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YIC2013
Second ECCOMAS Young Investigators Conference
2–6 September 2013, Bordeaux, France

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Abstract. *We present an Isogeometric Analysis study of for the strain gradient fracture problem. In particular we consider an anti-plane formulation (Mode III). The numerical approximation we resort to a standard Galerkin isogeometric formulation and the results are in agreement with the theoretically expected behavior.*

Keywords: isogeometric analysis; nurbs; fracture; anti-plane.

1 INTRODUCTION

Many engineering materials exhibit a micro-structure, as it is the case, e.g, of composite materials, MEMS and, more in general in all situations, where a local effect shows up, as it is typical in fracture problems. To model this kind of problems it is desiderable to use an enriched continuum; an example is the Cosserat continuum where the couples added to the equilibrium equations generate a new internal length-scale; another relevant example of analysis of micro-structured continuum was proposed in [8]. Recent studies developed in [5] introduced the micro scale using a strain gradient formulation where the potential energy is a function of displacement, strain, and strain gradient. The goal of this work is to numerically analyze the particular case of anti-plane fracture. For a given specif problem, the focus of the paper is the determination of the process zone where the effect of the second gradient is significative. This is accomplished via isogeometric analysis [7, 2], that is an analysis framework characterized by high-regularity functions, allowing to tackle high-order PDEs in primal form, within a standard Galerkin formulation (see [6] for a first application of isogeometric analysis to strain gradient elasticity).

2 STRONG FORM GOVERNING EQUATIONS

The governing equation of the second gradient anti plane problem reported in [3] is:

$$\Delta w + l_s^2 \Delta\Delta w = 0 \tag{1}$$

where the variable w is a transverse scalar displacement field, while l_s is a shear length-scale that is related to strain gradient (see [5] for more details). In equation (1) it is possible to distinguish two parts; the first one related to the effect of first gradient and the second related to the second gradient. Due to the presence of the bilaplacian operator, Neumann boundary conditions may apply for distributed couples and tractions as well as concentrated forces. For this problem such boundary conditions are:

$$\begin{cases} \underline{\underline{\Pi}} & = \left(l_s^2 \Delta w - \frac{l_t^2}{2} \frac{\partial^2 w}{\partial \tau^2} \right) \underline{e}_3 \\ \underline{t} & = \left(\frac{\partial w}{\partial \nu} - l_s^2 \frac{\partial}{\partial \nu} (\Delta w) - \frac{l_t^2}{2} \frac{\partial^3 w}{\partial \nu \partial \tau^2} \right) \underline{e}_3 \\ \underline{f} & = \left(-l_t^2 \frac{\partial^2 w}{\partial \nu \partial \tau} \right) \underline{e}_3 \end{cases} \tag{2}$$

where $\underline{\underline{\Pi}}$, $\underline{t}$ and $\underline{f}$ are the distributed couple, and traction, and the concentrated force, respectively, while $\underline{e}_3$ is the out-of plane unit normal vector and $\frac{\partial}{\partial \nu}$, $\frac{\partial}{\partial \tau}$ are the normal and tangent derivatives.

3 WEAK FORMULATION

The weak formulation of the problem is obtained testing the strong form (1) and integrate by parts, as follows:

$$\begin{aligned} & \int_\Omega l_s^2 \underline{\underline{D}}^2 w : \underline{\underline{D}}^2 v \, d\Omega + \int_\Omega \underline{\nabla} w \cdot \underline{\nabla} v \, d\Omega \\ & - \int_\Gamma \left(\frac{\partial w}{\partial \nu} - l_s^2 \frac{\partial}{\partial \nu} \Delta w - l_s^2 \frac{\partial^3 w}{\partial \nu \partial \tau^2} \right) v \, d\Gamma \\ & - \int_\Gamma \left(l_s^2 \frac{\partial^2 w}{\partial \nu^2} \right) \frac{\partial v}{\partial \nu} \, d\Gamma - \int_{\partial\Gamma} \left[l_s^2 \frac{\partial^2 w}{\partial \nu \partial \tau} \right] v \, d\partial\Gamma \quad = \quad 0 \qquad \forall \, v \in H^2(\Omega) \end{aligned} \tag{3}$$

Finally, substituting the Neumann conditions (2), one gets:

$$\begin{aligned} & \int_\Omega l_s^2 \underline{\underline{D}}^2 w : \underline{\underline{D}}^2 v \, d\Omega + \int_\Omega \underline{\nabla} w \cdot \underline{\nabla} v \, d\Omega \\ & - \int_\Gamma \left(t + \left(\frac{l_t^2}{2} - l_s^2 \right) \frac{\partial^3 w}{\partial \nu \partial \tau^2} \right) v \, d\Gamma \\ & - \int_\Gamma \left(\Pi + \left(\frac{l_t^2}{2} - l_s^2 \right) \frac{\partial^2 w}{\partial \tau^2} \right) \frac{\partial v}{\partial \nu} \, d\Gamma \\ & - \int_{\partial\Gamma} \left(\frac{l_s^2}{l_t^2} f \right) v \, d\partial\Gamma \quad = \quad 0 \end{aligned} \tag{4}$$

where

$$\underline{\underline{D}}^2 w : \underline{\underline{D}}^2 v \quad = \quad \sum_{i,j=1}^{2} \frac{\partial^2 w}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}, \tag{5}$$

$$\underline{\nabla} w \cdot \underline{\nabla} v \quad = \quad \sum_{i=1}^{2} \frac{\partial w}{\partial x_i} \frac{\partial v}{\partial x_i} \tag{6}$$

4 DISCRETIZATION VIA ISOGEOMETRIC ANALYSIS

Discretization of problem (4) is then directly performed via a displacement-based isogeometric formulation [7, 2]. The function used for representing both geometry and displacements (within an isoparametric approach) are Non-Uniform Rational B-Spline and their higher regularity allows the discretization of the weak form (4) in primal form within a standard plain Galerkin fomulation (in a similar way as shown in [1]). The implementation has been done within the open source isogeometric software GeoPDEs [4].
The performed test is the one described in [3] as shown in figure (1). The displacement field is antisymmetric so that the analysis may be limited to half of the domain. As boundary conditions, we impose homogeneous Dirichlet conditions on the horizontal right edge and we prescribe the far field solution along the circular boundary :

$$w_D(r, \theta) = \Gamma \, r \, \sin\theta + C_{III} \, r^{3/2} \left(\frac{3 \, \sin\theta/2}{16(l_s/l_t)^2 - 3} - \sin\frac{3\theta}{2} \right), \tag{7}$$

where Γ and C_{III} are two arbitrary constants. Within the numerical simulations Γ and C_{III} are set equal to 1 and l_s and l_t are selected to be 0.2. Figure (2) shows the adopted mesh, while figures (3) and (4) show the leap and

its zoommed version. One may notice the effect of the second gradient in this zone. In figure (5) we show the displacement leap in the case of first gradient effect only (blue line), the effect of the second gradient (green line) and the global effect (red line). The obtained numerical results are in good agreement with the theoretically expected behavior (cf. [3]).

5 CONCLUSIONS

In this paper some results on the numerical approximation of anti-plane fracture problems are presented. In particular, a good qualitative agreement with the theoretical results presented in [3] are obtained. More comparisons and test will be performed and reported.

ACKNOWLEDGEMENT

The support of the European Research Council through the FP7 Ideas Starting Grant no. 259229 "ISOBIO" is gratefully acknowledged.

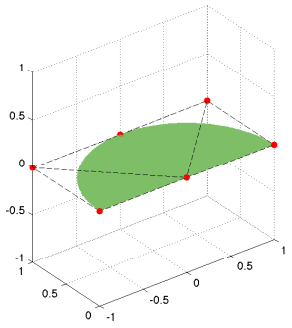

Figure 1: Problem geometry and initial control net.

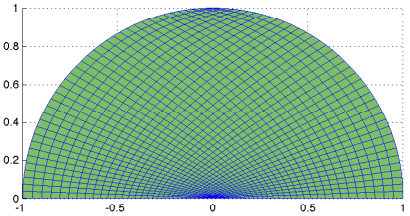

Figure 2: Adopted mesh.

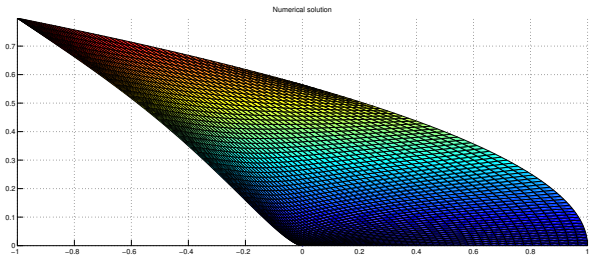

Figure 3: Frontal View Solution.

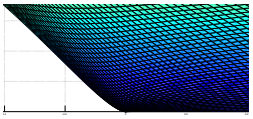

Figure 4: Zoom on the Leap Solution.

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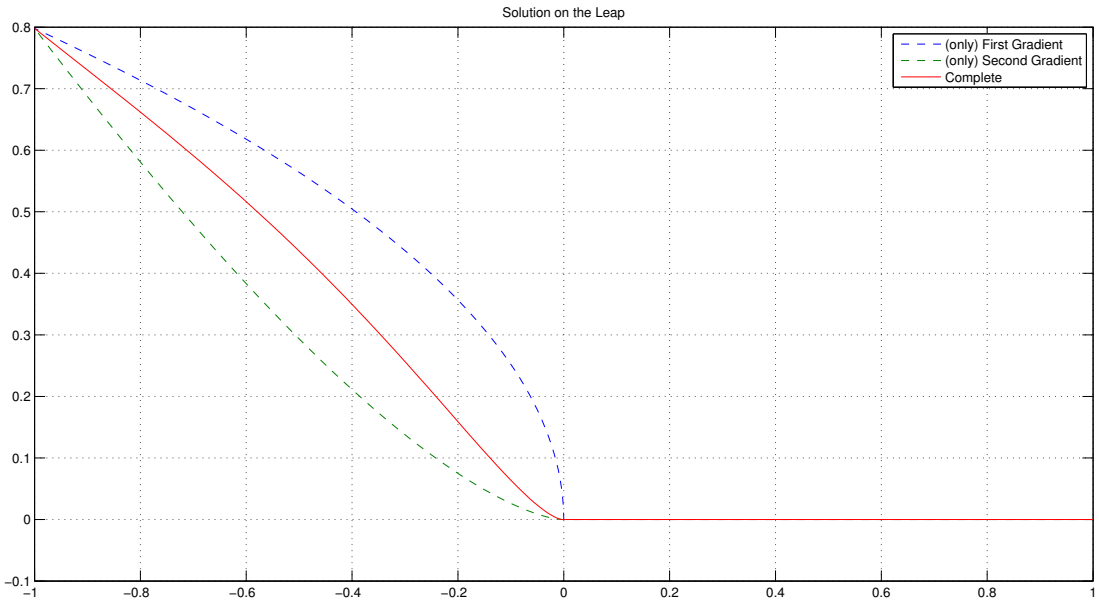

Figure 5: Solution on The Leap

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