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The adjoint-weighted equation method for the direct solution of inverse problems of elasticity

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Abstract. *We apply the adjoint weighted equation method (AWE) for the direct solution of inverse problems of elasticity where the shear modulus distribution is reconstructed from given strain data. We demonstrate that while continuous and noise-free data provides accurate reconstructions, for discontinuous or noisy data the reconstruction is challenging. For those cases, we show that data smoothing and regularization of the solution offer a dramatic improvement. These are necessary when considering real-world measured strain data.*

Keywords: inverse problems; elasticity; direct solution approaches.

1 INVERSE PROBLEMS OF ELASTICITY

Inverse problems arise in many fields of physics and often involve the recovery of the physical characteristics of a material from its response to external loads. One example is inverse problems of elasticity. Here mechanical properties of elastic bodies are sought based on measured displacement fields that arise from predetermined applied loads.

Elastography [1] is a new emerging imaging modality. In elastography, images of mechanical properties of tissues are generated for diagnostic purposes. The underlying assumption is that tissue pathology is often accompanied by significant changes in mechanical properties. This is especially true for malignancies such as breast and prostate cancers, but is also found in other diseases such as liver fibrosis. Elastography often involves two stages: First, a displacement field inside a tissue is generated and measured. Then, using the measured displacements, an inverse problem is solved for the elastic properties of the tissues (usually the shear modulus).

Inverse problems are typically solved using iterative optimization techniques [2]. This approach is robust to noise and can handle partial displacement data, but is computationally expensive. When full field data are available, direct solution approaches can be considered. Here the computational effort is relatively low, offering potential applications in real time imaging. The direct approach however is sensitive to noise, and therefore must be carefully applied.

2 THE ADJOINT-WEIGHT EQUATION METHOD

The adjoint-weight equation method (AWE) is a variational framework for the direct solution of inverse problems, motivated by stabilized solution methods for advection-dominated problems. The method was first proposed by Barbone et al. [3] for inverse problems of heat conduction where the thermal conductivities are sought.

In this work we extend the AWE formulation for inverse problems of elasticity. First, we derive the necessary conditions that guarantee a unique solution to the inverse problem of incompressible three-dimensional elasticity. This completes previous work on the uniqueness of inverse problems of two-dimensional elasticity. We then develop the AWE formulation for the inverse problem of incompressible plane stress elasticity [4], followed by a general

framework for elasticity which we specialize to several problems including incompressible plane strain and three-dimensional elasticity [5]. Finally we consider methods to improve the performance of the AWE formulation in the presence of noise and discontinuous solutions [6].

Inverse problems of elasticity involve partial differential equations (PDE's). Uniqueness of PDE's is typically guaranteed via appropriate boundary conditions. For inverse problems of elasticity however, these are often unavailable and other means to constrain the solution are necessary. One method to achieve this is by incorporating additional displacement fields. These provide additional equations which serve to constrain the solution space. We investigate the inverse problem of isotropic incompressible three-dimensional elasticity and demonstrate that with two independent displacement fields, the solution space to the inverse problem can involve, at most, five arbitrary constants. In comparison, the solution space of the single-field problem may involve arbitrary functions. This has practical significance in the context of biomedical imaging. The reconstruction of material properties when the solution space involves arbitrary functions is often impractical. On the other hand, reconstruction is likely to be achievable when only arbitrary constants exist in the solution.

3 COMPUTATIONAL EXAMPLES

Amongst the inverse problems of elasticity, the inverse problem of incompressible plane stress is one of the simplest to solve due to its favorable mathematical properties. Problems involving incompressibility typically involve a pressure term. For the inverse problem of incompressible plane stress the pressure term can be eliminated, significantly simplifying the solution to the inverse problem. The resulting inverse problem involves a single unknown elastic coefficient, e.g. the shear modulus. For this case, a single full-field displacement field provides two equations for a single unknown, and a unique solution is guaranteed by prescribing the shear modulus at a single point. Figure 1 depicts the solution to two inclusion problems in a homogeneous background, resembling problems arising in biomedical imaging (e.g. tumors inside healthy tissues). The problems are solved in a unit square domain, and the problem data is composed of a single noise-free displacement field. Figure 1-top presents the results for an inclusion involving a smoothly-varying shear modulus, and Figure 1-bottom presents a stepped inclusion involving a discontinuity in the shear modulus. The inverse problem is solved using the AWE method and a least-squares formulation for comparison. The AWE formulation provides superior results for both inclusion profiles, and especially for the stepped inclusion where the results of the least-squares formulation are very poor.

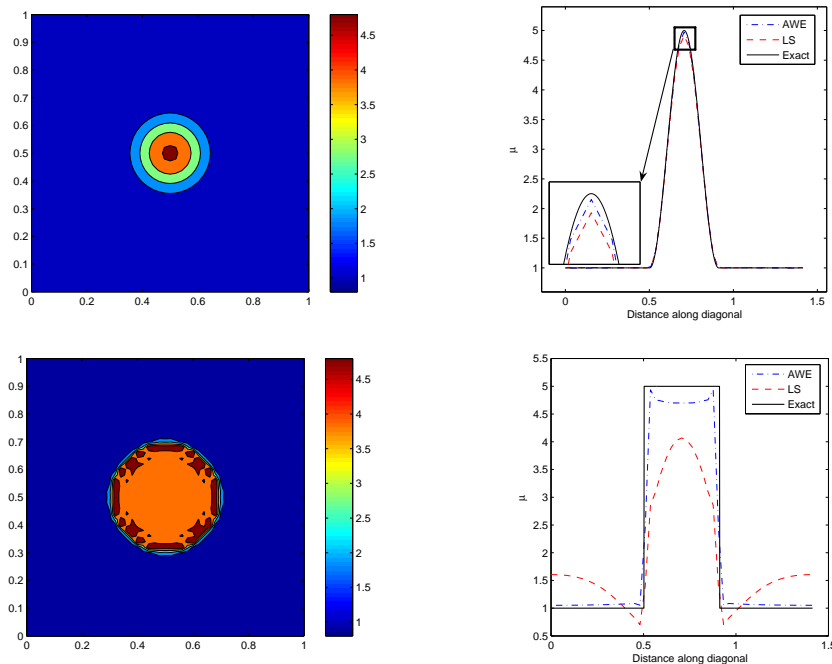


Figure 1: Inverse problem of incompressible plane stress. Inclusion problem with a smooth profile (top row) and a stepped profile (bottom row). Shear modulus distribution in the entire region for the AWE method (left column) and along the diagonal (right column).

A more challenging problem to solve is that of incompressible plane strain elasticity, since the pressure terms can not be eliminated. Here, with two displacements fields, the solution space involves four arbitrary constants for the shear modulus [7]. To guarantee a unique solution to this problem, we assume that the shear modulus is available in a small calibration region at the bottom portion of the square domain. We enforce the value of the shear modulus in the calibration region in a least-squares sense using the penalty method. Figure 2 depicts the reconstructed shear modulus solved using the AWE method. The results for the smooth profile are very accurate, but for the stepped profile only about 80% of contrast is recovered. These trends are also observed for the incompressible plane stress case.

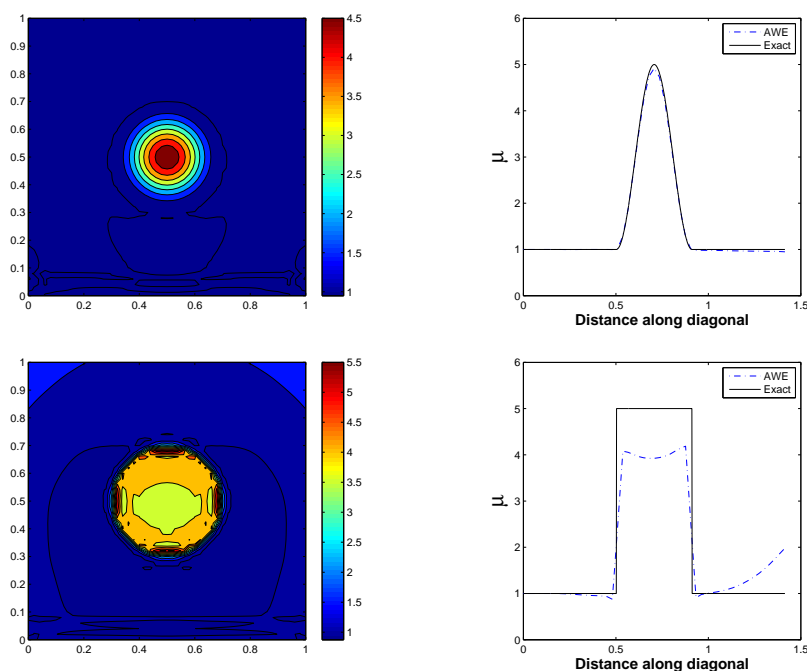


Figure 2: Inverse problem of incompressible plane strain. Inclusion problem with smooth profile (top row) and stepped profile (bottom row). Shear modulus distribution in the entire region (left column) and along the diagonal (right column).

In practical applications the measured displacement data always involves noise. The AWE method, being a direct solution approach, is sensitive to noise due to differentiation of the measured noisy data. To improve the reconstruction in the presence of noise, we consider smoothing the data and regularizing the solution. Smoothing is performed using a simple averaging technique, and regularization is performed using total variation (TV) regularization which is especially suited for preserving sharp layers (e.g. stepped inclusions). Figure 3 presents the reconstruction from data that involves 30% white Gaussian noise measured on the strains. This level of noise can be associated with measurements performed using ultrasound equipment. Figure 3-left depicts the results obtained using the AWE method directly without smoothing or regularization. The results are very poor and can not be considered useful. Figure 3-center presents the results with smoothing applied. The improvement is dramatic and the results are fairly close to the target solution. With regularization added (Figure 3-right) the results are even further improved.

4 CONCLUSIONS

The AWE method, as a direct solution approach, can provide fast reconstruction of material properties compared to conventional optimization approaches. We find that with some adaptations, the method can also handle noise making it a potential tool for real time imaging. The availability of this and related direct inversion approaches motivates the development of techniques to measure all components of the displacement field.

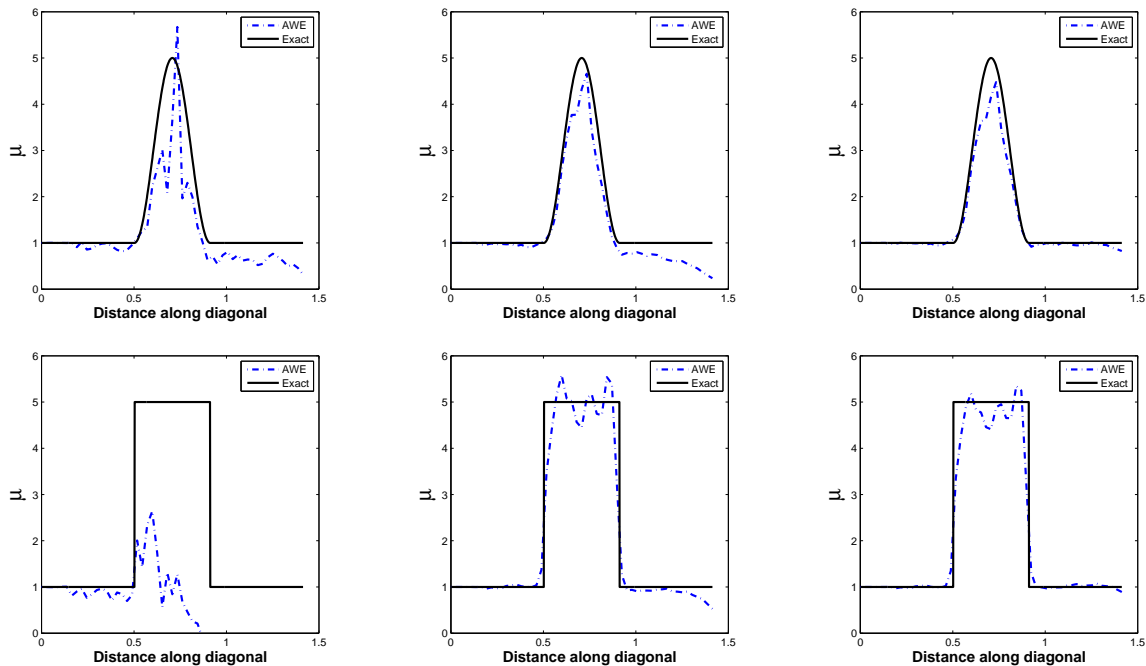


Figure 3: Inverse problem of incompressible plane strain with 30% noise. Inclusion problem with smooth profile (top row) and stepped profile (bottom row). AWE solution without smoothing or regularization (left column) AWE with displacement smoothing (center column) and AWE with smoothing and regularization (right column).

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