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A force-based formulation for the 2nd order elastoplastic analysis of frames

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Abstract. A force-based formulation for a nonlinear optimization program that performs 2nd order analysis of plane frames is derived, under assumptions for elastic-perfectly plastic material and non-holonomic plastic hinges of zero length. The problem is solved in an incremental manner using sequential quadratic programming (SQP). Results are compared with existing literature and/or widely accepted commercial packages, proving the formulation efficient and much faster than the direct stiffness method.

Keywords: force method; plastic hinges; second order analysis; nonlinear analysis; inelastic analysis; P-Δ effects; P-δ effects; framed structures; planar frames; mathematical programming; sequential quadratic programming.

1 INTRODUCTION

An important part of structural design and/or evaluation is to estimate the behaviour of structures beyond the elastic limit. Especially when a structure is flexible enough, resulting displacements due to external loading may severely exceed the expected ones derived by structural analysis of the 1st order. The importance of introducing the effects the structure’s deformed geometry has on the structure’s stress components is therefore evident, and has been acknowledged by the engineering community for many years now.

Based on the assumptions made, there are two main approaches; distributed plasticity (Hibbit et al. [1], Bathe et al.[2]) and concentrated plasticity (Liew & Chen [3],[4]). For both cases, an enormous amount of bibliography has been presented in the past years. Recent work in distributed plasticity has been done by Alemdar and White [5]. Also, Wang [6] did an interesting work implementing shear and warping effects in 2nd order analysis, and showed that shear deformations should be taken into account, especially for structures with large height to span ratios. Recent work based on concentrated plasticity was presented by Ngo–Huu et al. in [7]. Among the two approaches, the latter is preferable for practical purposes, due to the much lower computational cost required.

Both approaches are based on the displacement method (e.g. Wempner [8], Riks [9]) and require a certain number of iterations within each step, as well as modification of the problem’s parameters within subsequent steps (e.g. Crisfield [10]). However, in some occasions they may miss the critical point(s) and lead to convergence problems.

On the other hand, Maier [11] has proved that mathematical programming offers an ideal framework for non–holonomic plasticity. Indeed, local unstressing is a more realistic model for the real behaviour of structures, especially in the case of cyclic loading, where plastic hinges open and close subsequently, depending on the direction of the applied loads; nevertheless, local unstressing may occur even in cases of monotonic loading. Also, the force method offers a better alternative to the displacement method, mainly due to the smaller number of unknowns [12]; yet, up to now, the displacement method has been used exclusively because it is considered as easier to automate. However, efficient and easy to program techniques have been developed for automating the force method, for both 2D [13] and 3D frames [14]. Moreover, optimization algorithms have evolved in the past years, and are able to solve for non-linear objective functions with linear constraints [15]. Thus, the tools for the development of a new, promising, force-based approach are readily available. A brief presentation follows below.
2 PROBLEM FORMULATION

For the sake of simplicity and briefness, we focus on plane frames consisting of elastic–perfectly plastic material with non–holonomic plastic hinges of zero length (Figures 1a and 1b). Only nodal loads are considered, and each member has two critical sections (hinges), one at every end (Figure 1b). Distributed loading may also be implemented, either by splitting each member to smaller ones and applying equivalent point loads at the intermediate connecting nodes, or by adding the elastic rotations due to distributed loading to the corresponding elastic free rotations, according to [16]. The frames are subject to predefined patterns of static loads (Figure 2). Furthermore, we assume that complete lateral restraining holds (2D analysis).

A fully automated flexibility matrix formulation for the problem is derived through equilibrium arguments; it is based on the compatibility, the static admissibility, and the complementarity conditions. Solution is derived in an incremental manner. Equations are derived for a typical step “k” of analysis.

\[ \Delta \mathbf{m}_k = \mathbf{B}_k \cdot \mathbf{\Delta p}_k + \mathbf{\Delta y}_k \cdot \mathbf{B}_0 \cdot \mathbf{r}_p \]

\[ \Rightarrow \mathbf{Q}_k = \mathbf{B} \cdot \mathbf{\Delta p}_k + \mathbf{\Delta y}_k \cdot \mathbf{B}_0 \cdot \mathbf{r}_p \]
The equilibrium matrices ($\mathbf{\bar{B}}, \mathbf{\bar{B}}_e$) are formed in an automated way [17], using techniques and algorithms based on graph theory; a simple example may be found in [18]. However, for analysis with “P-Δ” effects, equilibrium matrices must be derived by taking into consideration the deformed state of both the statically determinate bases and the self-equilibrating systems, respectively. Thus, equations for $\mathbf{\bar{B}}_0$ and $\mathbf{\bar{B}}$ are modified as follows (for simplicity, the equations below are presented for the cases of a single load, and a standalone mesh, respectively):

\[
\begin{align*}
(m_{0i}) &= \pm \begin{bmatrix} (x_i + \Delta x_o) - (x_i + \Delta x_i) & (y_i + \Delta y_o) - (y_i + \Delta y_i) \end{bmatrix} \begin{bmatrix} p_{0x} \\ p_{0y} \end{bmatrix} \\
(m_{ei}) &= \pm \begin{bmatrix} (y_o + \Delta y_o) - (y_i + \Delta y_i) & (x_i + \Delta x_o) - (x_i + \Delta x_i) \end{bmatrix} \begin{bmatrix} p_{ex} \\ p_{ey} \end{bmatrix}
\end{align*}
\]  

Where the 2nd order displacements “$\Delta x$” and “$\Delta y$” of each critical section are computed using the principle of virtual work: By applying a unit virtual load (horizontal for “$\Delta x$”, vertical for “$\Delta y$”) at the critical section of interest “i”, the virtual stress diagram of a statically determinate basis is evaluated; this basis is defined by the shortest path between the point of application of the virtual load and the boundary node of the corresponding cantilever (for $\mathbf{\bar{B}}_e$), or the end node of the corresponding mesh (for $\mathbf{\bar{B}}$). The equilibrium matrices of the virtual cantilevers are computed using only the 1st order terms in equation (2). Also, for the formulation of $\mathbf{\bar{B}}$ of the 2nd order, the horizontal ($p_x$) and vertical ($p_y$) unit value redundant stress components of every mesh must be considered to lie along the direction of the conceivable line that joins the two displaced ends of the imposed cut.

The “P-Δ” effects are implemented in the problem’s formulation using the useful functions derived by A. Berry, back in 1916. A way to derive A. Berry’s functions may be found in [19]; herein, we use the first two terms of a Taylor approximation of these functions around zero:

\[
\begin{align*}
\frac{\partial \theta_{pi}}{\partial \theta_{p0}} &= \frac{\partial \theta_{pi}}{\partial \theta_{p0}} \frac{\partial \theta_{p0}}{\partial \theta_{p0}} \frac{\partial \theta_{p0}}{\partial \theta_{p0}} \frac{\partial \theta_{p0}}{\partial \theta_{p0}} \\
\Delta \theta_{pi} &= \left[ \frac{\partial \theta_{pi}}{\partial \theta_{p0}} \right] \Delta \theta_{p0} \\
\end{align*}
\]

The compatibility condition, as was first proposed by Maxwell, is:

\[
N^T \cdot Q_{k-1} + N^T \cdot \Delta Q_k + Y = e \Rightarrow N^T \cdot Q_{k-1} + N^T \cdot \mathbf{\bar{B}} \cdot \Delta p_k + \Delta y_k \cdot N^T \cdot \mathbf{\bar{B}}_0 \cdot r_p + Y = e \tag{5}
\]

Where the incidence matrix $\mathbf{\bar{N}}$ collects the linear coefficients $(s_{ij},s_{kj})$ of each line $j$ used to approximate the convex curve of the failure criterion, for all four quadrants, with respect to the plastic bearing capacities ($m_r,n_r$) of each member, according to: $m_r + n_r^2 = 1 \Rightarrow s_{ij} \cdot m_r + s_{k} \cdot n_r = 1, \quad (j = 1,2, \ldots)$ (Figure 3). Vector $Y$, collects the plastic potentials of every critical section which are by definition- positive quantities for both positive and negative plasticization (Figure 1a, Figure 3), and the vector $e^T = \{1 \quad 1 \quad \ldots \quad 1\}$. For the plastic deformations of a critical section, the axiom of maximization of plastic work yields: $\Delta q^T = \left\{\Delta \theta_{pi} / \Delta \theta_{p0} \right\} = \Delta \lambda_{k} \cdot \Delta y_k = \Delta \lambda_{k} \cdot \Delta y_k \cdot \Delta \lambda_{k}$, as occurs from (5). Furthermore, with the help of “$\cdot Y$,” and “$\cdot \Delta \lambda_{k}$,” non-holonomic plasticity may be expressed by the complementarity condition (Figure 1a):

\[
\begin{align*}
Y^T \cdot \Delta \lambda &= 0 \\
Y \geq 0, \quad \Delta \lambda \geq 0
\end{align*}
\]  

The compatibility condition, as was first proposed by Maxwell, is:

\[
\mathbf{B}^T \cdot \Delta q_{total} = \mathbf{B}^T \cdot (\Delta q_{pl} + \Delta q_{pl}) \tag{6}
\]

By substituting (1) into (4), then substituting (5) into (7) and combining both into (8), we have:

\[
\mathbf{B}^T \cdot (\mathbf{F} + \mathbf{F}_{\theta}) \cdot \mathbf{\bar{B}} \cdot \Delta p_k + \Delta y_k \cdot \mathbf{B}^T \cdot (\mathbf{F} + \mathbf{F}_{\theta}) \cdot \mathbf{B}_0 \cdot r_p + (\mathbf{B}^T \cdot \mathbf{N}) \cdot \Delta \lambda_{k} = 0 \tag{9}
\]

Equation (9), together with (5) and (7), may be seen as the Karush–Kuhn–Tucker (KKT) conditions (which are an equivalent form of the Fritz–John conditions) of the following optimization problem:

\[
\begin{align*}
\min f(\Delta p_k) &= f(\Delta p_k \cdot \mathbf{B}^T \cdot (\mathbf{F} + \mathbf{F}_{\theta}) \cdot \mathbf{B}) \cdot d(\Delta p_k) + \Delta y_k \cdot f((\mathbf{B}_0 \cdot r_p) \cdot (\mathbf{F} + \mathbf{F}_{\theta}) \cdot \mathbf{B}) \cdot d(\Delta p_k) \\
\text{s. t.:} & \quad (\mathbf{N}^T \cdot \mathbf{B}) \cdot \Delta p_k \leq e - \mathbf{N}^T \cdot Q_{k-1} - \Delta y_k \cdot \mathbf{N}^T \cdot \mathbf{B}_0 \cdot r_p
\end{align*}
\]

The way to solve the non-linear problem described by equation (10) is to apply a small load increment at the beginning of every analysis step, and then determine the direction on which the solution lies, using any SQP algorithm (e.g. [15]); the true load factor “$\Delta y_k$” that activates a new plastic hinge is then found using a well-established procedure which may be found in [20].
3 EXAMPLE

A 6-storey, 1-bay skeleton with storey height of 3.0m and bay width of 6.0m, made of steel S220, serves as a benchmark for the method; the structure’s mechanical properties are summarized in Table 1; beams are subject to a uniform load of 9kN/m. A comparative nonlinear static analysis including P-Δ effects and large displacements was performed using SAP2000 [21] (see Figure 4). It is worth mentioning that, using SAP2000, total time for 2nd order analysis took a little more than 22 minutes; using the proposed method, it took no more than 60 seconds in total.

Table 1: Mechanical Properties of 6-Storey Frame.

<table>
<thead>
<tr>
<th>Members</th>
<th>Length</th>
<th>Section</th>
<th>Elast. Modulus</th>
<th>Mom. of Inertia (m^4)</th>
<th>Area (m^2)</th>
<th>M_p (kNm)</th>
<th>N_p (kN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Columns</td>
<td>3.0m</td>
<td>HEM260</td>
<td>2.0E+08 kPa</td>
<td>3.131E-04</td>
<td>2.200E-02</td>
<td>555.28</td>
<td>4840.0</td>
</tr>
<tr>
<td>Beams</td>
<td>6.0m</td>
<td>HEB160</td>
<td>2.0E+08 kPa</td>
<td>2.492E-05</td>
<td>5.430E-03</td>
<td>77.88</td>
<td>1194.6</td>
</tr>
</tbody>
</table>

Figure 4: Static pushover curves for both 1st and 2nd order analyses using both SAP2000 and the force method.

4 CONCLUSIONS

A new, direct, force-based formulation for the inelastic 2nd order analysis of planar frames was presented. Nonholonomic behaviour is contained directly inside the solution of the sequential quadratic program. Equilibrium is effectively determined until the critical point beyond which instability phenomena start to occur. It renders robust and faster than the equivalent direct stiffness method, thus, it is most appealing for practical purposes (e.g. structural evaluation). The method may be extended to the analysis of 3D structures; material hardening/softening may also be implemented; work is being done towards these directions and will be presented in the near future.

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6 REFERENCES