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Blending verification in domain decomposition methods to achieve high quality simulations without over-solving

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Abstract. *This paper investigates the possibility to estimate the discretization error of large finite element problems solved by BDD or FETI domain decomposition methods. We show that a simple processing of the fields available during the iterations enables us to compute strict error bounds where we tell the difference between the effects of the FE discretization and the effects of the interface incompatibility. We then can stop iterations when the former prevail over the later and trigger mesh adaptation procedures.*

Keywords: Verification; domain decomposition methods; FETI; BDD; convergence criterion.

1 INTRODUCTION

Virtual testing is the industrialist's aim to minimize the number of experimental tests by replacing them by numerical simulations even for part of the certification process. To achieve that objective, we must propose computational methods for large systems which incorporate verification and adaptation tools.

A classical strategy for the parallel resolution of large systems is to employ solvers based on domain decomposition methods (DD), in particular we focus on FETI and BDD [3, 9, 10] methods which are now well mastered and remain scalable even for very complex problems [6, 15]. Schematically these methods consist in satisfying exactly the finite element equations inside subdomains whereas the continuity of the displacement and the balance of reactions on the interfaces are iteratively sought.

The a posteriori estimation of the error due to the finite element (FE) discretization can be done according several principles: equilibrium residual [1], flux projection [16] and error in constitutive relation [7]. A solution to obtain a strict estimation (an upper bound for the error), without involving hard to compute constants (related to the shape of the domain), is to make use of statically admissible fields (equilibrated residuals) which can be done by several techniques: in [5] a full dual analysis is conducted leading to an optimal estimator whereas in [11, 8, 13] the finite element stress is postprocessed which leads to lower quality estimations but at a much lower cost.

In [12] we proposed a way to process the fields obtained during FETI or BDD resolution [4] in order to enable a parallel reconstruction of (kinematic and static) admissible fields and a full parallel error estimation based on the error in constitutive equation. This estimation was possible even before convergence of the iterations associated to domain decomposition. We observed that the total error is due to the contribution of both the discretization inside subdomains and the discontinuities at the interface but that the latter quickly became negligible with respect to the former long before the domain decomposition method converged according to classical criterion.

Thanks to the results of [2] we are now able to propose a new strict error estimator for FETI and BDD domain decomposition methods which is the sum of a term purely associated to the interface error and an error dominated by the discretization error. This result enables us to propose an unbiased stopping criterion for domain decomposition. The rest of this short paper is organized as follow: we first introduce notations and give the main results then we show the performance of the various estimators on a classical example.

2 NOTATIONS AND RESULTS

Let us consider the static equilibrium of a (polyhedral) structure which occupies the open domain $\Omega \subset \mathbb{R}^d$ and which is subjected to given body force \underline{f} within Ω , to given traction force \underline{g} on $\partial_g \Omega$ and to given displacement field \underline{u}_d on the complementary part of the boundary $\partial_u \Omega \neq \emptyset$. We assume that the structure undergoes small perturbations and that the material is linear elastic, characterized by Hooke's elasticity tensor \mathbb{H} . Let \underline{u} be the unknown displacement field, $\underline{\underline{\varepsilon}}(\underline{u})$ the symmetric part of the gradient of \underline{u} , $\underline{\underline{\sigma}}$ the Cauchy stress tensor. Let $\text{KA} = \{\underline{u} \in \text{H}^1(\Omega), \underline{u} = \underline{u}_d \text{ on } \partial_u \Omega\}$ the set of admissible displacements and KA^0 the associated vector space. The problem to solve writes:

$$\begin{aligned} & \text{Find } \underline{u}_{ex} \in \text{KA}, \underline{\underline{\sigma}} \in (\text{L}^2(\omega))_{\text{sym}}^{d \times d} \\ & \forall \underline{v} \in \text{KA}^0(\Omega), \int_{\Omega} \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}(\underline{v}) \, d\Omega = \int_{\Omega} \underline{f} \cdot \underline{v} \, d\Omega + \int_{\partial_g \Omega} \underline{g} \cdot \underline{v} \, dS \\ & \underline{\underline{\sigma}} = \mathbb{H} : \underline{\underline{\varepsilon}}(\underline{u}_{ex}) \end{aligned} \quad (1)$$

Let \underline{u}_H be a finite element approximation of previous problem, $\underline{\underline{\sigma}}_H = \mathbb{H} : \underline{\underline{\varepsilon}}(\underline{u}_H)$ be the associated stress and $\hat{\underline{\underline{\sigma}}}_H$ be a post processed field which satisfies all equations except the constitutive equation, we then have the following bound for the energy norm of the error [7]:

$$\|\underline{u}_{ex} - \underline{u}_H\|_{\Omega}^2 \leq e_{\text{CR}\Omega}^2(\underline{u}_H, \hat{\underline{\underline{\sigma}}}_H) = \int_{\Omega} \left(\mathbb{H} : \underline{\underline{\varepsilon}}(\underline{u}_H) - \hat{\underline{\underline{\sigma}}}_H \right) : \mathbb{H}^{-1} : \left(\mathbb{H} : \underline{\underline{\varepsilon}}(\underline{u}_H) - \hat{\underline{\underline{\sigma}}}_H \right) \quad (2)$$

Let us now consider the resolution of previous finite element problem by a domain decomposition method. Let $(\Omega^{(s)})$ be a partition of Ω . The principle of domain decomposition method is to satisfy at any time the conservation of momentum independently on subdomains whereas the continuity of displacement $\underline{u}^{(s)} = \underline{u}^{(s')}$ and the action-reaction principle $\underline{\underline{\sigma}}^{(s)} \cdot \underline{n}^{(s)} + \underline{\underline{\sigma}}^{(s')} \cdot \underline{n}^{(s')} = 0$ on $\partial\Omega^{(s)} \cap \partial\Omega^{(s')}$ are alternatively satisfied. During any iteration of the DD solver we can deduce the following fields:

- \underline{u}_D : a displacement field continuous at the interface, hence kinematically admissible on the whole domain Ω , which solves local Dirichlet problems.
- \underline{u}_N : a displacement field so that the associated stress field $\underline{\underline{\sigma}}_N$ is equilibrated on the interface, which solves local Neumann problems and which can be the input to the parallel computation of a stress field satisfying the global equilibrium $\hat{\underline{\underline{\sigma}}}_N$.

We then have the following bounds [12, 14]:

$$\begin{aligned} \|\underline{u}_{ex} - \underline{u}_D\|_{\Omega} & \leq \sqrt{\sum_s e_{\text{CR}\Omega}^2(\underline{u}_D^{(s)}, \hat{\underline{\underline{\sigma}}}_N^{(s)})} \\ \|\underline{u}_{ex} - \underline{u}_N\|_{\Omega} & \leq \sqrt{\sum_s e_{\text{CR}\Omega}^2(\underline{u}_N^{(s)}, \hat{\underline{\underline{\sigma}}}_N^{(s)})} + \|\underline{u}_D - \underline{u}_N\|_{\Omega} \end{aligned} \quad (3)$$

The second term of the second expression can be expressed in term of a specific norm of the residual of the domain decomposition method which we note $\sqrt{\mathbf{r}^T \mathbf{z}}$.

3 ASSESSMENTS

Let us consider the structure of Figure 1 first proposed in [11]. We impose homogeneous Dirichlet boundary conditions in the central hole and on the base. The smaller hole is subjected to a constant unit pressure p_0 . A unit traction force \underline{g} is applied normally to the surface on the left upper part. The remaining boundaries are traction-free. A crack is also initiated from the smaller hole. The structure is homogeneous linear elastic (unit Young modulus) with Poisson coefficient of 0.3. We mesh the structure with first order triangular linear elements and create 16 subdomains by an automatic splitting of the mesh by Metis library, as shown in Figure 2. We use the FETI algorithm to solve the substructured problem with a Dirichlet preconditioner. The construction of statically admissible fields is performed using the Element Equilibration Technique (EET) with p+2-refinement.

Figure 3(a) presents the estimator of [12] and the new estimator during the DD iterations. They both converge close to the sequential estimator. On Figure 3(b) the two contributions to the new estimator are plotted, the DD-residual

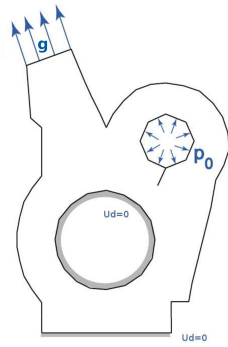


Figure 1: Cracked structure model problem



Figure 2: Decomposition in 16 subdomains

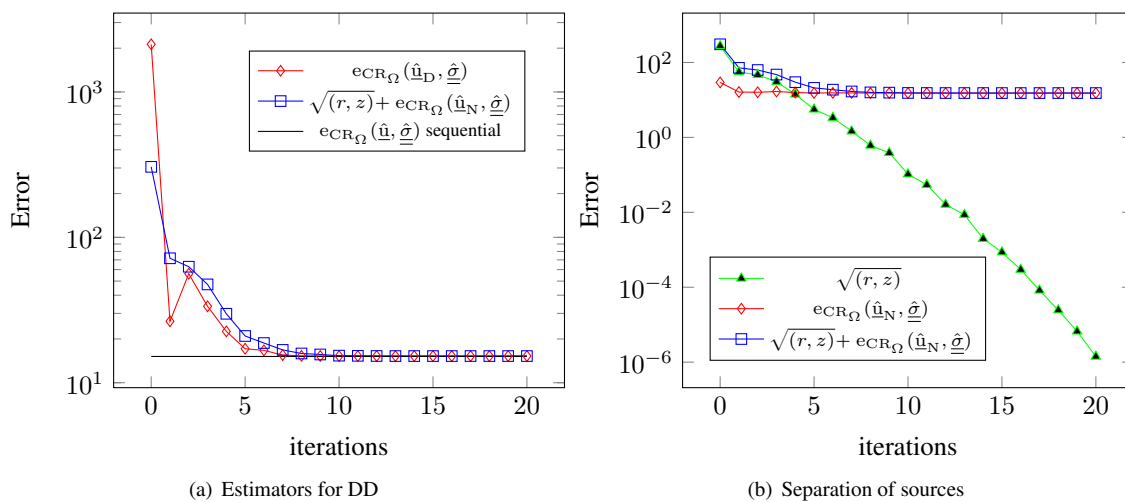


Figure 3: Error estimates

decreases whereas the contribution bearing the discretization error is almost constant. We observe that the DD-iterations after the 6th do not improve the global solution while the DD-residual is only 100 times smaller than the initial residual (a classical stopping criterion would correspond to a ratio of 10^6). Moreover Figure 4 enables us to verify that the map of the element contribution to the error estimator is converged when the algebraic error is negligible (10 times smaller) with respect to the discretization error: the maps represent the difference between the final element contributions (iteration 20) and the current element contributions at iterations 1 (initialization) and 7 (algebraic error is negligible). This means that the map at iteration 7 is sufficient to be used as an input to a mesh adaptation procedure.

4 CONCLUSION

Verification can be easily conducted in parallel when using a FETI or BDD domain decomposition method. The quality of the parallel estimator is very close to the sequential one with CPU cost divided by the number of processors. Moreover, a new bound enables to separate the contributions of the discretization and of the DD residual. This enables us to give an unbiased stopping criterion for the DD when the global error is no more minimized by the iterations and mesh adaptation is required.

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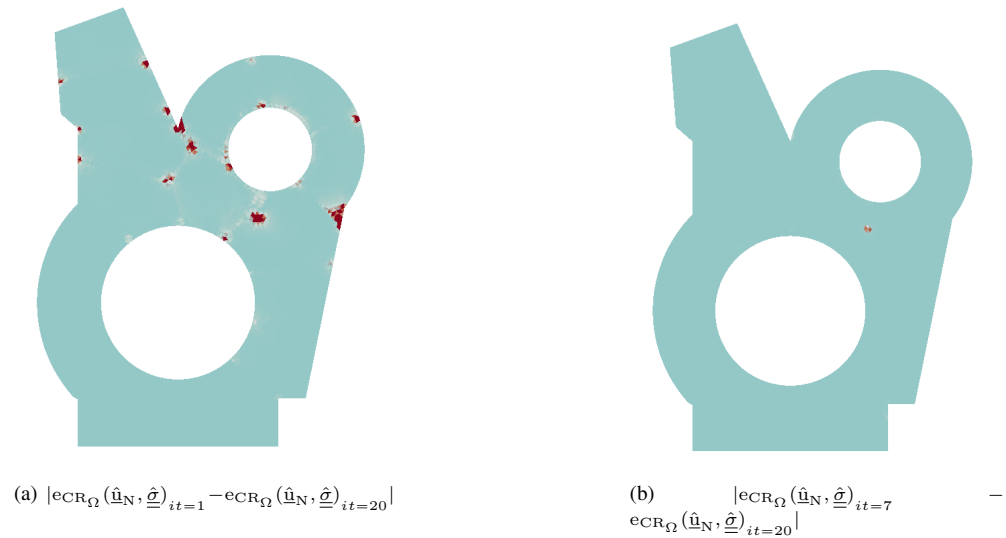


Figure 4: Convergence of the element contribution to the estimator

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