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YIC2013
Second ECCOMAS Young Investigators Conference
2–6 September 2013, Bordeaux, France

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Abstract. *A stabilised low-order finite element methodology is presented for the numerical simulation of a mixed conservation law formulation in fast solid dynamics. The mixed formulation, where the unknowns are linear momentum, deformation gradient and total energy, can be cast in the form of a system of first order hyperbolic equations. The difficulty associated with locking effects commonly encountered in traditional displacement formulations is addressed by treating the deformation gradient as one of the primary variables. Such formulation is first discretised in space by using a stabilised Petrov-Galerkin (PG) methodology, a generalisation of the Variational Multi-Scale (VMS) approach. The semi-discretised system of equations are then evolved in time by employing a Total Variation Diminishing Runge-Kutta (TVD-RK) time integrator. The resulting formulation achieves optimal convergence with equal orders in velocity (or displacement) and stresses. A series of numerical examples are used to assess the performance of the proposed algorithm. The new formulation is proven to be very efficient in nearly incompressible and bending-dominated scenarios.*

Keywords: Petrov-Galerkin; Variational Multi-Scale; Low-order; Tetrahedra; Locking; Stabilisation; Runge-Kutta

1 INTRODUCTION

Dynamic displacement-based finite element codes, combining 8-noded underintegrated hexahedron element and explicit time marching schemes, are commonly used for the simulation of large strain impact problems. However, many practical applications involve geometries that are far too complex to be meshed using this type of discretisation. The presence of large deformations accompanied by severe mesh distortion may lead to poorly shaped elements unless some form of adaptive remeshing is used. At present, the possibilities of employing mesh adaptation with hexahedral elements are very limited. On the contrary, tetrahedral mesh generators and related mesh adaptivity procedures are readily available.

From the spatial discretisation standpoint, the use of linear finite element interpolation leads to second order convergence for the primary variables (i.e. displacements) but one order less for derived variables (i.e. strains and stresses). Efforts to develop effective linear tetrahedral elements in both nearly incompressible and bending deformations have only been partially successful suffering from spurious mechanisms similar to hourglassing [1]. From the time discretisation point of view, the use of Newmark-type integrators is known to be very inefficient in shock-dominated problems.

To resolve these issues, we introduce a stabilised Petrov-Galerkin (PG) formulation in conjunction with a Total Variation Diminishing Runge-Kutta (TVD-RK) time integrator for the numerical analysis of fast dynamics. A mixed methodology is presented in the form of a system of first order conservation laws, where the linear momentum, the deformation gradient and the total energy are regarded as the three main conservation variables of this mixed approach [2]. The use of linear tetrahedral elements achieves second order convergence in velocity and stresses, and seems to be very effective in nearly incompressible, bending and shock dominated scenarios.

2 GOVERNING EQUATIONS FOR REVERSIBLE ELASTODYNAMICS

The motion of a continuum body can be described by a mapping $\boldsymbol{\phi}$ established from an undeformed (or material) configuration $\boldsymbol{X} \in V \subset \mathbb{R}^3$ to a deformed (or spatial) configuration $\boldsymbol{x} \in v(t) \subset \mathbb{R}^3$ at time t, namely $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{X}, t)$. The mixed conservation law formulation for Lagrangian solid dynamics is generally presented as follows [2],

$$\frac{\partial \boldsymbol{p}}{\partial t} - \boldsymbol{\nabla}_0 \cdot \boldsymbol{P} = \rho_0 \boldsymbol{b}, \tag{1a}$$

$$\frac{\partial \boldsymbol{F}}{\partial t} - \boldsymbol{\nabla}_0 \cdot (\boldsymbol{v} \otimes \boldsymbol{I}) = \boldsymbol{0}, \tag{1b}$$

$$\frac{\partial E_T}{\partial t} - \boldsymbol{\nabla}_0 \cdot (\boldsymbol{P}^T \boldsymbol{v} - \boldsymbol{Q}) = s, \tag{1c}$$

where $\boldsymbol{p} = \rho_0 \boldsymbol{v}$ is the linear momentum per unit of undeformed volume, ρ_0 is the material density, $\boldsymbol{v}$ is the velocity field, $\boldsymbol{b}$ is the body force per unit mass, $\boldsymbol{F}$ is the deformation gradient, $\boldsymbol{P}$ is the first Piola-Kirchhoff stress tensor, E_T is the total energy per unit of undeformed volume, $\boldsymbol{I}$ is the identity (or Kronecker) tensor, $\boldsymbol{Q}$ is the heat flux vector, s is the heat source and $\boldsymbol{\nabla}_0$ describes the material gradient operator in undeformed space. The above laws can be combined into a system of first order conservation equations as

$$\frac{\partial \boldsymbol{\mathcal{U}}}{\partial t} + \frac{\partial \boldsymbol{\mathcal{F}}_I}{\partial X_I} = \boldsymbol{\mathcal{S}}, \qquad \forall\, I = 1, 2, 3. \tag{2}$$

To complete the coupled system (1), a closure equation is introduced by means of an appropriate frame-indifferent constitutive model [3] relating deformation to stress field (i.e. $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{F})$).

3 STABILISED NUMERICAL METHOD

A stabilised Petrov-Galerkin (PG) methodology incorporating a suitable numerical stabilisation into the Bubnov-Galerkin formulation is presented [4]. To establish a weak (or variational) statement under isothermal considerations (enabling the energy equation (1c) to be decoupled from the linear momentum and compatibility balance principles (1a-1b)), we multiply the expressions (1a-1b) by an appropriate stabilised conjugate virtual field $\delta\boldsymbol{\mathcal{V}}^{st}$ and integrate over the volume V of the body to give

$$\delta W(\boldsymbol{\mathcal{U}}, \delta\boldsymbol{\mathcal{V}}^{st}) = \int_V \delta\boldsymbol{\mathcal{V}}^{st} \cdot \boldsymbol{\mathcal{R}}\, dV = 0; \qquad \delta\boldsymbol{\mathcal{V}}^{st} = \delta\boldsymbol{\mathcal{V}} + \boldsymbol{\tau}^T \boldsymbol{\mathcal{A}}_I^T \frac{\partial \delta\boldsymbol{\mathcal{V}}}{\partial X_I}; \qquad \boldsymbol{\mathcal{R}} = \begin{bmatrix} \boldsymbol{\mathcal{R}}_{\boldsymbol{p}} \\ \boldsymbol{\mathcal{R}}_{\boldsymbol{F}} \end{bmatrix}; \qquad \delta\boldsymbol{\mathcal{V}}^{st} = \begin{bmatrix} \delta\boldsymbol{v}^{st} \\ \delta\boldsymbol{P}^{st} \end{bmatrix}. \tag{3}$$

Or in complete form

$$\delta W(\boldsymbol{\mathcal{U}}, \delta\boldsymbol{v}^{st}, \delta\boldsymbol{P}^{st}) = \int_V \delta\boldsymbol{v}^{st} \cdot \underbrace{\left(\frac{\partial \boldsymbol{p}}{\partial t} - \boldsymbol{\nabla}_0 \cdot \boldsymbol{P} - \rho_0 \boldsymbol{b}\right)}_{\boldsymbol{\mathcal{R}}_p} dV + \int_V \delta\boldsymbol{P}^{st} : \underbrace{\left(\frac{\partial \boldsymbol{F}}{\partial t} - \boldsymbol{\nabla}_0 \boldsymbol{v}\right)}_{\boldsymbol{\mathcal{R}}_{\boldsymbol{F}}} dV = 0. \tag{4}$$

Note that $\delta\boldsymbol{v}^{st}$ is the stabilised virtual velocity, $\delta\boldsymbol{P}^{st}$ is the stabilised virtual first Piola-Kirchhoff stress, $\boldsymbol{\mathcal{A}}_I = \frac{\partial \boldsymbol{\mathcal{F}}_I}{\partial \boldsymbol{\mathcal{U}}}$ is the flux Jacobian matrix [2], $\boldsymbol{\tau}$ is the stabilisation parameter (an intrinsic time scale) and $\boldsymbol{\mathcal{R}}_{\boldsymbol{p}}$ and $\boldsymbol{\mathcal{R}}_{\boldsymbol{F}}$ are the residuals of the linear momentum and deformation gradient balance principles. Pairs such as $\{\delta\boldsymbol{v}^{st}, \boldsymbol{\mathcal{R}}_{\boldsymbol{p}}\}$ and $\{\delta\boldsymbol{P}^{st}, \boldsymbol{\mathcal{R}}_{\boldsymbol{F}}\}$ are said to be work conjugate with respect to the initial volume V in the sense that their inner product yields work rate per unit of undeformed volume.
The stabilised virtual field $\delta\boldsymbol{\mathcal{V}}^{st}$ can be particularised for the set of relevant variables as follows

$$\delta\boldsymbol{p}^{st} = \delta\boldsymbol{p} - \tau_{\boldsymbol{p}} \boldsymbol{\nabla}_0 \cdot \delta\boldsymbol{P}; \qquad \delta\boldsymbol{P}^{st} = \delta\boldsymbol{P} - \tau_{\boldsymbol{F}}\, \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla}_0 \delta\boldsymbol{v}; \qquad \delta\boldsymbol{P} = \boldsymbol{\mathcal{C}} : \delta\boldsymbol{F}. \tag{5}$$

Substituting equations (5a) and (5b) into equation (4), together with $\delta\boldsymbol{p}^{st} = \rho_0 \delta\boldsymbol{v}^{st}$, yields the following variational statements

$$\delta W_{\delta\boldsymbol{v}}(\boldsymbol{\mathcal{U}}, \delta\boldsymbol{v}) = \int_V \left(\delta\boldsymbol{v} \cdot \boldsymbol{\mathcal{R}}_{\boldsymbol{p}} - \tau_{\boldsymbol{F}} \left(\boldsymbol{\mathcal{C}} : \boldsymbol{\nabla}_0 \delta\boldsymbol{v}\right) : \boldsymbol{\mathcal{R}}_{\boldsymbol{F}}\right) dV = 0, \tag{6a}$$

$$\delta W_{\delta\boldsymbol{P}}(\boldsymbol{\mathcal{U}}, \delta\boldsymbol{P}) = \int_V \left(\delta\boldsymbol{P} : \boldsymbol{\mathcal{R}}_{\boldsymbol{F}} - \frac{\tau_{\boldsymbol{p}}}{\rho_0} \left(\boldsymbol{\nabla}_0 \cdot \delta\boldsymbol{P}\right) \cdot \boldsymbol{\mathcal{R}}_{\boldsymbol{p}}\right) dV = 0. \tag{6b}$$

Introducing the interpolation for $\delta \boldsymbol{v}$ given by $\delta \boldsymbol{v} = \sum_a N_a \delta \boldsymbol{v}_a$ and the use of Gauss divergence theorem renders [3]

$$\int_V N_a \dot{\boldsymbol{p}}\, dV = \int_{\partial V} N_a \boldsymbol{t}^B \, dA + \int_V N_a \rho_0 \boldsymbol{b} \, dV - \int_V \underbrace{\left[\boldsymbol{P} + \tau_{\boldsymbol{F}} \boldsymbol{\mathcal{C}} : (\boldsymbol{\nabla}_0 \boldsymbol{v} - \dot{\boldsymbol{F}})\right]}_{\boldsymbol{P}^{st}} \boldsymbol{\nabla}_0 N_a \, dV; \qquad \boldsymbol{t}^B = \boldsymbol{P}\hat{\boldsymbol{N}}, \quad (7)$$

where $\boldsymbol{t}^B$ is the traction vector and $\hat{\boldsymbol{N}}$ being the material outward unit normal on the boundary ∂V. Observe that the square brackets term on the right-hand side of (7) describes a stabilised first Piola-Kirchhoff stress $\boldsymbol{P}^{st}$. More generally, this term can be reinterpreted as the first Piola-Kirchhoff stress of a stabilised deformation gradient defined by

$$\boldsymbol{P}^{st} := \boldsymbol{P}\left(\boldsymbol{F}^{st}\right); \qquad \boldsymbol{F}^{st} := \boldsymbol{F} + \tau_{\boldsymbol{F}} \left(\boldsymbol{\nabla}_0 \boldsymbol{v} - \dot{\boldsymbol{F}}\right). \quad (8)$$

Above expression (8b) can be further enhanced by adding a time-integrated stabilisation term $\int_t \tau_{\boldsymbol{F}} (\boldsymbol{\nabla}_0 \boldsymbol{v} - \dot{\boldsymbol{F}})\, dt$ aims at diffusing spurious curl-error mechanisms [5] generated by compatibility conditions, resulting into a new definition for $\boldsymbol{F}^{st}$ as follows

$$\boldsymbol{F}^{st} := \boldsymbol{F} + \tau_{\boldsymbol{F}} \left(\boldsymbol{\nabla}_0 \boldsymbol{v} - \dot{\boldsymbol{F}}\right) + \alpha \left(\boldsymbol{\nabla}_0 \boldsymbol{x} - \boldsymbol{F}\right), \quad (9)$$

where α is a non-dimensional stabilisation parameter, typically between 0 and 0.5. Substituting equations (8) and (9) into the square brackets term of equation (7) and noting that $\dot{\boldsymbol{p}} = \sum_b N_b \dot{\boldsymbol{p}}_b$ to result in

$$\sum_b \boldsymbol{\mathcal{M}}_{ab} \dot{\boldsymbol{p}}_b = \int_{\partial V} N_a \boldsymbol{t}^B \, dA + \int_V N_a \rho_0 \boldsymbol{b} \, dV - \int_V \boldsymbol{P}\left(\boldsymbol{F}^{st}\right) \boldsymbol{\nabla}_0 N_a \, dV, \quad (10)$$

where $\boldsymbol{\mathcal{M}}_{ab} = \left(\int_V N_a N_b \, dV\right) \boldsymbol{I}$ is the consistent mass matrix contribution.
An identical discretisation procedure can now be followed for $\delta W_{\delta \boldsymbol{P}}$ (6b) by employing a standard finite element expansion $\delta \boldsymbol{P} = \sum_a N_a \delta \boldsymbol{P}_a$ and $\dot{\boldsymbol{F}} = \sum_b N_b \dot{\boldsymbol{F}}_b$ resulting in

$$\sum_b \boldsymbol{\mathcal{M}}_{ab} \dot{\boldsymbol{F}}_b = \int_{\partial V} N_a (\boldsymbol{v}^B \otimes \hat{\boldsymbol{N}})\, dA - \int_V \boldsymbol{v}^{st} \otimes \boldsymbol{\nabla}_0 N_a \, dV; \qquad \boldsymbol{v}^{st} = \frac{1}{\rho_0}\left[\boldsymbol{p} + \tau_{\boldsymbol{p}} \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{P} + \rho_0 \boldsymbol{b} - \dot{\boldsymbol{p}}\right)\right]. \quad (11)$$

Both traction and velocity vectors at the boundary (i.e. $\boldsymbol{t}^B$ and $\boldsymbol{v}^B$) are computed from prescribed boundary conditions. Notice that the proposed discrete approximations ((10) and (11)) can also be identified as a general case of Variational Multi-Scale (VMS) analysis [6], relying on the idea that unstable numerical methodology can be stabilised by enriching the discrete solution approximation by means of fine (or subgrid) scales.
These semi-discrete expressions (10) and (11) may then be integrated in time by employing the explicit two-stage Total Variation Diminishing Runge-Kutta (TVD-RK) time stepping procedure due to its excellent TVD qualities [2].

4 NUMERICAL EXAMPLES

A two-dimensional example is first presented to model the sloshing of a liquid in a unit squared container constrained with rollers at the bottom and on the left and right hand sides. In this problem the gravitational force is a source term that must be added into the equations. It is clear that the stabilised Petrov-Galerkin (PG) formulation eliminates both the volumetric locking effects and the appearance of spurious pressure instabilities in the case of near incompressibility (see Figure 1). Another interesting examples are three-dimensional short column with a unit square cross section initially loaded by various types of prescribed velocities (i.e. angular velocity and linear variation of velocity) are also examined, illustrating the bending capability of the proposed algorithm using linear tetrahedral meshes within the context of large deformation behaviours. Figure 2 shows that the resulting formulation performs effectively in bending-dominated scenario and eliminates the appearance of non-physical curl-error modes in the solution over a long-term response.

5 CONCLUSIONS

In this paper, a stabilised computational framework is proposed for the numerical analysis of fast transient phenomena within the context of large deformations. The mixed conservation law formulation, in the form of a system of first order hyperbolic equations, performs extremely well in both nearly incompressible and bending dominated scenarios. Both velocities (or displacements) and stresses display the same rate of convergence, which proves ideal in the case of linear finite element interpolation oftenly used in industry. It has been shown that the proposed formulation overcomes excessive artificial stiffness and pressure checkerboard modes, typical of linear triangular and tetrahedral meshes in bending-dominated problems. The proposed numerical algorithm provides a good balance between accuracy and speed of computation, showing very promising results, benchmarked against the recently proposed Finite Volume methodology [2]. A discontinuity-capturing operator with the purpose of enhancing shock resolution in the vicinity of sharp spatial gradients is the next step of the work. Finally, the methodology opens a door to the introduction of higher-order basis functions.

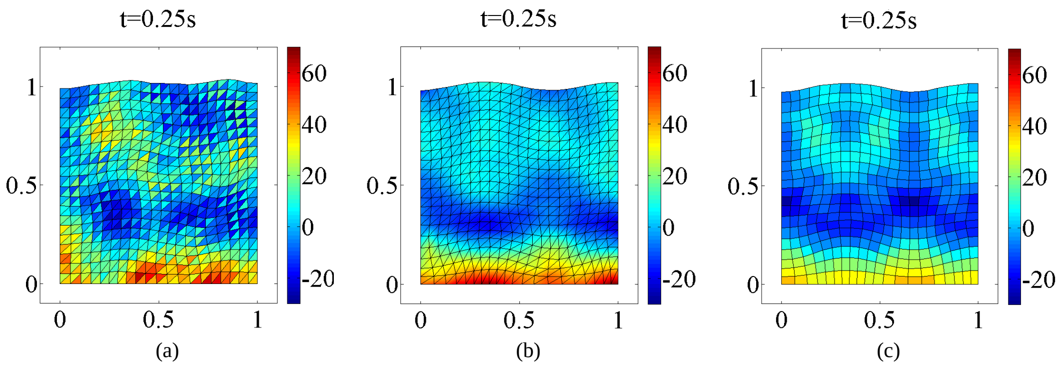

Figure 1: Pressure distribution of deformed shapes at a particular time using: (a) Standard FEM procedure; (b) Stabilised Petrov-Galerkin; and (c) Mean dilatation approach.

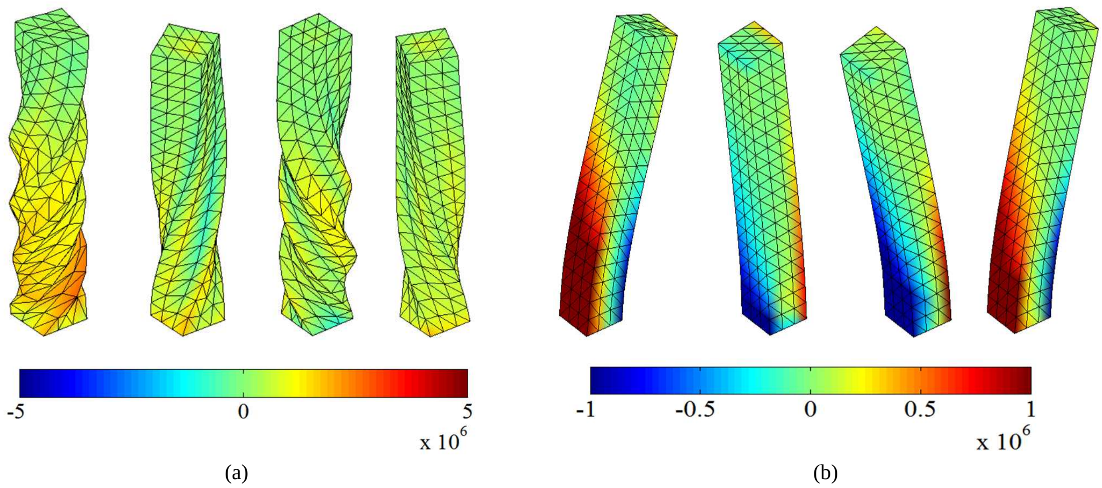

Figure 2: Sequence of pressure distribution of deformed shapes: (a) Twisting column; and (b) Bending column.

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