On spectral approximation algorithm for thermoelastic shells in the nonclassical theory of thermoelasticity with three phase-lags

Gia A. Avalishvili, Mariam A. Avalishvili

YIC2013
Second ECCOMAS Young Investigators Conference
2–6 September 2013, Bordeaux, France

On spectral approximation algorithm for thermoelastic shells in the nonclassical theory of thermoelasticity with three phase-lags

G. Avalishvili [a,*], M. Avalishvili [b]

[a]Faculty of Exact and Natural Sciences, Iv. Javakhishvili Tbilisi State University
3 I. Tchavtchavadze Ave., 0179, Tbilisi, Georgia

[b] School of Informatics, Engineering and Mathematics, University of Georgia
77a M. Kostava Str., 0175, Tbilisi, Georgia

[*] *gavalish@yahoo.com*

Abstract. *In this paper nonclassical dynamical model for thermoelastic bodies with three phase-lags is studied. Applying variational formulation of the general three-dimensional initial-boundary value problem existence and uniqueness of solution in suitable spaces is proved. Spectral algorithm of approximation of the three-dimensional initial-boundary value problem for thermoelastic shell by a sequence of two-dimensional ones is constructed, convergence of the algorithm in corresponding spaces is proved and the rate of convergence is estimated.*

Keywords: Thermoelastic shells; nonclassical thermoelasticity; spectral approximation methods; error estimates.

1 INTRODUCTION

The classical dynamical theory of thermoelasticity is based on Fourier's law of heat conduction, which leads to parabolic equation for the temperature field, and predicts that a thermal disturbance at some point of a thermoelastic body will be felt instantly at all other points of the body. This behavior, which is often called the paradox of heat conduction is physically unrealistic since it implies that thermal signal propagates with infinite speed. In various modern engineering constructions, such as high speed aircrafts, nuclear reactors, recently developed ultra-fast pulsed lasers [1], temperatures and temperature gradients are extremely high whereas the operation time period is of the order of picoseconds, that cause thermal shocks and cannot be successfully described by the classical theory of thermoelasticity. Several experimental studies indicate that at low temperatures heat propagates as a thermal wave [2]. Therefore, motivated by experiments and the above mentioned non-causal aspect of the classical theory, numerous nonclassical theories of thermoelasticity have been proposed. The first nonclassical model for thermoelastic bodies was proposed by Lord and Shulman [3], where instead of the classical Fourier's law of heat conduction Maxwell-Cattaneo law was used, which is a generalization of Fourier's law and depends on one relaxation time parameter. Hence, the equation corresponding to the temperature field is hyperbolic and as a consequence the paradox of infinite speed of propagation of thermoelastic waves is eliminated. The second nonclassical theory of thermoelasticity, which also eliminates the paradox of heat conduction was developed by Green and Lindsay [4], where, in comparison to the classical linear system of thermoelasticity, the constitutive relations for the stress tensor and the entropy are generalized by introducing two different relaxation times. Later on, Tzou [5] proposed a dual-phase-lag heat conduction model, where the phase-lag corresponding to the temperature gradient is caused by microstructural interactions such as phonon scattering or phonon-electron interactions and the second phase-lag is interpreted as the relaxation time due to fast-transient effects of thermal inertia. Further, Chandrasekharaiah [6] constructed nonclassical model for thermoelastic bodies, where the classical Fourier's law of heat conduction was replaced with its generalization proposed by Tzou. Note that the Chandrasekharaiah-Tzou model is an extension of the Lord-Shulman nonclassical model for thermoelastic bodies, which depends on one phase-lag.

Note that a different approach to mathematical modeling of thermoelastic bodies was originated by Green and Naghdi [7-9]. Their theory includes three different types of heat conduction, which leads to three models for

thermoelastic bodies. The linearized version of the first model coincides with the corresponding model of the classical theory. The second model does not include any energy dissipation and as opposed to the Lord-Shulman and Green-Lindsay nonclassical models admits only propagation of undamped thermoelastic waves, and therefore it is also referred to as the theory without energy dissipation. The third model is the most general case as it includes the previous two models as limiting cases and it eliminates the disadvantage of the classical theory, since it permits propagation of thermal waves at finite speed.

The present paper is devoted to investigation of general three-dimensional initial-boundary value problem for thermoelastic bodies, and construction and investigation of spectral algorithm of approximation of three-dimensional dynamical model for thermoelastic shells by two-dimensional problems within the framework of the nonclassical theory of thermoelasticity with three phase-lags, which was proposed by Roychoudhuri [10]. This theory is an extension of Lord-Shulman, Green-Naghdi and Chandrasekharaiah-Tzou theories, where Fourier's law of heat conduction is replaced by an approximation to a modification of Fourier's law, which includes three different phase-lags for the heat flux vector, the temperature gradient and the thermal displacement gradient. The stability of the nonclassical model of heat conduction with three phase-lags and the relations between the three parameters are investigated in [11]. The theory of thermoelasticity with three phase-lags was employed to study problems of thermoelastic interactions for functionally graded orthotropic hollow sphere and homogeneous viscoelastic isotropic spherical shell [12,13], and in the context of this theory the problem of propagation of harmonic plane waves was investigated in [14].

2 FORMULATION OF THE PROBLEM

Let us consider thermoelastic body with initial configuration $\Omega \subset \mathbf{R}^3$, which consists of homogeneous and isotropic thermoelastic material. The body is clamped along a part Γ_0 of the boundary $\Gamma = \partial\Omega$ and the temperature θ vanishes along a part $\Gamma_0^\theta \subset \Gamma$. The body is subjected to applied body force with density $\mathbf{f} = (f_i) : \Omega \times (0,T) \to \mathbf{R}^3$, applied surface force with density $\mathbf{g} = (g_i) : \Gamma_1 \times (0,T) \to \mathbf{R}^3$ given along a part $\Gamma_1 = \Gamma \setminus \overline{\Gamma_0}$ of the boundary, heat sources with density $f^\theta : \Omega \times (0,T) \to \mathbf{R}$ and heat flux with density $g^\theta : \Gamma_1^\theta \times (0,T) \to \mathbf{R}$ given along $\Gamma_1^\theta = \Gamma \setminus \overline{\Gamma_0^\theta}$.

The nonclassical dynamical linear three-dimensional model of stress-strain state of the thermoelastic body Ω that includes three different phase-lags is given by the following initial-boundary value problem in differential form:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left(\lambda \sum_{p=1}^{3} e_{pp}(\mathbf{u}) \delta_{ij} + 2\mu e_{ij}(\mathbf{u}) - \eta \theta \delta_{ij} \right) + f_i \quad \text{in } \Omega \times (0,T), \quad i = 1,2,3, \tag{1}$$

$$\begin{aligned} \chi \left(\frac{\partial^2 \theta}{\partial t^2} + \tau_0 \frac{\partial^3 \theta}{\partial t^3} + \frac{\tau_0^2}{2} \frac{\partial^4 \theta}{\partial t^4} \right) &= \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left(\kappa_1 \frac{\partial}{\partial x_j} \left(\theta + \tau_1 \frac{\partial \theta}{\partial t} \right) \right) + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left(\kappa_2 \frac{\partial}{\partial x_j} \left(\frac{\partial \theta}{\partial t} + \tau_2 \frac{\partial^2 \theta}{\partial t^2} \right) \right) \\ &- \Theta_0 \eta \sum_{p=1}^{3} e_{pp} \left(\frac{\partial^2 \mathbf{u}}{\partial t^2} + \tau_0 \frac{\partial^3 \mathbf{u}}{\partial t^3} + \frac{\tau_0^2}{2} \frac{\partial^4 \mathbf{u}}{\partial t^4} \right) + \left(\frac{\partial f^\theta}{\partial t} + \tau_0 \frac{\partial^2 f^\theta}{\partial t^2} + \frac{\tau_0^2}{2} \frac{\partial^3 f^\theta}{\partial t^3} \right) \quad \text{in } \Omega \times (0,T), \end{aligned} \tag{2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_0 \times (0,T), \quad \sum_{j=1}^{3} \left(\lambda \sum_{p=1}^{3} e_{pp}(\mathbf{u}) \delta_{ij} + 2\mu e_{ij}(\mathbf{u}) - \eta \theta \delta_{ij} \right) \nu_j = g_i \quad \text{on } \Gamma_1 \times (0,T), \; i = 1,2,3, \tag{3}$$

$$\theta = 0 \quad \text{on } \Gamma_0^\theta \times (0,T), \quad \sum_{j=1}^{3} \left(\kappa_1 \frac{\partial}{\partial x_j} \left(\theta + \tau_1 \frac{\partial \theta}{\partial t} \right) + \kappa_2 \frac{\partial}{\partial x_j} \left(\frac{\partial \theta}{\partial t} + \tau_2 \frac{\partial^2 \theta}{\partial t^2} \right) \right) \nu_j = g^\theta \quad \text{on } \Gamma_1^\theta \times (0,T), \tag{4}$$

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \; \frac{\partial \mathbf{u}}{\partial t}(x,0) = \mathbf{u}_1(x), \; \theta(x,0) = \theta_0(x), \; \frac{\partial \theta}{\partial t}(x,0) = \theta_1(x), \; \frac{\partial^2 \theta}{\partial t^2}(x,0) = \theta_2(x), \; \frac{\partial^3 \theta}{\partial t^3}(x,0) = \theta_3(x), \; x \in \Omega, \tag{5}$$

where $\mathbf{v}=(v_i)_{i=1}^3$ is the outward unit normal to Γ, $e_{ij}(\mathbf{u})$ is the linearized strain tensor, λ,μ are Lamé constants, ρ is the mass density, κ_1 is the thermal conductivity rate and κ_2 is the thermal conductivity, $\chi>0$ is the specific heat at zero strain, η is the thermoelastic constant, $\Theta_0>0$ is a constant reference temperature, $\tau_0\geq 0,\tau_1\geq 0$ and $\tau_2\geq 0$ stand for the heat flux, thermal displacement gradient and temperature gradient phase-lags, respectively. Note that in the case of $\tau_0=\tau_1=\tau_2=0$ the nonclassical three-dimensional model (1)-(5) coincides with the third Green-Naghdi model.

We study the three-dimensional initial-boundary value problem in the spaces of vector-valued distributions and employ the following variational formulation, which is equivalent to the problem (1)-(5) in the spaces of smooth enough functions: Find $\mathbf{u},\mathbf{u}',\mathbf{u}'',\mathbf{u}'''\in C^0([0,T];\mathbf{V}(\Omega))$, $\mathbf{u}^{(4)}\in L^2(0,T;\mathbf{V}(\Omega))$, $\theta,\theta',\theta''\in C^0([0,T];V^\theta(\Omega))$, $\theta'''\in L^2(0,T;V^\theta(\Omega))\cap L^\infty(0,T;L^2(\Omega))$, $\theta^{(4)}\in L^2(0,T;L^2(\Omega))$, which satisfy the following equations in the sense of distributions on $(0,T)$,

$$\rho(\mathbf{u}'',\mathbf{v})_{(L^2(\Omega))^3}+a(\mathbf{u},\mathbf{v})-\eta\left(\theta,\sum_{p=1}^3\frac{\partial v_p}{\partial x_p}\right)_{L^2(\Omega)}=(\mathbf{f},\mathbf{v})_{(L^2(\Omega))^3}+(\mathbf{g},\mathbf{v})_{(L^2(\Gamma_1))^3},\qquad \forall\mathbf{v}\in\mathbf{V}(\Omega), \tag{6}$$

$$\begin{aligned}&\chi\left(\theta''+\tau_0\theta'''+\frac{\tau_0^2}{2}\theta^{(4)},\varphi\right)_{L^2(\Omega)}+a_1^\theta(\theta+\tau_1\theta',\varphi)+\Theta_0\eta\left(\sum_{p=1}^3\frac{\partial}{\partial x_p}\left(u_p''+\tau_0u_p'''+\frac{\tau_0^2}{2}u_p^{(4)}\right),\varphi\right)_{L^2(\Omega)}\\&+a_2^\theta(\theta'+\tau_2\theta'',\varphi)=\left(\frac{\partial f^\theta}{\partial t}+\tau_0\frac{\partial^2 f^\theta}{\partial t^2}+\frac{\tau_0^2}{2}\frac{\partial^3 f^\theta}{\partial t^3},\varphi\right)_{L^2(\Omega)}+(g^\theta,\varphi)_{L^2(\Gamma_1^\theta)},\qquad\forall\varphi\in V^\theta(\Omega),\end{aligned} \tag{7}$$

together with the initial conditions

$$\mathbf{u}(0)=\mathbf{u}_0,\ \mathbf{u}'(0)=\mathbf{u}_1,\ \ \theta(0)=\theta_0,\ \ \theta'(0)=\theta_1,\ \ \theta''(0)=\theta_2,\ \ \theta'''(0)=\theta_3, \tag{8}$$

where $\mathbf{V}(\Omega)=\{\mathbf{v}\in(H^1(\Omega))^3;tr(v_i)=0\ \text{on}\ \Gamma_0,i=1,2,3\}$, $V^\theta(\Omega)=\{\varphi\in H^1(\Omega);tr(\varphi)=0\ \text{on}\ \Gamma_0^\theta\}$, $H^1(\Omega)$ denotes the first order Sobolev space based on $L^2(\Omega)$, $tr:H^1(\Omega)\to L^2(\Gamma)$ is the trace operator, and

$$a(\tilde{\mathbf{v}},\mathbf{v})=\int_\Omega\left(\lambda\sum_{p=1}^3e_{pp}(\tilde{\mathbf{v}})\sum_{q=1}^3e_{qq}(\mathbf{v})+2\mu\sum_{i,j=1}^3e_{ij}(\tilde{\mathbf{v}})e_{ij}(\mathbf{v})\right)dx,\qquad\forall\mathbf{v},\tilde{\mathbf{v}}\in\mathbf{V}(\Omega),$$

$$a_1^\theta(\tilde\varphi,\varphi)=\kappa_1\int_\Omega\sum_{j=1}^3\frac{\partial\tilde\varphi}{\partial x_j}\frac{\partial\varphi}{\partial x_j}dx,\quad a_2^\theta(\tilde\varphi,\varphi)=\kappa_2\int_\Omega\sum_{j=1}^3\frac{\partial\tilde\varphi}{\partial x_j}\frac{\partial\varphi}{\partial x_j}dx,\qquad\forall\varphi,\tilde\varphi\in V^\theta(\Omega).$$

For Banach space X, $C^0([0,T];X)$ denotes the space of continuous vector-functions on $[0,T]$ with values in X, $L^m(0,T;X)$, $1\leq m\leq\infty$, is the space of such vector-functions $g:(0,T)\to X$ that $\|g\|_X\in L^m(0,T)$ and the generalized derivative of g we denote by $g'=dg/dt$.

In order to construct an algorithm of approximation of the three-dimensional problem (6)-(8) by two-dimensional problems we consider a general thermoelastic shell with the following initial configuration $\overline{\Omega}^*=\boldsymbol{\theta}(\overline{\Omega})\subset\mathbf{R}^3$,

$$\Omega=\{x=(x_1,x_2,x_3)\in\mathbf{R}^3;\ h^-(x_1,x_2)<x_3<h^+(x_1,x_2),\ (x_1,x_2)\in\omega\subset\mathbf{R}^2\},$$

$\Omega\subset\mathbf{R}^3$ is a Lipschitz domain and $\boldsymbol{\theta}$ is a C^2 diffeomorphism of $\overline{\Omega}$ onto $\overline{\Omega}^*$, so that the vectors $\boldsymbol{g}_i(x)=\partial_i\boldsymbol{\theta}(x)$ are linearly independent at all points of $\overline{\Omega}$. The coordinates x_i of $x\in\overline{\Omega}$ are the curvilinear coordinates of $x^*=\boldsymbol{\theta}(x)\in\overline{\Omega}^*$. We assume, that $h^+(x_1,x_2)=h^-(x_1,x_2)$, for $(x_1,x_2)\in\tilde\gamma\subset\partial\omega$ and $h^+(x_1,x_2)>h^-(x_1,x_2)$,

for $(x_1, x_2) \in \omega \cup (\partial\omega \setminus \widetilde{\gamma})$. The upper Γ^+ and the lower Γ^- faces of the cylindrical domain Ω, given by the equations $x_3 = h^+(x_1, x_2)$ and $x_3 = h^-(x_1, x_2)$, $(x_1, x_2) \in \omega$, respectively, define the face surfaces $\theta(\Gamma^+)$ and $\theta(\Gamma^-)$ of the shell with variable thickness.

3 CONCLUSIONS

The three-dimensional initial-boundary value problem corresponding to the dynamical model for thermoelastic bodies with three phase-lags is investigated in suitable spaces of vector-valued distributions. Applying variational formulation (6)-(8) and suitable a priori estimates the existence and uniqueness of solution is proved.

To simplify algorithms of numerical solution of three-dimensional problem in the case of general thermoelastic shells a sequence of two-dimensional initial-boundary value problems is constructed applying spectral approximation method, which is a generalization and extension of the dimensional reduction method proposed by I. Vekua [15] in the classical theory of elasticity for plates with variable thickness. Later on, various two-dimensional and one-dimensional models were constructed and investigated for problems of the theory of elasticity and mathematical physics applying I. Vekua's reduction method and similar spectral methods (see [16] and the references given therein).

The problem (6)-(8) is rewritten in curvilinear coordinates defined by diffeomorphism θ and by semidiscretization of the obtained three-dimensional problem in the transverse direction of the shell a hierarchy of two-dimensional initial-boundary value problems is constructed, which are investigated in suitable weighted Sobolev spaces. Moreover, the relationship between the constructed two-dimensional problems and the original three-dimensional one is studied. It is proved, that the sequence of approximate solutions of three space variables, constructed by means of the solutions of the two-dimensional problems, converges in corresponding spaces to the exact solution of the original three-dimensional problem and under suitable regularity conditions of the solution estimates of the rate of convergence are obtained.

Note that the results obtained in this paper and the spectral approximation algorithm constructed for the nonclassical model of thermoelastic shells with three phase-lags can be used for solution of various engineering problems and for investigation of other nonclassical models, which describe not only mechanical and thermal, but also other physical properties of elastic structures.

ACKNOWLEDGEMENT

The work of G. Avalishvili has been supported by the Presidential Grant for Young Scientists (Contract No. 12/62).

REFERENCES

[1] Zhu, X., Vileneuve, D.M., Naumov, A.Y., Nikumb, S., Korkum, P.B.. Experimental study of drilling sub-10 μm holes in thin metal foils with femtosecond laser pulses. *Applied Surface Science* 152: 138–148, 1999.

[2] Dreyer, W., Struchtrup, H.. Heat-pulse experiments revisited. *Continuum Mechanics and Thermodynamics* 5: 3-50, 1993.

[3] Lord, H.W., Shulman, Y.. A generalized dynamical theory of thermoelasticity. *Journal of the Mechanics and Physics of Solids* 15: 229-309, 1967.

[4] Green, A.E., Lindsay, K.A.. Thermoelasticity. *Journal of Elasticity* 2: 1-7, 1972.

[5] Tzou, D.Y.. *Macro to micro-scale heat transfer: The lagging behavior*. Taylor & Francis, 1996.

[6] Chandrasekharaiah, D.S.. Hyperbolic thermoelasticity: a review of recent literature. *Applied Mechanics Reviews* 51: 705-729, 1998.

[7] Green, A.E., Naghdi, P.M.. A re-examination of the basic postulates of thermomechanics. *Proceedings of the Royal Society A* 432: 171-194, 1991.

[8] Green, A.E., Naghdi, P.M.. On undamped heat waves in an elastic solid. *Journal of Thermal Stresses* 15: 253-264, 1992.

[9] Green, A.E., Naghdi, P.M.. Thermoelasticity without energy dissipation. *Journal of Elasticity* 31: 189-208, 1993.

[10] Roychoudhuri, S.K.. On a thermoelastic three-phase-lag model. *Journal of Thermal Stresses* 30: 231-238, 2007.

[11] Quintanilla, R., Racke, R.. A note on stability in three-phase lag heat conduction. *International Journal of Heat and Mass Transfer* 51: 24-29, 2008.

[12] Kar, A., Kanoria, M.. Generalized thermoelastic functionally graded orthotropic hollow sphere under thermal shock with three-phase-lag effect. *European Journal of Mechanics A/Solids* 28: 757-767, 2009.

[13] Kar, A., Kanoria, M.. Generalized thermo-visco-elastic problem of a spherical shell with three-phase-lag effect. *Applied Mathematical Modelling* 33: 3287-3298, 2009.

[14] Kumar, R., Mukhopadhyay, S.. Analysis of the effects of phase-lags on propagation of harmonic plane waves in thermoelastic media. *Computational Methods in Science and Technology* 16: 19-28, 2010.

[15] Vekua, I.N.. *Shell theory: General methods of construction*. Pitman Advanced Publishing Program, 1985.

[16] Avalishvili, G., Avalishvili, M., Gordeziani, D., Miara, B.. Hierarchical modeling of thermoelastic plates with variable thickness, *Analysis and Applications* 8: 125-159, 2010.