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## A Non intrusive reduced basis method : application to computational fluid dynamics

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**Abstract.** *The reduced basis method recent progress have permitted to make the computations reliable thanks to a posteriori estimators . However, it may not always be possible to use the code to perform all the “off-line” computations required for an efficient performance of the reduced basis method. We propose here an alternating approach based on a coarse grid finite element, the convergence of which is accelerated through the reduced basis and an improved post processing.*

**Keywords:** reduced basis method; finite element method; Navier-Stokes equations

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## 1 INTRODUCTION

For the real-time or many-query context classical discretization techniques such as finite element methods are generally too expensive. The reduced basis method [6, 8, 10, 12] exploits the parametric structure of the governing PDEs to construct rapidly, convergent and computationally efficient approximations. Previous work on the reduced basis method in numerical fluid dynamics has been carried out by [7, 9, 13] and more particularly for the steady Navier-Stokes equations [3, 4, 11, 14] which requires treatment of non-linearities and non-affine parametric dependence. These methods rely on the fact that when the parameters vary, the set of solutions is often of small (Kolmogorov) dimension. In this instance, there exists a set of parameters  $(\mu_1, \dots, \mu_N)$  in the parameter space  $\mathcal{D}$ , from which one can build a basis. This basis, called reduced basis, is made of the solutions  $(u(\mu_1), \dots, u(\mu_N))$  and can approach any solution  $u(\mu)$ ,  $\mu \in \mathcal{D}$ . Thus, when the  $\mu_i$  are well chosen, the size of the reduced basis is quite smaller compared to the number of degrees of freedom of the problem discretized by a classical method (finite element, finite volume, or other). In an industrial framework, for optimization processes for instance these reduced basis methods have a great potential. One of the keys of this technique is the decomposition of the computational work into an off-line and on-line stage. However in some situation, it's not possible to perform all the off-line computations required with an efficient performance of the reduced method. For example when the simulation code is used as a black box, one won't be able to perform a very fast and cheap online stage. For this reason, in [1, 2] we proposed an alternative method. The aim of this work is to provide tests to validate and generalize our method to fluid dynamics problems.

## 2 GOVERNING EQUATIONS

The governing equations are the incompressible steady state Navier-Stokes equations and are given by :

$$-\nu \Delta \vec{u} + \frac{1}{\rho} \nabla p + (\vec{u} \cdot \nabla) \vec{u} = \vec{F}_e \quad \text{and} \quad \text{div}(\vec{u}) = 0 \quad (1)$$

where is  $\nu$  the kinematic viscosity,  $\rho$  the density and  $\vec{F}_e$  the external forces.

Let us consider a physical domain  $\Omega$  with its boundary  $\partial\Omega = \Gamma_{in} \cup \Gamma_{wall} \cup \Gamma_{out}$ . On  $\Gamma_{in}$ , a velocity  $\vec{v}_{in}$  is prescribed, which has a parabolic profile :  $\vec{v}_{in} = V_{in} f(\vec{x}) \vec{n}|_{\Gamma_{in}}$ . On  $\Gamma_{wall}$  we impose homogeneous Dirichlet boundary condition and on  $\Gamma_{out}$  we impose homogeneous Neumann boundary condition.

Here, we are interested in studying the non intrusive reduced method [1, 2] applied to the steady-state Navier-Stokes equation parametrized by the velocity magnitude  $V_{in}$  of the inlet flow. We will denote by  $\mu$  the parameter and  $\mathcal{D}$  the parameter space. Let be  $X \subset H^1(\Omega)$ ,  $V \subset [H^1(\Omega)]^2$ ,  $M \subset L^2(\Omega)$ , and  $\{\mathcal{T}_h\}_h$  a family of regular triangulation of  $\Omega$ , we denote by  $X_h$ ,  $V_h$  and  $M_h$  the following finite element spaces :

$$X_h = \{v \in X, \forall K \in \mathcal{T}_h, v|_K \in \mathbb{P}_2(K)\}, \quad V_h = X_h \times X_h \quad \text{and} \quad M_h = \{v \in M, \forall K \in \mathcal{T}_h, v|_K \in \mathbb{P}_1(K)\}.$$

The finite element (FE) formulation of the problem (1) is : given  $\mu \in \mathcal{D}$ , for all  $\vec{v}_h \in V_h$  and  $q_h \in M_h$ , find  $\vec{u}_h(\mu) \in V_h$  and  $p_h(\mu) \in M_h$  such that

$$\begin{cases} \int_{\Omega} \left( \vec{u}_h(\mu) \cdot \nabla \overline{u_h(\mu)} \right) \vec{v}_h + \nu \int_{\Omega} \nabla \vec{u}_h(\mu) : \nabla \vec{v}_h + \frac{1}{\rho} \int_{\Omega} \nabla p_h(\mu) \cdot \vec{v}_h = \int_{\Omega} \vec{F}_e \cdot \vec{v}_h + \mathcal{L}_V(\vec{v}_h; \mu), \\ \int_{\Omega} \nabla q_h \cdot \vec{u}_h = 0 \end{cases} \quad (2)$$

where the linear form  $\mathcal{L}_V(\vec{v}_h; \mu)$  is due to the non homogenous Dirichlet boundary conditions on  $\Gamma_{in}$ .

## 3 NON INTRUSIVE REDUCED BASIS METHOD

The reduced basis method rely on the fact that when the parameters vary, the set of solutions is often of small Kolomgorov dimension. In that case, for any  $\varepsilon > 0$  there exist a set of parameters  $(\mu_1, \mu_2, \dots, \mu_N) \in \mathcal{D}$  such that

$$\forall \mu \in \mathcal{D}, \quad \exists (\sigma_i(\mu)) \in \mathbb{R}^N, \quad \|\vec{u}(\mu) - \sum_{i=1}^N \sigma_i(\mu) \vec{u}(\mu_i)\|_Y \leq \varepsilon. \quad (3)$$

We denote by  $V_h^N = \text{span}\{\vec{u}_h(\mu_1), \dots, \vec{u}_h(\mu_N)\}$  the reduced basis space. The reduced basis approximation of (2) is obtained by using a Galerkin approach on  $V_h^N$ . For a stable implementation of the reduced basis method, it is required to build a better basis than the one composed with the  $\vec{u}_h(\mu_i)$ , usually by a Gramm-Schmidt method. Here we replace it by the resolution of an eigenvalue problem that will provide  $L^2$  and  $H^1$  orthogonal functions.

Let denotes by  $\{\vec{\phi}_i^N\}_{i=1, \dots, N}$  these orthonormalized basis of  $V_h^N$ , the reduced basis solution  $\vec{u}_h^N(\mu)$  can be expressed as a linear combination of the basis functions  $\{\vec{\phi}_i^N\}$  :  $\vec{u}_h^N(\mu) = \sum_{i=1}^N \sigma_i^h(\mu) \vec{\phi}_i^N$ .

Let the matricial formulation of the Galerkin approach of (2) on  $V_h^N$  be : find  $\mathbf{U}_h^N(\mu)$  such that

$$A_h^N(\mathbf{U}_h^N(\mu)) \mathbf{U}_h^N(\mu) = B_h^N(\mu). \quad (4)$$

Considering that the dimension of the reduced basis space is quite smaller compared to the finite element space's size, solving the previous system (4) is less expensive than the true finite element problem (2). Indeed, all the expensive computations are done off-line which allows us to have online computations of small complexity.

However, since the construction of the matrix  $A_h^N$  and the vector  $B_h^N$  has to be done for each new value of  $\mu$ , to perform efficiently the online stage, one has to be able to isolate their parametric contributions so that all  $\mu$  independent matrices and vectors can be build only once and saved during the offline stage. This part of the offline stage require to enter in the simulation code used to compute the truth finite element approximations. Unfortunately, when this code is used as a black box, which is often the case in the industrial framework, the parametric decomposition is not possible, which prevent us from build each new matrix quickly for a new value of  $\mu$ . This take away the benefit of the reduced basis method, thus to overcome it, we proposed an alternative method : *a non intrusive reduced basis method (NIRB)*. The standard reduced basis method aims at evaluating the coefficients  $\sigma_i^h(\mu)$  intervening in the decomposition of  $\vec{u}_h^N(\mu)$  in the basis of the  $\vec{\phi}_i^N$ , those can appears as a substitute to the optimal coefficients  $\gamma_i^h(\mu) = \langle \vec{u}_h^N(\mu), \vec{\phi}_i^N \rangle$  intervening in the decomposition of the  $L^2$ -projection of  $\vec{u}_h^N(\mu)$  on  $Y_h^N$ . Our alternative method consists in proposing an other surrogate to the  $\gamma_i^h(\mu)$  defined by  $\gamma_i^H(\mu) = \langle \vec{u}_H^N(\mu), \vec{\phi}_i^N \rangle$ . Since

the computation of  $\vec{u}_H^N(\mu)$ , for  $H \gg h$  is less expensive than the one of  $\vec{u}_h^N(\mu)$ , using the industrial code with the mesh size  $H$  (chosen adequately) to construct the  $\gamma_i^H(\mu)$  is still cheap enough. Then for each new value of  $\mu$ , one can compute the coefficients  $\gamma_i^H(\mu)$  and build a new approximation of the solution of (2)

$$\vec{u}_{H,h}^N(\mu) = \sum_{i=1}^N \gamma_i^H(\mu) \phi_i^N. \quad (5)$$

Besides, to improve even further the accuracy of this technique we propose to do a simple post-processing of the results. This treatment will insure that for each new value of the parameters  $\mu_i, i = 1, \dots, N$ , used in the construction of the reduced basis, the method return exactly the  $L^2$ -projection of  $\vec{u}_h(\mu_i)$  on the  $V_h^N$ . Indeed, contrarily to the  $\vec{u}_h(\mu)$ , that we don't want to compute for a large number of values of  $\mu$ , the truth solutions  $\vec{u}_h(\mu_i)$  have been actually already computed to build the basis. To do this, let consider the following linear application  $\mathcal{R}$  defined by:

$$\mathcal{R} : \mathbb{R}^N \rightarrow \mathbb{R}^N$$

$$\left( \gamma_1^H(\mu_i), \dots, \gamma_N^H(\mu_i) \right)^t \mapsto \left( \gamma_1^h(\mu_i), \dots, \gamma_N^h(\mu_i) \right)^t,$$

where the values of the parameters  $\mu_i$  are the one used to build the reduced basis. We denote by  $T^N$  the matrix associated to this transformation such that:

$$\left( T^N \right) \left( \begin{array}{c|c|c} \gamma_1^H(\mu_1) & \cdots & \gamma_1^H(\mu_N) \\ \vdots & \vdots & \vdots \\ \gamma_N^H(\mu_1) & \cdots & \gamma_N^H(\mu_N) \end{array} \right) = \left( \begin{array}{c|c|c} \gamma_1^h(\mu_1) & \cdots & \gamma_1^h(\mu_N) \\ \vdots & \vdots & \vdots \\ \gamma_N^h(\mu_1) & \cdots & \gamma_N^h(\mu_N) \end{array} \right).$$

For each new value of  $\mu$ , we will replace the  $\gamma_i^H(\mu)$  coefficients by  $\tilde{\gamma}_i^H(\mu) = \sum_{k=1}^N T_{ik}^N \gamma_k^H(\mu)$ .

#### 4 NUMERICAL EXPERIMENT

We are interested in the evaluation of the velocity vector  $\vec{u}(\mu) = (u_x, u_y)$  for any set of parameters  $\mu = V_{in}$ .

The OFFLINE procedure is divided in 3 stages :

- 1: Construction of a reduced approximation's space for the velocity.
- 2: Orthonormalisation in  $L^2$  and  $H^1$ -norm of the reduced basis functions.
- 3: Preparation for the post-processing.

The ONLINE procedure is divided in 3 stages :

- 1: Using the black box software to solve the problem on a coarser mesh and extract the velocity vector and temperature.
- 2: Compute the coefficient  $\gamma_i^H(\mu)$  for any values of  $\mu$
- 3: Apply the post-processing on the  $\gamma_i^H(\mu)$ .

Let denote by  $\vec{u}_h^N(\mu)$  velocity solutions of the discretized problem solved on  $\mathcal{T}_h$ . We compared this solutions for a fixed  $\mu$  with different reduced solutions (see FIGURE 1).

- **Case 1:** In this example, we want to see the error due only to the reduced basis size  $N$ . We build a NIRB solution using the finite element solution computed on reference mesh, which the projection of the finite element solution on the reduced basis space  $V_h^N$ . For a given  $N$ , those solutions are the best approximation that we can expect in the reduced basis.
- **Case 2, 3 and 4:** In those examples, we wanted to see how the choice of the coarse mesh  $\mathcal{T}_H$  affect the NIRB method. We build three coarse meshes  $\mathcal{T}_{H_i}, i = 1, 2, 3$  used to build the reduced solution in respectively the case 2, 3 and 4. We notice that as  $N$  goes larger the error between the different reduced solution and the finite element solution goes smaller to finally reach a threshold. This is due to the fact that the finite element's error become more significant than the reduced basis size's error.
- **Case 2 + PP, 3 + PP and 4 +PP:** In those examples we wanted to see the influence of the post-processing on the reduced solution. We observe that with this post - processing we were able to reach the same accuracy as if we have projected the reference finite element solution on the reduced basis space, even with the coarsest mesh.

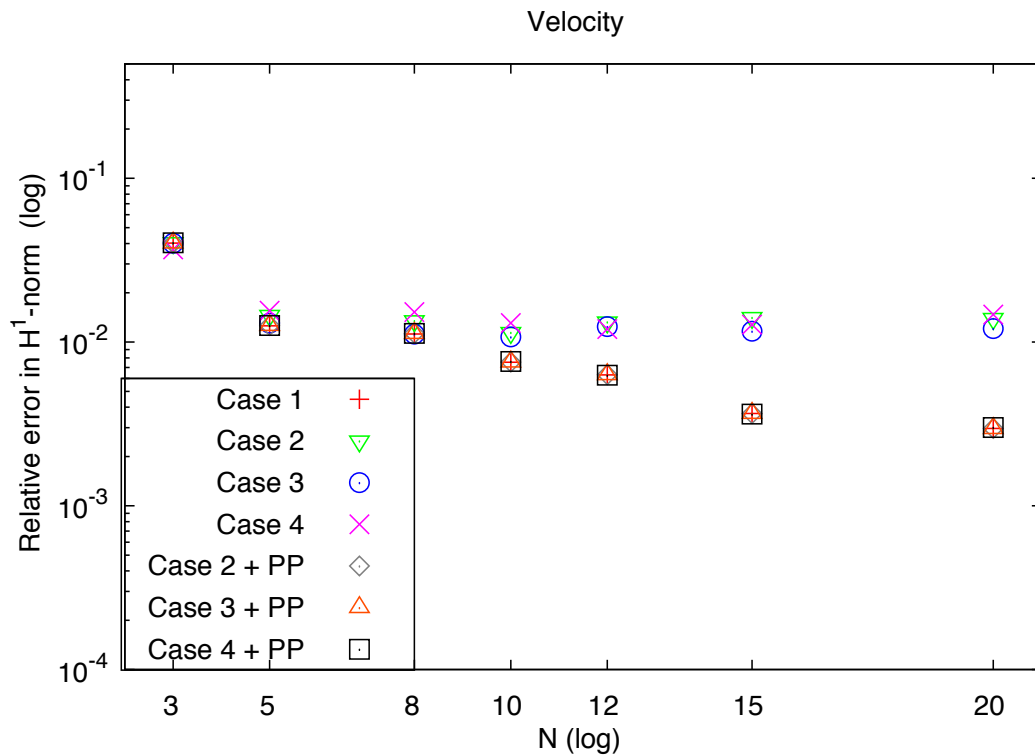


Figure 1: Relative error between the finite element solution of reference obtained on the fine mesh and the various reduced solutions measured in  $H^1$ -norm.

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