



HAL
open science

Quasi-Gasdynamical Algorithm for Magnetohydrodynamic Shallow Water Equations

Tatiana Gennadyevna Elizarova, Maria Istomina

► **To cite this version:**

Tatiana Gennadyevna Elizarova, Maria Istomina. Quasi-Gasdynamical Algorithm for Magnetohydrodynamic Shallow Water Equations. 2nd ECCOMAS Young Investigators Conference (YIC 2013), Sep 2013, Bordeaux, France. hal-00855879

HAL Id: hal-00855879

<https://hal.science/hal-00855879>

Submitted on 30 Aug 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Quasi-Gasdynamical Algorithm for Magnetohydrodynamic Shallow Water Equations

T.G. Elizarova, M.A. Istomina
Moscow, Keldysh Institute of Applied Mathematics,
e-mail: telizar@mail.ru, m_ist@mail.ru

Abstract In this paper we present the regularized form of the shallow water equations for ideal magnetohydrodynamics, that allows its efficient numerical solution using central-difference approximation for all space derivatives. The non-negative form of the dissipative function for the new system is shown together with the examples of numerical simulations of two test problems.

1. Magnetohydrodynamic equations for the shallow water and its regularization form

These equations were first introduced by Gilman [1] in order to obtain a simplified set of equations from the full MHD equations. This system can be used as a mathematical model in certain applications, such as in the dynamics of the solar tachocline. According to [2] the system of the MHD equations in the shallow water approximation has a form

$$\begin{aligned}\frac{\partial h}{\partial t} + \operatorname{div}(h\mathbf{u}) &= 0, \\ \frac{\partial h\mathbf{u}}{\partial t} + \operatorname{div}(h\mathbf{u} \otimes \mathbf{u}) - \operatorname{div}(h\mathbf{B} \otimes \mathbf{B}) + \nabla \left(\frac{gh^2}{2} \right) &= \operatorname{div} \Pi_{NS} - gh\nabla b, \\ \frac{\partial h\mathbf{B}}{\partial t} + \operatorname{div}(h\mathbf{u} \otimes \mathbf{B}) - \operatorname{div}(h\mathbf{B} \otimes \mathbf{u}) &= 0, \\ \operatorname{div}(h\mathbf{B}) &= 0,\end{aligned}\tag{1}$$

where $h(x, y, t)$ is a fluid level, \mathbf{u} and \mathbf{B} are velocity and magnetic field strength vectors correspondingly, $b(x, y)$ is a bottom profile, Π_{NS} is a deformation velocity tensor

$$\Pi_{NS} = 2\mu\hat{\sigma}(\mathbf{u}) = 2h\nu\hat{\sigma}(\mathbf{u}),\tag{2}$$

where $\hat{\sigma}(\mathbf{u}) = \frac{1}{2} [(\nabla \otimes \mathbf{u}) + (\nabla \otimes \mathbf{u})^T]$, μ – dynamic viscosity coefficient, ν – kinematic viscosity coefficient. The expression $(\mathbf{u} \otimes \mathbf{u})$ denotes a tensor product of \mathbf{u} and \mathbf{u} vectors.

The regularized system of the shallow water equations with magnetic field can be obtained in the same way, as quasi-gas dynamic systems and regularized shallow water

equations were obtained earlier, e.g. [3] and [4]. For this procedure one can take a small time interval Δt and calculate the average values in $(t, t + \Delta t)$ interval.

All averaged variables denoted here by $(^*)$ are expanded in Taylor series where second time derivatives and terms of order $O(\tau^2)$ and $O(\tau\nu)$ are neglected. Here τ is a time smoothing parameter.

$$\begin{aligned}
h^* &= h + \tau \frac{\partial h}{\partial t} = h - \tau \operatorname{div}(h\mathbf{u}), \\
\mathbf{u}^* &= \mathbf{u} + \tau \frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} - \mathbf{w}^1, \\
\mathbf{w}^1 &= \tau((\mathbf{u}\nabla)\mathbf{u} - (\mathbf{B}\nabla)\mathbf{B} + g\nabla(b + h)), \\
(h\mathbf{u})^* &= h\mathbf{u} + \tau \frac{\partial h\mathbf{u}}{\partial t} = h(\mathbf{u} - \mathbf{w}) = \mathbf{j}, \\
\mathbf{w} &= \frac{\tau}{h}(\operatorname{div}(h\mathbf{u} \otimes \mathbf{u}) - \operatorname{div}(h\mathbf{B} \otimes \mathbf{B}) + gh\nabla(b + h)), \\
(h\mathbf{B})^* &= h\mathbf{B} + \tau \frac{\partial h\mathbf{B}}{\partial t} = h\mathbf{B} + \tau(\operatorname{div}(h\mathbf{B} \otimes \mathbf{u}) - \operatorname{div}(h\mathbf{u} \otimes \mathbf{B})) = h\mathbf{B} + \boldsymbol{\beta}, \\
\boldsymbol{\beta} &= \tau \frac{\partial h\mathbf{B}}{\partial t} = \tau(\operatorname{div}(h\mathbf{B} \otimes \mathbf{u}) - \operatorname{div}(h\mathbf{u} \otimes \mathbf{B})), \\
\mathbf{B}^* &= \mathbf{B} + \tau \frac{\partial \mathbf{B}}{\partial t} = \mathbf{B} + \tau((\mathbf{B}\nabla)\mathbf{u} - (\mathbf{u}\nabla)\mathbf{B}) = \mathbf{B} + \boldsymbol{\gamma}, \\
\boldsymbol{\gamma} &= \tau \frac{\partial \mathbf{B}}{\partial t} = \tau((\mathbf{u}\nabla)\mathbf{B} - (\mathbf{B}\nabla)\mathbf{u}).
\end{aligned} \tag{3}$$

Then the regularized system of the MHD shallow water equations takes a form

$$\begin{aligned}
\frac{\partial h}{\partial t} + \operatorname{div} \mathbf{j} &= 0, \\
\frac{\partial h\mathbf{u}}{\partial t} + \operatorname{div}(\mathbf{j} \otimes \mathbf{u}) - \operatorname{div}(h\mathbf{B} \otimes \mathbf{B}) + \frac{1}{2}g\nabla h^2 &= \operatorname{div}(h\mathbf{u} \otimes \mathbf{w}^1) + \operatorname{div}(\boldsymbol{\beta} \otimes \mathbf{B}) + \\
+ \operatorname{div}(h\mathbf{B} \otimes \boldsymbol{\gamma}) + g\nabla(\tau h \operatorname{div} h\mathbf{u}) - g\nabla b(h - \tau \operatorname{div} h\mathbf{u}) + \operatorname{div} \Pi_{NS}, & \\
\frac{\partial h\mathbf{B}}{\partial t} + \operatorname{div}(\mathbf{j} \otimes \mathbf{B}) - \operatorname{div}(h\mathbf{B} \otimes \mathbf{u}) &= \operatorname{div}(\boldsymbol{\beta} \otimes \mathbf{u}) - \operatorname{div}(h\mathbf{u} \otimes \boldsymbol{\gamma}) - \operatorname{div}(h\mathbf{B} \otimes \mathbf{w}^1), \\
\operatorname{div} h\mathbf{B} + \operatorname{div} \boldsymbol{\beta} &= 0.
\end{aligned} \tag{4}$$

The strongly non-linear terms in τ have the form of the second order space derivatives and may be regarded as regularizators. In the limit $\tau \rightarrow 0$ regularized system (4) goes to (1).

2. Equation for the specific entropy

The growth of the specific entropy for the regularized equation system is the important feature of the proposed model. It was shown, that the main energetic equity takes place

for the system (4). It was obtained using the approach from [6] and has a form

$$\begin{aligned} \frac{\partial}{\partial t} \left(h \frac{\mathbf{u}^2}{2} + h \frac{\mathbf{B}^2}{2} + g \frac{h^2}{2} \right) + \operatorname{div} \left[\mathbf{j} \left(\frac{\mathbf{u}^2}{2} + \frac{\mathbf{B}^2}{2} + gh \right) - 2h\nu(\hat{\sigma}\mathbf{u}) - g\tau h\mathbf{u} \operatorname{div}(h\mathbf{u}) - \right. \\ \left. - h\mathbf{u}(\mathbf{w}^1\mathbf{u}) + h\mathbf{B}(\mathbf{w}^1\mathbf{B}) - h\mathbf{B}(\mathbf{u}\boldsymbol{\gamma}) + h\mathbf{u}(\mathbf{B}\boldsymbol{\gamma}) - h\mathbf{B}(\mathbf{u}\mathbf{B}) - \boldsymbol{\beta}(\mathbf{u}\mathbf{B}) \right] = -\Phi, \end{aligned} \quad (5)$$

where

$$\Phi = 2h\nu(\hat{\sigma} : \hat{\sigma}) + h \frac{(\mathbf{w}^1)^2}{\tau} + h \frac{(\boldsymbol{\gamma})^2}{\tau} + g\tau (\operatorname{div}(h\mathbf{u}))^2 \quad (6)$$

is a non-negative dissipative function of the regularized system of the shallow water equations with magnetic field.

Here

$$(\hat{\sigma} : \hat{\sigma}) = \sum_{i,j=1}^2 \sigma_{ij}\sigma_{ij} = \sum_{i,j=1}^2 \sigma_{ij}^2 = \frac{1}{4} \sum_{i,j=1}^2 \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 \quad (7)$$

is an inner tensor product of symmetric matrix σ_{ij} by itself.

3. 1D-test-cases

We use explicit schemes with central differences. The regularization parameter is calculated as

$$\tau = \alpha \frac{h_x}{\sqrt{B_1^2 + gh(x)}}, \quad \mu = h\nu = \tau \frac{gh^2}{2}, \quad (8)$$

where h_x is the step of spatial grid. We take the regularization coefficient α from 0.3 to 0.5.

Test №1 corresponds to [2], where initial conditions are

$$\mathbf{U}_L^T = 1, 0, 0, 1, 0, \quad \mathbf{U}_R^T = 2, 0, 0, 0.5, 1. \quad (9)$$

Time calculation is $t = 0.4$ sec. This problem corresponds to a decay of a strong discontinuity and is similar to one presented in [7], where time calculation is $t = 0.5$ sec.

Test №2 corresponds to [7], where initial conditions are

$$\mathbf{U}_L^T = 1, 0, 1, 1, 1, \quad \mathbf{U}_R^T = \mathbf{U}_L^T + 10^{-4} (-1, 0, 0, (1 - 10^{-4})^{-1}, 2). \quad (10)$$

Time calculation is $t = 0.5$ sec. This problem corresponds to a decay of a weak discontinuity, Fig.1.

The convergence of the numerical solution for tests cases from [2] and [7] by refinement of a spatial grid was revealed in all cases.

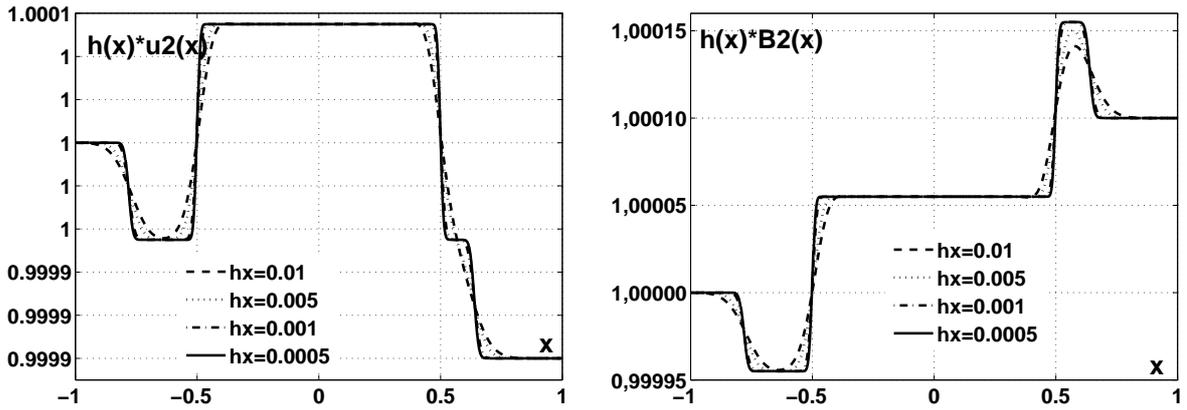


Figure 1: Test 2. $h(x)u_2(x)$ and $h(x)B_2(x)$ on the refined grids.

References

- [1] *P.A. Gilman* Magneto-hydrodynamic "shallow water" equations for the solar tachocline. *Astrophys. J.*, 544: L79-L82, 2000.
- [2] *James A. Rossmannith* A wave propagation method with constrained transport for ideal and shallow water magnetohydrodynamics / Ph.D. Dissertation, 2002.
- [3] *T.G. Elizarova* Time averaging as the approximate method of building quasi-gasdynamics and quasi-hydrodynamic equations. *Journal of Computational Mathematics and Mathematical Physics*, 2011, vol. 51, №11, p. 2096-2105.
- [4] *T.G. Elizarova, O.V. Bulatov* Regularized shallow water equations and a new method of simulation of the open channel flows. *Comp. Fluids*. 2011, N 46, p.206-211.
- [5] *T.G. Elizarova, S.D. Ustyugov, M.A. Istomina* Quasi-gasdynamics solution algorithm for the magneto-hydrodynamic shallow water equations/ Preprints of Keldysh Institute of Applied Mathematics, 2012, №64. <http://library.keldysh.ru/preprint.asp?id=2012-64> [in Russian]
- [6] *A.A. Suhomozgiy, U.V. Sheretov* Solution uniqueness of the regularised Sen-Venan equation in the linear approximation. Tver: Tver University, N 1(24), pp. 5 - 17, 2012.
- [7] *H. De Sterck* Hyperbolic theory of the "shallow water" magneto-hydrodynamics equations / *Physics of plasma*, 2001, Vol. 8, № 7. pp. 3293-3304.