}{N+\beta+1-2\alpha} = p_{\alpha,\beta}^*. \end{aligned}$$

When  $\beta = 0$ ,

$$\begin{aligned} \lim_{\gamma \rightarrow \frac{N}{q}^+} \frac{Np}{N-(2\alpha-\gamma-1)p} &= \frac{Np}{N-(2\alpha-N\frac{p-1}{p}-1)p} \\ &= \frac{N}{N+1-2\alpha} = p_{\alpha,0}^*. \end{aligned}$$

Then the mapping  $\nu \mapsto |\nabla \mathbb{G}_\alpha[\nu]|$  is compact from  $L^1(\Omega, \rho^\beta(x)dx)$  into  $L^q(\Omega)$  for any  $q \in [1, p_{\alpha,\beta}^*)$ . Inequality (2.11) follows by (2.12) and the continuity of the embedding of  $W^{2\alpha-\gamma-1,p}(\Omega)$  into  $L^q(\Omega)$ .  $\square$

*Remark.* If  $\nu \in L^1(\Omega, \rho^\beta dx)$  with  $0 \leq \beta < 2\alpha - 1$  and  $u$  is the solution of

$$\begin{aligned} (-\Delta)^\alpha u &= \nu \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \Omega^c, \end{aligned}$$

there holds, for any  $q \in [1, p_{\alpha,\beta}^*)$ ,

$$\left( \int_\Omega |\nabla u|^q dx \right)^{\frac{1}{q}} \leq c_{10} \int_\Omega |\nu(x)| \rho^\beta(x) dx.$$

## 2.2 Classical solutions

In this section we consider the question of existence of classical solutions to problem

$$\begin{aligned} (-\Delta)^\alpha u + h(|\nabla u|) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \Omega^c. \end{aligned} \quad (2.13)$$

We denote that  $h_+(s) = \max\{h(s), 0\}$  and  $h_-(s) = \max\{-h(s), 0\}$ .

**Theorem 2.1** *Assume  $h \in C^\theta(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$  for some  $\theta \in (0, 1]$  and  $f \in C^\theta(\bar{\Omega})$ . Then problem (2.13) admits a unique classical solution  $u$ . Moreover,*

- (i) *if  $f - h(0) \geq 0$  in  $\Omega$ , then  $u \geq 0$ ;*
- (ii) *the mappings  $h \mapsto u$  and  $f \mapsto u$  are respectively nonincreasing and nondecreasing.*

*Proof of Theorem 2.1.* We divide the proof into several steps.

*Step 1. Existence.* We define the operator  $T$  by

$$Tu = \mathbb{G}_\alpha [f - h(|\nabla u|)], \quad \forall u \in W_0^{1,1}(\Omega).$$

Using (2.7) with  $s = 0$  yields

$$\begin{aligned} \|Tu\|_{W^{1,1}(\Omega)} &\leq \|\mathbb{G}_\alpha[f]\|_{W^{1,1}(\Omega)} + \|\mathbb{G}_\alpha[h(|\nabla u|)]\|_{W^{1,1}(\Omega)} \\ &\leq (\|f\|_{L^\infty(\Omega)} + \|h(|\nabla u|)\|_{L^\infty(\Omega)}) \left\| \int_\Omega G_\alpha(\cdot, y) dy \right\|_{W^{1,1}(\Omega)} \\ &= c_{10} (\|f\|_{L^\infty(\Omega)} + \|h\|_{L^\infty(\mathbb{R}_+)}), \end{aligned} \quad (2.14)$$

where  $c_{10} = \left\| \int_\Omega G_\alpha(\cdot, y) dy \right\|_{W^{1,1}(\Omega)}$ . Thus  $T$  maps  $W_0^{1,1}(\Omega)$  into itself. Clearly, if  $u_n \rightarrow u$  in  $W_0^{1,1}(\Omega)$  when  $n \rightarrow \infty$ , then  $h(|\nabla u_n|) \rightarrow h(|\nabla u|)$  in  $L^1(\Omega)$ , thus  $T$  is continuous. We claim that  $T$  is a compact operator. In fact, for  $u \in W_0^{1,1}(\Omega)$ , we see that  $f - h(|\nabla u|) \in L^1(\Omega)$  and then, by Proposition 2.4, it implies that  $Tu \in W_0^{2\alpha-\gamma,p}(\Omega)$  where  $\gamma = \frac{N(p-1)}{p}$  and  $2\alpha - \gamma > 1$  for  $p \in (1, \frac{N}{N-2\alpha})$ . Since the embedding  $W_0^{2\alpha-\gamma,p}(\Omega) \hookrightarrow W_0^{1,1}(\Omega)$  is a compact,  $T$  is a compact operator.

Let  $\mathcal{O} = \{u \in W_0^{1,1}(\Omega) : \|u\|_{W^{1,1}(\Omega)} \leq c_{10}(\|f\|_{L^\infty(\Omega)} + \|h\|_{L^\infty(\mathbb{R}_+)})\}$ , which is a closed and convex set of  $W_0^{1,1}(\Omega)$ . Combining with (2.14), there holds

$$T(\mathcal{O}) \subset \mathcal{O}.$$

It follows by Schauder's fixed point theorem that there exists some  $u \in W_0^{1,1}(\Omega)$  such that  $Tu = u$ .

Next we show that  $u$  is a classical solution of (2.13). Let open set  $O$  satisfy  $O \subset \bar{O} \subset \Omega$ . By Proposition 2.3 in [26], for any  $\sigma \in (0, 2\alpha)$ , there exists  $c_{11} > 0$  such that

$$\|u\|_{C^\sigma(O)} \leq c_{11} \{ \|h(|\nabla u|)\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)} \},$$

and by choosing  $\sigma = \frac{2\alpha+1}{2} \in (1, 2\alpha)$ , then

$$\|\nabla u\|_{C^{\sigma-1}(O)} \leq c_{11} \{ \|h(|\nabla u|)\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)} \},$$

and then applied [26, Corollary 2.4],  $u$  is  $C^{2\alpha+\epsilon_0}$  locally in  $\Omega$  for some  $\epsilon_0 > 0$ . Then  $u$  is a classical solution of (2.13). Moreover, from [13], we have

$$\int_\Omega [u(-\Delta)^\alpha \xi + h(|\nabla u|)\xi] dx = \int_\Omega \xi f dx, \quad \forall \xi \in \mathbb{X}_\alpha. \quad (2.15)$$

*Step 2. Proof of (i).* If  $u$  is not nonnegative, then there exists  $x_0 \in \Omega$  such that

$$u(x_0) = \min_{x \in \Omega} u(x) < 0,$$

then  $\nabla u(x_0) = 0$  and  $(-\Delta)^\alpha u(x_0) < 0$ . Since  $u$  is the classical solution of (2.13),  $(-\Delta)^\alpha u(x_0) = f(x_0) - h(0) \geq 0$ , which is a contradiction.

*Step 3. Proof of (ii).* We just give the proof of the first argument, the proof of the second being similar. Let  $h_1$  and  $h_2$  satisfy our hypotheses for  $h$  and  $h_1 \leq h_2$ . Denote  $u_1$  and  $u_2$  the solutions of (2.13) with  $h$  replaced by  $h_1$  and  $h_2$  respectively. If there exists  $x_0 \in \Omega$  such that

$$(u_1 - u_2)(x_0) = \min_{x \in \Omega} \{(u_2 - u_1)(x)\} < 0.$$

Then

$$(-\Delta)^\alpha (u_1 - u_2)(x_0) < 0, \quad \nabla u_1(x_0) = \nabla u_2(x_0).$$

This implies

$$(-\Delta)^\alpha (u_1 - u_2)(x_0) + h_1(|\nabla u_1(x_0)|) - h_2(|\nabla u_2(x_0)|) < 0. \quad (2.16)$$

However,

$$(-\Delta)^\alpha (u_1 - u_2)(x_0) + h_1(|\nabla u_1(x_0)|) - h_2(|\nabla u_2(x_0)|) = f(x_0) - f(x_0) = 0,$$

contradiction. Then  $u_1 \geq u_2$ .

Uniqueness follows from Step 4.  $\square$

### 3 Proof of Theorems 1.1 and 1.2

#### 3.1 The absorption case

In this subsection, we prove the existence of a weak solution to (1.1) when  $\epsilon = 1$ . We give below an auxiliary lemma.

**Lemma 3.1** *Assume that  $g : [0, \infty) \mapsto [0, \infty)$  is continuous, nondecreasing and (1.10) holds with  $p_{\alpha, \beta}^*$ . Then*

$$\lim_{s \rightarrow \infty} g(s) s^{-p_{\alpha, \beta}^*} = 0.$$

*Proof.* Since

$$\int_s^{2s} g(t) t^{-1-p_{\alpha, \beta}^*} dt \geq g(s) (2s)^{-1-p_{\alpha, \beta}^*} \int_s^{2s} dt = 2^{-1-p_{\alpha, \beta}^*} g(s) s^{-p_{\alpha, \beta}^*}$$

and by (1.10),

$$\lim_{s \rightarrow \infty} \int_s^{2s} g(t) t^{-1-p_{\alpha, \beta}^*} dt = 0,$$

the claim follows.  $\square$

*Proof of Theorem 1.1. Existence.* We define

$$C_\beta(\bar{\Omega}) = \{\zeta \in C(\bar{\Omega}) : \rho^{-\beta}\zeta \in C(\bar{\Omega})\}$$

endowed with the norm

$$\|\zeta\|_{C_\beta(\bar{\Omega})} = \|\rho^{-\beta}\zeta\|_{C(\bar{\Omega})}.$$

Let  $\{\nu_n\} \subset C^1(\bar{\Omega})$  be a sequence of nonnegative functions such that  $\nu_n \rightarrow \nu$  in sense of duality with  $C_\beta(\bar{\Omega})$ , that is

$$\lim_{n \rightarrow \infty} \int_{\bar{\Omega}} \zeta \nu_n dx = \int_{\bar{\Omega}} \zeta d\nu \quad \forall \zeta \in C_\beta(\bar{\Omega}). \quad (3.1)$$

By the Banach-Steinhaus Theorem  $\|\nu_n\|_{\mathfrak{M}(\Omega)}$  is bounded independently of  $n$ . We consider a sequence  $\{g_n\}$  of  $C^1$  nonnegative functions defined on  $\mathbb{R}_+$  such that  $g_n(0) = 0$  and

$$g_n \leq g_{n+1} \leq g, \quad \sup_{s \in \mathbb{R}_+} g_n(s) = n \quad \text{and} \quad \lim_{n \rightarrow \infty} \|g_n - g\|_{L_{loc}^\infty(\mathbb{R}_+)} = 0. \quad (3.2)$$

By Theorem 2.1, there exists a unique *nonnegative* solution  $u_n$  of (1.1) with data  $\nu_n$  and  $g_n$  instead of  $\nu$  and  $g$ , and there holds

$$\int_{\Omega} (u_n + g_n(|\nabla u_n|)\eta_1) dx \leq \int_{\Omega} \nu_n \eta_1 dx \leq C \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}. \quad (3.3)$$

Therefore  $\|g_n(|\nabla u_n|)\|_{\mathfrak{M}(\Omega, \rho^\alpha)}$  is bounded independently of  $n$ . Set  $\eta_1 = \mathbb{G}_\alpha[1]$  and for  $\varepsilon > 0$ ,  $\xi_\varepsilon = (\eta_1 + \varepsilon)^{\frac{\beta}{\alpha}} - \varepsilon^{\frac{\beta}{\alpha}} \in \mathbb{X}_\alpha$ , which is concave in the interval  $[0, \eta_1(\bar{\omega})]$  where  $\eta_1(\bar{\omega}) = \max_{x \in \Omega} \eta_1(x)$ . By [13, Lemma 2.3 -(ii)], we see that

$$\begin{aligned} (-\Delta)^\alpha \xi_\varepsilon &= \frac{\beta}{\alpha} (\eta_1 + \varepsilon)^{\frac{1}{\alpha}} (-\Delta)^\alpha \eta_1 - \frac{\beta(\beta - \alpha)}{\alpha^2} (\eta_1 + \varepsilon)^{\frac{\beta - 2\alpha}{\alpha}} \int_{\Omega} \frac{(\eta_1(y) - \eta_1(x))^2}{|y - x|^{N+2\alpha}} dy \\ &\geq \frac{\beta}{\alpha} (\eta_1 + \varepsilon)^{\frac{\beta - \alpha}{\alpha}}, \end{aligned}$$

and  $\xi_\varepsilon \in \mathbb{X}_\alpha$ . Since

$$\int_{\Omega} (u_n (-\Delta)^\alpha \xi_\varepsilon + g_n(|\nabla u_n|)\xi_\varepsilon) dx = \int_{\Omega} \xi_\varepsilon \nu_n dx,$$

we obtain

$$\int_{\Omega} \left( \frac{\beta}{\alpha} u_n (\eta_1 + \varepsilon)^{\frac{\beta - \alpha}{\alpha}} + g_n(|\nabla u_n|)\xi_\varepsilon \right) dx \leq \int_{\Omega} \xi_\varepsilon \nu_n dx.$$

If we let  $\varepsilon \rightarrow 0$ , it yields

$$\int_{\Omega} \left( \frac{\beta}{\alpha} u_n \eta_1^{\frac{\beta - \alpha}{\alpha}} + g_n(|\nabla u_n|)\eta_1^{\frac{\beta}{\alpha}} \right) dx \leq \int_{\Omega} \eta_1^{\frac{\beta}{\alpha}} \nu_n dx.$$

Using [13, Lemma 2.3], we derive the estimate

$$\int_{\Omega} \left( u_n \rho^{\beta-\alpha} + g_n(|\nabla u_n|) \rho^{\beta} \right) dx \leq c_{12} \|\nu_n\|_{\mathfrak{M}(\Omega, \rho^{\beta})} \leq c_{13} \|\nu\|_{\mathfrak{M}(\Omega, \rho^{\beta})}. \quad (3.4)$$

Thus  $\{g_n(|\nabla u_n|)\}$  is bounded in  $L^1(\Omega, \rho^{\beta} dx)$ . Since  $u_n = \mathbb{G}[\nu_n - g_n(|\nabla u_n|)]$ , there holds

$$\begin{aligned} \|\nabla u_n\|_{M^{p_{\alpha}^*}(\Omega, \rho^{\alpha} dx)} &\leq \|\nu_n\|_{\mathfrak{M}(\Omega, \rho^{\beta})} + \|g_n(|\nabla u_n|)\|_{\mathfrak{M}(\Omega, \rho^{\beta})} \\ &\leq c_{14} \|\nu\|_{\mathfrak{M}(\Omega, \rho^{\beta})}. \end{aligned}$$

Since  $\nu_n - g_n(|\nabla u_n|)$  is uniformly bounded in  $L^1(\Omega, \rho^{\beta} dx)$ , we use Proposition 2.4 to obtain that the sequences  $\{u_n\}$ ,  $\{|\nabla u_n|\}$  are relatively compact in  $L^q(\Omega)$  for  $q \in [1, \frac{N}{N-2\alpha+\beta})$  and  $q \in [1, p_{\alpha, \beta}^*)$ , respectively. Thus, there exist a sub-sequence  $\{u_{n_k}\}$  and some  $u \in L^q(\Omega)$  with  $q \in [1, \frac{N}{N-2\alpha+\beta})$  such that

- (i)  $u_{n_k} \rightarrow u$  a.e. in  $\Omega$  and in  $L^q(\Omega)$  with  $q \in [1, \frac{N}{N-2\alpha+\beta})$ ;
- (ii)  $|\nabla u_{n_k}| \rightarrow |\nabla u|$  a.e. in  $\Omega$  and in  $L^q(\Omega)$  with  $q \in [1, p_{\alpha, \beta}^*)$ .

Therefore,  $g_{n_k}(|\nabla u_{n_k}|) \rightarrow g(|\nabla u|)$  a.e. in  $\Omega$ . For  $\lambda > 0$ , we denote

$$S_{\lambda} = \{x \in \Omega : |\nabla u_{n_k}(x)| > \lambda\} \text{ and } \omega(\lambda) = \int_{S_{\lambda}} \rho^{\alpha}(x) dx.$$

Then for any Borel set  $E \subset \Omega$ , we have

$$\begin{aligned} \int_E g_{n_k}(|\nabla u_{n_k}|) \rho^{\alpha}(x) dx &\leq \int_E g(|\nabla u_{n_k}|) \rho^{\alpha}(x) dx \\ &\leq \int_{E \cap S_{\lambda}^c} g(|\nabla u_{n_k}|) \rho^{\alpha}(x) dx + \int_{E \cap S_{\lambda}} g(|\nabla u_{n_k}|) \rho^{\alpha}(x) dx \\ &\leq \tilde{g}(\lambda) \int_E \rho^{\alpha} dx + \int_{S_{\lambda}} g(|\nabla u_{n_k}|) \rho^{\alpha}(x) dx \\ &\leq \tilde{g}(\lambda) \int_E \rho^{\alpha} dx - \int_{\lambda}^{\infty} g(s) d\omega(s), \end{aligned}$$

where  $\tilde{g}(s) = \max_{t \in [0, s]} \{g(t)\}$ . But

$$\int_{\lambda}^{\infty} g(s) d\omega(s) = \lim_{T \rightarrow \infty} \int_{\lambda}^T g(s) d\omega(s).$$

Since  $|\nabla u_{n_k}| \in M^{p_{\alpha}^*}(\Omega, \rho^{\alpha} dx)$ ,  $\omega(s) \leq c_{15} s^{-p_{\alpha}^*}$  and

$$\begin{aligned} - \int_{\lambda}^T g(s) d\omega(s) &= - \left[ g(s) \omega(s) \right]_{s=\lambda}^{s=T} + \int_{\lambda}^T \omega(s) dg(s) \\ &\leq g(\lambda) \omega(\lambda) - g(T) \omega(T) + c_{15} \int_{\lambda}^T s^{-p_{\alpha}^*} dg(s) \\ &\leq g(\lambda) \omega(\lambda) - g(T) \omega(T) + c_{15} \left( T^{-p_{\alpha}^*} g(T) - \lambda^{-p_{\alpha}^*} g(\lambda) \right) \\ &\quad + \frac{c_{15}}{p_{\alpha}^* + 1} \int_{\lambda}^T s^{-1-p_{\alpha}^*} g(s) ds. \end{aligned}$$

By assumption (1.10) and Lemma 3.1, it follows

$$\lim_{T \rightarrow \infty} T^{-p_\alpha^*} g(T) = 0. \quad (3.5)$$

Along with  $g(\lambda)\omega(\lambda) \leq c_{15}\lambda^{-p_\alpha^*}g(\lambda)$ , we have

$$-\int_\lambda^\infty g(s)d\omega(s) \leq \frac{c_{15}}{p_\alpha^* + 1} \int_\lambda^\infty s^{-1-p_\alpha^*}g(s)ds.$$

Notice that the above quantity on the right-hand side tends to 0 when  $\lambda \rightarrow \infty$ . It implies that for any  $\epsilon > 0$  there exists  $\lambda > 0$  such that

$$\frac{c_{15}}{p_\alpha^* + 1} \int_\lambda^\infty s^{-1-p_\alpha^*}g(s)ds \leq \frac{\epsilon}{2},$$

and  $\delta > 0$  such that

$$\int_E \rho^\alpha(x)dx \leq \delta \implies g(\lambda) \int_E dx \leq \frac{\epsilon}{2}.$$

This proves that  $\{g_{n_k}(|\nabla u_{n_k}|)\}$  is uniformly integrable in  $L^1(\Omega, \rho^\alpha dx)$ . Then  $g_{n_k}(|\nabla u_{n_k}|) \rightarrow g(|\nabla u|)$  in  $L^1(\Omega, \rho^\alpha dx)$  by Vitali convergence theorem. Letting  $n_k \rightarrow \infty$  in the identity

$$\int_\Omega (u_{n_k}(-\Delta)^\alpha \xi + g_{n_k}(|\nabla u_{n_k}|)\xi) dx = \int_\Omega \nu_{n_k} \xi dx, \quad \forall \xi \in \mathbb{X}_\alpha,$$

it infers that  $u$  is a weak solution of (1.1). Since  $u_{n_k}$  is nonnegative, so is  $u$ . Estimate (1.12) is a consequence of positivity and

$$u_{n_k} \leq \mathbb{G}_\alpha[\nu_{n_k}].$$

Since  $\lim_{n_k \rightarrow \infty} u_{n_k} = u$ , (1.12) follows.  $\square$

### 3.2 The source case

In this subsection we study the existence of solutions to problem 1.1) when  $\epsilon = -1$ .

*Proof of Theorem 1.2.* Let  $\{\nu_n\}$  be a sequence of  $C^2$  nonnegative functions converging to  $\nu$  in the sense of (3.1),  $\{g_n\}$  a sequence of  $C^1$ , nonnegative functions defined on  $\mathbb{R}_+$  satisfying (3.2). We set  $p_0 = \frac{p+p_{\alpha,\beta}^*}{2} \in (p, p_{\alpha,\beta}^*)$ , where  $p_{\alpha,\beta}^*$  is given by (1.13) and  $p < p_{\alpha,\beta}^*$  is the maximal growth rate of  $g$  which satisfies (1.14), and

$$M(v) = \left( \int_\Omega |\nabla v|^{p_0} dx \right)^{\frac{1}{p_0}}.$$

We may assume that  $\|\nu_n\|_{L^1(\Omega, \rho^\beta dx)} \leq 2\|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}$  for all  $n$ . Let  $\{u_n\}$  be the sequence of functions defined by  $u_0 = 0$  and, for  $n \geq 1$ ,

$$\begin{aligned} (-\Delta)^\alpha u_n &= g_{n-1}(|\nabla u_{n-1}|) + \nu_n & \text{in } \Omega, \\ u_n &= 0 & \text{in } \Omega^c. \end{aligned}$$

Assume that  $u_{n-1}$  is  $C^{2\alpha+\gamma_{n-1}}$  locally in  $\Omega$  with  $\gamma_{n-1} \in (0, 2\alpha - 1)$ , then  $g_{n-1}(|\nabla u_{n-1}|) + \nu_n$  is  $C^{\tilde{\theta}}$  where  $\tilde{\theta} \leq \min\{\gamma_{n-1}, \theta\}$ , then by [26, Corollary 2.4], we have that  $u_n$  is  $C^{2\alpha+\gamma_n}$  with  $\gamma_n \in (0, \gamma_{n-1})$ . Jointly with [13, Lemmas 2.1, 2.2], it yields

$$\int_{\Omega} u_n (-\Delta)^\alpha \xi = \int_{\Omega} g_{n-1}(|\nabla u_{n-1}|) \xi dx + \int_{\Omega} \xi \nu_n dx, \quad \forall \xi \in \mathbb{X}_\alpha. \quad (3.6)$$

Since  $u_0 = 0$ , we assume by induction that  $u_{n-1} \geq 0$ , then  $g_{n-1}(|\nabla u_{n-1}|) + \nu_n \geq 0$  and  $u_n \geq 0$  by the maximum principle. Therefore  $\{u_n\}$  is a sequence of nonnegative functions.

*Step 1: we claim that  $M(u_n)$  is bounded.* By (1.14) and Proposition 2.4, we have

$$\begin{aligned} \left( \int_{\Omega} |\nabla u_n|^{p_0} dx \right)^{\frac{1}{p_0}} &\leq c_2 \|g(|\nabla u_{n-1}|) + \nu_n\|_{L^1(\Omega, \rho^\beta dx)} \\ &\leq c_2 [\|g(|\nabla u_{n-1}|)\|_{L^1(\Omega, \rho^\beta dx)} + 2\|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}] \\ &\leq c_2 c_1 \int_{\Omega} |\nabla u_{n-1}|^p \rho^\beta dx + c_{16} \sigma_0 + 2c_2 \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}, \end{aligned} \quad (3.7)$$

where  $c_{16} = c_2 \int_{\Omega} \rho^\beta dx$ . Then we use Hölder inequality to obtain that

$$\left( \int_{\Omega} |\nabla u_n|^p \rho^\beta dx \right)^{\frac{1}{p}} \leq \left( \int_{\Omega} \rho^{\frac{\beta p_0}{p_0-p}} dx \right)^{\frac{1}{p} - \frac{1}{p_0}} \left( \int_{\Omega} |\nabla u_n|^{p_0} dx \right)^{\frac{1}{p_0}}, \quad (3.8)$$

where  $\int_{\Omega} \rho^{\frac{\beta p_0}{p_0-p}} dx$  is bounded, since  $\frac{\beta p_0}{p_0-p} \geq 0$ . Along with (3.7) and (3.8), we derive

$$M(u_n) \leq c_{17} M(u_{n-1})^p + c_{18} \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)} + c_{16} \sigma_0, \quad (3.9)$$

where  $c_{17} = c_2 c_1 \left( \int_{\Omega} \rho^{\frac{\beta p_0}{p_0-p}} dx \right)^{\frac{1}{p} - \frac{1}{p_0}} > 0$  and  $c_{18} > 0$  independent of  $n$ . Therefore, if we assume that  $M(u_{n-1}) \leq \lambda$ , inequality (3.9) implies

$$M(u_n) \leq c_{17} \lambda^p + c_{18} \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)} + c_{16} \sigma_0. \quad (3.10)$$

Let  $\bar{\lambda} > 0$  be the largest root of the equation

$$c_{17} \lambda^p + c_{18} \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)} + c_{16} \sigma_0 = \lambda, \quad (3.11)$$



This root exists if one of the following condition holds:

(i)  $p \in (0, 1)$ , in which case (3.11) admits only one root:

(ii)  $p = 1$  and  $c_{17} < 1$ , and again (3.11) admits only one root;

(iii)  $p \in (1, p_\alpha^*)$ , and there exists  $\varepsilon_0 > 0$  such that  $\max \left\{ \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}, \sigma_0 \right\} \leq \varepsilon_0$ .

In that case (3.11) admits usually two positive roots.

If we suppose that one of the above conditions holds, the definition of  $\bar{\lambda} > 0$  implies that it is the largest  $\lambda > 0$  such that

$$c_{17}\lambda^p + c_{18}\|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)} + c_{16}\sigma_0 \leq \lambda, \quad (3.12)$$

We assume  $M(u_k) \leq \bar{\lambda}$  for any  $k \leq n - 1$ , which holds obviously for  $n = 1$  since  $M(u_0) = 0$ . Then (3.9)-(3.12), implies

$$M(u_n) \leq c_{17}\bar{\lambda}^p + c_{18}\|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)} + c_{16}\sigma_0 \leq \bar{\lambda}.$$

By the assumptions of Theorem 1.2  $\bar{\lambda}$  exists and it is larger than  $M(u_n)$ . Therefore

$$\int_{\Omega} |\nabla u_n|^{p_0} dx \leq \bar{\lambda}^{p_0}, \quad \forall n \geq 0. \quad (3.13)$$

*Step 2: Convergence.* By (3.13) and (3.8),  $g_n(|\nabla u_n|)$  is uniformly bounded in  $L^1(\Omega, \rho^\beta dx)$ . By Proposition 2.3,  $\{u_n\}$  is bounded in  $W_0^{2\alpha-\gamma, q}(\Omega)$  where  $q \in (1, \frac{N}{N-2\alpha})$  and  $2\alpha - \gamma > 1$ . By compactness, there exist a subsequence  $\{u_{n_k}\}$  and  $u$  such that  $u_{n_k} \rightarrow u$  a.e. in  $\Omega$  and in  $L^1(\Omega)$ , and  $|\nabla u_{n_k}| \rightarrow |\nabla u|$  a.e. in  $\Omega$  and in  $L^q(\Omega)$  for any  $q \in [1, p_{\alpha, \beta}^*)$ . By Proposition 2.4 and assumption (G),  $g_{n_k}(|\nabla u_{n_k}|) \rightarrow g(|\nabla u|)$  in  $L^1(\Omega, \rho^\beta dx)$ . Letting  $n_k \rightarrow \infty$  to have that

$$\int_{\Omega} u(-\Delta)^\alpha \xi = \int_{\Omega} g(|\nabla u|)\xi dx + \int_{\Omega} \xi d\nu(x), \quad \forall \xi \in \mathbb{X}_\alpha,$$

thus  $u$  is a weak solution of (1.1) which is nonnegative as the  $u_n$  are. Furthermore (1.15) follows from the positivity of  $g(|\nabla u_n|)$  and the maximum principle.  $\square$

## 4 The case of the Dirac mass

In this section we assume that  $\Omega$  is a bounded and  $C^2$  domain containing 0 and  $u$  a nonnegative weak solution of

$$\begin{aligned} (-\Delta)^\alpha u + |\nabla u|^p &= \delta_0 & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c, \end{aligned} \quad (4.1)$$

where  $\delta_0$  the Dirac measure at 0. We recall the following result dealing with the convolution operator  $*$  in Lorentz spaces  $L^{p, q}(\mathbb{R}^N)$  (see [25]).

**Proposition 4.1** *Let  $1 \leq p_1, q_1, p_2, q_2 \leq \infty$  and suppose  $\frac{1}{p_1} + \frac{1}{p_2} > 1$ . If  $f \in L^{p_1, q_1}(\mathbb{R}^N)$  and  $g \in L^{p_2, q_2}(\mathbb{R}^N)$ , then  $f * g \in L^{r, s}(\mathbb{R}^N)$  with  $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} - 1$ ,  $\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s}$  and there holds*

$$\|f * g\|_{L^{r, s}(\mathbb{R}^N)} \leq 3r \|f\|_{L^{p_1, q_1}(\mathbb{R}^N)} \|g\|_{L^{p_2, q_2}(\mathbb{R}^N)}. \quad (4.2)$$

*In the particular case of Marcinkiewicz spaces  $L^{p, \infty}(\mathbb{R}^N) = M^p(\mathbb{R}^N)$ , the result takes the form*

$$\|f * g\|_{M^r(\mathbb{R}^N)} \leq 3r \|f\|_{M^{p_1}(\mathbb{R}^N)} \|g\|_{M^{p_2}(\mathbb{R}^N)}. \quad (4.3)$$

**Proposition 4.2** *Assume  $0 < p < p_\alpha^*$  and  $u$  is a nonnegative weak solution of (4.1). Then*

$$0 \leq u \leq \mathbb{G}_\alpha[\delta_0], \quad (4.4)$$

$\nabla u \in L_{loc}^\infty(\Omega \setminus \{0\})$  and  $u$  is a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u + |\nabla u|^p &= 0 & \text{in } \Omega \setminus \{0\}, \\ u &= 0 & \text{in } \Omega^c. \end{aligned} \quad (4.5)$$

*Proof.* Since  $0 < p < p_\alpha^*$ , (4.1) admits a solution. Estimate (4.4) is a particular case of (1.12). We pick a point  $a \in \Omega \setminus \{0\}$  and consider a finite sequence  $\{r_j\}_{j=0}^\kappa$  such that  $0 < r_\kappa < r_{\kappa-1} < \dots < r_0$  and  $\overline{B_{r_0}}(a) \subset \Omega \setminus \{0\}$ . We set  $d_j = r_{j-1} - r_j$ ,  $j = 1, \dots, \kappa$ . By (3.4) with  $\beta = 0$

$$\int_\Omega (u + |\nabla u|^p) dx \leq c_{19} \quad (4.6)$$

Let  $\{\eta_n\} \subset C_0^\infty(\mathbb{R}^N)$  be a sequence of radially decreasing and symmetric mollifiers such that  $\text{supp}(\eta_n) \subset B_{\varepsilon_n}$  and  $\varepsilon_n \leq \rho(a) - r_0$  and  $u_n = u * \eta_n$ . Since

$$\eta_n * (-\Delta)^\alpha \xi = (-\Delta)^\alpha (\xi * \eta_n)$$

by Fourier analysis and

$$\int_{\mathbb{R}^N} (u(-\Delta)^\alpha (\xi * \eta_n) + \xi * \eta_n |\nabla u|^p) dx = \int_{\mathbb{R}^N} (u * \eta_n (-\Delta)^\alpha \xi + \eta_n * |\nabla u|^p \xi) dx$$

because  $\eta_n$  is radially symmetric, it follows that  $u_n$  is a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u_n + |\nabla u|^p * \eta_n &= \eta_n, & \text{in } \Omega_n \\ u_n &= 0 & \text{in } \Omega_n^c \end{aligned} \quad (4.7)$$

where  $\Omega_n = \{x \in \mathbb{R}^N : \text{dist}(x, \overline{\Omega}) < \varepsilon_n\}$ . We denote by  $G_{\alpha, n}(x, y)$  the Green kernel of  $(-\Delta)^\alpha$  in  $\Omega_n$  and by  $\mathbb{G}_{\alpha, n}$  the Green operator. Set  $f_n =$

$\eta_n - |\nabla u|^p * \eta_n$ , then  $u_n = \mathbb{G}_{\alpha,n}[f_n]$ . If we set  $f_{n_0} = f_n \chi_{B_{r_0}(a)}$ ,  $f'_{n_0} = f_n - f_{n_0}$ , we have

$$\begin{aligned} \partial_{x_i} u_n(x) &= \int_{\Omega_n} \partial_{x_i} G_{\alpha,n}(x, y) f_n(y) dy \\ &= \int_{\Omega_n} \partial_{x_i} G_{\alpha,n}(x, y) f_{n_0}(y) dy + \int_{\Omega_n} \partial_{x_i} G_{\alpha,n}(x, y) f'_{n_0}(y) dy \\ &= v_{n_0}(x) + v'_{n_0}(x), \end{aligned}$$

with

$$v_{n_0}(x) = \int_{B_{r_0}(a)} \partial_{x_i} G_{\alpha,n}(x, y) f_n(y) dy = - \int_{B_{r_0}(a)} \partial_{x_i} G_{\alpha,n}(x, y) |\nabla u|^p * \eta_n(y) dy$$

if we assume that  $\varepsilon_n < |a| - r_0$ , and

$$v'_{n_0}(x) = \int_{\Omega_n \setminus B_{r_0}(a)} \partial_{x_i} G_{\alpha,n}(x, y) f_n(y) dy.$$

We set  $\rho_n(x) = \text{dist}(x, \Omega_n^c)$ , then by (2.7)

$$|\partial_{x_i} G_{\alpha,n}(x, y)| \leq c_5 \max \left\{ \frac{\rho_n^\alpha(y)}{\rho_n^\alpha(x) |x - y|^{N-2\alpha+1}}, \frac{\rho_n^{2\alpha-1}(y)}{\rho_n^\alpha(x) |x - y|^{N-2\alpha}} \right\},$$

where  $c_5$  is independent of  $n$ . Thus, if  $x \in B_{r_1}(a)$ ,

$$|v'_{n_0}(x)| \leq c_{20} \int_{\Omega_n \setminus B_{r_0}(a)} f_n(y) \rho_n^{2\alpha-1}(y) dy \leq c_{19} c_{20} \rho_\Omega^{2\alpha-1} \quad (4.8)$$

where  $c_{20}$  depends on  $d_1$  and  $\rho_\Omega$  (and the structural constants  $N$  and  $\alpha$ ). Furthermore, if  $x \in B_{r_1}(a)$  and  $y \in B_{r_0}(a)$ ,

$$|\partial_{x_i} G_{\alpha,n}(x, y)| \leq \frac{c_{21}}{|x - y|^{N-2\alpha+1}} \quad (4.9)$$

We have already use the fact that  $y \mapsto |y|^{2\alpha-N-1} \in L^{p_\alpha^*}(\mathbb{R}^N)$ . Since  $f_n$  is uniformly bounded in  $L^1(\Omega)$ , there exists  $c_{22}$  such that

$$\|v_{n_0}\|_{M^{p_\alpha^*}(B_{r_1}(a))} \leq c_{22}. \quad (4.10)$$

Combined with (4.8), it yields

$$\| |\nabla u|^p * \eta_n \|_{M^{\frac{p_\alpha^*}{p}}(B_{r_1}(a))} \leq c_{23}. \quad (4.11)$$

Next we set  $f_{n_1} = f_n \chi_{B_{r_1}(a)}$  and  $f'_{n_1} = f_n - f_{n_1}$ . Then

$$\partial_{x_i} u_n = v_{n_1} + v'_{n_1}$$

where

$$v_{n_1}(x) = \int_{B_{r_1}(a)} \partial_{x_i} G_\alpha(x, y) f_n(y) dy = - \int_{B_{r_1}(a)} \partial_{x_i} G_\alpha(x, y) |\nabla u|^p * \eta_n(y) dy$$

and

$$v'_{n_1}(x) = \int_{\Omega_n \setminus B_{r_1}(a)} \partial_{x_i} G_\alpha(x, y) f_n(y) dy$$

Clearly  $v'_{n_1}(x)$  is uniformly bounded in  $B_{r_2}(a)$  by a constant  $c_{24}$  depending on the structural constants and  $d_2 = r_1 - r_2$ . Estimate (4.9) holds if we assume  $x \in B_{r_2}(a)$  and  $y \in B_{r_1}(a)$ , except that the new constant, still denoted by  $c_{21}$  depends on  $d_2$ . Therefore

$$|v_{n_1}(x)| \leq c_{21} \int_{B_{r_1}(a)} \frac{|\nabla u|^p * \eta_n(y)}{|x - y|^{N-2\alpha+1}} dy.$$

We derive from Proposition 4.1

$$\|v_{n_1}\|_{M^{q_2}(B_{r_2}(a))} \leq c_{24} \| |\nabla u|^p * \eta_n \|_{M^{\frac{p_\alpha^*}{p}}(B_{r_1}(a))},$$

with

$$\frac{1}{q_2} = \frac{p}{p_\alpha^*} + \frac{1}{p_\alpha^*} - 1. \quad (4.12)$$

Notice that  $q_2 > p_\alpha^*$ . Therefore

$$\| |\nabla u|^p * \eta_n \|_{M^{\frac{q_2}{p}}(B_{r_2}(a))} \leq c_{25}. \quad (4.13)$$

We iterate this construction and obtain the existence of constants  $c_j$  such that

$$\| |\nabla u|^p * \eta_n \|_{M^{\frac{q_j}{p}}(B_{r_j}(a))} \leq c_j \quad \forall j = 1, 2, \dots \quad (4.14)$$

where  $q_1 = p_\alpha^*$  and

$$\frac{1}{q_{j+1}} = \frac{p}{q_j} + \frac{1}{p_\alpha^*} - 1. \quad (4.15)$$

Set  $\ell = \frac{p_\alpha^* - 1}{p_\alpha^*(p-1)}$ , then  $\ell = p\ell + \frac{1}{p_\alpha^*} - 1$ , thus

$$\begin{aligned} \frac{1}{q_{j+1}} &= \ell + p^{j-1} \left( \frac{1}{q_1} - \ell \right) \\ &= \ell + p^{j-1} \left( \frac{1}{p_\alpha^*} - \ell \right) \\ &= \ell - p^{j-1} \frac{p_\alpha^* - p}{p_\alpha^*(p-1)}. \end{aligned} \quad (4.16)$$

Therefore there exists  $j_0$  such that  $q_{j_0} > 0$  and  $q_{j_0+1} \leq 0$ . This implies

$$\| |\nabla u|^p * \eta_n \|_{L^s(B_{r_{j_0+1}}(a))} \leq c_{26} \quad \forall s < \infty. \quad (4.17)$$

Since  $y \mapsto |y|^{2\alpha-N-1} \in L^t_{loc}(\mathbb{R}^N)$  for all  $t < p^*_\alpha$  it yields

$$\| |\nabla u|^p * \eta_n \|_{L^\infty(B_{r_{j_0+2}}(a))} \leq c_{27}, \quad (4.18)$$

with  $c_{27}$  independent of  $n$ . Letting  $n \rightarrow \infty$  infers

$$\|\nabla u\|_{L^\infty(B_{r_{j_0+2}}(a))} \leq c_{27}^{\frac{1}{p}}, \quad (4.19)$$

Combining this estimate with (4.4) and using [26, Corollary 2.5] which states

$$\begin{aligned} \|u\|_{C^\beta(B_{r_{j_0+3}}(a))} &\leq c \left( \|u\|_{L^1(\mathbb{R}^N, (1+|x|)^{-N-2\alpha} dx} \right. \\ &\quad \left. + \|u\|_{L^\infty(B_{r_{j_0+2}}(a))} + \|\nabla u\|_{L^\infty(B_{r_{j_0+2}}(a))} \right), \end{aligned} \quad (4.20)$$

for any  $\beta < 2\alpha$ , we obtain that  $u$  remains bounded in  $C^{1+\varepsilon}(K)$  for any compact set  $K \subset \Omega \setminus \{0\}$  and some  $\varepsilon > 0$ . Using now [26, Corollary 2.4], we obtain that  $C^{2\alpha+\varepsilon'}(\Omega \setminus \{0\})$  for  $0 < \varepsilon' < \varepsilon$ . Furthermore  $u$  is continuous up to  $\partial\Omega$ . As a consequence it is a strong solution in  $\Omega \setminus \{0\}$ .  $\square$

In the next result we give a pointwise estimate of  $\nabla u$  for a positive solution  $u$  of (4.1).

**Proposition 4.3** *Assume  $R > 0$  is such that  $\overline{B}_R \subset \Omega$ . If  $p < p^*_\alpha$  and  $u$  is a nonnegative weak solution of (4.1), then there exists  $c_{20} > 0$  depending on  $R$ ,  $p$  and  $\alpha$  such that*

$$|\nabla u(x)| \leq c_{20} |x|^{2\alpha-N-1} \quad \forall x \in \overline{B}_{R/4} \setminus \{0\}. \quad (4.21)$$

*Proof.* Up to a change of variable we can assume that  $R = 1$ . For  $0 < |x| \leq 1$ , there exists  $b \in (0, 1)$  such that  $b/2 \leq |x| \leq b$ . We set

$$u_b(y) = b^{N-2\alpha} u(by).$$

Then

$$(-\Delta)^\alpha u_b + b^{N+p(2\alpha-N-1)} |\nabla u_b|^p = 0 \quad \text{in } \Omega_b := b^{-1}\Omega.$$

Using [26, Corollary 2.5] with  $\beta < 2\alpha$ , for any  $a$  such that  $|a| = 3/4$ , there holds

$$\begin{aligned} \|u_b\|_{C^\beta(B_{\frac{3}{16}}(a))} &\leq c_{18} \left( \|u_b\|_{L^1(\mathbb{R}^N, (1+|y|)^{-N-2\alpha})} + \|u_b\|_{L^\infty(B_{\frac{3}{8}}(a))} \right. \\ &\quad \left. + b^{N+p(2\alpha-N-1)} \|\nabla u_b\|_{L^\infty(B_{\frac{3}{8}}(a))} \right). \end{aligned} \quad (4.22)$$

Furthermore, by the same argument as in Proposition 4.2,

$$\|\nabla u_b\|_{L^\infty(B_{\frac{3}{8}}(a))}^p \leq c_{21} \int_{\Omega_b} |\nabla u_b|^p(y) dy = c_{21} b^{p(N+1-2\alpha)-N} \int_{\Omega} |\nabla u|^p(x) dx, \quad (4.23)$$

and from (4.4) and (2.4)

$$u(x) \leq G_\alpha(x, 0) \leq \frac{c_4}{|x|^{N-2\alpha}} \implies u_b(y) \leq \frac{c_4}{|y|^{N-2\alpha}}.$$

Then

$$\|u_b\|_{L^1(\mathbb{R}^N, (1+|y|)^{-N-2\alpha})} \leq c_4 \int_{\mathbb{R}^N} \frac{dy}{|y|^{N-2\alpha}(1+|y|)^{N+2\alpha}} = c_{22}.$$

If we take  $\beta = 1$ , which is possible since  $\alpha > 1/2$ , we derive

$$|\nabla u_b(a)| \leq c_{23} \implies |\nabla u(ba)| \leq c_{23}^{-1} b^{2\alpha-N-1}$$

In particular, with  $|b| = 4|x|/3$  we derive (4.21) with  $c_{20} = c_{23}^{-1} (\frac{4}{3})^{2\alpha-N-1}$ .  $\square$

We denote

$$c_{N,\alpha} = \lim_{x \rightarrow 0} |x|^{N-2\alpha} G_\alpha(x, 0). \quad (4.24)$$

It is well known that  $c_{N,\alpha} > 0$  does not depend on the domain  $\Omega$  and, by the maximum principle,  $G_\alpha(x, 0) \leq c_{N,\alpha} |x|^{2\alpha-N}$  in  $\Omega \setminus \{0\}$ .

**Theorem 4.1** *Let  $\Omega$  be a bounded  $C^2$  domain containing 0,  $\alpha \in (\frac{1}{2}, 1)$  and  $0 < p < p_\alpha^*$ . If  $u$  is a positive solution of problem (??) and  $\overline{B}_R \subset \Omega$ , it satisfies*

(i) if  $\frac{2\alpha}{N-2\alpha+1} < p < p_\alpha^*$ ,

$$0 < \frac{c_{N,\alpha}}{|x|^{N-2\alpha}} - u(x) \leq \frac{c_{24}}{|x|^{(N-2\alpha+1)p-2\alpha}}, \quad x \in B_{R/4}(0) \setminus \{0\};$$

(ii) if  $p = \frac{2\alpha}{N-2\alpha+1}$ ,

$$0 < \frac{c_{N,\alpha}}{|x|^{N-2\alpha}} - u(x) \leq -c_{24} \ln(|x|), \quad x \in B_{R/4}(0) \setminus \{0\};$$

(iii) if  $0 < p < \frac{2\alpha}{N-2\alpha+1}$ ,

$$0 < \frac{c_{N,\alpha}}{|x|^{N-2\alpha}} - u(x) \leq c_{24}, \quad x \in B_{R/4}(0) \setminus \{0\},$$

where  $c_{24}$  depends on  $N, p, \alpha, R$  and  $u$ . Furthermore, if  $1 \leq p < p_\alpha^*$ , this solution is unique and  $c_{24}$  is independent of  $u$ .

*Proof.* The existence of a nonnegative weak solution is a consequence of the subcriticality assumption; the fact that this solution is a classical solution in  $\Omega \setminus \{0\}$  derives from Proposition 4.2. It follows from the equation and (4.4) that for any  $x \in \Omega \setminus \{0\}$ ,

$$\begin{aligned} \frac{c_{N,\alpha}}{|x|^{N-2\alpha}} - u(x) &\leq \int_{\Omega} G_\alpha(x, y) |\nabla u(y)|^p dy \\ &\leq c_{20}^p c_4 \int_{B_{\frac{R}{4}}} |x-y|^{2\alpha-N} |y|^{p(2\alpha-N-1)} dy + c_{25}, \end{aligned} \quad (4.25)$$

where  $c_{25}$  depends on  $N, p, \alpha, \text{diam}(\Omega), R$  and  $\sup\{|\nabla u(z)| : z \in \Omega \setminus B_{\frac{R}{4}}\}$ .

Next we assume  $0 < |x| \leq \frac{R}{16}$ .

*Case:*  $\frac{2\alpha}{N-2\alpha+1} < p < p_\alpha^*$ . We can write

$$\int_{B_{\frac{R}{4}}} |x-y|^{2\alpha-N} |y|^{p(2\alpha-N-1)} dy = E_1 + E_2$$

with

$$E_1 = \int_{B_{\frac{R}{4}} \setminus B_{\frac{R}{8}}} |x-y|^{2\alpha-N} |y|^{p(2\alpha-N-1)} dy \leq c_{25}$$

where  $c_{26} > 0$  depends on  $N, \alpha, p$  and  $R$  and

$$\begin{aligned} E_2 &= \int_{B_{\frac{R}{8}}} |x-y|^{2\alpha-N} |y|^{p(2\alpha-N-1)} dy \\ &= |x|^{2\alpha-p(N+1-2\alpha)} \int_{B_{\frac{R}{8|x|}}} |\xi-\zeta|^{2\alpha-N} |\zeta|^{p(2\alpha-N-1)} d\zeta \\ &\leq \int_{|\zeta|>2} |\xi-\zeta|^{2\alpha-N} |\zeta|^{p(2\alpha-N-1)} d\zeta \end{aligned}$$

with  $\xi = x/|x|$ . Since  $2\alpha - N < 0$ ,  $|\xi - \zeta|^{2\alpha-N} \leq (|\zeta| - 1)^{2\alpha-N}$ , thus

$$E_2 \leq c_N \int_2^\infty (r-1)^{2\alpha-N} r^{p(2\alpha-N-1)+N-1} dr = c_{27},$$

Thus (i) follows.

*Case:*  $\frac{2\alpha}{N-2\alpha+1} = p$ . We see that

$$E_2 = \int_{B_{\frac{R}{8|x|}}} |\xi-\zeta|^{2\alpha-N} |\zeta|^{-2\alpha} d\zeta$$

Then clearly

$$E_2 = -\ln|x| + o(1) \quad \text{when } |x| \rightarrow 0.$$

Thus (ii) follows

*Case:*  $0 < p < \frac{2\alpha}{N-2\alpha+1}$ . Then

$$E_2 = \int_{B_{\frac{R}{8|x|}}} |\xi-\zeta|^{2\alpha-N} |\zeta|^{-2\alpha} d\zeta = c_{28}|x|^{p(N+1-2\alpha)-2\alpha} + o(1) \quad \text{when } |x| \rightarrow 0.$$

Thus (iii) follows.

Uniqueness in the case  $1 \leq p < p_\alpha^*$ , is very standard, since if  $u_1$  and  $u_2$  are two positive solutions of (??), they satisfies

$$\lim_{x \rightarrow 0} \frac{u_1(x)}{u_2(x)} = 1.$$

Then, for any  $\varepsilon > 0$ ,  $u_{1,\varepsilon} := (1 + \varepsilon)u_1$  is a supersolution which dominates  $u_2$  near 0, it follows by the maximum principle that  $w := u_2 - (1 + \varepsilon)u_1$  satisfies

$$(-\Delta)^\alpha w + |\nabla u_2|^p - |\nabla u_{1,\varepsilon}|^p \leq 0$$

since  $w$  is negative near 0 and vanishes on  $\partial\Omega$ , if it is not always negative, there would exist  $x_0 \in \Omega \setminus \{0\}$  such that  $w(x_0)$  reaches a maximum and  $|\nabla u_2(x_0)| = |\nabla u_{1,\varepsilon}(x_0)|$ , thus  $(-\Delta)^\alpha w(x_0) \leq 0$ , contradiction.  $\square$

*Remark.* If  $0 < p < 1$  the nonlinearity is not convex and uniqueness does hold only if two solutions  $u_1$  and  $u_2$  satisfy

$$\lim_{x \rightarrow 0} (u_1(x) - u_2(x)) = 0.$$

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