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Fractional elliptic equation with gradient nonlinearity and measure data

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Abstract

In this paper, we study fractional elliptic equation (E1) $(-\Delta)^\alpha u + g(|\nabla u|) = \nu$ in a bounded regular domain Ω of \mathbb{R}^N ($N \geq 2$), satisfying (E2) $u = 0$ in Ω^c , where $(-\Delta)^\alpha$ denotes the fractional Laplacian with $\alpha \in (0, 1)$, ν is a Radon measure and $g : \mathbb{R} \mapsto \mathbb{R}$ is a continuous function satisfying. We prove the existence of weak solutions for problem (E1)-(E2) when ν is Radon measure and g is subcritical. Furthermore, the asymptotic behaviors and uniqueness of solutions are analyzed when ν is Dirac measure.

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Key words: Fractional Laplacian, Radon measure, Green kernel, Dirac measure.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be an open bounded C^2 domain and $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a continuous and nondecreasing function. The purpose of this paper is to study the existence of weak solutions to the semilinear fractional elliptic problem

$$\begin{aligned} (-\Delta)^\alpha u + g(|\nabla u|) &= \nu & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c, \end{aligned} \tag{1.1}$$

where $\alpha \in (1/2, 1)$ and $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, 2\alpha - 1]$. Here $\rho(x) = \text{dist}(x, \Omega^c)$ and $\mathfrak{M}(\Omega, \rho^\beta)$ denotes the space of Radon measures in Ω satisfying

$$\int_{\Omega} \rho^\beta d|\nu| < +\infty. \tag{1.2}$$

In particular, we denote that $\mathfrak{M}(\Omega, \rho^0) = \mathfrak{M}^b(\Omega)$ is the space of bounded Radon measures. The operator $(-\Delta)^\alpha$ is the fractional Laplacian defined as

$$(-\Delta)^\alpha u(x) = \lim_{\epsilon \rightarrow 0^+} (-\Delta)_\epsilon^\alpha u(x),$$

where for $\epsilon > 0$,

$$(-\Delta)_\epsilon^\alpha u(x) = - \int_{\mathbb{R}^N} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} \chi_\epsilon(|x - z|) dz \tag{1.3}$$

and

$$\chi_\epsilon(t) = \begin{cases} 0, & \text{if } t \in [0, \epsilon], \\ 1, & \text{if } t > \epsilon. \end{cases}$$

In the pioneering work, Brezis [6] studied the existence and uniqueness of the solution to the semilinear Dirichlet elliptic problem

$$\begin{aligned} -\Delta u + h(u) &= \nu & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

where ν is a bounded measure in Ω and the function h is nondecreasing, positive on $(0, +\infty)$ and satisfies that

$$\int_1^{+\infty} (h(s) - h(-s)) s^{-2\frac{N-1}{N-2}} ds < +\infty.$$

Also see Benilan and Brezis [1]. Later, Véron [27] improved this result in replacing the Laplacian by more general uniformly elliptic second order differential operator, where $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, 1]$ and h is a nondecreasing function satisfying that

$$\int_1^{+\infty} (h(s) - h(-s)) s^{-2\frac{N+\beta-1}{N+\beta-2}} ds < +\infty.$$

The general semilinear elliptic equations involving measures have been intensively studied, such as the equations involving measures boundary data which was initiated by Gmira and Véron [16] and then this subject has being extended in various ways, see [4, 5, 18, 19, 20, 21] for details. In a recent work, Phuoc and Véron [23] considered the existence of solutions to problem

$$\begin{aligned} -\Delta u + h(|\nabla u|) &= \nu & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{aligned} \tag{1.5}$$

where ν is a positive bounded measure in Ω , h is a continuous nondecreasing function vanishing at 0 and satisfying that

$$\int_1^{+\infty} h(s)s^{-1-\frac{N}{N-1}} ds < +\infty.$$

During the last years there has also been a renewed and increasing interest in the study of linear and nonlinear integro-differential operators, especially, the fractional Laplacian, motivated by great applications and by important advances on the theory of nonlinear partial differential equations, and we mention the references [7, 10, 11, 12, 9, 14, 24, 26, 25]. Recently, Chen and Véron [12] studied semilinear fractional elliptic equation

$$\begin{aligned} (-\Delta)^\alpha u + h(u) &= \nu & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c, \end{aligned} \tag{1.6}$$

where $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, \alpha]$. They proved the existence and uniqueness of the solution to (1.6) when the function h satisfies that

$$\int_1^{+\infty} (h(s) - h(-s))s^{-1-k_{\alpha,\beta}} ds < +\infty,$$

where

$$k_{\alpha,\beta} = \begin{cases} \frac{N}{N-2\alpha}, & \text{if } \beta \in [0, \frac{N-2\alpha}{N}\alpha], \\ \frac{N+\alpha}{N-2\alpha+\beta}, & \text{if } \beta \in (\frac{N-2\alpha}{N}\alpha, \alpha]. \end{cases} \tag{1.7}$$

Motivated by these results and in view of the nonlocal character of the fractional Laplacian we are interested in the existence of weak solutions to fractional equation involving gradient term and measure. Before stating our main theorem we make precise the notion of weak solution used in this article.

Definition 1.1 *We say that u is a weak solution of (1.1), if $u \in L^1(\Omega)$, $g(|\nabla u|) \in L^1(\Omega, \rho^\alpha dx)$ and*

$$\int_\Omega [u(-\Delta)^\alpha \xi + g(|\nabla u|)\xi] dx = \int_\Omega \xi d\nu, \quad \forall \xi \in \mathbb{X}_\alpha, \tag{1.8}$$

where $\mathbb{X}_\alpha \subset C(\mathbb{R}^N)$ is the space of functions ξ satisfying:

- (i) $\text{supp}(\xi) \subset \bar{\Omega}$,
- (ii) $(-\Delta)^\alpha \xi(x)$ exists for all $x \in \Omega$ and $|(-\Delta)^\alpha \xi(x)| \leq C$ for some $C > 0$,
- (iii) there exist $\varphi \in L^1(\Omega, \rho^\alpha dx)$ and $\epsilon_0 > 0$ such that $|(-\Delta)_\epsilon^\alpha \xi| \leq \varphi$ a.e. in Ω , for all $\epsilon \in (0, \epsilon_0]$.

In order to state our main result, we introduce G_α the Green kernel of $(-\Delta)^\alpha$ in Ω and $\mathbb{G}_\alpha[\cdot]$ the Green operator defined as

$$\mathbb{G}_\alpha[\nu](x) = \int_{\Omega} G_\alpha(x, y) d\nu(y), \quad \forall \nu \in \mathfrak{M}(\Omega, \rho^\alpha). \quad (1.9)$$

In section 2, we obtain some estimates by the properties of $\mathbb{G}_\alpha[\nu]$ which play an important role in the analysis of the existence of weak solutions. Now we state the main result when g is a absorption.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be an open bounded C^2 domain, $\alpha \in (1/2, 1)$ and $p_\alpha^* = \frac{N}{N-2\alpha+1}$. Assume that $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous function satisfying*

$$\int_1^{+\infty} g(s) s^{-1-p_\alpha^*} ds < +\infty. \quad (1.10)$$

Then for any nonnegative Radon measure $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, 2\alpha - 1]$, problem (1.1) admits a nonnegative weak solution u_ν such that

$$\mathbb{G}_\alpha[\nu] - \mathbb{G}_\alpha[g(|\nabla \mathbb{G}_\alpha[\nu]|)] \leq u_\nu \leq \mathbb{G}_\alpha[\nu]. \quad (1.11)$$

Remark 1.1 *We note that the critical value p_α^* is independent of β .*

Remark 1.2 *We don't know if weak solution of (1.1) is unique. Because we don't find some inequality which plays the role of Kato's inequality to deal with the gradient term $g(|\nabla u|)$.*

Another interest is to consider the solution of (1.1) when g is a source, i.e. $g : \mathbb{R}_- \rightarrow \mathbb{R}_-$ is a continuous function. Then we have following result:

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be an open bounded C^2 domain, $\alpha \in (1/2, 1)$, $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, 2\alpha - 1]$ and $p_\alpha^* = \frac{N}{N-2\alpha+1}$. Assume that $g : \mathbb{R}_+ \rightarrow \mathbb{R}_-$ is a continuous function such that there exists $p < p_\alpha^*$*

$$-g(s) \leq c_1 s^p + \sigma_0 \quad \text{for } s \geq 0, \quad (1.12)$$

for some $c_1 > 0$ and $\sigma_0 > 0$.

If (i) $p \in (1, p_\alpha^)$, σ_0 and $\|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}$ small enough; or (ii) $p = 1$, c_1 small enough; or (iii) $p \in (0, 1)$, then problem (1.1) admits a weak solution u_ν such that*

$$\mathbb{G}_\alpha[\nu] \leq u_\nu \leq \mathbb{G}_\alpha[\nu] - \mathbb{G}_\alpha[g(|\nabla \mathbb{G}_\alpha[\nu]|)]. \quad (1.13)$$

We note that Theorem 1.2 deals with for general Radon measure ν , while ν is nonnegative when g is an absorption.

The paper is organized as follows. In Section 2, we present some preliminaries to obtain the estimate

$$\|\nabla G_\alpha[\nu]\|_{M^{p_\alpha^*}(\Omega, \rho^\alpha(x)dx)} \leq c_2 \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}, \quad c_2 > 0$$

and investigate the related classical solution.

Section 3 is devoted to prove the main theorems. In Section 4, we analyze the asymptotic behaviour of weak solutions to (1.1) when ν is a Dirac mass and g is a power absorption.

2 Preliminary

2.1 Marcinkiewicz type estimate

In this subsection, we recall some definition and basic properties of Marcinkiewicz spaces.

Definition 2.1 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain and μ be a positive Borel measure in Ω . For $\kappa > 1$, $\kappa' = \kappa/(\kappa - 1)$ and $u \in L^1_{loc}(\Omega, d\mu)$, we set*

$$\|u\|_{M^\kappa(\Omega, d\mu)} = \inf\{c \in [0, \infty] : \int_E |u| d\mu \leq c \left(\int_E d\mu \right)^{\frac{1}{\kappa'}}, \forall E \subset \Omega \text{ Borel set}\} \quad (2.1)$$

and

$$M^\kappa(\Omega, d\mu) = \{u \in L^1_{loc}(\Omega, d\mu) : \|u\|_{M^\kappa(\Omega, d\mu)} < \infty\}. \quad (2.2)$$

$M^\kappa(\Omega, d\mu)$ is called the Marcinkiewicz space of exponent κ or weak L^κ space and $\|\cdot\|_{M^\kappa(\Omega, d\mu)}$ is a quasi-norm. The following property holds.

Proposition 2.1 *[2, 8] Assume $1 \leq q < \kappa < \infty$ and $u \in L^1_{loc}(\Omega, d\mu)$. Then there exists $c_3(q, \kappa) > 0$ such that*

$$\int_E |u|^q d\mu \leq c_3(q, \kappa) \|u\|_{M^\kappa(\Omega, d\mu)} \left(\int_E d\mu \right)^{1-q/\kappa},$$

for any Borel set E of Ω .

Proposition 2.2 *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be an open bounded C^2 domain and $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, 2\alpha - 1]$. Then*

$$\|\nabla G_\alpha[|\nu|]\|_{M^{p_\alpha^*}(\Omega, \rho^\alpha(x)dx)} \leq c_2 \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}, \quad (2.3)$$

where $\nabla G_\alpha[|\nu|](x) = \int_\Omega \nabla_x G_\alpha(x, y) d|\nu|(y)$ and

$$p_\alpha^* = \frac{N}{N - 2\alpha + 1}. \quad (2.4)$$

Proof. For $\lambda > 0$ and $y \in \Omega$, we denote

$$\omega_\lambda(y) = \{x \in \Omega \setminus \{y\} : |\nabla_x G_\alpha(x, y)| \rho^\alpha(x) > \lambda \rho^\beta(y)\}, \quad m_\lambda(y) = \int_{\omega_\lambda(y)} dx.$$

From [10], there exists $c_4 > 0$ such that for any $(x, y) \in \Omega \times \Omega$ with $x \neq y$,

$$G_\alpha(x, y) \leq c_4 \min \left\{ \frac{1}{|x-y|^{N-2\alpha}}, \frac{\rho^\alpha(x)}{|x-y|^{N-\alpha}}, \frac{\rho^\alpha(y)}{|x-y|^{N-\alpha}} \right\}, \quad (2.5)$$

$$G_\alpha(x, y) \leq c_4 \frac{\rho^\alpha(y)}{\rho^\alpha(x)|x-y|^{N-2\alpha}}, \quad (2.6)$$

and by Corollary 3.3 in [3], we have

$$|\nabla_x G_\alpha(x, y)| \leq N G_\alpha(x, y) \max \left\{ \frac{1}{|x-y|}, \frac{1}{\rho(x)} \right\}, \quad (2.7)$$

which implies that for some $c_5 > 0$,

$$|\nabla_x G_\alpha(x, y)| \leq \max \left\{ \frac{c_5 \rho^\alpha(y)}{\rho^\alpha(x)|x-y|^{N-2\alpha+1}}, \frac{c_5 \rho^{\alpha-1-s}(x) \rho^s(y)}{|x-y|^{N-\alpha}} \right\}, \quad (2.8)$$

where $s \in [0, 2\alpha - 1]$ will be chosen later.

Therefore, for $2\alpha - 1 - s \geq 0$, we have

$$\begin{aligned} |\nabla_x G_\alpha(x, y)| \rho^\alpha(x) &\leq \max \left\{ \frac{c_5 \rho^\alpha(y)}{|x-y|^{N-2\alpha+1}}, \frac{c_5 \rho^{2\alpha-1-s}(x) \rho^s(y)}{|x-y|^{N-\alpha}} \right\} \\ &\leq \max \left\{ \frac{c_5 \rho^\alpha(y)}{|x-y|^{N-2\alpha+1}}, \frac{c_5 \rho^s(y)}{|x-y|^{N-\alpha}} \right\}. \end{aligned}$$

Since $N - 2\alpha + 1 > N - \alpha$ and $2\alpha - 1 - s \geq 0$, then there exists some $c_6 > 0$ such that

$$\omega_\lambda(y) \subset \left\{ x \in \Omega : |x-y| \leq c_6 \max \left\{ \rho^{\frac{\alpha-\beta}{N-2\alpha+1}}(y) \lambda^{-\frac{1}{N-2\alpha+1}}, \rho^{\frac{s-\beta}{N-\alpha}}(y) \lambda^{-\frac{1}{N-\alpha}} \right\} \right\}.$$

Choosing $s = \beta \in [0, 2\alpha - 1]$ and then for $\lambda > 1$,

$$\omega_\lambda(y) \subset \{x \in \Omega : |x-y| \leq c_6 \lambda^{-\frac{1}{N-2\alpha+1}}\}. \quad (2.9)$$

As a consequence,

$$m_\lambda(y) \leq c_7 \lambda^{-p_\alpha^*},$$

where $c_7 > 0$ independent of y and λ .

For any Borel set E of Ω and $\lambda > 1$, we have

$$\int_E |\nabla_x G_\alpha(x, y)| \rho^\alpha(x) dx \leq \int_{\omega_\lambda(y)} |\nabla_x G_\alpha(x, y)| \rho^\alpha(x) dx + \lambda \rho^\beta(y) \int_E dx$$

and

$$\begin{aligned}
\int_{\omega_\lambda(y)} |\nabla_x G_\alpha(x, y)| \rho^\alpha(x) dx &= - \int_{\lambda \rho^\beta(y)}^\infty s dm_s(y) \\
&= \lambda \rho^\beta(y) m_\lambda(y) + \int_{\lambda \rho^\beta(y)}^\infty m_s(y) ds \\
&\leq c_8 \rho^\beta(y) \lambda^{1-p_\alpha^*},
\end{aligned}$$

for some $c_8 > 0$. Thus,

$$\int_E |\nabla_x G_\alpha(x, y)| \rho^\alpha(x) dx \leq \rho^\beta(y) \left(c_8 \lambda^{1-p_\alpha^*} + \lambda \int_E dx \right).$$

Choosing $\lambda = \left(\int_E dx \right)^{-\frac{1}{p_\alpha^*}}$, we have

$$\int_E |\nabla_x G_\alpha(x, y)| \rho^\alpha(x) dx \leq c_8 \rho^\beta(y) \left(\int_E dx \right)^{\frac{p_\alpha^*-1}{p_\alpha^*}}, \quad \forall y \in \Omega.$$

Therefore,

$$\begin{aligned}
\int_E |\nabla \mathbb{G}_\alpha[|\nu|](x)| \rho^\alpha(x) dx &= \int_\Omega \int_E |\nabla_x G_\alpha(x, y)| \rho^\alpha(x) dx d|\nu(y)| \\
&\leq \int_\Omega \rho^\beta(y) d|\nu(y)| \sup_{x \in K} \rho^{-\beta}(y) \int_E |\nabla_x G_\alpha(x, y)| \rho^\alpha(x) dx \\
&\leq c_8 \int_\Omega \rho^\beta(y) d|\nu(y)| \left(\int_E dx \right)^{\frac{p_\alpha^*-1}{p_\alpha^*}} \\
&\leq c_8 \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)} \left(\int_E dx \right)^{\frac{p_\alpha^*-1}{p_\alpha^*}}.
\end{aligned}$$

As a consequence,

$$\|\nabla \mathbb{G}_\alpha[|\nu|]\|_{MP_\alpha^*(\Omega, \rho^\alpha dx)} \leq c_8 \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}.$$

We complete the proof. \square

The next proposition is a higher order regularity result

Proposition 2.3 [12] *Assume that $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $0 \leq \beta \leq \alpha$. Then for $q \in (1, \frac{N}{N+\beta-2\alpha})$ there exists $c_9 > 0$ such that for any $\nu \in L^1(\Omega, \rho^\beta dx)$*

$$\|\mathbb{G}_\alpha[\nu]\|_{W^{2\alpha-\gamma, q}(\Omega)} \leq c_9 \|\nu\|_{L^1(\Omega, \rho^\beta dx)} \quad (2.10)$$

where $q' = \frac{q}{q-1}$, $\gamma = \beta + \frac{N}{q'}$ if $\beta > 0$ and $\gamma > \frac{N}{q'}$ if $\beta = 0$.

Proposition 2.4 *Assume that $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $0 \leq \beta \leq \alpha$, then the mapping $\nu \mapsto |\nabla \mathbb{G}_\alpha[\nu]|$ is compact from $L^1(\Omega, \rho^\beta dx)$ into $L^q(\Omega)$ for any $q \in [1, \frac{N}{N+\beta+1-2\alpha})$.*

Proof. From the previous proposition we see that $\mathbb{G}_\alpha[\nu] \in W^{2\alpha-\gamma,p}(\Omega)$, where $2\alpha - \gamma > 1$ for $p \in (1, \frac{N}{N+\beta-2\alpha})$. Therefore, $|\nabla \mathbb{G}_\alpha[\nu]| \in W^{2\alpha-\gamma-1,p}(\Omega)$ and

$$\|\nabla \mathbb{G}_\alpha[\nu]\|_{W^{2\alpha-\gamma-1,p}(\Omega)} \leq c_9 \|\nu\|_{L^1(\Omega, \rho^\beta dx)}.$$

By [22, Th 6.5], the embedding of $W^{2\alpha-\gamma-1,p}(\Omega)$ into $L^q(\Omega)$ is compact. We complete the proof. \square

2.2 Classical solution

In order to prove our main theorems, we consider the existence of classical solution to problem

$$\begin{aligned} (-\Delta)^\alpha u + g(|\nabla u|) &= f & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c. \end{aligned} \quad (2.11)$$

We denote that $g_+ = \max\{g, 0\}$ and $g_- = \max\{-g, 0\}$.

Theorem 2.1 *Suppose that Ω is an open bounded C^2 domain of \mathbb{R}^N ($N \geq 2$), $\alpha \in (1/2, 1)$ and the function $g : \mathbb{R} \rightarrow \mathbb{R}$ is bounded. Assume more that there exists $\theta > 0$ such that $g \in C^\theta(\mathbb{R})$ and $f \in C^\theta(\bar{\Omega})$. Then problem (2.11) admits a unique classical solution u such that*

$$\mathbb{G}_\alpha[f] - \mathbb{G}_\alpha[g_+(|\nabla \mathbb{G}_\alpha[f]|)] \leq u \leq \mathbb{G}_\alpha[f] + \mathbb{G}_\alpha[g_-(|\nabla \mathbb{G}_\alpha[f]|)] \quad (2.12)$$

and if f is nonnegative, then $u \geq 0$.

Furthermore, the mapping $g \mapsto u$ is decreasing and $f \mapsto u$ is increasing.

To this end, we first introduce following Schauder fixed point theorem.

Theorem 2.2 [15] *Assume that X is a Banach space and $T : X \rightarrow X$ is a compact operator such that*

$$T(\mathcal{O}) \subset \mathcal{O},$$

where $\mathcal{O} \neq \emptyset$ is a closed bounded convex set of X . Then there exists fixed point $x \in \mathcal{O}$ of T , i.e.

$$Tx = x.$$

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. We divide the proof into several steps.

Step 1. To prove the existence. We define the operator T as

$$Tu = \mathbb{G}_\alpha[f - g(|\nabla u|)], \quad u \in W_0^{1,1}(\Omega),$$

where $W_0^{1,1}(\Omega)$ is the closure of $\{w \in C^\infty(\mathbb{R}^N) : \text{supp}(w) \subset \bar{\Omega}\}$ under the norm of $W^{1,1}(\Omega)$. We claim first that T is compact operator. In fact, for $u \in W_0^{1,1}(\Omega)$, we have that $f - g(|\nabla u|) \in L^1(\Omega)$ and then by using Proposition

2.4, implies that $Tu \in W_0^{2\alpha-\gamma,p}(\Omega)$ where $\gamma = \frac{p-1}{p}N$ and $2\alpha - \gamma > 1$ for $p \in (1, \frac{N}{N-2\alpha})$. Then the embedding $W_0^{2\alpha-\gamma,p}(\Omega) \hookrightarrow W_0^{1,1}(\Omega)$ is compact, so T is compact operator. Moreover, using (2.8) with $s = 0$, we have that

$$\begin{aligned} \|Tu\|_{W^{1,1}(\Omega)} &\leq \|\mathbb{G}_\alpha[f]\|_{W^{1,1}(\Omega)} + \|\mathbb{G}_\alpha[g(|\nabla u|)]\|_{W^{1,1}(\Omega)} \\ &\leq (\|f\|_{L^\infty(\Omega)} + \|g(|\nabla u|)\|_{L^\infty(\Omega)}) \left\| \int_\Omega G_\alpha(\cdot, y) dy \right\|_{W^{1,1}(\Omega)} \\ &= c_{10} (\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\mathbb{R}_+)}), \end{aligned} \quad (2.13)$$

where $c_{10} = \left\| \int_\Omega G_\alpha(\cdot, y) dy \right\|_{W^{1,1}(\Omega)}$.

Let $\mathcal{O} = \{u \in W_0^{1,1}(\Omega) : \|u\|_{W^{1,1}(\Omega)} \leq c_{10}(\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\mathbb{R}_+)})\}$, which is a closed and convex set of $W_0^{1,1}(\Omega)$. Combining with (2.13), then

$$T(\mathcal{O}) \subset \mathcal{O},$$

and using Theorem 2.2 to obtain a fixed point $u \in \mathbb{Y}$, that is, $Tu = u$.

Next we show that u is a classical solution of (2.11). Let open set O satisfy $O \subset \bar{O} \subset \Omega$. By Proposition 2.3 in [24], for any $\sigma \in (0, 2\alpha)$, there exists $c_{11} > 0$ such that

$$\|u\|_{C^\sigma(O)} \leq c_{11} \{ \|g(|\nabla u|)\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)} \},$$

and by choosing $\sigma = \frac{2\alpha+1}{2} \in (1, 2\alpha)$, then

$$\|\nabla u\|_{C^{\sigma-1}(O)} \leq c_{11} \{ \|g(|\nabla u|)\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)} \},$$

and then applied Corollary 2.4 in [24], u is $C^{2\alpha+\epsilon_0}$ locally in Ω for some $\epsilon_0 > 0$. Then u is a classical solution of (2.11). Moreover, from [12], we have

$$\int_\Omega [u(-\Delta)^\alpha \xi + g(|\nabla u|)\xi] dx = \int_\Omega \xi f dx, \quad \forall \xi \in \mathbb{X}_\alpha. \quad (2.14)$$

Step 2. To prove (2.12). Denote by $U_- = \mathbb{G}_\alpha[f] - \mathbb{G}_\alpha[g_+(\nabla \mathbb{G}_\alpha[f])]$ and $U_+ = \mathbb{G}_\alpha[f] + \mathbb{G}_\alpha[g_-(\nabla \mathbb{G}_\alpha[f])]$. Then $U_\pm = 0$ in Ω^c and

$$(-\Delta)^\alpha U_- + g_+(\nabla \mathbb{G}_\alpha[f]) = f \quad \text{in } \Omega$$

and

$$(-\Delta)^\alpha U_+ = f + g_-(\nabla \mathbb{G}_\alpha[f]) \quad \text{in } \Omega$$

We claim that

$$U_- \leq u \leq U_+ \quad \text{in } \Omega. \quad (2.15)$$

We prove the first inequality of (2.15). If not, there exists $x_0 \in \Omega$ such that

$$(U_- - u)(x_0) = \max_{x \in \Omega} \{(U_- - u)(x)\} > 0.$$

Then

$$(-\Delta)^\alpha(U_- - u)(x_0) > 0 \quad \text{and} \quad \nabla U_-(x_0) = \nabla u(x_0).$$

This implies that

$$(-\Delta)^\alpha(U_- - u)(x_0) + g_+(|\nabla U(x_0)|) - g(|\nabla u(x_0)|) > 0. \quad (2.16)$$

However, since u is the solutions of (2.11), then we have

$$(-\Delta)^\alpha(U_- - u)(x_0) + g_+(|\nabla U_-(x_0)|) - g(|\nabla u(x_0)|) = f(x_0) - f(x_0) = 0.$$

This is impossible with (2.16). Therefore, (2.12) follows by (2.15). Use the same way to prove the second inequality of (2.15).

In particular, for $f \geq 0$, since the function $\lim_{x \rightarrow \partial\Omega} U(x) = 0$, if u is not nonnegative, then there exists $x_0 \in \Omega$ such that

$$u(x_0) = \min_{x \in \Omega} u(x) < 0,$$

then we see that $\nabla u(x_0) = 0$ and $(-\Delta)^\alpha u(x_0) < 0$, This contradicts that u is the solution of (2.11).

Step 3. To show that the mapping $g \mapsto u$ is decreasing and $f \mapsto u$ is increasing. We just give the proof of the first argument, the second is similar to prove. Let g_1 and g_2 satisfy our hypotheses for g and $g_1 \leq g_2$. Denote u_1 and u_2 the solution of (2.11) with g_1 and g_2 respectively.

If there exists $x_0 \in \Omega$ such that

$$(u_1 - u_2)(x_0) = \min_{x \in \Omega} \{(u_2 - u_1)(x)\} < 0.$$

Then we have

$$(-\Delta)^\alpha(u_1 - u_2)(x_0) < 0, \quad \nabla u_1(x_0) = \nabla u_2(x_0).$$

This implies that

$$(-\Delta)^\alpha(u_1 - u_2)(x_0) + g_1(|\nabla u_1(x_0)|) - g_2(|\nabla u_2(x_0)|) < 0. \quad (2.17)$$

However,

$$(-\Delta)^\alpha(u_1 - u_2)(x_0) + g_1(|\nabla u_1(x_0)|) - g_2(|\nabla u_2(x_0)|) = f(x_0) - f(x_0) = 0$$

and the contradiction is obvious. Then $u_1 \geq u_2$.

Finally. The uniqueness follows by the monotonicity in Step 3. The proof is complete. \square

3 Proof of main theorems

3.1 The absorption nonlinearity

In this subsection, we prove the existence of weak solution to (1.1) when g is a absorption term, that is, $g(s) \geq 0$ for all $s \geq 0$.

Proof of Theorem 1.1. Existence. We define

$$C_\beta(\bar{\Omega}) = \{\zeta \in C(\bar{\Omega}) : \rho^{-\beta}\zeta \in C(\bar{\Omega})\}$$

endowed with the norm

$$\|\zeta\|_{C_\beta(\bar{\Omega})} = \|\rho^{-\beta}\zeta\|_{C(\bar{\Omega})}.$$

We consider a sequence nonnegative functions $\{\nu_n\} \subset C^1(\bar{\Omega})$ such that $\nu_n \rightarrow \nu$ in the duality sense with $C_\beta(\bar{\Omega})$, which means

$$\lim_{n \rightarrow \infty} \int_{\bar{\Omega}} \zeta \nu_n dx = \int_{\bar{\Omega}} \zeta d\nu \quad (3.1)$$

for all $\zeta \in C_\beta(\bar{\Omega})$. It follows from the Banach-Steinhaus theorem that $\|\nu_n\|_{\mathfrak{M}(\Omega)}$ is bounded independently of n . We denote by $\{g_n\}$ defined on \mathbb{R}_+ a sequence of C^θ and nonnegative functions such that

$$g_n \leq g_{n+1} \leq g, \quad \sup_{s \in \mathbb{R}_+} g_n(s) = n \quad \text{and} \quad \lim_{n \rightarrow \infty} \|g_n - g\|_{L_{loc}^\infty(\mathbb{R}_+)} = 0. \quad (3.2)$$

By Proposition 2.1, there exists a unique nonnegative solution u_n of (1.1) with ν_n and g_n , therefore

$$\int_{\Omega} (u_n + g_n(|\nabla u_n|)\eta_1) dx \leq \int_{\Omega} \nu_n \eta_1 dx \leq C \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}. \quad (3.3)$$

Therefore $\|g_n(|\nabla u_n|)\|_{\mathfrak{M}(\Omega, \rho^\alpha)}$ is bounded independently of n . For $\epsilon > 0$, set $\xi_\epsilon = (\eta_1 + \epsilon)^{\frac{\beta}{\alpha}} - \epsilon^{\frac{\beta}{\alpha}}$, which is concave in the interval $\eta(\bar{\omega})$. By Lemma 2.3 part (ii) in [12], we see that

$$\begin{aligned} (-\Delta)^\alpha \xi_\epsilon &= \frac{\beta}{\alpha} (\eta_1 + \epsilon)^{\frac{1}{\alpha}} (-\Delta)^\alpha \eta_1 - \frac{\beta(\beta - \alpha)}{\alpha^2} (\eta_1 + \epsilon)^{\frac{\beta - 2\alpha}{\alpha}} \int_{\Omega} \frac{(\eta_1(y) - \eta_1(x))^2}{|y - x|^{N+2\alpha}} dy \\ &\geq \frac{\beta}{\alpha} (\eta_1 + \epsilon)^{\frac{\beta - \alpha}{\alpha}}, \end{aligned}$$

and $\xi_\epsilon \in \mathbb{X}_\alpha$. Since

$$\int_{\Omega} (|u_n| (-\Delta)^\alpha \xi_\epsilon + g_n(|\nabla u_n|) \xi_\epsilon) dx \leq \int_{\Omega} \xi_\epsilon d|\nu_n|,$$

we obtain

$$\int_{\Omega} \left(\frac{\beta}{\alpha} |u_n| (\eta_1 + \epsilon)^{\frac{\beta - \alpha}{\alpha}} + g_n(|\nabla u_n|) \xi_\epsilon \right) dx \leq \int_{\Omega} \xi_\epsilon d|\nu_n|.$$

If we let $\epsilon \rightarrow 0$, we obtain

$$\int_{\Omega} \left(\frac{\beta}{\alpha} |u_n| \eta_1^{\frac{\beta-\alpha}{\alpha}} + g_n(|\nabla u_n|) \eta_1^{\frac{\beta}{\alpha}} \right) dx \leq \int_{\Omega} \eta_1^{\frac{\beta}{\alpha}} d|\nu_n|.$$

By Lemma 2.3 in [12], we derive the estimate

$$\int_{\Omega} \left(|u_n| \rho^{\beta-\alpha} + g_n(|\nabla u_n|) \rho^{\beta} \right) dx \leq c_{12} \|\nu_n\|_{\mathfrak{M}(\Omega, \rho^{\beta})} \leq c_{13} \|\nu\|_{\mathfrak{M}(\Omega, \rho^{\beta})}. \quad (3.4)$$

Then $\nu_n, g_n(|\nabla u_n|)$ is uniformly bounded in $\mathfrak{M}(\Omega, \rho^{\beta})$. Since $u_n = \mathbb{G}[\nu_n - g_n(|\nabla u_n|)]$, then

$$\begin{aligned} \|\nabla u_n\|_{M^{p_{\alpha}^*}(\Omega, \rho^{\alpha} dx)} &\leq \|\nu_n\|_{\mathfrak{M}(\Omega, \rho^{\beta})} + \|g_n(|\nabla u_n|)\|_{\mathfrak{M}(\Omega, \rho^{\beta})} \\ &\leq c_{14} \|\nu\|_{\mathfrak{M}(\Omega, \rho^{\beta})}. \end{aligned}$$

By Corollary 2.4 the sequence $\{u_n\}, \{|\nabla u_n|\}$ are relatively compact in the $L^q(\Omega)$ for $1 \leq q < \frac{N}{N-2\alpha}$ and $1 \leq q < \frac{N}{N-2\alpha+1}$, respectively. Therefore there exist a sub-sequence $\{u_{n_k}\}$ and some $u \in L^q(\Omega)$ such that $|\nabla u_{n_k}| \rightarrow |\nabla u|$ in $L^q(\Omega)$ and almost every where in Ω . Furthermore $g_{n_k}(|\nabla u_{n_k}|) \rightarrow g(|\nabla u|)$ almost every where in Ω . For $\lambda > 0$, we set $S_{\lambda} = \{x \in \Omega : |\nabla u_{n_k}(x)| > \lambda\}$ and $\omega(\lambda) = \int_{S_{\lambda}} \rho^{\alpha}(x) dx$. Then for any Borel set $E \subset \Omega$, we have

$$\begin{aligned} \int_E g_{n_k}(|\nabla u_{n_k}|) \rho^{\alpha}(x) dx &\leq \int_E g(|\nabla u_{n_k}|) \rho^{\alpha}(x) dx \\ &= \int_{E \cap S_{\lambda}^c} g(|\nabla u_{n_k}|) \rho^{\alpha}(x) dx + \int_{E \cap S_{\lambda}} g(|\nabla u_{n_k}|) \rho^{\alpha}(x) dx \\ &\leq \tilde{g}(\lambda) \int_E \rho^{\alpha} dx + \int_{S_{\lambda}} g(|\nabla u_{n_k}|) \rho^{\alpha}(x) dx \\ &\leq \tilde{g}(\lambda) \int_E \rho^{\alpha} dx - \int_{\lambda}^{\infty} g(s) d\omega(s), \end{aligned}$$

where $\tilde{g}(s) = \max_{t \in [0, s]} \{g(t)\}$. But

$$\int_{\lambda}^{\infty} g(s) d\omega(s) = \lim_{T \rightarrow \infty} \int_{\lambda}^T g(s) d\omega(s).$$

Since $|\nabla u_{n_k}| \in M^{p_{\alpha}^*}(\Omega, \rho^{\alpha} dx)$, $\omega(s) \leq c_{15} s^{-p_{\alpha}^*}$ and

$$\begin{aligned} - \int_{\lambda}^T g(s) d\omega(s) &= - \left[g(s) \omega(s) \right]_{s=\lambda}^{s=T} + \int_{\lambda}^T \omega(s) dg(s) \\ &\leq g(\lambda) \omega(\lambda) - g(T) \omega(T) + c_{15} \int_{\lambda}^T s^{-p_{\alpha}^*} dg(s) \\ &\leq g(\lambda) \omega(\lambda) - g(T) \omega(T) + c_{15} \left(T^{-p_{\alpha}^*} g(T) - \lambda^{-p_{\alpha}^*} g(\lambda) \right) \\ &\quad + \frac{c_{15}}{p_{\alpha}^* + 1} \int_{\lambda}^T s^{-1-p_{\alpha}^*} g(s) ds. \end{aligned}$$

By assumption (1.10) there exists $\{T\} \rightarrow \infty$ such that $T^{-p_\alpha^*}g(T) \rightarrow 0$ when $n \rightarrow \infty$. Furthermore $g(\lambda)\omega(\lambda) \leq c\lambda^{-p_\alpha^*}g(\lambda)$, therefore

$$-\int_\lambda^\infty g(s)d\omega(s) \leq \frac{c_{15}}{p_\alpha^* + 1} \int_\lambda^\infty s^{-1-p_\alpha^*}g(s)ds.$$

Notice that the above quantity on the right-hand side tends to 0 when $\lambda \rightarrow \infty$. The conclusion follows: for any $\epsilon > 0$ there exists $\lambda > 0$ such that

$$\frac{c_{15}}{p_\alpha^* + 1} \int_\lambda^\infty s^{-1-p_\alpha^*}g(s)ds \leq \frac{\epsilon}{2}$$

and $\delta > 0$ such that

$$\int_E \rho^\alpha(x)dx \leq \delta \implies g(\lambda) \int_E dx \leq \frac{\epsilon}{2}.$$

This proves that $\{g_{n_k}(|\nabla u_{n_k}|)\}$ is uniformly integrable in $L^1(\Omega, \rho^\alpha(x)dx)$. Then $g_{n_k}(|\nabla u_{n_k}|) \rightarrow g(|\nabla u|)$ in $L^1(\Omega, \rho^\alpha(x)dx)$ by Vitali convergence theorem. Letting $n_k \rightarrow \infty$ in the identity

$$\int_\Omega (u_{n_k}(-\Delta)^\alpha \xi + \xi g_{n_k}(|\nabla u_{n_k}|)) dx = \int_\Omega \nu_{n_k} \xi dx$$

where $\xi \in \mathbb{X}_\alpha$, it infers that u is a weak solution of (1.1). Since u_{n_k} is nonnegative, so is u .

To prove (1.11). In the proof of the existence, we have know that u_{n_k} satisfies (2.12), that is

$$\mathbb{G}_\alpha[\nu_{n_k}] - \mathbb{G}_\alpha[g_{n_k}(|\nabla \mathbb{G}_\alpha[\nu_{n_k}]|)] \leq u_{n_k} \leq \mathbb{G}_\alpha[\nu_{n_k}].$$

We observe that $\mathbb{G}_\alpha[\nu_{n_k}] \rightarrow \mathbb{G}_\alpha[\nu]$ a.e. in Ω , and $g_{n_k}(|\nabla \mathbb{G}_\alpha[\nu_{n_k}]|) \rightarrow g(|\nabla \mathbb{G}_\alpha[\nu]|)$ in $L^1(\Omega, \rho^\alpha dx)$. Then

$$\mathbb{G}_\alpha[\nu_{n_k}] - \mathbb{G}_\alpha[g_{n_k}(|\nabla \mathbb{G}_\alpha[\nu_{n_k}]|)] \rightarrow \mathbb{G}_\alpha[\nu] - \mathbb{G}_\alpha[g(|\nabla \mathbb{G}_\alpha[\nu]|)] \quad \text{a.e. in } \Omega.$$

Therefore, the solution u obtained as the limit of u_{n_k} satisfies (1.11). \square

3.2 The source nonlinearity

The purpose of this section is to study the existence of weak solutions to the semilinear fractional elliptic problem with gradient source term g satisfying $g(s) \leq 0$ for $s \geq 0$.

Proof of Theorem 1.2. Let $\{\nu_n\}$ be a sequence of C^2 functions converging to ν in the sense of (3.1), $\{g_n\}$ defined on \mathbb{R}_+ a sequence of C^θ , negative functions satisfying

$$g_n \geq g_{n+1} \geq g, \quad \inf_{s \in \mathbb{R}_+} g_n(s) = -n \quad \text{and} \quad \lim_{n \rightarrow \infty} \|g_n - g\|_{L_{loc}^\infty(\mathbb{R}_+)} = 0$$

and

$$M(v) = \left(\int_{\Omega} |\nabla v|^p \rho^\beta dx \right)^{\frac{1}{p}}.$$

We may assume that $\|\nu_n\|_{\mathfrak{M}(\Omega, \rho^\beta)} \leq 2\|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}$. Denote $u_1 = 0$ and

$$(-\Delta)^\alpha u_n = -g_{n-1}(|\nabla u_{n-1}|) + \nu_n.$$

On the one hand, by (1.12) and Proposition 2.2,

$$\begin{aligned} \left(\int_{\Omega} |\nabla u_n|^{p_\alpha^*} \rho^\alpha dx \right)^{\frac{1}{p_\alpha^*}} &\leq c_2 \| -g(|\nabla u_{n-1}|) + \nu_n \|_{\mathfrak{M}(\Omega, \rho^\beta)} \\ &\leq c_2 [\| -g(|\nabla u_{n-1}|) \|_{L^1(\Omega, \rho^\beta dx)} + 2\|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}] \\ &\leq c_2 c_1 \int_{\Omega} |\nabla u_{n-1}|^p \rho^\beta dx + c_{16} \sigma_0 + 2c_2 \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}, \end{aligned} \quad (3.5)$$

where $c_{16} = c_2 \int_{\Omega} \rho^\beta dx$.

On the other hand,

$$\left(\int_{\Omega} |\nabla u_n|^p \rho^\beta dx \right)^{\frac{1}{p}} \leq \left(\int_{\Omega} \rho^{\frac{\beta p_\alpha^* - \alpha}{p_\alpha^* - p}} dx \right)^{\frac{1}{p} - \frac{1}{p_\alpha^*}} \left(\int_{\Omega} |\nabla u_n|^{p_\alpha^*} \rho^\alpha dx \right)^{\frac{1}{p_\alpha^*}}, \quad (3.6)$$

where $\int_{\Omega} \rho^{\frac{\beta p_\alpha^* - \alpha}{p_\alpha^* - p}} dx$ is bounded, since $\frac{\beta p_\alpha^* - \alpha}{p_\alpha^* - p} > -1$.

Along with (3.5) and (3.6), we have

$$M(u_n) \leq c_{17} M(u_{n-1})^p + c_{18} \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)} + c_{16} \sigma_0,$$

where $c_{17} = c_2 c_1 \left(\int_{\Omega} \rho^{\frac{\beta p_\alpha^* - \alpha}{p_\alpha^* - p}} dx \right)^{\frac{1}{p} - \frac{1}{p_\alpha^*}} > 0$ and $c_{18} > 0$ independent of n . Then by induction, there exists $\lambda > 0$ such that $M(u_n) \leq \lambda$ for any $n \geq 1$, which holds if

$$c_{17} \lambda^p + c_{18} \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)} \leq \lambda. \quad (3.7)$$

When $p \in (1, p_\alpha^*)$, (3.7) holds if $\|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}$ is small enough; when $p \in (0, 1)$, (3.7) holds with any restriction; when $p = 1$, (3.7) holds if $c_{17} < 1$. \square

4 Further study for g an absorption and ν Dirac mass

4.1 Regularity

In order to tackle the uniqueness, we have to make use of the properties of classical solution, since Kato's inequality is invalid to obtain uniqueness for gradient nonlinear term. To this end, we start by

Proposition 4.1 *Assume that $0 \in \Omega$ is an open, bounded and C^2 domain, $\alpha \in (\frac{1}{2}, 1)$, $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous. and u is a weak nonnegative solution of (1.1). If $g(s) \geq 0$ for $s \geq s_0$ with $s_0 \geq 0$, then u be a classical solution of*

$$\begin{aligned} (-\Delta)^\alpha u + g(|\nabla u|) &= 0 & \text{in } \Omega \setminus \{0\}, \\ u &= 0 & \text{in } \Omega^c \end{aligned} \quad (4.1)$$

and

$$0 \leq u \leq \mathbb{G}_\alpha[\delta_0]. \quad (4.2)$$

Proof. Let $\Omega_k = (1 + \frac{1}{k})\Omega$, then there exists $k_0 > 0$ such that Ω_k is C^2 domain for $k \geq k_0$ and $\{v_k\}$ be a sequence functions in $C^2(\mathbb{R}^N)$ with $\text{supp}(v_k) \subset \Omega_k$ such that

$$v_k \rightarrow u \text{ in } L^1(\Omega) \quad \text{and} \quad g(|\nabla v_k|) \rightarrow g(|\nabla u|) \quad \text{in } L^1(\Omega, \rho^\alpha dx).$$

On the one hand, $(-\Delta)^\alpha v_k$ and $g(|\nabla v_k|)$ are sequences of $C^{2-2\alpha}(\Omega_k)$ positive functions and denote

$$f_k = (-\Delta)^\alpha v_k + g(|\nabla v_k|) \quad x \in \Omega_k.$$

On the other hand, for any $\xi \in \mathbb{X}_\alpha(\Omega)$, then $\xi \circ k \in \mathbb{X}_\alpha(\Omega_k)$, where $\mathbb{X}_\alpha(\Omega) = \mathbb{X}_\alpha$ is defined (1.8) with respect to Ω , $\mathbb{X}_\alpha(\Omega_k)$ does with to Ω_k and

$$\xi \circ k(x) = \xi\left(\left(1 + \frac{1}{k}\right)^{-1}x\right).$$

Combining with Lemma 2.1 (ii) and Lemma 2.3 in [12], we have

$$\int_{\Omega_k} [v_k(-\Delta)^\alpha \xi \circ k + g(|\nabla v_k|)\xi \circ k] dx = \int_{\Omega_k} \xi \circ k f_k dx, \quad \forall \xi \in \mathbb{X}_\alpha. \quad (4.3)$$

We observe that for $\xi \in \mathbb{X}_\alpha(\Omega)$, $(-\Delta)^\alpha \xi \circ k \rightarrow (-\Delta)^\alpha \xi$ in L^∞ and $\xi \circ k \rightarrow \xi$ uniformly in \mathbb{R}^N as $k \rightarrow \infty$, thus we have

$$\int_{\Omega_k} [v_k(-\Delta)^\alpha \xi \circ k + g(|\nabla v_k|)\xi \circ k] dx \rightarrow \int_{\Omega} [u(-\Delta)^\alpha \xi + g(|\nabla u|)\xi] dx \quad \text{as } k \rightarrow \infty.$$

Then we have $f_k \rightarrow \delta_0$ as $k \rightarrow \infty$ in the distribution sense. Therefore, $\mathbb{G}_{\alpha,k}[f_k] \rightarrow \mathbb{G}_\alpha[\delta_0]$ in $\mathbb{R}^N \setminus \{0\}$ as $k \rightarrow \infty$ and there exists $c_{19} > 0$ such that for all $k > N_0$, $|\mathbb{G}_{\alpha,k}[f_k]| \leq c_{19}\mathbb{G}_{\alpha,k}[\delta_0]$ in $\mathbb{R}^N \setminus \{0\}$ as where $\mathbb{G}_{\alpha,k}$ is the Green operator in (1.9) with to Ω_k .

As a consequence,

$$\begin{aligned} 0 \leq v_k &= \mathbb{G}_{\alpha,k}[f_k] - \mathbb{G}_{\alpha,k}[g(|\nabla v_k|)] \\ &\leq c_{20}\mathbb{G}_{\alpha,k}[\delta_0] + \max_{s \in [0, s_0]} |g(s)|. \end{aligned} \quad (4.4)$$

Denote by O an open set satisfying $\bar{O} \subset \Omega \setminus B_r$ with $r > 0$ and $O_k = (1 + \frac{1}{k})O$, then there exists $c_{21} > 0$ independent of k such that

$$\|v_k\|_{L^\infty(O_k)} \leq c_{21} \|\mathbb{G}_{\alpha,k}[\delta_0]\|_{L^\infty(O)} + \max_{s \in [0, s_0]} |g(s)| \quad \text{and}$$

$$\|v_k\|_{L^1(\Omega_k)} \leq c_{21} \|\mathbb{G}_{\alpha,k}[\delta_0]\|_{L^1(\Omega)} + \max_{s \in [0, s_0]} |g(s)|.$$

By Corollary 2.4 in [24] and Lemma 3.1 in [13], for some $\epsilon > 0$ and $\beta \in (0, 2\alpha)$, we have

$$\begin{aligned} & \|v_k\|_{C^{2\alpha+\epsilon}(O_k)} \\ & \leq c_{22} \{ \|v_k\|_{L^\infty(\Omega_k \setminus B_{\frac{r}{2}})}^p + \|f_k\|_{L^\infty(\Omega_k \setminus B_{\frac{r}{2}})} + \|v_k\|_{C^\beta(\Omega_k \setminus B_{\frac{3r}{4}})} \} \\ & \leq c_{23} \{ \|v_k\|_{L^\infty(\Omega_k \setminus B_{\frac{r}{2}})}^p + \|v_k\|_{L^\infty(\Omega_k \setminus B_{\frac{r}{2}})} + \|f_k\|_{L^\infty(\Omega_k \setminus B_{\frac{r}{2}})} + \|v_k\|_{L^1(\Omega_k)} \} \\ & \leq c_{24} \{ \|\mathbb{G}_{\alpha,k}[\delta_0]\|_{L^\infty(\Omega_k \setminus B_{\frac{r}{2}})}^p + \|\mathbb{G}_{\alpha,k}[\delta_0]\|_{L^\infty(\Omega_k \setminus B_{\frac{r}{2}})} \\ & \quad + \|\mathbb{G}_{\alpha,k}[\delta_0]\|_{L^1(\Omega_k)} + \max_{s \in [0, s_0]} |g(s)| \} \\ & \leq c_{25} \{ \|\mathbb{G}_\alpha[\delta_0]\|_{L^\infty(\Omega \setminus B_{\frac{r}{2}})}^p + \|\mathbb{G}_\alpha[\delta_0]\|_{L^\infty(\Omega \setminus B_{\frac{r}{2}})} \\ & \quad + \|\mathbb{G}_\alpha[\delta_0]\|_{L^1(\Omega)} + \max_{s \in [0, s_0]} |g(s)| \}, \end{aligned}$$

where constants $c_{23}, c_{24}, c_{25} > 0$ independent of k . Therefore, together with (2.13) and Arzela-Ascoli Theorem, we have that $u \in C^{2\alpha+\frac{\epsilon}{2}}(O)$, which implies that u is $C^{2\alpha+\frac{\epsilon}{2}}$ locally in $\Omega \setminus \{0\}$. Therefore, u is a classical solution of (4.1). By (4.4), we have

$$0 \leq u \leq \mathbb{G}_\alpha[\delta_0].$$

□

Corollary 4.1 *Assume that $0 \in \Omega$ is an open, bounded and C^2 domain, $\alpha \in (\frac{1}{2}, 1)$ and $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous. Let u is a weak solution of (1.1) such that u_- is uniformly bounded in any compact set of $\Omega \setminus \{0\}$. Then u is a classical solution of (4.1).*

Proof. The proof proceeds the same as the one of Proposition 4.1 just replacing (4.4) by

$$-C\mathbb{G}_{\alpha,k}[\delta_0] \leq v_k \leq C\mathbb{G}_{\alpha,k}[\delta_0],$$

for some $C > 0$.

□

4.2 Uniqueness and asymptotic behavior with Dirac mass

This section is devoted to consider the behavior of the solution of (1.1) in particular case. We consider the following problem

$$\begin{aligned} (-\Delta)^\alpha u + |\nabla u|^p &= \delta_0 & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c, \end{aligned} \quad (4.5)$$

where $0 \in \Omega$ and δ_0 is Dirac mass.

Theorem 4.1 *Assume that $0 \in \Omega$ is an open, bounded and C^2 domain, $\alpha \in (\frac{1}{2}, 1)$ and $0 < p < p_\alpha^*$. Then problem (4.5) admits a solution u such that*

(i) if $p \in (\frac{2\alpha}{N-2\alpha+1}, p_\alpha^*)$,

$$\frac{C^{-1}}{|x|^{(N-2\alpha+1)p-2\alpha}} < u(x) - \frac{C_0}{|x|^{N-2\alpha}} \leq \frac{C}{|x|^{(N-2\alpha+1)p-2\alpha}}, \quad x \in B_r(0) \setminus \{0\};$$

(ii) if $p = \frac{2\alpha}{N-2\alpha+1}$,

$$-C^{-1} \ln(|x|) < u(x) - \frac{C_0}{|x|^{N-2\alpha}} \leq -C \ln(|x|), \quad x \in B_r(0) \setminus \{0\};$$

(iii) if $p < \frac{2\alpha}{N-2\alpha+1}$,

$$C^{-1} < u(x) - \frac{C_0}{|x|^{N-2\alpha}} \leq C, \quad x \in B_r(0) \setminus \{0\},$$

where $r = \frac{\min\{\rho(0), 1\}}{4}$, $C_0 = \lim_{x \rightarrow 0} G_\alpha(x, 0)|x|^{N-2\alpha}$ and $C > 0$.

Moreover, the uniqueness holds in the sense of

$$\lim_{|x| \rightarrow 0^+} u(x)|x|^{N-2\alpha} = C_0. \quad (4.6)$$

We observe that if $g(s) = s^p$ with $p < p_\alpha^*$, (1.10) holds. Then the asymptotic behavior of solution obtained by Theorem 1.1 is controlled by $\mathbb{G}_\alpha[\delta_0]$ and $\mathbb{G}_\alpha[\delta_0] - \mathbb{G}_\alpha[|\nabla \mathbb{G}_\alpha[\delta_0]|^p]$. Therefore, we have to do estimate of the behavior of $\mathbb{G}_\alpha[|\nabla \mathbb{G}_\alpha[\delta_0]|^p]$.

Lemma 4.1 *Assume the hypotheses of Theorem 4.1 holds. Then there exists a positive constant $C = C(N, \alpha, \Omega) > 1$ such that*

(i) if $p \in (\frac{2\alpha}{N-2\alpha+1}, p_\alpha^*)$,

$$\frac{1}{C} \leq \mathbb{G}_\alpha[|\nabla \mathbb{G}_\alpha[\delta_0]|^p] |x|^{-2\alpha+(N-2\alpha+1)p} \leq C, \quad x \in B_r(0) \setminus \{0\}; \quad (4.7)$$

(ii) if $p = \frac{2\alpha}{N-2\alpha+1}$,

$$\frac{1}{C} \leq \mathbb{G}_\alpha[|\nabla \mathbb{G}_\alpha[\delta_0]|^p] (-\ln|x|)^{-1} \leq C, \quad x \in B_r(0) \setminus \{0\}; \quad (4.8)$$

(iii) if $p < \frac{2\alpha}{N-2\alpha+1}$,

$$\frac{1}{C} \leq \mathbb{G}_\alpha[|\nabla \mathbb{G}_\alpha[\delta_0]|^p] \leq C, \quad x \in B_r(0) \setminus \{0\}, \quad (4.9)$$

where $r = \frac{1}{4} \min\{1, \rho(0)\}$.

Proof. (i) In the case of $\frac{2\alpha}{N-2\alpha+1} < p < p_\alpha^*$. On one side, by the fact of $p < \frac{1}{1-\alpha}$ and $\mathbb{G}_\alpha[\delta_0] = G(x, 0)$, then for $x \in B_r(0) \setminus \{0\}$,

$$\begin{aligned} \mathbb{G}_\alpha[|\nabla \mathbb{G}_\alpha[\delta_0]|^p](x) &= \int_{\Omega} G(x, y) |\nabla_y G(y, 0)|^p dy \\ &\leq \int_{\Omega} \frac{C_0}{|x-y|^{N-2\alpha}} \frac{\rho^{(\alpha-1)p}(y)}{\rho^{\alpha p}(0)} \frac{C_0^p}{|y|^{(N-2\alpha+1)p}} dy \\ &\leq C \left[\int_{B_r} \frac{1}{|x-y|^{N-2\alpha}} \frac{1}{|y|^{(N-2\alpha+1)p}} dy + \int_{\Omega \setminus B_r} \rho^{(\alpha-1)p}(y) dy \right] \\ &=: C[E_1(x) + E_2(x)]. \end{aligned}$$

Since $N - 2\alpha < N$ and $(N - 2\alpha + 1)p < N$, then

$$\begin{aligned} E_1(x) &= |x|^{2\alpha-(N-2\alpha+1)p} \int_{B_{\frac{r}{|x|}}} \frac{1}{|e_x - y|^{N-2\alpha}} \frac{1}{|y|^{(N-2\alpha+1)p}} dy \\ &\leq C_1 |x|^{2\alpha-(N-2\alpha+1)p} \left[1 + \int_1^{\frac{r}{|x|}} s^{2\alpha-1-(N-2\alpha+1)p} ds \right] \\ &\leq C_2 |x|^{2\alpha-(N-2\alpha+1)p}, \quad x \in B_r(0) \setminus \{0\}. \end{aligned} \quad (4.10)$$

where $e_x = \frac{x}{|x|}$. It is obvious that

$$E_2(x) \leq C, \quad x \in B_r(0) \setminus \{0\}.$$

by the fact of $p < p_\alpha^* \leq \frac{1}{1-\alpha}$. Then

$$\mathbb{G}_\alpha[|\nabla \mathbb{G}_\alpha[\delta_0]|^p](x) \leq C_2 |x|^{2\alpha-(N-2\alpha+1)p}, \quad x \in B_r(0) \setminus \{0\}. \quad (4.11)$$

On the other side, for $x \in B_r(0)$ with $r = \rho(0)/4$, from [3], we have

$$|\nabla_x G(x, 0)| \geq \frac{C_0}{|x|^{N-2\alpha+1}}, \quad x \in \Omega.$$

Then for $x \in B_r(0) \setminus \{0\}$, we have

$$\begin{aligned} \mathbb{G}_\alpha[|\nabla \mathbb{G}_\alpha[\delta_0]|^p](x) &\geq C \int_{B_r(0)} \frac{1}{|y-x|^{N-2\alpha}} \frac{1}{|y|^{(N-2\alpha)p}} dy \\ &\geq C |x|^{2\alpha-(N-2\alpha)p}, \end{aligned} \quad (4.12)$$

for some $C > 0$. As a consequence, we deduce (4.7) by (4.11) and (4.12).

(ii) In the case of $p = \frac{2\alpha}{N-2\alpha+1}$. It is obvious that (4.10) becomes

$$\mathbb{G}_\alpha[|\nabla\mathbb{G}_\alpha[\delta_0]|^p](x) \leq C + \int_1^{\frac{r}{|x|}} s^{-1} ds \leq C(1 - \ln|x|)$$

and (4.12) becomes

$$\mathbb{G}_\alpha[|\nabla\mathbb{G}_\alpha[\delta_0]|^p](x) \geq C + \int_1^{\frac{D}{|x|}} s^{-1} ds \leq C(1 - \ln|x|).$$

(iii) In the case of $p < \frac{2\alpha}{N-2\alpha+1}$. In this case, we only need to prove that $\int_\Omega \frac{1}{|y-x|^{N-2\alpha}|y|^{(N-2\alpha+1)p}} dy$ is bounded. Indeed, for $x \in B_r(x_0)$ with $r = \frac{\min\{1, d(x_0)\}}{8}$, by Young inequality and the fact of $(N-2\alpha)(p+1) < N$, we obtain that

$$\begin{aligned} & \int_{B_{2r}(0)} \frac{1}{|y-x|^{N-2\alpha}|y|^{(N-2\alpha+1)p}} dy \\ & \leq \int_{B_{2r}(0)} \left[\frac{1}{|y-x|^{(N-2\alpha)(p+1)}} + \frac{1}{|y|^{(N-2\alpha+1)(p+1)}} \right] dy \\ & \leq \int_{B_{4r}(x)} \frac{1}{|y-x|^{(N-2\alpha)(p+1)}} dy + \int_{B_{2r}(0)} \frac{1}{|y|^{(N-2\alpha+1)(p+1)}} dy \\ & \leq C. \end{aligned}$$

Then (4.12) becomes

$$\mathbb{G}_\alpha[|\nabla\mathbb{G}_\alpha[\delta_0]|^p](x) \geq C.$$

We complete the proof. \square

Proof of Theorem 4.1. The existence follows Theorem 1.1 and the solution u satisfies that

$$\mathbb{G}_\alpha[\delta_0] - \mathbb{G}_\alpha[|\nabla\mathbb{G}_\alpha[\delta_0]|^p] \leq u \leq \mathbb{G}_\alpha[\delta_0].$$

We deduce the asymptotic behavior of solution u by $\mathbb{G}_\alpha[\delta_0] = G_\alpha(x, 0)$ and Lemma 4.1.

To prove uniqueness. Let u and \tilde{u} be two solutions of (4.5), then u and \tilde{u} are classical solution of (4.1). For any $\epsilon > 0$, let $u_\epsilon = (1 + \epsilon)u$, we claim that

$$\tilde{u} \leq u_\epsilon. \quad (4.13)$$

In fact, since $\lim_{x \rightarrow 0} \tilde{u}|x|^{N-2\alpha} = C_0$ and $\lim_{x \rightarrow 0} u_\epsilon|x|^{N-2\alpha} = (1 + \epsilon)C_0$, if not, there exists some point $x_0 \in \Omega \setminus \{0\}$ such that

$$(\tilde{u} - u_\epsilon)(x_0) = \max_{x \in \Omega} (\tilde{u} - u_\epsilon)(x) > 0.$$

This implies that $(-\Delta)^\alpha(\tilde{u} - u_\epsilon)(x_0) < 0$ and $\nabla\tilde{u}(x_0) = \nabla u_\epsilon(x_0)$. Then on the one hand, we have

$$(-\Delta)^\alpha(\tilde{u} - u_\epsilon)(x_0) + |\nabla\tilde{u}|^p(x_0) - |\nabla u_\epsilon|^p(x_0) = 0 \quad (4.14)$$

On the other hand, $(-\Delta)^\alpha\tilde{u}(x_0) + |\nabla\tilde{u}|^p(x_0) = 0$ and $(-\Delta)^\alpha u_\epsilon(x_0) = -(1 + \epsilon)|\nabla u|^p$ and $|\nabla u_\epsilon|^p(x_0) = (1 + \epsilon)^p|\nabla u|^p$ and then

$$(-\Delta)^\alpha(\tilde{u} - u_\epsilon)(x_0) + |\nabla\tilde{u}|^p(x_0) - |\nabla u_\epsilon|^p(x_0) > 0,$$

which is contradiction with (4.14). Take $\epsilon \rightarrow 0$ in (4.13), then we have $\tilde{u} \leq u$. Similarly, $u \leq \tilde{u}$. The proof is complete. \square

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