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Geometrically exact multi-layer beams with a rigid interconnection

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Abstract. *In this work, a finite-element formulation for geometrically exact multi-layer beams without considering the interlayer slip and uplift is proposed. Numerical examples indicate that, in comparison with the existing geometrically non-linear sandwich beam models, the 2D plane-stress elements and the analytical results from the theory of elasticity, the multi-layer beam model is very efficient for modelling thick beams where warping of cross-sections has to be considered.*

Keywords: multi-layer beam; geometrically exact theory; non-linear analysis; cross-sectional warping.

1 INTRODUCTION

Research and application of layered composite structures using beam elements in many areas of engineering has increased considerably over the past couple of decades and continues to be a topic of undiminished interest in the computational mechanics community [1, 2, 3, 4, 5, 6, 7]. This work introduces a finite-element formulation for a geometrically exact multi-layer beam element. The number of layers (n) is arbitrary and they are assembled in a composite beam with the interlayer connection allowing only for the occurrence of independent rotations of each layer. In other words, interlayer slip and uplift effects are not considered. Vu-Quoc et al. [2] also proposed a formulation for a geometrically exact multi-layer beam and they used the Galerkin projection to obtain the computational formulation of the resulting non-linear equations of equilibrium in the static case, while in the present work the equilibrium equations are derived from the principle of virtual work. While the resulting numerical procedure is of necessity equal, here we focus on the actual transformation of the displacement vector for each layer to the displacement vector of the beam reference line and show that it may be written in a remarkably elegant form allowing for simple numerical implementation. Furthermore, we specifically analyse the problems with large number of layers and on the thick beam problems with pronounced cross-sectional warping compare the performance of the elements derived to the analytical results and the finite-element results obtained using 2D plane-stress elements. Detailed analysis of this problem is given in [8].

2 PROBLEM DESCRIPTION

Consider an initially straight composite beam of length L and a cross-section composed of n parts with heights h_i and areas A_i , where $i \in [1, n]$ is an arbitrary layer. Layers are made of linear elastic material with E_i and G_i acting as Young's and shear moduli of each layer's material. The reference axes of all layers in the initial undeformed state are defined by the unit vector \mathbf{t}_{01} which closes an angle ψ with respect to the axis defined by the base vector \mathbf{e}_1 of the spatial co-ordinate system. During the deformation the cross-sections of the layers remain planar but not necessarily orthogonal to their reference axes (Timoshenko beam theory with the Bernoulli hypothesis).

3 GOVERNING EQUATIONS

In the assembly equations, the displacements of each layer (\mathbf{u}_i) are expressed in terms of the basic unknown functions \mathbf{u} and θ_i . For each layer, the kinematic and constitutive equations are derived. The equilibrium equations are

derived from the principle of virtual work. The displacement of each layer can be, according to Fig. 1, expressed in terms of the displacement of an arbitrarily chosen main layer α (denoted by \mathbf{u}) and corresponding rotations θ_i as [8]

$$\mathbf{u}_i = \mathbf{u} + d_{i,\zeta}(\mathbf{t}_{\zeta,2} - \mathbf{t}_{0,2}) + d_{i,\xi}(\mathbf{t}_{\xi,2} - \mathbf{t}_{0,2}) + \sum_{s=\zeta+1}^{\xi-1} d_{i,s}(\mathbf{t}_{s,2} - \mathbf{t}_{0,2}), \quad (1)$$

where ζ represents the bottom and ξ the upper layer between layers i and α , while $d_{i,j}$, ($j = \zeta, \xi, s$), are the distances depending on the mutual position between layers i and α . The reference axis of the layer α then becomes the reference axis of the composite beam and \mathbf{u} , θ_i , $i \in [1, n]$, become the basic unknown functions of the problem which are assembled in a vector as $\mathbf{p}_f^T = \langle \mathbf{u} \quad \theta_1 \quad \dots \quad \theta_n \rangle^T$.

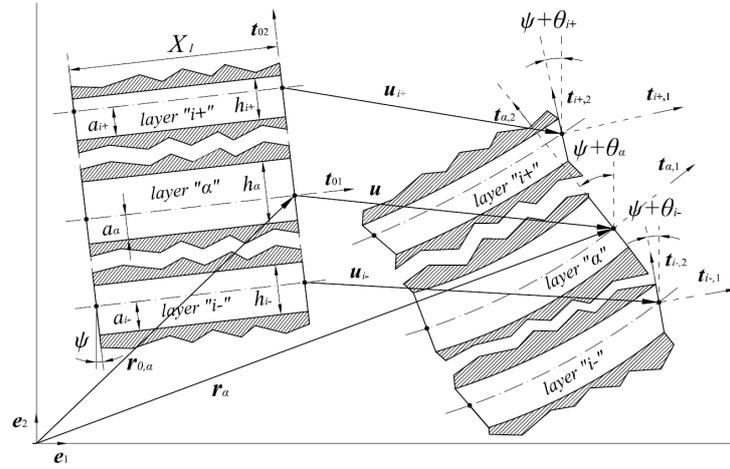


Figure 1: Undeformed and deformed state of the multilayer composite beam

Non-linear kinematic equations are defined according to Reissner's beam theory [9] as

$$\gamma_i = \begin{Bmatrix} \epsilon_i \\ \gamma_i \end{Bmatrix} = \mathbf{\Lambda}_i^T \mathbf{r}'_i - \mathbf{e}_1 = \mathbf{\Lambda}_i^T (\mathbf{t}_{01} + \mathbf{u}'_i) - \mathbf{e}_1, \quad \kappa_i = \theta'_i, \quad (2)$$

where ϵ_i , γ_i , κ_i are the axial strain, shear strain and curvature, respectively, with respect to the reference axis of the i -th layer. The constitutive law is given as

$$\begin{Bmatrix} N_i \\ T_i \\ M_i \end{Bmatrix} = \begin{Bmatrix} N_i \\ M_i \end{Bmatrix} = \mathbf{C}_i = \begin{bmatrix} E_i A_i & 0 & -E_i S_i \\ 0 & G_i k_i A_i & 0 \\ -E_i S_i & 0 & E_i I_i \end{bmatrix} \begin{Bmatrix} \epsilon_i \\ \gamma_i \\ \kappa_i \end{Bmatrix} = \mathbf{C}_i \begin{Bmatrix} \gamma_i \\ \kappa_i \end{Bmatrix}, \quad (3)$$

where N_i , T_i , M_i are the axial force, shear force and bending moment with respect to the reference axis of layer i , respectively, while k_i is the shear correction coefficient. S_i and I_i are the first and the second moment of area of the cross-section of layer i , respectively, and ϵ_i , γ_i and κ_i are the axial strain, shear strain and the curvature. Equilibrium equations are derived using the principle of virtual work for a static problem ($G \equiv G_i - G_e \equiv 0$), which for a multilayer beam composed of n layers after a series of transformations becomes [8]

$$G = \sum_{i=1}^n \left[\int_0^L \bar{\mathbf{p}}_f^T \mathbf{B}_i^T \left(\mathbf{D}_i^T \mathbf{L}_i^T \mathbf{C}_i \begin{Bmatrix} \gamma_i \\ \kappa_i \end{Bmatrix} - \begin{Bmatrix} \mathbf{f}_i \\ w_i \end{Bmatrix} \right) dX_1 - \bar{\mathbf{p}}_{f,0}^T \mathbf{B}_{i,0}^T \begin{Bmatrix} \mathbf{F}_{i,0} \\ W_{i,0} \end{Bmatrix} - \bar{\mathbf{p}}_{f,L}^T \mathbf{B}_{i,L}^T \begin{Bmatrix} \mathbf{F}_{i,L} \\ W_{i,L} \end{Bmatrix} \right] \equiv 0. \quad (4)$$

Matrix \mathbf{B}_i transforms the vector of displacements and rotation of the layer i to the basic the unknown functions, while the matrices \mathbf{L}_i and \mathbf{D}_i transform the vector of virtual strains and curvature to the vector of virtual displacements and rotation of the layer i . Indices 0 and L represent the beam ends where the boundary point forces $\mathbf{F}_{j,0}$, $\mathbf{F}_{j,L}$ and bending moments $W_{j,0}$, $W_{j,L}$ are applied. The distributed force and moment loads are denoted by \mathbf{f}_j and w_j . Finally, $\bar{\mathbf{p}}_f$ is the vector of the virtual basic unknown functions.

4 SOLUTION PROCEDURE

The governing equations of the problem are highly non-linear and cannot be solved in a closed form. Thus, it is necessary to choose in advance the shape of test functions $(\bar{\mathbf{u}}, \bar{\theta}_i)$, and later also the shape of trial functions (\mathbf{u}, θ_i) , where $i \in [1, \dots, n]$ as

$$\bar{\mathbf{p}}_f = \sum_{j=1}^N \Psi_j(X_1) \bar{\mathbf{p}}_j, \quad \Delta \mathbf{p}_f = \sum_{k=1}^N \Psi_k(X_1) \Delta \mathbf{p}_k, \quad (5)$$

where Ψ_j are the matrices of interpolation functions, $\bar{\mathbf{p}}_j$ is the vector containing nodal displacements and rotations of node j , while $\Delta \mathbf{p}_f$ and $\Delta \mathbf{p}_j$ are the vectors of the increments of the basic unknown functions and nodal increments of the displacements and rotations of the node j , respectively. From expression (4) we can easily obtain the vector of residual forces for the node j as

$$G \equiv \sum_{j=1}^N \bar{\mathbf{p}}_j^T \mathbf{g}_j = 0. \quad (6)$$

After the linearisation of the nodal vector of residual forces

$$\mathbf{g}_j + \Delta \mathbf{g}_j = \mathbf{g}_j + \sum_{k=1}^N \mathbf{K}_{j,k} \Delta \mathbf{p}_k = \mathbf{0}, \quad j = 1, \dots, N. \quad (7)$$

the nodal tangent stiffness matrices are obtained as

$$\mathbf{K}_{j,k} = \sum_{i=1}^n \left(\int_0^L \Psi_j^T \mathbf{J}_{i,1} \Psi_k dX_1 - \delta_{j,1} \delta_{k,1} \mathbf{J}_{i,0} - \delta_{j,N} \delta_{k,N} \mathbf{J}_{i,L} \right), \quad (8)$$

which are of dimension $(2+n) \times (2+n)$ and are assembled into an element tangent stiffness matrix of dimension $N(2+n) \times N(2+n)$. Matrices $\mathbf{J}_{i,l}$, ($l = 1, 0, L$) are produced in the process of linearisation of the vector of residual forces (see [8] for details). For integration in (8) we use the Gaussian quadrature with $N-1$ integration points in order to avoid shear-locking [10]. From (7) we finally obtain

$$\Delta \mathbf{p} = -\mathbf{K}^{-1} \mathbf{g}, \quad (9)$$

where vectors $\Delta \mathbf{p}_k$ and \mathbf{g}_j are assembled into vectors $\Delta \mathbf{p}$ and \mathbf{g} , respectively. The solution is obtained iteratively using Newton-Raphson method until a satisfying accuracy is achieved.

5 NUMERICAL EXAMPLES

5.1 Roll-up Maneuver

A comparison of the presented formulation with [1] and [2] is given for the roll-up maneuver (a moment $M = 2EI\pi/L$ at the beam tip which bends the cantilever beam into an exact circle). Both for a single layer beam and for a sandwich beam with three identical layers using the so-called "normal" moment distribution over the layers ($M_1 : M_2 : M_3 = 7 : 13 : 7$) the present formulation shows excellent accordance with the results from [1, 2] and the analytical results [8].

5.2 Thick beam tests

The presented multi-layer beam model is further compared to a homogeneous beam divided into a finite number of equal laminae (layers) with identical geometrical and material properties and no interlayer slip and uplift. The independent cross-sectional rotations of each layer allow the cross-section to deform in a piecewise linear form. This is compared to the results from the theory of elasticity [11] where the cross-sectional warping occurs in the deformed state. For a one-point-clamped thick cantilever beam with a narrow rectangular cross-section of unit width subjected to a transverse force F at the free end the solution, for a geometrically linear case, the multilayer formulation is compared to a two-dimensional finite element mesh. The results (see Fig.2) show that the multilayer formulation gives considerably better results with less degrees of freedom in comparison with the two-dimensional finite element meshes. A more detailed analysis is provided in [8].

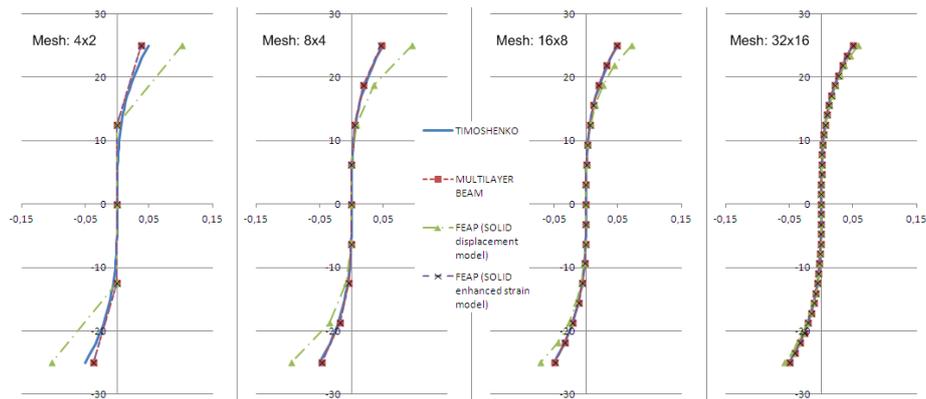


Figure 2: A comparison between the warped cross-section of the left-hand end of the beam according to the theory of elasticity and the linear-pieewise cross-sections obtained by the multi-layer beam model and the two-dimensional finite element models for different meshes

6 CONCLUSIONS

In this work we have presented a geometrically exact multi-layer beam element with rigid connection between the layers and arbitrary position of the layers' and the composite beam's reference axes, thus allowing for arbitrary position of the applied loading. We have shown that the kinematic constraint relating the displacement vector of an arbitrary layer and the displacement vector of the beam reference line may be written in a unique way regardless of the positions of the layer and the beam reference axes. The element has been verified against the results in [1, 2] and its capabilities tested on a thick beam example against analytical and numerical results coming from 2D elasticity. While the beam theory utilised obviously cannot recognise the existence of the transverse normal stresses and strains, it shows remarkable ability to capture the cross-sectional warping effect and give good approximation of 2D elasticity results using far less degrees of freedom.

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