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Amplification of pulses in nonlinear geometric optics

Jean-François COULOMBEL*, Mark WILLIAMS†

August 30, 2013

Abstract

In this companion paper to our study of amplification of wavetrains [CGW13], we study weakly stable semilinear hyperbolic boundary value problems with pulse data. Here weak stability means that exponentially growing modes are absent, but the so-called uniform Lopatinskii condition fails at some boundary frequency in the hyperbolic region. As a consequence of this degeneracy there is again an amplification phenomenon: outgoing pulses of amplitude $O(\varepsilon^2)$ and wavelength ε give rise to reflected pulses of amplitude $O(\varepsilon)$, so the overall solution has amplitude $O(\varepsilon)$. Moreover, the reflecting pulses emanate from a radiating pulse that propagates in the boundary along a characteristic of the Lopatinskii determinant.

In the case of $N \times N$ systems considered here, a single outgoing pulse produces on reflection a family of incoming pulses traveling at different group velocities. Unlike wavetrains, pulses do not interact to produce resonances that affect the leading order profiles. However, pulse interactions do affect lower order profiles and so these interactions have to be estimated carefully in the error analysis. Whereas the error analysis in the wavetrain case dealt with small divisor problems by approximating periodic profiles by trigonometric polynomials (which amounts to using a high frequency cutoff), in the pulse case we approximate decaying profiles with nonzero moments by profiles with zero moments (a low frequency cutoff). Unlike the wavetrain case, we are now able to obtain a rate of convergence in the limit describing convergence of approximate to exact solutions.

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1 Introduction

In this paper we study the reflection of pulses in weakly stable semilinear hyperbolic boundary value problems. The problems are weakly stable in the sense that exponentially growing modes are absent, but the uniform Lopatinskii condition fails at a boundary frequency β in the hyperbolic region¹ \mathcal{H} . As a consequence of this degeneracy in the boundary conditions, there is an amplification phenomenon: boundary data of amplitude ε in the problem (1.2) below gives rise to a response of amplitude $O(1)$.

On $\overline{\mathbb{R}}_+^{d+1} = \{x = (x', x_d) = (t, y, x_d) = (t, x'') : x_d \geq 0\}$, consider the $N \times N$ semilinear hyperbolic boundary problem for $v = v_\varepsilon(x)$, where $\varepsilon > 0^2$:

$$(1.1) \quad \begin{aligned} (a) \quad & L(\partial)v + f_0(v) = 0 \\ (b) \quad & \phi(v) = \varepsilon^2 G(x', \frac{x' \cdot \beta}{\varepsilon}) \text{ on } x_d = 0 \\ (c) \quad & v = 0 \text{ and } G = 0 \text{ in } t < 0, \end{aligned}$$

where $L(\partial) := \partial_t + \sum_{j=1}^d B_j \partial_j$, the matrix B_d is invertible, and both $f_0(v)$ and $\phi(v)$ vanish at $v = 0$. The function $G(x', \theta_0)$ is assumed to have polynomial decay in θ_0 , and the frequency $\beta \in \mathbb{R}^d \setminus \{0\}$ is taken to be a boundary frequency at which the uniform Lopatinskii condition fails. A consequence of this failure is that the choice of the factor ε^2 in (1.1)(b) corresponds to the weakly nonlinear regime for this problem. The leading order profile is coupled to the next order profile in the nonlinear system (1.36) derived below.

Before proceeding we write the problem in an equivalent form that is better adapted to the boundary. After multiplying the first equation by $(B_d)^{-1}$ we obtain

$$\begin{aligned} \tilde{L}(\partial)v + f(v) &= 0 \\ \phi(v) &= \varepsilon^2 G(x', \frac{x' \cdot \beta}{\varepsilon}) \text{ on } x_d = 0 \\ v &= 0 \text{ and } G = 0 \text{ in } t < 0, \end{aligned}$$

¹See Definition 1.3 and Assumption 1.5 for precise statements.

²We usually suppress the subscript ε .

where we have set

$$\tilde{L}(\partial) := \partial_d + \sum_{j=0}^{d-1} A_j \partial_j \text{ with } A_j := B_d^{-1} B_j \text{ for } j = 0, \dots, d-1, \quad \partial_0 := \partial_t.$$

Setting $v = \varepsilon u$ and writing $f(v) = D(v)v$, $\phi(v) = \psi(v)v$, we get the problem for $u = u_\varepsilon(x)$

$$(1.2) \quad \begin{aligned} (a) \quad & \tilde{L}(\partial)u + D(\varepsilon u)u = 0 \\ (b) \quad & \psi(\varepsilon u)u = \varepsilon G(x', \frac{x' \cdot \beta}{\varepsilon}) \text{ on } x_d = 0 \\ (c) \quad & u = 0 \text{ in } t < 0. \end{aligned}$$

For this problem we pose the two basic questions of rigorous nonlinear geometric optics:

(1) Does an exact solution u_ε of (1.2) exist for $\varepsilon \in (0, 1]$ on a fixed time interval $[0, T_0]$ independent of ε ?

(2) Suppose the answer to the first question is yes. If we let u_ε^{app} denote an approximate solution on $[0, T_0]$ constructed by the methods of nonlinear geometric optics (that is, solving eikonal equations for phases and suitable transport equations for profiles), how well does u_ε^{app} approximate u_ε for ε small? For example, is it true that

$$(1.3) \quad \lim_{\varepsilon \rightarrow 0} |u_\varepsilon - u_\varepsilon^{app}|_{L^\infty} \rightarrow 0?$$

The amplification phenomenon was studied in a formal way for several different quasilinear problems in [AM87, MA88, MR83]. In [MR83] amplification was studied in connection with Mach stem formation in reacting shock fronts, while [AM87] explored a connection to the formation of instabilities in compressible vortex sheets. Both papers derived equations for profiles using an ansatz that exhibited amplification; however, neither of the two questions posed above were addressed. The first rigorous amplification results were proved in [CG10] for *linear* wavetrains. That paper provided positive answers to the above questions by making use of approximate solutions of high order, and showed in particular that the limit (1.3) holds.

An amplification result for wavetrains in weakly stable *semilinear* problems was proved in [CGW13], and in this paper we prove the analogous result for pulses. Our approximate solution has the form

$$(1.4) \quad u_\varepsilon^{app}(x) = \mathcal{U}^0(x, \theta_0, \xi_d) \Big|_{\theta_0 = \frac{\phi_0}{\varepsilon}, \xi_d = \frac{x_d}{\varepsilon}},$$

where³

$$(1.5) \quad \mathcal{U}^0(x, \theta_0, \xi_d) = \sum_{m \in \mathcal{I}} \sum_{k=1}^{\nu_{k_m}} \sigma_{m,k}(x, \theta_0 + \omega_m \xi_d) r_{m,k}.$$

Here each pulse profile $\sigma_{m,k}(x, \theta)$ is polynomially decaying in $\theta \in \mathbb{R}$, rather than periodic in θ as in the wavetrain case. From the explicit and rather simple equations satisfied by the profiles appearing in \mathcal{U}^0 , we read off the main amplification phenomenon for solutions of (1.2): in $t > 0$ pulses of amplitude $O(1)$, which emanate from a pulse in the boundary that propagates along a

³The notation in (1.4) and (1.5) is explained in sections 1.1 and 1.2.

characteristic of the Lopatinskii determinant, radiate into the interior at distinct group velocities determined by β .

In the case of $N \times N$ systems considered here, a single outgoing pulse produces on reflection a family of incoming pulses traveling at different group velocities. Unlike wavetrains, pulses do not interact to produce resonances that affect the leading order profiles. One effect of this is that we do not need to use Nash-Moser iteration to construct the leading pulse profiles as we did in the wavetrain case. The construction of exact solutions is again based on the study of an associated singular system (1.23). This system is solved by a Nash-Moser iteration argument based on a tame estimate for the linearized singular system (3.6). The proof of that estimate and the Nash-Moser iteration can be carried over *verbatim* from the corresponding argument in [CGW13] for wavetrains⁴. The estimate for the linearized singular system also plays a key role in the error analysis here.

The main differences between the analysis of this paper and that of [CGW13] occur in the study of the profile equations and in the error analysis. Although pulses do not produce resonances, pulse interactions do affect lower order profiles and so these interactions have to be estimated carefully in the error analysis. Unlike the wavetrain case, we are now able to obtain a rate of convergence in the limit (1.3). Whereas the error analysis in the wavetrain case dealt with small divisor problems by approximating periodic profiles by trigonometric polynomials (which amounts to using a high frequency cutoff), in the pulse case we approximate decaying profiles with nonzero moments by profiles with zero moments (using a low frequency cutoff)⁵. We provide a more detailed summary of the argument after stating our main assumptions.

1.1 Assumptions

We make the following hyperbolicity assumption on the system (1.1):

Assumption 1.1. *There exist an integer $q \geq 1$, some real functions $\lambda_1, \dots, \lambda_q$ that are analytic on $\mathbb{R}^d \setminus \{0\}$ and homogeneous of degree 1, and there exist some positive integers ν_1, \dots, ν_q such that:*

$$\forall \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d \setminus \{0\}, \quad \det \left[\tau I + \sum_{j=1}^d \xi_j B_j \right] = \prod_{k=1}^q (\tau + \lambda_k(\xi))^{\nu_k}.$$

Moreover the eigenvalues $\lambda_1(\xi), \dots, \lambda_q(\xi)$ are semi-simple (their algebraic multiplicity equals their geometric multiplicity) and satisfy $\lambda_1(\xi) < \dots < \lambda_q(\xi)$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$.

For simplicity, we restrict our analysis to noncharacteristic boundaries and therefore make the following:

Assumption 1.2. *The matrix B_d is invertible and the matrix $B := \psi(0)$ has maximal rank, its rank p being equal to the number of positive eigenvalues of B_d (counted with their multiplicity). Moreover, the integer p satisfies $1 \leq p \leq N - 1$.*

In the normal modes analysis for (1.2), one first performs a Laplace transform in the time variable t and a Fourier transform in the tangential space variables y . We let $\tau - i\gamma \in \mathbb{C}$ and

⁴Thus, we do not repeat those (lengthy) arguments here.

⁵We borrow the idea of introducing low frequency cutoffs from [AR03] where single phase pulses in free space are studied.

$\eta \in \mathbb{R}^{d-1}$ denote the dual variables of t and y . We introduce the symbol

$$\mathcal{A}(\zeta) := -i B_d^{-1} \left((\tau - i\gamma) I + \sum_{j=1}^{d-1} \eta_j B_j \right), \quad \zeta := (\tau - i\gamma, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1}.$$

For future use, we also define the following sets of frequencies:

$$\begin{aligned} \Xi &:= \left\{ (\tau - i\gamma, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1} \setminus (0, 0) : \gamma \geq 0 \right\}, & \Sigma &:= \left\{ \zeta \in \Xi : \tau^2 + \gamma^2 + |\eta|^2 = 1 \right\}, \\ \Xi_0 &:= \left\{ (\tau, \eta) \in \mathbb{R} \times \mathbb{R}^{d-1} \setminus (0, 0) \right\} = \Xi \cap \{\gamma = 0\}, & \Sigma_0 &:= \Sigma \cap \Xi_0. \end{aligned}$$

Two key objects in our analysis are the hyperbolic region and the glancing set that are defined as follows:

Definition 1.3. • *The hyperbolic region \mathcal{H} is the set of all $(\tau, \eta) \in \Xi_0$ such that the matrix $\mathcal{A}(\tau, \eta)$ is diagonalizable with purely imaginary eigenvalues.*

- *Let \mathbf{G} denote the set of all $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d$ such that $\xi \neq 0$ and there exists an integer $k \in \{1, \dots, q\}$ satisfying:*

$$\tau + \lambda_k(\xi) = \frac{\partial \lambda_k}{\partial \xi_d}(\xi) = 0.$$

If $\pi(\mathbf{G})$ denotes the projection of \mathbf{G} on the d first coordinates (in other words $\pi(\tau, \xi) = (\tau, \xi_1, \dots, \xi_{d-1})$ for all (τ, ξ)), the glancing set \mathcal{G} is $\mathcal{G} := \pi(\mathbf{G}) \subset \Xi_0$.

We recall the following result that is due to Kreiss [Kre70] in the strictly hyperbolic case (when all integers ν_j in Assumption 1.1 equal 1) and to Métivier [Mét00] in our more general framework:

Proposition 1.4 ([Kre70, Mét00]). *Let Assumptions 1.1 and 1.2 be satisfied. Then for all $\zeta \in \Xi \setminus \Xi_0$, the matrix $\mathcal{A}(\zeta)$ has no purely imaginary eigenvalue and its stable subspace $\mathbb{E}^s(\zeta)$ has dimension p . Furthermore, \mathbb{E}^s defines an analytic vector bundle over $\Xi \setminus \Xi_0$ that can be extended as a continuous vector bundle over Ξ .*

For all $(\tau, \eta) \in \Xi_0$, we let $\mathbb{E}^s(\tau, \eta)$ denote the continuous extension of \mathbb{E}^s to the point (τ, η) . The analysis in [Mét00] shows that away from the glancing set $\mathcal{G} \subset \Xi_0$, $\mathbb{E}^s(\zeta)$ depends analytically on ζ , and the hyperbolic region \mathcal{H} does not contain any glancing point.

To treat the case when the boundary operator in (1.2)(b) is independent of u , meaning $\psi(\varepsilon u) \equiv \psi(0) =: B$, we make the following *weak stability assumption* on the problem $(L(\partial), B)$.

Assumption 1.5. • *For all $\zeta \in \Xi \setminus \Xi_0$, $\text{Ker} B \cap \mathbb{E}^s(\zeta) = \{0\}$.*

- *The set $\Upsilon_0 := \{\zeta \in \Sigma_0 : \text{Ker} B \cap \mathbb{E}^s(\zeta) \neq \{0\}\}$ is nonempty and included in the hyperbolic region \mathcal{H} .*
- *For all $\underline{\zeta} \in \Upsilon_0$, there exists a neighborhood \mathcal{V} of $\underline{\zeta}$ in Σ , a real valued \mathcal{C}^∞ function σ defined on \mathcal{V} , a basis $E_1(\zeta), \dots, E_p(\zeta)$ of $\mathbb{E}^s(\zeta)$ that is of class \mathcal{C}^∞ with respect to $\zeta \in \mathcal{V}$, and a matrix $P(\zeta) \in \text{GL}_p(\mathbb{C})$ that is of class \mathcal{C}^∞ with respect to $\zeta \in \mathcal{V}$, such that*

$$\forall \zeta \in \mathcal{V}, \quad B \begin{pmatrix} E_1(\zeta) & \dots & E_p(\zeta) \end{pmatrix} = P(\zeta) \text{diag} (\gamma + i\sigma(\zeta), 1, \dots, 1).$$

For comparison and later reference we recall the following definition.

Definition 1.6 ([Kre70]). *As before let p be the number of positive eigenvalues of B_d . The problem $(\tilde{L}(\partial), B)$ is said to be uniformly stable or to satisfy the uniform Lopatinskii condition if*

$$B : \mathbb{E}^s(\zeta) \longrightarrow \mathbb{C}^p$$

is an isomorphism for all $\zeta \in \Sigma$.

Remark 1.7. Observe that if $(\tilde{L}(\partial), B)$ satisfies the uniform Lopatinskii condition, continuity implies that this condition still holds for $(\tilde{L}(\partial), B + \dot{\psi})$, where $\dot{\psi}$ is any sufficiently small perturbation of B . Hence the uniform Lopatinskii condition is a convenient framework for nonlinear perturbation. The analogous statement may not be true when $(\tilde{L}(\partial), B)$ is only weakly stable. Remarkably, weak stability persists under perturbation in the so-called WR class exhibited in [BGRSZ02], and Assumption 1.5 above is a convenient equivalent definition of the WR class (see [CG10, Appendix B] or [BGS07, Chapter 8]). In order to handle general nonlinear boundary conditions as in (1.2) we shall strengthen Assumption 1.5 in Assumption 1.12 below.

Boundary and interior phases. We consider a planar real phase ϕ_0 defined on the boundary:

$$(1.6) \quad \phi_0(t, y) := \underline{\tau}t + \underline{\eta} \cdot y, \quad (\underline{\tau}, \underline{\eta}) \in \Xi_0.$$

As follows from earlier works, see e.g. [MA88], oscillations or pulses on the boundary associated with the phase ϕ_0 give rise to oscillations or pulses in the interior associated with some planar phases ϕ_m . These phases are characteristic for the hyperbolic operator $L(\partial)$ and their trace on the boundary $\{x_d = 0\}$ equals ϕ_0 . For now we make the following:

Assumption 1.8. *The phase ϕ_0 defined by (1.6) satisfies $(\underline{\tau}, \underline{\eta}) \in \Upsilon_0$. In particular $(\underline{\tau}, \underline{\eta}) \in \mathcal{H}$.*

Thanks to Assumption 1.8, we know that the matrix $\mathcal{A}(\underline{\tau}, \underline{\eta})$ is diagonalizable with purely imaginary eigenvalues. These eigenvalues are denoted $i\omega_1, \dots, i\omega_M$, where the ω_m 's are real and pairwise distinct. The ω_m 's are the roots (and all the roots are real) of the dispersion relation:

$$\det \left[\underline{\tau}I + \sum_{j=1}^{d-1} \underline{\eta}_j B_j + \omega B_d \right] = 0.$$

To each root ω_m there corresponds a unique integer $k_m \in \{1, \dots, q\}$ such that $\underline{\tau} + \lambda_{k_m}(\underline{\eta}, \omega_m) = 0$. We can then define the following real⁶ phases and their associated group velocities:

$$(1.7) \quad \forall m = 1, \dots, M, \quad \phi_m(x) := \phi_0(t, y) + \omega_m x_d, \quad \mathbf{v}_m := \nabla \lambda_{k_m}(\underline{\eta}, \omega_m).$$

Let us observe that each group velocity \mathbf{v}_m is either incoming or outgoing with respect to the space domain \mathbb{R}_+^d : the last coordinate of \mathbf{v}_m is nonzero. This property holds because $(\underline{\tau}, \underline{\eta})$ does not belong to the glancing set \mathcal{G} . We can therefore adopt the following classification:

Definition 1.9. *The phase ϕ_m is incoming if the group velocity \mathbf{v}_m is incoming (that is, when $\partial_{\xi_d} \lambda_{k_m}(\underline{\eta}, \omega_m) > 0$), and it is outgoing if the group velocity \mathbf{v}_m is outgoing ($\partial_{\xi_d} \lambda_{k_m}(\underline{\eta}, \omega_m) < 0$).*

⁶If $(\underline{\tau}, \underline{\eta})$ does not belong to the hyperbolic region \mathcal{H} , some of the phases φ_m may be complex, see e.g. [Her12, Wil00, Les07, Mar10]. Moreover, glancing phases introduce a new scale $\sqrt{\varepsilon}$ as well as boundary layers.

In all that follows, we let \mathcal{I} denote the set of indices $m \in \{1, \dots, M\}$ such that ϕ_m is an incoming phase, and \mathcal{O} denote the set of indices $m \in \{1, \dots, M\}$ such that ϕ_m is an outgoing phase. If $p \geq 1$, then \mathcal{I} is nonempty, while if $p \leq N - 1$, \mathcal{O} is nonempty (see Lemma 1.11 below). We will use the notation:

$$L(\tau, \xi) := \tau I + \sum_{j=1}^d \xi_j B_j, \quad \tilde{L}(\beta, \omega_m) := \omega_m I + \sum_{k=0}^{d-1} \beta_k A_k,$$

$$\beta := (\underline{\tau}, \underline{\eta}), \quad x' = (t, y), \quad \phi_0(x') = \beta \cdot x'.$$

For each phase ϕ_m , $d\phi_m$ denotes the differential of the function ϕ_m with respect to its argument $x = (t, y, x_d)$. It follows from Assumption 1.1 that the eigenspace of $\mathcal{A}(\beta)$ associated with the eigenvalue $i\omega_m$ coincides with the kernel of $L(d\phi_m)$ and has dimension ν_{km} . The following result is a direct consequence of the diagonalizability of $\mathcal{A}(\beta)$, see [CG10] for the proof:

Lemma 1.10. *The space \mathbb{C}^N admits the decomposition:*

$$(1.8) \quad \mathbb{C}^N = \bigoplus_{m=1}^M \ker L(d\phi_m)$$

and each vector space in (1.8) admits a basis of real vectors. If we let P_1, \dots, P_M denote the projectors associated with the decomposition (1.8), then there holds $\text{Im } \tilde{L}(d\phi_m) = \ker P_m$ for all $m = 1, \dots, M$.

The following well-known lemma, whose proof is also recalled in [CG10], gives a useful decomposition of \mathbb{E}^s in the hyperbolic region.

Lemma 1.11. *The stable subspace $\mathbb{E}^s(\beta)$ admits the decomposition:*

$$(1.9) \quad \mathbb{E}^s(\beta) = \bigoplus_{m \in \mathcal{I}} \ker L(d\phi_m),$$

and each vector space in the decomposition (1.9) admits a basis of real vectors.

By Assumption 1.5 we know that the vector space $\ker B \cap \mathbb{E}^s(\beta)$ is one-dimensional; moreover, it admits a real basis because B has real coefficients and $\mathbb{E}^s(\beta)$ has a real basis. This vector space is therefore spanned by a vector $e \in \mathbb{R}^N \setminus \{0\}$ that we can decompose in a unique way by using Lemma 1.11:

$$(1.10) \quad \ker B \cap \mathbb{E}^s(\beta) = \text{Span } \{e\}, \quad e = \sum_{m \in \mathcal{I}} e_m, \quad P_m e_m = e_m.$$

Each vector e_m in (1.10) has real components. We also know that the vector space $B\mathbb{E}^s(\beta)$ is $(p - 1)$ -dimensional. We can therefore write it as the kernel of a real linear form:

$$(1.11) \quad B\mathbb{E}^s(\beta) = \left\{ X \in \mathbb{C}^p, \quad b \cdot X = \sum_{j=1}^p b_j X_j = 0 \right\},$$

for a suitable vector $b \in \mathbb{R}^p \setminus \{0\}$.

To formulate our last assumption we observe first that for every point $\underline{\zeta} \in \mathcal{H}$ there is a neighborhood \mathcal{V} of $\underline{\zeta}$ in Σ and a C^∞ conjugator $Q_0(\zeta)$ defined on \mathcal{V} such that

$$(1.12) \quad Q_0(\zeta)\mathcal{A}(\zeta)Q_0^{-1}(\zeta) = \begin{pmatrix} i\omega_1(\zeta)I_{n_1} & & 0 \\ & \ddots & \\ 0 & & i\omega_J(\zeta)I_{n_J} \end{pmatrix} =: \mathbb{D}_1(\zeta),$$

where the ω_j are real when $\gamma = 0$ and there is a constant $c > 0$ such that either

$$\operatorname{Re}(i\omega_j) \leq -c\gamma \text{ or } \operatorname{Re}(i\omega_j) \geq c\gamma \text{ for all } \zeta \in \mathcal{V}.$$

In view of Lemma 1.11 we can choose the first p columns of $Q_0^{-1}(\zeta)$ to be a basis of $\mathbb{E}^s(\zeta)$, and we write

$$Q_0^{-1}(\zeta) = [Q_{in}(\zeta) \ Q_{out}(\zeta)].$$

Choose J' so that the first J' blocks of \mathbb{D}_1 lie in the first p columns, and the remaining blocks in the remaining $N - p$ columns. Thus, $\operatorname{Re}(i\omega_j) \leq -c\gamma$ for $1 \leq j \leq J'$.

Observing that the linearization of the boundary condition in (1.2) is

$$\dot{u} \mapsto \psi(\varepsilon u)\dot{u} + [d\psi(\varepsilon u) \dot{u}] \varepsilon u,$$

we define the operator

$$(1.13) \quad \mathcal{B}(v_1, v_2) \dot{u} := \psi(v_1)\dot{u} + [d\psi(v_1) \dot{u}] v_2,$$

which appears in Assumption 1.12 below. For later use we also define

$$(1.14) \quad \mathcal{D}(v_1, v_2) \dot{u} := D(v_1)\dot{u} + [dD(v_1) \dot{u}] v_2,$$

as well as

$$(1.15) \quad \mathcal{B}(v_1) := \mathcal{B}(v_1, v_1), \quad \mathcal{D}(v_1) := \mathcal{D}(v_1, v_1).$$

We now state the weak stability assumption that we make when considering the general case of nonlinear boundary conditions in (1.2).

Assumption 1.12. • *There exists a neighborhood \mathcal{O} of $(0, 0) \in \mathbb{R}^{2N}$ such that for all $(v_1, v_2) \in \mathcal{O}$ and all $\zeta \in \Xi \setminus \Xi_0$, $\ker \mathcal{B}(v_1, v_2) \cap \mathbb{E}^s(\zeta) = \{0\}$. For each $(v_1, v_2) \in \mathcal{O}$ the set $\Upsilon(v_1, v_2) := \{\zeta \in \Sigma_0 : \ker \mathcal{B}(v_1, v_2) \cap \mathbb{E}^s(\zeta) \neq \{0\}\}$ is nonempty and included in the hyperbolic region \mathcal{H} . Moreover, if we set $\Upsilon := \cup_{(v_1, v_2) \in \mathcal{O}} \Upsilon(v_1, v_2)$, then $\overline{\Upsilon} \subset \mathcal{H}$ (closure in Σ_0).*

- *For every $\underline{\zeta} \in \overline{\Upsilon}$ there exists a neighborhood \mathcal{V} of $\underline{\zeta}$ in Σ and a C^∞ function $\sigma(v_1, v_2, \zeta)$ on $\mathcal{O} \times \mathcal{V}$ such that for all $(v_1, v_2, \zeta) \in \mathcal{O} \times \mathcal{V}$ we have $\ker \mathcal{B}(v_1, v_2) \cap \mathbb{E}^s(\zeta) \neq \{0\}$ if and only if $\zeta \in \Sigma_0$ and $\sigma(v_1, v_2, \zeta) = 0$.*

Moreover, there exist matrices $P_i(v_1, v_2, \zeta) \in \operatorname{GL}_p(\mathbb{C})$, $i = 1, 2$, of class C^∞ on $\mathcal{O} \times \mathcal{V}$ such that $\forall (v_1, v_2, \zeta) \in \mathcal{O} \times \mathcal{V}$

$$(1.16) \quad P_1(v_1, v_2, \zeta)\mathcal{B}(v_1, v_2)Q_{in}(\zeta)P_2(v_1, v_2, \zeta) = \operatorname{diag}(\gamma + i\sigma(v_1, v_2, \zeta), 1, \dots, 1).$$

For nonlinear boundary conditions, the phase ϕ_0 in (1.6) is assumed to satisfy $(\underline{\tau}, \underline{\eta}) \in \Upsilon(0, 0)$, or in other words the intersection $\ker B \cap \mathbb{E}^s(\underline{\tau}, \underline{\eta})$ is not reduced to $\{0\}$. The set Υ_0 in Assumption 1.5 is a short notation for $\Upsilon(0, 0)$. The phases ϕ_m are still defined by (1.7) and thus only depend on $L(\partial)$ and B and not on the nonlinear perturbations f_0 and $\psi(\varepsilon u) - \psi(0)$ added in (1.2).

Remark 1.13. 1) The properties stated in Assumption 1.12 are just a convenient description of the requirements for belonging to the WR class of [BGRSZ02]. Like the uniform Lopatinskii condition, Assumption 1.12 can in practice be verified by hand via a ‘‘constant-coefficient’’ computation. More precisely, for (v_1, v_2) near $(0, 0) \in \mathbb{R}^{2N}$ and $\zeta \in \Sigma$, one can define (see, e.g., [BGS07, chapter 4]) a Lopatinskii determinant $\Delta(v_1, v_2, \zeta)$ that is C^∞ in (v_1, v_2) , analytic in $\zeta = (\tau - i\gamma, \eta)$ on $\Sigma \setminus \mathcal{G}$, continuous on Σ , and satisfies

$$\Delta(v_1, v_2, \zeta) = 0 \text{ if and only if } \text{Ker } \mathcal{B}(v_1, v_2) \cap \mathbb{E}^s(\zeta) \neq \{0\}.$$

In particular, $\Delta(v_1, v_2, \cdot)$ is real-analytic on \mathcal{H} .

It is shown in Remark 1.13 of [CGW13] that Assumption 1.12 holds provided

$$(1.17) \quad \emptyset \neq \{\zeta \in \Sigma : \Delta(0, 0, \zeta) = 0\} \subset \mathcal{H} \text{ and } \Delta(0, 0, \underline{\zeta}) = 0 \Rightarrow \partial_\tau \Delta(0, 0, \underline{\zeta}) \neq 0,$$

and it thus only involves a weak stability property for the linearized problem at $(v_1, v_2) = (0, 0)$. Instead of assuming (1.17), we have stated Assumption 1.12 in a form that is more directly applicable to the error analysis of Theorem 4.10.

2) To prove the basic estimate for the linearized singular system, Proposition 3.3, and to construct the exact solution U_ε to the singular system (1.23) below, it is enough to require that the analogue of Assumption 1.12 holds when $\mathcal{B}(v_1, v_2)$ is replaced by $\mathcal{B}(v_1)$ given in (1.15). However, for the error analysis of section 4 in the case of nonlinear boundary conditions, we need Assumption 1.12 as stated.

1.2 Main results

For each $m \in \{1, \dots, M\}$ we let

$$r_{m,k}, \quad k = 1, \dots, \nu_{k_m}$$

denote a basis of $\ker L(d\phi_m)$ consisting of real vectors. In section 4 we shall construct a ‘‘corrected’’ approximate solution u_ε^c of (1.2) of the form

$$(1.18) \quad u_\varepsilon^c(x) = \left[\mathcal{U}^0(x, \theta_0, \xi_d) + \varepsilon \mathcal{U}^1(x, \theta_0, \xi_d) + \varepsilon^2 \mathcal{U}_p^2(x, \theta_0, \xi_d) \right] \Big|_{\theta_0 = \frac{\phi_0}{\varepsilon}, \xi_d = \frac{x_d}{\varepsilon}},$$

where

$$(1.19) \quad \begin{aligned} \mathcal{U}^0(x, \theta_0, \xi_d) &= \sum_{m \in \mathcal{I}} \sum_{k=1}^{\nu_{k_m}} \sigma_{m,k}(x, \theta_0 + \omega_m \xi_d) r_{m,k}, \\ \mathcal{U}^1(x, \theta_0, \xi_d) &= \sum_{m \in \mathcal{I}} \sum_{k=1}^{\nu_{k_m}} \tau_{m,k}(x, \theta_0 + \omega_m \xi_d) r_{m,k} - \mathbf{R}_\infty \left(\tilde{L}(\partial) \mathcal{U}^0 + D(0) \mathcal{U}^0 \right), \end{aligned}$$

and \mathbf{R}_∞ is the integral operator defined in (1.32). The $\sigma_{m,k}$ ’s and $\tau_{m,k}$ ’s are scalar functions decaying in $\theta_m := \theta_0 + \omega_m \xi_d$. The $\sigma_{m,k}$ ’s satisfy

$$\int_{\mathbb{R}} \sigma_{m,k}(x, \theta) d\theta = 0, \quad (\text{moment zero})$$

and describe the propagation of oscillations with phase ϕ_m and group velocity \mathbf{v}_m . The second corrector $\varepsilon^2 \mathcal{U}_p^2(x, \theta_0, \xi_d)$ in (1.18) is a more complex object constructed in the error analysis of section 4.

The following theorem, our main result, is an immediate corollary of the more precise Theorem 4.10. Here we let $\Omega_T := \{(x, \theta_0) = (t, y, x_d, \theta_0) \in \mathbb{R}^{d+1} \times \mathbb{R}^1 : x_d \geq 0, t < T\}$, and $b\Omega_T := \{(t, y, \theta_0) \in \mathbb{R}^d \times \mathbb{R}^1 : t < T\}$. The spaces E_T^s appearing in the theorem are

$$(1.20) \quad E_T^s := C(\mathbb{R}_{x_d}, H^s(b\Omega_T)) \cap L^2(\mathbb{R}_{x_d}, H^{s+1}(b\Omega_T)),$$

while the Sobolev spaces with θ -weights Γ_T^s and $b\Gamma_T^s$ are defined in Definition 2.6. With $[r]$ denoting the smallest integer greater than r , we define

$$(1.21) \quad a_0 := \left\lceil \frac{d+1}{2} \right\rceil, \quad a_1 := \left\lceil \frac{d+1}{2} \right\rceil + M_0, \quad a := \max(2a_0 + 3, a_1 + 1), \quad \tilde{a} := 2a - a_0,$$

where $M_0 := 3d + 5$ is fixed according to the regularity requirements in the singular pseudodifferential calculus (see Appendix B).

Theorem 1.14. *We make Assumptions 1.1, 1.2, 1.5, and 1.8 when the boundary condition in (1.2) is linear ($\psi(\varepsilon u) \equiv \psi(0)$); in the general case we substitute Assumption 1.12 for Assumption 1.5. Fix $T > 0$, set $M_0 := 3d + 5$, and let a and \tilde{a} be as in (1.21).*

Consider the semilinear boundary problem (1.2), where $G \in b\Gamma_T^{\tilde{a}}$. There exists $\varepsilon_0 > 0$ such that if $\langle G \rangle_{b\Gamma_T^{a+2}}$ is small enough, there exists a unique function $U_\varepsilon(x, \theta_0) \in E_T^{a-1}$ satisfying the singular system (1.23) on Ω_T such that

$$u_\varepsilon(x) := U_\varepsilon \left(x, \frac{\phi_0}{\varepsilon} \right)$$

is an exact solution of (1.2) on $(-\infty, T] \times \overline{\mathbb{R}}_+^d$ for $0 < \varepsilon \leq \varepsilon_0$. In addition there exists a profile $\mathcal{U}^0(x, \theta_0, \xi_d)$ as in (1.19), whose components $\sigma_{m,k}$ lie in Γ_T^{a+1} , such that the approximate solution defined by

$$u_\varepsilon^{app} := \mathcal{U}^0 \left(x, \frac{\phi_0}{\varepsilon}, \frac{x_d}{\varepsilon} \right)$$

satisfies

$$\lim_{\varepsilon \rightarrow 0} |u_\varepsilon - u_\varepsilon^{app}|_{L^\infty} = 0 \quad \text{on } (-\infty, T] \times \overline{\mathbb{R}}_+^d.$$

In fact we have the rate of convergence

$$|u_\varepsilon - u_\varepsilon^{app}|_{L^\infty} \leq C \varepsilon^{\frac{1}{2M_1+3}}, \quad \text{where } M_1 := \left\lceil \frac{d}{2} + 3 \right\rceil.$$

Observe that although the boundary data in the problem (1.2) is of size $O(\varepsilon)$, the approximate solution u_ε^{app} is of size $O(1)$, exhibiting an amplification due to the weak stability at frequency β . The main information provided by Theorem 1.14 is that this amplification does not rule out existence of a smooth solution on a fixed time interval, that is it does not trigger a violent instability, at least in this weakly nonlinear regime. Although the first and second correctors \mathcal{U}^1 and \mathcal{U}_p^2 do not appear in the statement of Theorem 1.14, they are essential in the error analysis. As an application of Theorem 1.14, an easy analysis of the equations for the leading profile \mathcal{U}^0 (see (2.11) or (5.2)) gives the qualitative information about the exact solution described below in Remark 1.15(a).

Remark 1.15. a) In order to avoid some technicalities we have stated our main result for the problem (1.2) where all data is 0 in $t < 0$. This result easily implies a similar result in which outgoing pulses defined in $t < 0$ of amplitude $O(\varepsilon)$ and wavelength ε give rise to reflected pulses of amplitude $O(1)$. In either formulation, inspection of the profile equations shows that the pulses of amplitude $O(1)$ emanate from a radiating pulse that propagates in the boundary along a characteristic of the Lopatinskii determinant.

b) We have decided to fix $T > 0$ at the start and choose data small enough so that a solution to the nonlinear problem exists up to time T . With only minor changes one can fix the data in the problem (G in (1.2)) at the start, and then choose T small enough so that a solution to the nonlinear problem exists up to time T .

In the remainder of this introduction, we discuss the construction of exact solutions, the construction of the approximate solution U^0 , and the error analysis. The proofs are given in Sections 2, 3, and 4. We hope that the analysis developed in this article will be useful in justifying quasilinear amplification phenomena such as the Mach stems or kink modes formation of [MR83, AM87].

1.3 Exact solutions and singular systems

The theory of weakly stable hyperbolic initial boundary value problems fails to provide a solution of the system (1.2) that exists on a fixed time interval independent of ε^7 . In order to obtain such an exact solution to the system (1.2) we adopt the strategy of studying an associated singular problem first used by [JMR95] for an initial value problem in free space. We look for a solution of the form

$$(1.22) \quad u_\varepsilon(x) = U_\varepsilon(x, \theta_0)|_{\theta_0 = \frac{\phi_0(x')}{\varepsilon}},$$

where $U_\varepsilon(x, \theta_0)$ satisfies the singular system derived by substituting (1.22) into the problem (1.2). Recalling that $\tilde{L}(\partial) = \partial_d + \sum_{j=0}^{d-1} A_j \partial_j$, we obtain:

$$(1.23) \quad \begin{aligned} \partial_d U_\varepsilon + \sum_{j=0}^{d-1} A_j \left(\partial_j + \frac{\beta_j \partial_{\theta_0}}{\varepsilon} \right) U_\varepsilon + D(\varepsilon U_\varepsilon) U_\varepsilon &=: \\ \partial_d U_\varepsilon + \mathbb{A} \left(\partial_{x'} + \frac{\beta \partial_{\theta_0}}{\varepsilon} \right) U_\varepsilon + D(\varepsilon U_\varepsilon) U_\varepsilon &= 0 \\ \psi(\varepsilon U_\varepsilon) U_\varepsilon|_{x_d=0} &= \varepsilon G(x', \theta_0), \\ U_\varepsilon &= 0 \text{ in } t < 0. \end{aligned}$$

In the wavetrain case the function $U_\varepsilon(x, \theta_0)$ is taken to be periodic in θ_0 , but in the case of pulses the variable $\theta_0 \in \mathbb{R}$ is treated as just another tangential variable on an equal footing with x' . The special difficulties presented by such singular problems when there is a boundary are described in detail in the introductions to [Wil02] and [CGW11]; these difficulties are present for both wavetrains and pulses. In particular we mention: (a) symmetry assumptions on the matrices B_j appearing in the problem (1.1) equivalent to (1.2) are generally of no help in obtaining an L^2 estimate for (1.23) (boundary conditions satisfying Assumption 1.5 can not be maximally dissipative, see [CG10]); (b) one cannot control L^∞ norms just by estimating tangential derivatives $\partial_{(x', \theta_0)}^\alpha U_\varepsilon$ because (1.23) is not a hyperbolic problem in the x_d direction⁸; moreover, even if one has estimates of tangential

⁷This would be true even for problems $(\tilde{L}(\partial), B)$ that are uniformly stable in the sense of Definition 1.6.

⁸For initial value problems in free space, one *can* control L^∞ norms just by estimating enough derivatives tangent to time slices $t = c$.

derivatives uniform with respect to ε , because of the factors $1/\varepsilon$ in (1.23) one cannot just use the equation to control $\partial_d U_\varepsilon$ and thereby control L^∞ norms.

In [CGW12] a class of singular pseudodifferential operators, acting on functions $U(x', \theta_0)$ and having the form

$$(1.24) \quad p_D U(x', \theta_0) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^d \times \mathbb{R}} e^{ix' \cdot \xi' + i\theta_0 k} p\left(\varepsilon V(x', \theta_0), \xi' + \frac{k\beta}{\varepsilon}, \gamma\right) \widehat{U}(\xi', k) d\xi' dk, \quad \gamma \geq 1,$$

was introduced to deal with these difficulties. Observe that the differential operator \mathbb{A} appearing in (1.23) can be expressed in this form. Operators like (1.24), but with the integral over k replaced by a sum over $k \in \mathbb{Z}$, were first used in [Wil02] to prove uniform estimates for quasilinear singular systems arising in the uniformly stable wavetrain case. The calculi of [CGW12] are inspired by the calculus of [Wil02], but incorporate improvements needed for the study of amplification in both the wavetrain and pulse cases.

Under the weak stability assumptions of the present paper, Assumption 1.5 or Assumption 1.12, the basic L^2 estimate for the problem $(\tilde{L}(\partial), B)$ and for its singular analogue both exhibit loss of one tangential derivative. To derive energy estimates with a loss of derivative, one needs the remainder in the composition of two zero-order singular operators to be a smoothing operator, and not just bounded on L^2 with small norm as in [Wil02]. The calculi constructed in [CGW12] have this smoothing property. The improved wavetrain calculus was used in [CGW13] to study amplification of wavetrains. As noted in that paper, the arguments used to prove the main L^2 estimate for the linearized singular problem and the higher order tame estimates carry over *verbatim* to the pulse case. The Nash-Moser iteration used in [CGW13] to construct an exact solution $U_\varepsilon(x, \theta_0)$ of (1.23) on Ω_T also carries over verbatim to the pulse case. In section 3 we state the main estimates for the linearized singular problem in the pulse case along with the existence theorem for the nonlinear singular system (1.23), and refer to [CGW13] for the proofs. The tame estimate for the linearized singular problem is given in terms of E_T^s spaces (recall (1.20)), and these spaces are used as well in the Nash-Moser iteration. The singular pulse calculus summarized in Appendix B must, of course, be substituted for the wavetrain calculus used in [CGW13].

1.4 Derivation of the profile equations

At first we work formally, looking for a corrected approximate solution of the form

$$u_\varepsilon^c(x) = \left[\mathcal{U}^0(x, \theta_0, \xi_d) + \varepsilon \mathcal{U}^1(x, \theta_0, \xi_d) + \varepsilon^2 \mathcal{U}^2(x, \theta_0, \xi_d) \right] \Big|_{\theta_0 = \frac{\phi_0}{\varepsilon}, \xi_d = \frac{x_d}{\varepsilon}}.$$

Plugging into the system (1.2) and setting coefficients of successive powers of ε equal to zero, we obtain interior profile equations

$$(1.25) \quad \begin{aligned} \varepsilon^{-1} : \quad & \tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d}) \mathcal{U}^0 = 0 \\ \varepsilon^0 : \quad & \tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d}) \mathcal{U}^1 + \tilde{L}(\partial) \mathcal{U}^0 + D(0) \mathcal{U}^0 = 0 \\ \varepsilon^1 : \quad & \tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d}) \mathcal{U}^2 + \tilde{L}(\partial) \mathcal{U}^1 + D(0) \mathcal{U}^1 + (dD(0) \mathcal{U}^0) \mathcal{U}^0 = 0, \end{aligned}$$

where

$$\tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d}) := \partial_{\xi_d} + \sum_{j=0}^{d-1} \beta_j A_j \partial_{\theta_0} = \partial_{\xi_d} + i\mathcal{A}(\beta) \partial_{\theta_0}.$$

Similarly, we obtain the boundary equations on $\{x_d = 0, \xi_d = 0\}$,

$$(1.26) \quad \begin{aligned} \varepsilon^0 : B\mathcal{U}^0 &= 0 \\ \varepsilon^1 : B\mathcal{U}^1 + (d\psi(0)\mathcal{U}^0)\mathcal{U}^0 &= G(x', \theta_0). \end{aligned}$$

For ease of exposition we will begin by studying these equations in the 3×3 strictly hyperbolic case, which contains all the main difficulties. In section 5 we describe the (minor) changes needed to treat the general case. We define the boundary phase $\phi_0 := \beta \cdot x'$, eigenvalues ω_j , and interior phases $\phi_j := \phi_0 + \omega_j x_d$ as in Section 1.1, where $\beta \in \mathcal{H}$. We suppose ω_1 and ω_3 are incoming (or causal) and ω_2 is outgoing. The corresponding right and left eigenvectors of $i\mathcal{A}(\beta)$ are r_j and l_j , $j = 1, 2, 3$, normalized so that $l_j \cdot r_k = \delta_{jk}$.

A general function $W(x, \theta_0, \xi_d)$ valued in \mathbb{R}^3 can be written

$$W = W_1(x, \theta_0, \xi_d)r_1 + W_2(x, \theta_0, \xi_d)r_2 + W_3(x, \theta_0, \xi_d)r_3$$

where we refer to W_i , $i = 1, 3$ as the *incoming* components and W_2 as the *outgoing* component of W . With $F(x, \theta_0, \xi_d) = \sum_{i=1}^3 F_i(x, \theta_0, \xi_d)r_i$ valued in \mathbb{R}^3 consider now an equation of the form

$$\tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d})W = F.$$

Using the matrix $[r_1 \ r_2 \ r_3]$ to diagonalize $i\mathcal{A}(\beta)$, we find that the W_i must satisfy

$$(1.27) \quad (\partial_{\xi_d} - \omega_i \partial_{\theta_0})W_i(x, \theta_0, \xi_d) = l_i \cdot F := F_i(x, \theta_0, \xi_d), \quad i = 1, 2, 3.$$

The general solution to (1.27) is

$$(1.28) \quad W_i(x, \theta_0, \xi_d) = \tau_i^*(x, \theta_0 + \omega_i \xi_d) + \int_0^{\xi_d} F_i(x, \theta_0 + \omega_i(\xi_d - s), s) ds,$$

where τ_i^* is arbitrary. This can be rewritten

$$(1.29) \quad \begin{aligned} W_i(x, \theta_0, \xi_d) &= \tau_i^*(x, \theta_0 + \omega_i \xi_d) + \int_{\mathbb{R}^+} F_i(x, \theta_0 + \omega_i(\xi_d - s), s) ds - \int_{\xi_d}^{+\infty} F_i(x, \theta_0 + \omega_i(\xi_d - s), s) ds \\ &:= \tau_i(x, \theta_0 + \omega_i \xi_d) - \int_{\xi_d}^{+\infty} F_i(x, \theta_0 + \omega_i(\xi_d - s), s) ds, \end{aligned}$$

provided the integrals in (1.29) exist.

Definition 1.16 (Functions of type \mathcal{F}). *Suppose*

$$(1.30) \quad F(x, \theta_0, \xi_d) = \sum_{i=1}^3 F_i(x, \theta_0, \xi_d)r_i,$$

where each F_i has the form

$$(1.31) \quad F_i(x, \theta_0, \xi_d) = \sum_{k=1}^3 f_k^i(x, \theta_0 + \omega_k \xi_d) + \sum_{l \leq m=1}^3 g_{l,m}^i(x, \theta_0 + \omega_l \xi_d) h_{l,m}^i(x, \theta_0 + \omega_m \xi_d),$$

where the functions $f_k^i(x, \theta)$, $g_{l,m}^i(x, \theta)$, $h_{l,m}^i(x, \theta)$ are real-valued, C^1 , and decay along with their first order partials at the rate $O(\langle \theta \rangle^{-2})$ uniformly with respect to x . We then say that F is of type \mathcal{F} . For such functions F , define

$$\mathbf{E}F(x, \theta_0, \xi_d) := \sum_{j=1}^3 \left(\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T l_j \cdot F(x, \theta_0 + \omega_j(\xi_d - s), s) ds \right) r_j.$$

Remark 1.17. For F as in (1.31), we have

$$\mathbf{E}F = \sum_{i=1}^3 \tilde{F}_i(x, \theta_0, \xi_d) r_i, \text{ where } \tilde{F}_i := f_i^i(x, \theta_0 + \omega_i \xi_d) + g_{i,i}^i(x, \theta_0 + \omega_i \xi_d) h_{i,i}^i(x, \theta_0 + \omega_i \xi_d).$$

Remark 1.18. The definition of \mathbf{E} can be extended to more general functions. For example, if

$$F = \sum_{i=1}^3 F_i(x, \theta_0 + \omega_i \xi_d) r_i,$$

where the $F_i(x, \theta)$ are arbitrary continuous functions (not necessarily decaying with respect to θ), the limits that define $\mathbf{E}F$ exist and we have $\mathbf{E}F = F$.

For another example, suppose F is of type \mathcal{F} and satisfies $\mathbf{E}F = 0$. Define

$$(1.32) \quad \mathbf{R}_\infty F(x, \theta_0, \xi_d) := - \sum_{i=1}^3 \left(\int_{\xi_d}^{+\infty} F_i(x, \theta_0 + \omega_i(\xi_d - s), s) ds \right) r_i.$$

Then the limits defining $\mathbf{R}_\infty F$ and $\mathbf{E}\mathbf{R}_\infty F$ exist and we have $\mathbf{E}\mathbf{R}_\infty F = 0$.

The following two results are proved in [CW13].

Proposition 1.19 ([CW13]). *Suppose F is of type \mathcal{F} and $\mathbf{E}F = 0$. Then $\mathbf{R}_\infty F$ is bounded and*

$$(1.33) \quad \tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d}) \mathbf{R}_\infty F = \mathbf{R}_\infty \tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d}) F = F = (I - \mathbf{E})F.$$

Proposition 1.20 ([CW13]). *Let $F(x, \theta_0, \xi_d)$ be a function of type \mathcal{F} . (a) Then the equation $\tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d}) \mathcal{U} = F$ has a solution bounded in (θ_0, ξ_d) if and only if $\mathbf{E}F = 0$.*

(b) When $\mathbf{E}F = 0$, every C^1 solution bounded in (θ_0, ξ_d) has the form

$$\mathcal{U} = \sum_{i=1}^3 \tau_i(x, \theta_0 + \omega_i \xi_d) r_i + \mathbf{R}_\infty F \text{ with } \tau_i(x, \theta) \in C^1 \text{ and bounded.}$$

Here $\mathbf{E}\mathcal{U} = \sum_{i=1}^3 \tau_i(x, \theta_0 + \omega_i \xi_d) r_i$ and $(I - \mathbf{E})\mathcal{U} = \mathbf{R}_\infty F$.

(c) If \mathcal{U} is of type \mathcal{F} then

$$(1.34) \quad \mathbf{E}\tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d}) \mathcal{U} = \tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d}) \mathbf{E}\mathcal{U} = 0.$$

By applying the operators \mathbf{E} and \mathbf{R}_∞ to the equations (1.25) formally (for now) using the properties (1.33), (1.34), we obtain:

$$(1.35) \quad \begin{aligned} (a) \quad & \mathbf{E}\mathcal{U}^0 = \mathcal{U}^0, \\ (b) \quad & \mathbf{E}(\tilde{\mathcal{L}}(\partial)\mathcal{U}^0 + D(0)\mathcal{U}^0) = 0, \\ (c) \quad & B\mathcal{U}^0 = 0 \text{ on } x_d = 0, \xi_d = 0, \\ (d) \quad & \mathcal{U}^0 = 0 \text{ in } t < 0 \end{aligned}$$

and

$$\begin{aligned}
(1.36) \quad & (a) (I - \mathbf{E})\mathcal{U}^1 + \mathbf{R}_\infty(\tilde{L}(\partial)\mathcal{U}^0 + D(0)\mathcal{U}^0) = 0, \\
& (b) \mathbf{E} \left(\tilde{L}(\partial)\mathcal{U}^1 + D(0)\mathcal{U}^1 + (dD(0)\mathcal{U}^0)\mathcal{U}^0 \right) = 0, \\
& (c) B\mathbf{E}\mathcal{U}^1 = G - (d\psi(0)\mathcal{U}^0)\mathcal{U}^0 - B(I - \mathbf{E})\mathcal{U}^1 \text{ on } x_d = 0, \xi_d = 0, \\
& (d) \mathcal{U}^1 = 0 \text{ in } t < 0.
\end{aligned}$$

To construct \mathcal{U}^1 we write $\mathcal{U}^1 = \mathbf{E}\mathcal{U}^1 + (I - \mathbf{E})\mathcal{U}^1$ where

$$(1.37) \quad (I - \mathbf{E})\mathcal{U}^1 = -\mathbf{R}_\infty(\tilde{L}(\partial)\mathcal{U}^0 + D(0)\mathcal{U}^0) = \mathbf{R}_\infty F^0,$$

and $\mathbf{E}\mathcal{U}^1$ satisfies

$$\begin{aligned}
(1.38) \quad & (a) \mathbf{E}(\tilde{L}(\partial) + D(0))\mathbf{E}\mathcal{U}^1 = -\mathbf{E}((dD(0)\mathcal{U}^0)\mathcal{U}^0) - \mathbf{E}(\tilde{L}(\partial) + D(0))\mathbf{R}_\infty F^0, \\
& (b) B\mathbf{E}\mathcal{U}^1 = G - (d\psi(0)\mathcal{U}^0)\mathcal{U}^0 - \mathbf{R}_\infty F^0 \text{ on } x_d = 0, \xi_d = 0 \\
& (c) \mathbf{E}\mathcal{U}^1 = 0 \text{ in } t < 0.
\end{aligned}$$

1.5 Error analysis

The error analysis is done by studying the singular problem (4.17) satisfied by the difference

$$W_\varepsilon(x, \theta_0) = U_\varepsilon(x, \theta_0) - \mathcal{U}_\varepsilon(x, \theta_0),$$

where U_ε is the exact solution of the semilinear singular problem (1.23) and \mathcal{U}_ε is a corrected approximate solution of the form

$$\mathcal{U}_\varepsilon(x, \theta_0) = \mathcal{U}^0(x, \theta_0, \frac{x_d}{\varepsilon}) + \varepsilon \mathcal{U}^1(x, \theta_0, \frac{x_d}{\varepsilon}) + \varepsilon^2 \mathcal{U}_p^2(x, \theta_0, \frac{x_d}{\varepsilon}).$$

Here \mathcal{U}^0 and \mathcal{U}^1 have the forms given in (1.19) and satisfy the profile equations (1.35), (1.36), and the second corrector \mathcal{U}_p^2 remains to be chosen.

Let us denote $\mathcal{U}_\varepsilon^0(x, \theta_0) := \mathcal{U}^0(x, \theta_0, \frac{x_d}{\varepsilon})$ and similarly define $\mathcal{U}_\varepsilon^1, \mathcal{U}_{p,\varepsilon}^2$. The idea is to apply the tame estimate of Proposition 3.6 to the linear singular problem (4.17) satisfied by W_ε . To apply this estimate we need the functions $\mathcal{U}_\varepsilon^0, \mathcal{U}_\varepsilon^1$, and $\mathcal{U}_{p,\varepsilon}^2$ to lie in E_T^s spaces for appropriate choices of s . Thus, roughly speaking, these functions must decay in (x', θ_0) “like functions in $H^s(x', \theta_0)$ ”.

The pulse profiles $\sigma_{m,k}(x, \theta_m)$ defining $\mathcal{U}^0(x, \theta_0, \xi_d)$ have good decay in θ_m , so it is not hard to see that $\mathcal{U}_\varepsilon^0$ lies in an E_T^s space (Proposition 4.3). Moreover, we show that the $\sigma_{m,k}(x, \theta_m)$ actually have moment zero, and this can be used to show that $\mathcal{U}_\varepsilon^1$, whose definition involves taking integrals as in (1.32) of an expression that is linear in \mathcal{U}^0 , also lies in an E_T^s space (see, for example, Lemma 2.9 and Proposition 2.10). If the $\sigma_{m,k}(x, \theta_m)$ did not have moment zero, $\mathcal{U}_\varepsilon^1$ would fail to lie in any E_T^s space.

However, the second corrector presents greater difficulties. Considering the ε^1 profile equation in (1.25), we see that a natural first choice for the second corrector would be

$$(1.39) \quad \mathcal{U}^2 = -\mathbf{R}_\infty(I - \mathbf{E}) \left(\tilde{L}(\partial)\mathcal{U}^1 + D(0)\mathcal{U}^1 + (dD(0)\mathcal{U}^0)\mathcal{U}^0 \right).$$

Since the primitive $-\int_\theta^{+\infty} f(s)ds$ of a function f that decays, say, like $|s|^{-2}$ as $|s| \rightarrow +\infty$, itself decays to zero as $|\theta| \rightarrow +\infty$ if and only if f has moment zero ($\int_{\mathbb{R}} f(s) ds = 0$), and since neither

\mathcal{U}^1 nor the term $(dD(0)\mathcal{U}^0)\mathcal{U}^0$ in (1.39) has moment zero⁹, we see that this choice of $\mathcal{U}_\varepsilon^2$ generally cannot lie in any E_T^s space. A natural idea then is to replace \mathcal{U}^0 and \mathcal{U}^1 by approximations that involve profiles having moment zero. More precisely, we take for example \mathcal{U}_p^0 to be as in (1.19), but where the profile $\sigma_{m,k}(x, \theta_m)$ is replaced by the moment-zero approximation $\sigma_{m,k,p}(x, \theta_m)$ whose Fourier transform in θ_m is defined by

$$\hat{\sigma}_{m,k,p}(x, \eta) = \chi_p(\eta)\hat{\sigma}_{m,k}(x, \eta).$$

Here $\chi_p(\eta)$ is a smooth cutoff function vanishing on a neighborhood of 0 of size $O(|p|)$ and equal to one outside a slightly larger neighborhood¹⁰. In place of (1.39) one could then try

$$(1.40) \quad \mathcal{U}_p^2 = -\mathbf{R}_\infty(I - \mathbf{E}) \left(\tilde{L}(\partial)\mathcal{U}_p^1 + D(0)\mathcal{U}_p^1 + (dD(0)\mathcal{U}_p^0)\mathcal{U}_p^0 \right).$$

The contributions to $\mathcal{U}_{p,\varepsilon}^2$ involving \mathcal{U}_p^1 now do lie in suitable E_T^s spaces, but some of the contributions from the quadratic term do not lie in any E_T^s space. The difficulty is caused by products of the form $\sigma_{m,k,p}\sigma_{m',k',p}$, and the resolution is to redefine \mathcal{U}_p^2 as in (1.40) after replacing these terms by $(\sigma_{m,k,p}\sigma_{m',k',p})_p$, see (4.10). We note that the contributions to $\mathcal{U}_{p,\varepsilon}^2$ from “transversal interactions” like $\sigma_{m,k,p}\sigma_{m',k',p}$, where $m \neq m'$ do already lie in appropriate E_T^s spaces, see Proposition A.1.

These moment-zero approximations introduce errors that blow up as $p \rightarrow 0$, of course, but taking $p = \varepsilon^b$ for an appropriate $b > 0$, one can hope to control these errors using the factor ε^2 in $\varepsilon^2\mathcal{U}_p^2$. This strategy works and is carried out in the proof of Theorem 4.10 in section 4.2. The proof relies on the machinery of moment-zero approximations developed in section 4.1 and the quadratic interaction estimates of Appendix A.

Remark 1.21. 1) We learned the idea of using moment-zero approximations to construct correctors in pulse problems from the paper [AR03].

2) Together with [CW13], which dealt with reflecting pulses in quasilinear uniformly stable boundary problems, this paper completes the first stage of our project to rigorously justify, when that is possible, the formal constructions of [HMR86, AM87, MA88, MR83] in boundary problems involving multiple interacting pulses. The operators \mathbf{E} , \mathbf{R}_∞ , and the machinery of moment-zero approximations developed in these papers provide a set of tools for rigorously constructing leading profiles and correctors. The estimates for weakly stable singular systems (1.23) given in [CGW13]¹¹ and for uniformly stable quasilinear singular systems in [CW13] provide the basis for showing that approximate solutions are close to exact solutions for ε small. The approach to error analysis based on singular systems is especially well-suited to situations involving several pulses traveling at distinct group velocities. In these situations “one-phase methods” that depend on constructing high order approximate solutions, or which use estimates of conormal or striated type, seem inapplicable. Finally, we stress that pulses do interact; the interactions do not produce resonances that affect leading order profiles as in the wavetrain case, but the interactions, whether of transversal or self-interaction type, do affect correctors to the leading order profiles. We have treated the analogous problems for wavetrains in [CGW11, CGW13].

⁹More precisely, we refer here to the profiles like $\tau_{m,k}(x, \theta_m)$ or products of profiles that appear in these terms.

¹⁰The cutoff renders harmless the small divisor that appears when one writes the Fourier transform of the θ_m -primitive of $\sigma_{m,k,p}$ in terms of $\hat{\sigma}_{m,k,p}(x, \eta)$.

¹¹Recall that the wavetrain estimate in [CGW13] applies verbatim to the pulse case, as long as one uses the singular pulse calculus instead of the singular wavetrain calculus in the proof. Both calculi are constructed in [CGW12].

2 Analysis of profile equations

For ease of exposition we continue as in Section 1.4 to consider the 3×3 strictly hyperbolic case. The minor changes needed to treat the general case are described in section 5.

2.1 The key subsystem for \mathcal{U}^0

Let us write \mathcal{U}^0 , the solution of the ε^{-1} equation in (1.25), as

$$\mathcal{U}^0 = \tilde{\sigma}_1(x, \theta_0, \xi_d) r_1 + \tilde{\sigma}_2(x, \theta_0, \xi_d) r_2 + \tilde{\sigma}_3(x, \theta_0, \xi_d) r_3.$$

As we saw in (1.28), the $\tilde{\sigma}_i$'s have the form

$$\tilde{\sigma}_i(x, \theta_0, \xi_d) = \sigma_i(x, \theta_0 + \omega_i \xi_d) \quad \text{for some } \sigma_i(x, \theta_i).$$

Recall that we assume that the phase ϕ_2 is outgoing while the other two phases ϕ_1, ϕ_3 are incoming (the case where only one phase is incoming and all other phases are outgoing is actually simpler). Our first task is to show that $\sigma_2 \equiv 0$. We will need the following version of a classical lemma by Lax [Lax57], see [CGW11, Lemma 2.11] for the proof.

Proposition 2.1. *Let $W(x, \theta_0, \xi_d) = \sum_{i=1}^3 w_i(x, \theta_0, \xi_d) r_i$ be any C^1 function. Then*

$$\tilde{L}(\partial)W = \sum_{i=1}^3 (X_{\phi_i} w_i) r_i + \sum_{i=1}^3 \left(\sum_{k \neq i} V_k^i w_k \right) r_i,$$

where X_{ϕ_i} is the characteristic vector field

$$X_{\phi_i} := \partial_{x_d} + \sum_{j=0}^{d-1} -\partial_{\xi_j} \omega_i(\beta) \partial_{x_j} = \frac{1}{\mathbf{v}_i \cdot \mathbf{e}_d} (\partial_t + \mathbf{v}_i \cdot \nabla_x),$$

and V_k^i , for $k \neq i$, is the tangential vector field

$$V_k^i := \sum_{l=0}^{d-1} (l_i A_l r_k) \partial_{x_l}.$$

The relation (1.35)(b) can be simplified by using Proposition 2.1. More precisely, we recall the notation $F^0 := -(\tilde{L}(\partial)\mathcal{U}^0 + D(0)\mathcal{U}^0)$, see (1.37). The next Proposition is an immediate consequence of Proposition 2.1.

Proposition 2.2. *The function F^0 reads $F^0 = \sum_{i=1}^3 F_i r_i$, with*

$$F_i(x, \theta_0, \xi_d) = -X_{\phi_i} \sigma_i - \sum_{k \neq i} V_k^i \sigma_k + \sum_{k=1}^3 e_k^i \sigma_k,$$

for some real constants e_k^i and vector fields V_k^i as in Proposition 2.1.

In particular, the requirement $\mathbf{E}F^0 = 0$ in (1.35)(b) reads

$$(2.1) \quad X_{\phi}\sigma_i - e_i^i \sigma_i = 0, \quad i = 1, 2, 3.$$

Since the outgoing mode σ_2 is zero in $t < 0$, this implies $\sigma_2 \equiv 0$ by integrating along the characteristics.

Using (1.26) (ε^0) and $\sigma_2 = 0$, we find the boundary condition

$$B(\sigma_1(x', 0, \theta_0) r_1 + \sigma_3(x', 0, \theta_0) r_3) = 0,$$

and we recall that the vectors r_1, r_3 span the stable subspace $\mathbb{E}^s(\beta)$, see Lemma 1.11. Thus, by (1.10) we have

$$\sigma_1(x', 0, \theta_0) r_1 + \sigma_3(x', 0, \theta_0) r_3 = a(x', \theta_0) e, \quad \text{for some scalar function } a(x', \theta_0).$$

Since $e = e_1 + e_3$, where $e_i \in \text{span}\{r_i\}$, we deduce

$$(2.2) \quad \sigma_i(x', 0, \theta_0) r_i = a(x', \theta_0) e_i, \quad i = 1, 3.$$

Using (2.1) and (2.2), we have derived the main subsystem governing \mathcal{U}^0 :

$$(2.3) \quad \begin{aligned} X_{\phi_i}\sigma_i - e_i^i \sigma_i &= 0 \text{ for } i = 1, 3 \\ \sigma_i(x', 0, \theta) r_i &= a(x', \theta) e_i, \\ \sigma_i &= 0 \text{ in } t < 0, \end{aligned}$$

where the scalar function $a(x', \theta)$ remains to be determined.

Boundary equation From the definition of F^0 , the fact that \mathcal{U}^0 is purely incoming ($\sigma_2 \equiv 0$), and Remark 1.17, we see that the outgoing component of the right side of (1.38)(a) is zero. Writing

$$\mathbf{E}\mathcal{U}^1 = \sum_{i=1}^3 \tau_i(x, \theta_0 + \omega_i \xi_d) r_i$$

and using Proposition 2.1, we find

$$X_{\phi_2}\tau_2 - e_2^2 \tau_2 = 0, \quad \tau_2 = 0 \text{ in } t < 0,$$

and thus $\tau_2 \equiv 0$.

Since $\mathbf{E}\mathcal{U}^1$ is purely incoming, it is valued in $\mathbb{E}^s(\beta)$. Therefore, by (1.11), the equation (1.38)(b) is solvable if and only if the condition

$$(2.4) \quad b \cdot [G - (d\psi(0)\mathcal{U}^0)\mathcal{U}^0 - B(I - \mathbf{E})\mathcal{U}^1] = 0 \text{ on } x_d = \xi_d = 0,$$

holds. Since $b \cdot B r_i = 0$, $i = 1, 3$, using (1.37) we see that

$$\begin{aligned} b \cdot B(I - \mathbf{E})\mathcal{U}^1(x', 0, \theta_0, 0) &= b \cdot B \mathbf{R}_{\infty} F^0 \\ &= -b \cdot B r_2 \left(\int_{\mathbb{R}^+} F_2(x', 0, \theta_0 - \omega_2 s, s) ds \right). \end{aligned}$$

Thus, the solvability condition (2.4) becomes

$$(2.5) \quad b \cdot \left[G - (d\psi(0)\mathcal{U}^0)\mathcal{U}^0 + B r_2 \left(\int_{\mathbb{R}^+} F_2(x', 0, \theta_0 - \omega_2 s, s) ds \right) \right] = 0.$$

We now rewrite this equation in terms of the function $a(x', \theta_0)$ appearing in (2.3). We will use the following fundamental fact:

Proposition 2.3 ([CG10]). *There holds*

$$(2.6) \quad b \cdot B \sum_{m \in \{1,3\}} R_m A_j e_m = \kappa \partial_{\eta_j} \sigma(\beta), \quad j = 0, \dots, d-1, \quad \text{where } \kappa \in \mathbb{R} \setminus 0, \quad \partial_\tau \sigma(\beta) = 1,$$

the function σ is defined in Assumption 1.5, and $R_m := \sum_{k \neq m} \frac{P_k}{\omega_m - \omega_k}$, so¹²

$$R_1 = \frac{P_2}{\omega_1 - \omega_2} + \frac{P_3}{\omega_1 - \omega_3}, \quad R_3 = \frac{P_1}{\omega_3 - \omega_1} + \frac{P_2}{\omega_3 - \omega_2}.$$

Thus,

$$(2.7) \quad b \cdot B \sum_{m \in \mathcal{I}} R_m \tilde{L}(\partial) e_m = \kappa \left(\partial_\tau \sigma(\beta) \partial_t + \sum_{j=1}^{d-1} \partial_{\eta_j} \sigma(\beta) \partial_{x_j} \right) := X_{Lop}.$$

Our main result in this paragraph reads as follows.

Proposition 2.4. *There are real constants c and d such that taking the θ_0 -derivative of equation (2.5) is equivalent to the following equation for the function $a(x', \theta_0)$ in (2.3):*

$$(2.8) \quad X_{Lop} a + c a + d \partial_{\theta_0} (a^2) = -b \cdot \partial_{\theta_0} G.$$

Proof. 1.) Let us first observe that $\mathcal{U}^0 = a(x', \theta_0) e$ on $x_d = \xi_d = 0$, so the bilinear term $(d\psi(0) \mathcal{U}^0) \mathcal{U}^0$ in (2.5) reads $a^2 (d\psi(0) e) e$, which gives an a^2 contribution (up to a real multiplicative constant). This term contributes to the "Burgers like" term $d \partial_{\theta_0} (a^2)$ in (2.8) after taking the θ_0 -derivative.

2.) Let us recall the definition $F^0 = -(\tilde{L}(\partial) \mathcal{U}^0 + D(0) \mathcal{U}^0)$, and $F_2 := l_2 \cdot F^0$. Another easy contribution to (2.5) is thus given when computing

$$\begin{aligned} & \int_{\mathbb{R}^+} l_2 \cdot D(0) (\sigma_1(x', 0, \theta_0 + (\omega_1 - \omega_2) s) r_1 + \sigma_3(x', 0, \theta_0 + (\omega_3 - \omega_2) s) r_3) ds \\ &= \int_{\mathbb{R}^+} l_2 \cdot D(0) (a(x', \theta_0 + (\omega_1 - \omega_2) s) e_1 + a(x', \theta_0 + (\omega_3 - \omega_2) s) e_3) ds. \end{aligned}$$

Taking the θ_0 derivative of this integral gives a contribution of the form $c a(x', \theta_0)$ for a suitable real constant c .

3.) It remains to examine the last integral term in (2.5), that is

$$\begin{aligned} I &:= \int_{\mathbb{R}^+} l_2 \cdot \tilde{L}(\partial) \mathcal{U}^0(x', 0, \theta_0 - \omega_2 s, s) ds \\ &= \int_{\mathbb{R}^+} l_2 \cdot \tilde{L}(\partial) (\sigma_1(x', 0, \theta_0 + (\omega_1 - \omega_2) s) r_1 + \sigma_3(x', 0, \theta_0 + (\omega_3 - \omega_2) s) r_3) ds. \end{aligned}$$

Since $l_2 \cdot r_i = 0$, $i = 1, 3$, the coefficient of the ∂_{x_d} term in the latter integral vanishes and we obtain

$$(2.9) \quad \begin{aligned} I r_2 &= \left[\sum_{j=0}^{d-1} (l_2 A_j e_1) \int_{\mathbb{R}^+} \partial_j a(x', \theta_0 + (\omega_1 - \omega_2) s) ds \right] r_2 \\ &+ \left[\sum_{j=0}^{d-1} (l_2 A_j e_3) \int_{\mathbb{R}^+} \partial_j a(x', \theta_0 + (\omega_3 - \omega_2) s) ds \right] r_2. \end{aligned}$$

¹²Recall $P_i X = (l_i \cdot X) r_i$.

Let $\mathcal{A}_m(x', \theta_0)$, $m = 1, 3$ be the unique antiderivative(s) of a in θ_0 such that

$$\mathcal{A}_m(x', (\omega_m - \omega_2)\infty) = 0.$$

The integrals in (2.9) can be computed and the function $I r_2$ can be rewritten

$$(2.10) \quad I r_2 = \sum_{j=0}^{d-1} (l_2 A_j e_1) \frac{\partial_j \mathcal{A}_1(x', \theta_0)}{\omega_1 - \omega_2} r_2 + \sum_{j=0}^{d-1} (l_2 A_j e_3) \frac{\partial_j \mathcal{A}_3(x', \theta_0)}{\omega_3 - \omega_2} r_2.$$

Using (2.6), (2.7), and the fact that $b \cdot B e_i = 0$, $i = 1, 3$, we obtain from (2.10) that the final contribution in taking the θ_0 -derivative of (2.5) reads

$$b \cdot B \partial_{\theta_0}(I r_2) = X_{Lop} a.$$

□

Combining (2.3) with Proposition 2.4, we have the following key subsystem satisfied by (σ_1, σ_3) :

$$(2.11) \quad \begin{aligned} (a) \quad & X_{\phi_1} \sigma_1 - e_1^1 \sigma_1 = 0 \\ (b) \quad & X_{\phi_3} \sigma_3 - e_3^3 \sigma_3 = 0 \\ (c) \quad & X_{Lop} a + c a + d \partial_{\theta_0}(a^2) = -b \cdot \partial_{\theta_0} G \text{ on } \{x_d = 0, \xi_d = 0\}. \end{aligned}$$

where

$$(2.12) \quad \sigma_i(x', 0, \theta_0) r_i = a(x', \theta_0) e_i$$

and all unknowns are zero in $t < 0$. Recall that the outgoing component σ_2 of \mathcal{U}^0 vanishes. In Proposition 2.8 we solve for a by Picard iteration, and then obtain σ_1 and σ_3 in the interior by simply integrating along the appropriate characteristics.

Remark 2.5. In the case of wavetrains the interior equation for τ_2 is coupled to σ_1 and σ_3 by a nonlinear interaction integral whenever there is a resonance of the form

$$n\phi_1 + n_3\phi_3 = n_2\phi_2, \text{ with } n_i \in \mathbb{Z}.$$

Thus, τ_2 , the outgoing component of $\mathbf{E}\mathcal{U}^1$ is not necessarily zero, and this leads to the presence of an extra term of the form $\partial_{\theta_0} \tau_2$ in the last equation of (2.11). That extra term leads to a loss of derivatives in the estimate for the linearized system, and forced us to use Nash-Moser iteration to solve for $(\sigma_1, \sigma_3, \tau_2)$ in [CGW13].

We shall work with profiles lying in a class of Sobolev spaces weighted in θ . Let

$$\Omega_T = \{(x, \theta) \in \overline{\mathbb{R}}_+^{d+1} \times \mathbb{R} : t < T\},$$

and introduce the following spaces.

Definition 2.6. For $s \in \mathbb{N}$, define the spaces

$$(2.13) \quad \Gamma_T^s := \{a(x, \theta) \in L^2(\Omega_T) : (\theta, \partial_x, \partial_\theta)^\beta a \in L^2(\Omega_T) \text{ for } |\beta| \leq s \text{ and } a = 0 \text{ in } t < 0\},$$

with norms

$$(2.14) \quad |a|_{s,T} = \sum_{|\beta|=|\beta_1,\beta_2,\beta_3|\leq s} |\theta^{\beta_1} \partial_x^{\beta_2} \partial_\theta^{\beta_3} a|_{L^2(\Omega_T)}.$$

The analogous norms of functions of $b(x',\theta)$ defined on $b\Omega_T := \{(x',\theta) \in \mathbb{R}^d \times \mathbb{R} : t < T\}$ are denoted with brackets: $\langle b \rangle_{s,T}$. We denote the corresponding spaces by $b\Gamma_T^s$.

We will let H_T^s denote the usual Sobolev space on Ω_T with norm defined just as in (2.14) but without the θ weights. These spaces and those below have the obvious meanings when a function is vector-valued.

Remark 2.7. By Sobolev embedding if the functions $f_k^i, g_{l,m}^i, h_{l,m}^i$ appearing in (1.31) lie in Γ_T^s for $s > \frac{d+2}{2} + 3$, then F as in (1.30) is of type \mathcal{F} .

Proposition 2.8. Let $T > 0$, $m > \frac{d+1}{2} + 1$ and suppose $G \in b\Gamma_T^{m+1}$ and vanishes in $t < 0$. Then provided $\langle G \rangle_{b\Gamma_T^{m+1}}$ is small enough, the system (2.11), (2.12) has a unique solution satisfying $a \in b\Gamma_T^m$ and $\sigma_i \in \Gamma_T^m$, $i = 1, 3$.

Proof. First one solves (2.11)(c) using the iteration scheme

$$X_{Lop} a^{n+1} + 2d a^n \partial_{\theta_0} a^{n+1} = -b \cdot \partial_{\theta_0} G - c a^n.$$

The standard proof showing convergence towards a solution $a \in H_T^m(x',\theta_0)$ is easily adapted to the weighted spaces $b\Gamma_T^m$. Details of a very similar argument are given in the proof of [CW13, Proposition 3.6]. The conclusion for σ_1 and σ_3 follows easily. \square

2.2 The first corrector \mathcal{U}^1

To understand the properties of \mathcal{U}^1 we need:

Lemma 2.9. The functions σ_1, σ_3 , and a all have moment zero.

Proof. Let $\underline{a}(x) = \int_{\mathbb{R}} a(x',\theta_0) d\theta_0$. Taking the moment of equation (2.11)(c), we obtain the problem for \underline{a} :

$$X_{Lop} \underline{a} + c_4 \underline{a} = 0, \quad \underline{a} = 0 \text{ in } t < 0.$$

Thus $\underline{a} = 0$ and so the equations for σ_i , $i = 1, 3$ imply their moments are zero as well. \square

Proposition 2.10. Let $\sigma(x,\theta) \in \Gamma_T^s$, $s > \frac{d}{2} + 3$, have moment zero and let $\sigma^*(x,\theta)$ denote the unique θ -primitive of σ that decays to zero as $|\theta| \rightarrow \infty$. Then $\sigma^* \in \Gamma_T^{s-1}$.

Proof. We have $\sigma \in \Gamma_T^s$ if and only if $\partial_m^k \partial_x^\alpha (m^p \hat{\sigma}(x,m)) \in L^2(x,m)$ for all k,α,p such that $k + |\alpha| + p \leq s$. Since $\hat{\sigma}(x,0) = 0$ we have

$$\hat{\sigma}(x,m) = m \int_0^1 (\partial_m \hat{\sigma})(x,sm) ds$$

so

$$\widehat{\sigma^*}(x,m) = \frac{1}{im} \hat{\sigma}(x,m) = -i \int_0^1 (\partial_m \hat{\sigma})(x,sm) ds,$$

and the result follows. \square

Proposition 2.11. *Suppose $G \in b\Gamma_T^{s+1}$, $s > \frac{d}{2} + 4$, and vanishes in $t < 0$. Then the first corrector \mathcal{U}^1 in (1.36) satisfies*

$$(2.15) \quad (I - \mathbf{E})\mathcal{U}^1(x, \theta_0, \xi_d) = \sum_{i=1}^3 W_i(x, \theta_0, \xi_d) r_i,$$

$$\text{where } W_i(x, \theta_0, \xi_d) = \sum_{k \neq i} b_k(x, \theta_0 + \omega_k \xi_d), \quad \text{with } b_k \in \Gamma_T^{s-2}.$$

Proof. We have $(I - \mathbf{E})\mathcal{U}^1 = \mathbf{R}_\infty F^0 = \mathbf{R}_\infty (I - \mathbf{E})F^0$, so using Proposition 1.37 we obtain

$$\begin{aligned} ((I - \mathbf{E})\mathcal{U}^1)_i &= - \sum_{k \neq i} e_k^i \int_{\xi_d}^{+\infty} \sigma_k(x, \theta_0 + \omega_i \xi_d + s(\omega_k - \omega_i)) ds \\ &\quad + \sum_{k \neq i} \int_{\xi_d}^{+\infty} V_k^i \sigma_k(x, \theta_0 + \omega_i \xi_d + s(\omega_k - \omega_i)) ds. \end{aligned}$$

Now $\sigma_k \in \Gamma_T^s$ has moment zero, so

$$\int_{\xi_d}^{+\infty} \sigma_k(x, \theta_0 + \omega_i \xi_d + s(\omega_k - \omega_i)) ds = \sigma_k^*(x, \theta_0 + \omega_k \xi_d) / (\omega_k - \omega_i),$$

where $\sigma_k^*(x, \theta_k) \in \Gamma_T^{s-1}$ by Proposition 2.10. The other terms are treated similarly using $V_k^i \sigma_k \in \Gamma_T^{s-1}$. \square

Proposition 2.12. *Suppose $G \in b\Gamma_T^{s+1}$, $s > \frac{d}{2} + 4$, and vanishes in $t < 0$. Then there exists a solution \mathcal{U}^1 to (1.36) that satisfies*

$$\mathbf{E}\mathcal{U}^1(x, \theta_0, \xi_d) = \tau_1(x, \theta_0 + \omega_1 \xi_d) r_1 + \tau_3(x, \theta_0 + \omega_3 \xi_d) r_3 \quad \text{with } \tau_k \in \Gamma_T^{s-3}.$$

Proof. We have already shown $\tau_2 = 0$. Since the $\sigma_i \in \Gamma_T^s$, $i = 1, 3$, have moment zero, and the functions b_k in (2.15) lie in Γ_T^{s-2} , it follows that the right side of (1.38)(a) has components given by functions in Γ_T^{s-3} . Similarly, the right side of the boundary equation (1.38)(b) has components given by functions in $b\Gamma_T^{s-3}$. The solvability condition (2.4) holds, so we may make a choice of boundary data $\mathbf{E}\mathcal{U}^1(x', 0, \theta_0, 0)$ satisfying (1.38)(b) whose components are elements of $b\Gamma_T^{s-3}$. Thus, solving the system (1.38), we obtain a solution $\mathbf{E}\mathcal{U}^1$ with components given by $\tau_k \in \Gamma_T^{s-3}$, $k = 1, 3$. \square

Definition 2.13. *Suppose $W(x, \theta_0, \xi_d) = \mathbf{E}W + (I - \mathbf{E})W$ and*

$$\begin{aligned} \mathbf{E}W(x, \theta_0, \xi_d) &= \sum_{i=1}^3 a_i(x, \theta_0 + \omega_i \xi_d) r_i, \quad \text{with } a_i(x, \theta_i) \in \Gamma_T^s; \\ (I - \mathbf{E})W(x, \theta_0, \xi_d) &= \sum_{i=1}^3 w_i r_i, \quad \text{where } w_i(x, \theta_0, \xi_d) = \sum_{k \neq i} b_k(x, \theta_0 + \omega_k \xi_d) \quad \text{with } b_k(x, \theta_k) \in \Gamma_T^s. \end{aligned}$$

Then we write $W \in \tilde{\Gamma}_T^s$.

We may summarize our construction of \mathcal{U}^0 and \mathcal{U}^1 as follows.

Theorem 2.14. *Suppose $G \in b\Gamma_T^{s+1}$, $s > \frac{d}{2} + 4$, and vanishes in $t < 0$. Then provided $\langle G \rangle_{b\Gamma_T^{s+1}}$ is small enough, there exist solutions $\mathcal{U}^0 \in \tilde{\Gamma}_T^s$, $\mathcal{U}^1 \in \tilde{\Gamma}_T^{s-3}$, satisfying the profile equations (1.35), (1.36). Moreover, \mathcal{U}^0 and $\mathbf{E}\mathcal{U}^1$ are purely incoming.*

3 The exact solution of the singular system

In this section we state the main estimates for the linearized singular system and also the existence theorem for the nonlinear singular system (1.23). We begin by gathering the notation for spaces and norms that is needed below.

Notations 3.1. Here we take $s \in \mathbb{N} = \{0, 1, 2, \dots\}$.

(a) Let $\Omega := \overline{\mathbb{R}}_+^{d+1} \times \mathbb{R}$, $\Omega_T := \Omega \cap \{-\infty < t < T\}$, $b\Omega := \mathbb{R}^d \times \mathbb{R}$, $b\Omega_T := b\Omega \cap \{-\infty < t < T\}$, and set $\omega_T := \overline{\mathbb{R}}_+^{d+1} \cap \{-\infty < t < T\}$.

(b) Let $H^s \equiv H^s(b\Omega)$, the standard Sobolev space with norm $\langle V(x', \theta_0) \rangle_s$. For $\gamma \geq 1$ we set $H_\gamma^s := e^{\gamma t} H^s$ and $\langle V \rangle_{s,\gamma} := \langle e^{-\gamma t} V \rangle_s$.

(c) $L^2 H^s \equiv L^2(\overline{\mathbb{R}}_+, H^s(b\Omega))$ with norm $|U(x, \theta_0)|_{L^2 H^s} \equiv |U|_{0,s}$ given by

$$|U|_{0,s}^2 = \int_0^\infty |U(x', x_d, \theta_0)|_{H^s(b\Omega)}^2 dx_d.$$

The corresponding norm on $L^2 H_\gamma^s$ is denoted $|V|_{0,s,\gamma}$.

(d) $CH^s \equiv C(\overline{\mathbb{R}}_+, H^s(b\Omega))$ denotes the space of continuous bounded functions of x_d with values in $H^s(b\Omega)$, with norm $|U(x, \theta_0)|_{CH^s} = |U|_{\infty,s} := \sup_{x_d \geq 0} |U(\cdot, x_d, \cdot)|_{H^s(b\Omega_T)}$. The corresponding norm on CH_γ^s is denoted $|V|_{\infty,s,\gamma}$.

(e) Let $M_0 := 3d + 5$ and define $C^{0,M_0} := C(\overline{\mathbb{R}}_+, C^{M_0}(b\Omega))$ as the space of continuous bounded functions of x_d with values in $C^{M_0}(b\Omega)$, with norm $|U(x, \theta_0)|_{C^{0,M_0}} := |U|_{L^\infty W^{M_0,\infty}}$. Here $L^\infty W^{M_0,\infty}$ denotes the space $L^\infty(\overline{\mathbb{R}}_+; W^{M_0,\infty}(b\Omega))$ ¹³.

(f) The corresponding spaces on Ω_T are denoted $L^2 H_T^s$, $L^2 H_{\gamma,T}^s$, CH_T^s , $CH_{\gamma,T}^s$ and C_T^{0,M_0} with norms $|U|_{0,s,T}$, $|U|_{0,s,\gamma,T}$, $|U|_{\infty,s,T}$, $|U|_{\infty,s,\gamma,T}$, and $|U|_{C_T^{0,M_0}}$ respectively. On $b\Omega_T$ we use the spaces H_T^s and $H_{\gamma,T}^s$ with norms $\langle U \rangle_{s,T}$ and $\langle U \rangle_{s,\gamma,T}$.

(g) All constants appearing in the estimates below are independent of ε , γ , and T unless such dependence is explicitly noted.

The following spaces appear in the statement of the main existence theorem and are used in its proof.

Definition 3.2. For $s \in \{0, 1, 2, \dots\}$, let

$$\begin{aligned} E_T^s &:= CH_T^s \cap L^2 H_T^{s+1}, \quad \text{with the norm } |U(x, \theta_0)|_{E_T^s} := |U|_{\infty,s,T} + |U|_{0,s+1,T}, \\ E_{\gamma,T}^s &:= CH_{\gamma,T}^s \cap L^2 H_{\gamma,T}^{s+1}, \quad \text{with the norm } |U(x, \theta_0)|_{E_{\gamma,T}^s} := |U|_{\infty,s,\gamma,T} + |U|_{0,s+1,\gamma,T}. \end{aligned}$$

The linearization of the singular problem (1.23) at $U(x, \theta_0)$ has the form

$$(3.1) \quad \begin{aligned} (a) \quad & \partial_d \dot{U}_\varepsilon + \mathbb{A} \left(\partial_{x'} + \frac{\beta \partial_{\theta_0}}{\varepsilon} \right) \dot{U}_\varepsilon + \mathcal{D}(\varepsilon U) \dot{U}_\varepsilon = f(x, \theta_0) \quad \text{on } \Omega, \\ (b) \quad & \mathcal{B}(\varepsilon U) \dot{U}_\varepsilon|_{x_d=0} = g(x', \theta_0), \\ (c) \quad & \dot{U}_\varepsilon = 0 \text{ in } t < 0, \end{aligned}$$

¹³The size of M_0 is determined by the requirements of the singular calculus described in Appendix B.

where the matrices $\mathcal{B}(\varepsilon U)$, $\mathcal{D}(\varepsilon U)$ are defined in (1.15)¹⁴. Instead of (3.1), consider the equivalent problem satisfied by $\dot{U}^\gamma := e^{-\gamma t}\dot{U}$:

$$(3.2) \quad \begin{aligned} \partial_d \dot{U}^\gamma + \mathbb{A} \left((\partial_t + \gamma, \partial_{x''}) + \frac{\beta \partial_{\theta_0}}{\varepsilon} \right) \dot{U}^\gamma + \mathcal{D}(\varepsilon U) \dot{U}^\gamma &= f^\gamma(x, \theta_0), \\ \mathcal{B}(\varepsilon U) \dot{U}^\gamma|_{x_d=0} &= g^\gamma(x', \theta_0), \\ \dot{U}^\gamma &= 0 \text{ in } t < 0. \end{aligned}$$

Below we let Λ_D denote the singular Fourier multiplier (see (B.2)) associated with the symbol

$$\Lambda(X, \gamma) := \left(\gamma^2 + \left| \xi' + \frac{k\beta}{\varepsilon} \right|^2 \right)^{1/2}, \quad X := \xi' + \frac{k\beta}{\varepsilon}.$$

The basic estimate for the linearized singular problem (3.2) is given in the next Proposition. Observe that the estimate (3.3) exhibits a loss of one ‘‘singular derivative’’ Λ_D . This is quite a high price to pay, which counts as a factor $1/\varepsilon$. In view of [CG10, Theorem 4.1], there is strong evidence that the loss below is optimal.

Proposition 3.3 (Main L^2 linear estimate). *We make the structural assumptions of Theorem 1.14, let $s_0 := \lfloor \frac{d+1}{2} \rfloor + 1$, and recall $M_0 = 3d + 5$. Fix $K > 0$ and suppose $|\varepsilon \partial_d U|_{C^{0, M_0-1}} + |U|_{C^{0, M_0}} + |U|_{CH^{s_0}} \leq K$ for $\varepsilon \in (0, 1]$. There exist positive constants $\varepsilon_0(K) > 0$, $C(K) > 0$ and $\gamma_0(K) \geq 1$ such that sufficiently smooth solutions \dot{U} of the linearized singular problem (3.1) satisfy¹⁵:*

$$(3.3) \quad |\dot{U}^\gamma|_{0,0} + \frac{\langle \dot{U}^\gamma \rangle_0}{\sqrt{\gamma}} \leq C(K) \left(\frac{|\Lambda_D f^\gamma|_{0,0} + |f^\gamma/\varepsilon|_{0,0}}{\gamma^2} + \frac{\langle \Lambda_D g^\gamma \rangle_0 + \langle g^\gamma/\varepsilon \rangle_0}{\gamma^{3/2}} \right)$$

for $\gamma \geq \gamma_0(K)$, $0 < \varepsilon \leq \varepsilon_0(K)$.

The same estimate holds if $\mathcal{B}(\varepsilon U)$ in (3.1) is replaced by $\mathcal{B}(\varepsilon U, \varepsilon \mathcal{U})$ and $\mathcal{D}(\varepsilon U)$ is replaced by $\mathcal{D}(\varepsilon U, \varepsilon \mathcal{U})$ as long as $|\varepsilon \partial_d(U, \mathcal{U})|_{C^{0, M_0-1}} + |U, \mathcal{U}|_{C^{0, M_0}} + |U, \mathcal{U}|_{CH^{s_0}} \leq K$ for $\varepsilon \in (0, 1]$.

The proof is identical to the proof of Proposition 2.2 in [CGW13]. Here our hypothesis on $|U|_{CH^{s_0}}$ is needed to allow the pulse calculus summarized in Appendix B to be used in exactly the same way that the wavetrain calculus was used in [CGW13]. The corollary below follows from Proposition 3.3 by the same argument used to derive Corollary 2.3 from Proposition 2.2 in [CGW13].

Corollary 3.4 (Main H_{tan}^1 linear estimate). *Under the same assumptions as in Proposition 3.3, smooth enough solutions \dot{U} of the linearized singular problem (3.1) satisfy:*

$$(3.4) \quad |\dot{U}^\gamma|_{\infty,0} + |\dot{U}^\gamma|_{0,1} + \frac{\langle \dot{U}^\gamma \rangle_1}{\sqrt{\gamma}} \leq C(K) \left(\frac{|\Lambda_D f^\gamma|_{0,1} + |f^\gamma/\varepsilon|_{0,1}}{\gamma^2} + \frac{\langle \Lambda_D g^\gamma \rangle_1 + \langle g^\gamma/\varepsilon \rangle_1}{\gamma^{3/2}} \right)$$

for $\gamma \geq \gamma_0(K)$, $0 < \varepsilon \leq \varepsilon_0(K)$.

The next proposition localizes the estimate (3.4) to Ω_T . By considering data εf , εg instead of f, g we can recast (3.4) in a form where the loss of a singular derivative is replaced by loss of an ordinary derivative. The proof is identical to that of [CGW13, Proposition 2.8]. Let us write the linearized operators on the left sides of (3.1)(a) and (b) as $\mathbb{L}'(\varepsilon U)\dot{U}$ and $\mathbb{B}'(\varepsilon U)\dot{U}$ respectively.

¹⁴Here and below we often suppress the subscript ε on \dot{U} .

¹⁵Note that the norms $|u|_{0,1}$ and $|\Lambda_D u|_{0,0}$ are not equivalent.

Proposition 3.5. *Let $s_0 = \lfloor \frac{d+1}{2} \rfloor + 1$, fix $K > 0$, and suppose $|\varepsilon \partial_d U|_{C_T^{0, M_0-1}} + |U|_{C_T^{0, M_0}} + |U|_{CH_T^{s_0}} \leq K$ for $\varepsilon \in (0, 1]$. There exist positive constants $\varepsilon_0(K)$, $\gamma_0(K)$ such that solutions of the singular problem*

$$\begin{aligned} \mathbb{L}'_\varepsilon(U) \dot{U} &= \varepsilon f & \text{in } \Omega_T, \\ \mathbb{B}'_\varepsilon(U) \dot{U} &= \varepsilon g & \text{on } b\Omega_T, \\ \dot{U} &= 0 & \text{in } t < 0, \end{aligned}$$

satisfy

$$|\dot{U}|_{\infty, 0, \gamma, T} + |\dot{U}|_{0, 1, \gamma, T} + \frac{\langle \dot{U}|_{x_d=0} \rangle_{1, \gamma, T}}{\sqrt{\gamma}} \leq C(K) \left(\frac{|f|_{0, 2, \gamma, T}}{\gamma^2} + \frac{\langle g \rangle_{2, \gamma, T}}{\gamma^{3/2}} \right)$$

for $0 < \varepsilon \leq \varepsilon_0(K)$, $\gamma \geq \gamma_0(K)$, and the constant $C(K)$ only depends on K .

The same estimate holds if $\mathcal{B}(\varepsilon U)$ in (3.1) is replaced by $\mathcal{B}(\varepsilon U, \varepsilon \mathcal{U})$ given in (1.13), and $\mathcal{D}(\varepsilon U)$ is replaced by $\mathcal{D}(\varepsilon U, \varepsilon \mathcal{U})$ given in (1.14), as long as there holds $|\varepsilon \partial_d(U, \mathcal{U})|_{C_T^{0, M_0-1}} + |U, \mathcal{U}|_{C_T^{0, M_0}} + |U, \mathcal{U}|_{CH_T^{s_0}} \leq K$ for $\varepsilon \in (0, 1]$.

We need the following higher derivative estimate in the error analysis. This estimate is also needed in the Nash-Moser iteration used to prove the nonlinear existence Theorem (3.7). We recall the definitions (1.21) for the indices a_0, a_1, a, \tilde{a} .

Proposition 3.6 (Tame estimate for the linearized system). *Fix $K > 0$ and suppose*

$$(3.5) \quad |\varepsilon \partial_d U|_{C_T^{0, M_0-1}} + |U|_{C_T^{0, M_0}} + |U|_{CH_T^{s_0}} \leq K \text{ for } \varepsilon \in (0, 1].$$

Let $\mu_0 := a_0 + 2$ and $s \in [0, \tilde{a}]$, where \tilde{a} is defined in (1.21). There exist positive constants $\gamma = \gamma(K)$, $\kappa_0(\gamma, T)$, ε_0 , and C such that if

$$|U|_{0, \mu_0, \gamma, T} + \langle U|_{x_d=0} \rangle_{\mu_0, \gamma, T} \leq \kappa_0,$$

then solutions \dot{U} of the linearized system (3.1) satisfy for $0 < \varepsilon \leq \varepsilon_0$:

$$\begin{aligned} & |\dot{U}|_{E_{\gamma, T}^s} + \langle \dot{U}|_{x_d=0} \rangle_{s+1, \gamma, T} \\ & \leq C \left[|f|_{0, s+2, \gamma, T} + \langle g \rangle_{s+2, \gamma, T} + (|f|_{0, \mu_0, \gamma, T} + \langle g \rangle_{\mu_0, \gamma, T}) (|U|_{0, s+2, \gamma, T} + \langle U|_{x_d=0} \rangle_{s+2, \gamma, T}) \right]. \end{aligned}$$

The same estimate holds for $0 < \varepsilon \leq \varepsilon_0$ if $\mathcal{B}(\varepsilon U)$ in (3.1) is replaced by $\mathcal{B}(\varepsilon U, \varepsilon \mathcal{U})$ and $\mathcal{D}(\varepsilon U)$ is replaced by $\mathcal{D}(\varepsilon U, \varepsilon \mathcal{U})$ as long as $|\varepsilon \partial_d(U, \mathcal{U})|_{C_T^{0, M_0-1}} + |U, \mathcal{U}|_{C_T^{0, M_0}} + |U, \mathcal{U}|_{CH_T^{s_0}} \leq K$ for $\varepsilon \in (0, 1]$ and

$$|U, \mathcal{U}|_{0, \mu_0, \gamma, T} + \langle U|_{x_d=0}, \mathcal{U}|_{x_d=0} \rangle_{\mu_0, \gamma, T} \leq \kappa_0.$$

The proof is exactly the same as the proof of Proposition 2.15 in [CGW13]. Finally, we state the existence theorem for solutions of the singular system (1.23).

Theorem 3.7. *Fix $T > 0$, define a , a_0 , and \tilde{a} as in (1.21), and suppose $G \in H^{\tilde{a}}(b\Omega_T)$. There exists $\varepsilon_0 > 0$ such that if $\langle G \rangle_{a+2}$ is small enough, there exists a solution U_ε of the system (1.23) on Ω_T for $0 < \varepsilon \leq \varepsilon_0$ with $U_\varepsilon \in E^{a-1}$, $U_\varepsilon|_{x_d=0} \in H^a$. This statement remains true if a is increased and if $\tilde{a} \geq 2a - a_0$.*

Proof. The proof is an exact repetition of the Nash-Moser argument used to prove Theorem 5.13 in [CGW13]. One fixes $T > 0$, $K > 0$ and $\gamma = \gamma(K)$ as in Proposition 3.6, and uses the scale of spaces $E_{\gamma,T}^s$ (Definition 3.2) with $\gamma T = 1$. The smoothing operators S_θ are defined just as in Lemma 5.12 of [CGW13], taking account in the obvious way for the fact that now functions $u(x, \theta_0) \in E_{\gamma,T}^s$ are defined for $\theta_0 \in \mathbb{R}$ instead of $\theta_0 \in \mathbb{T}$. The verification of the condition (3.5) needed to apply the tame estimate of Proposition 3.6 in the induction argument is done just as in Lemma 5.20 of [CGW13]. \square

4 Error analysis

4.1 Moment-zero approximations

When constructing a corrector to the leading part of the approximate solution we must take primitives in θ of functions $f(x, \theta)$ that decay to zero as $|\theta| \rightarrow \infty$. A difficulty is that such primitives do not necessarily decay to zero as $|\theta| \rightarrow \infty$, and this prevents us from using those primitives directly in the error analysis. The failure of the primitive to decay manifests itself on the Fourier transform side as a small divisor problem. To get around this difficulty we work with the primitive of a *moment-zero approximation*, because such a primitive does have the desired decay.

We will use the following spaces:

Definition 4.1. 1.) For $s \geq 0$, let E_T^s be given in Definition 3.2.

2.) Let $\mathcal{E}_T^s := \{\mathcal{U}(x, \theta_0, \xi_d) : |\mathcal{U}|_{\mathcal{E}_T^s} := \sup_{\xi_d \geq 0} |\mathcal{U}(\cdot, \cdot, \xi_d)|_{E_T^s} < \infty\}$.

Proposition 4.2. For $s > (d+1)/2$, the spaces E_T^s and \mathcal{E}_T^s are Banach algebras.

Proof. This is a consequence of Sobolev embedding and the fact that $L^\infty(b\Omega_T) \cap H^s(b\Omega_T)$ is a Banach algebra for $s \geq 0$. \square

The proofs of the following two propositions follow directly from the definitions.

Proposition 4.3. (a) For $s \geq 0$, let $\sigma(x, \theta) \in H_T^{s+1}$ and set $\tilde{\sigma}(x, \theta_0, \xi_d) := \sigma(x, \theta_0 + \omega \xi_d)$, $\omega \in \mathbb{R}$. Then there holds

$$|\tilde{\sigma}|_{\mathcal{E}_T^s} \leq C |\sigma|_{H_T^{s+1}}.$$

(b) Set $\tilde{\sigma}_\varepsilon(x, \theta_0) := \tilde{\sigma}(x, \theta_0, \frac{x_d}{\varepsilon})$. Then

$$|\tilde{\sigma}_\varepsilon|_{E_T^s} \leq |\tilde{\sigma}|_{\mathcal{E}_T^s}.$$

Definition 4.4 (Moment-zero approximations). Let $0 < p \leq 1$, and let $\phi \in C^\infty(\mathbb{R})$ have $\text{supp } \phi \subset \{m : |m| \leq 2\}$, with $\phi = 1$ on $|m| \leq 1$. Set $\phi_p(m) = \phi(\frac{m}{p})$ and $\chi_p = 1 - \phi_p$. For $\sigma(x, \theta) \in L^2$, define the moment zero approximation to σ , $\hat{\sigma}_p(x, \theta)$ by

$$(4.1) \quad \hat{\sigma}_p(x, m) := \chi_p(m) \hat{\sigma}(x, m),$$

where the hat denotes the Fourier transform in θ .

Proposition 4.5. For $s \geq 1$ suppose $\sigma(x, \theta) \in \Gamma^{s+2}$ and $\tilde{\sigma}(x, \theta_0, \xi_d) := \sigma(x, \theta_0 + \omega \xi_d)$. Then

$$(a) \quad |\tilde{\sigma} - \tilde{\sigma}_p|_{\mathcal{E}_T^s} \leq C |\sigma|_{\Gamma_T^{s+2}} \sqrt{p},$$

$$(b) \quad |\partial_{x_d} \tilde{\sigma} - \partial_{x_d} \tilde{\sigma}_p|_{\mathcal{E}_T^{s-1}} \leq C |\sigma|_{\Gamma_T^{s+2}} \sqrt{p}.$$

Proof. 1.) Recall that $\sigma \in \Gamma_T^s \Leftrightarrow \theta^{\beta_1} \partial_x^{\beta_2} \partial_\theta^{\beta_3} \sigma(x, \theta) \in L^2(x, \theta)$ for $|\beta| \leq s \Leftrightarrow \partial_m^{\beta_1} \partial_x^{\beta_2} m^{\beta_3} \hat{\sigma}(x, m) \in L^2(x, m)$ for $|\beta| \leq s$. It follows that

$$(4.2) \quad \sigma \in \Gamma_T^{s+2} \Rightarrow \hat{\sigma}(x, m) \in H_T^{s+2}(x, m) \subset H^1(m, H^{s+1}(x)) \subset L^\infty(m, H^{s+1}(x)).$$

2.) We have

$$\begin{aligned} |\sigma - \sigma_p|_{H_T^{s+1}}^2 &\sim \sum_{|\alpha|+k \leq s+1} \left| \partial_x^\alpha m^k \hat{\sigma}(x, m) (1 - \chi_p(m)) \right|_{L^2(x, m)}^2 \\ &= \sum_{|\alpha|+k \leq s+1} \int_{|m| \leq 2p} \int |\partial_x^\alpha m^k \hat{\sigma}(x, m) (1 - \chi_p(m))|^2 dx dm \\ &\leq C \int_{|m| \leq 2p} |\hat{\sigma}(x, m)|_{H^{s+1}(x)}^2 dm \leq C |\sigma|_{\Gamma_T^{s+2}}^2(2p), \end{aligned}$$

where the last inequality uses (4.2). The conclusion now follows from Proposition 4.3.

3.) The proof of part (b) is essentially the same. \square

Proposition 4.6. Let $\sigma(x, \theta) \in H_T^s$, $s \geq 0$, and let σ_p be a moment-zero approximation to σ . We have

$$\begin{aligned} (a) \quad &|\sigma_p|_{H_T^s} \leq C |\sigma|_{H_T^s}, \\ (b) \quad &\text{If } \sigma \in \Gamma_T^s, \text{ then } |\sigma_p|_{\Gamma_T^s} \leq \frac{C}{p^s} |\sigma|_{\Gamma_T^s}. \end{aligned}$$

Proof. Part (b) follows from (4.1). Indeed, for $|\beta| \leq s$, there holds

$$|\partial_m^{\beta_1} \partial_x^{\beta_2} m^{\beta_3} \hat{\sigma}_p(x, m)|_{L_T^2} \leq \frac{C}{p^{\beta_1}} |\sigma|_{\Gamma_T^s},$$

since $|\partial_m^{\beta_1} \chi_p| \leq C/p^{\beta_1}$. Taking $\beta_1 = 0$ we similarly obtain part (a). \square

Next we consider primitives of moment-zero approximations.

Proposition 4.7. Let $\sigma(x, \theta) \in \Gamma_T^s$, $s > \frac{d}{2} + 3$. Let $\sigma_p^*(x, \theta)$ be the unique primitive of σ_p in θ that decays to zero as $|\theta| \rightarrow \infty$. Then $\sigma_p^* \in \Gamma_T^s$ with moment zero, and

$$(4.3) \quad \begin{aligned} (a) \quad &|\sigma_p^*|_{H_T^s} \leq C \frac{|\sigma_p|_{H_T^s}}{p} \\ (b) \quad &|\sigma_p^*|_{\Gamma_T^s} \leq C \frac{|\sigma_p|_{\Gamma_T^s}}{p^{s+1}}. \end{aligned}$$

Proof. 1.) Since $\sigma_p(x, \theta) \in \Gamma_T^s$, $s > \frac{d}{2} + 3$, we have $|\sigma_p(x, \theta)| \leq C \langle \theta \rangle^{-2}$ for all (x, θ) . The unique θ -primitive of σ_p decaying to zero as $|\theta| \rightarrow \infty$ is thus

$$\sigma_p^*(x, \theta) = - \int_\theta^{+\infty} \sigma_p(x, s) ds = \int_{-\infty}^\theta \sigma_p(x, s) ds.$$

Moreover, we have

$$(4.4) \quad \partial_\theta \sigma_p^* = \sigma_p \Leftrightarrow im \widehat{\sigma_p^*} = \widehat{\sigma_p} = \chi_p \hat{\sigma}, \text{ so } \widehat{\sigma_p^*} = \frac{\chi_p \hat{\sigma}}{im}.$$

Since $|m| \geq p$ on the support of χ_p , this gives

$$|\widehat{\sigma}_p^*(x, m)| \leq C \frac{|\hat{\sigma}(x, m)|}{p}$$

and (4.3)(a) follows directly from this. From (4.4) we also obtain $\widehat{\sigma}_p^*(x, 0) = 0$.

2.) The proof of (4.3)(b) is almost the same, except that now one uses

$$\left| \partial_m^s \left(\frac{\chi_p}{m} \right) \right| \leq \frac{C}{p^{s+1}}.$$

□

Proposition 4.8. *Let $\sigma(x, \theta)$ and $\tau(x, \theta)$ belong to H_T^s , $s > \frac{d+2}{2}$. Then*

$$(4.5) \quad |\sigma \tau - (\sigma \tau)_p|_{H_T^s} \leq C |\sigma|_{H_T^s} |\tau|_{H_T^s} \sqrt{p}.$$

Proof. With $*$ denoting convolution in m we have

$$\begin{aligned} |\sigma \tau - (\sigma \tau)_p|_{H_T^s}^2 &\sim \sum_{|\alpha|+k \leq s+1} |\partial_x^\alpha m^k (\hat{\sigma} * \hat{\tau})(x, m) (1 - \chi_p(m))|_{L^2(x, m)}^2 \\ &\leq C \int_{|m| \leq 2p} |(\hat{\sigma} * \hat{\tau})(x, m)|_{H^s(x)}^2 dm \\ &\leq C \int_{|m| \leq 2p} \left(\int |\hat{\sigma}(x, m - m_1)|_{H^s(x)} |\hat{\tau}(x, m_1)|_{H^s(x)} dm_1 \right)^2 dm \\ &\leq C p |\hat{\sigma}(x, m)|_{L^2(m, H^s(x))}^2 |\hat{\tau}(x, m)|_{L^2(m, H^s(x))}^2 \leq C p |\sigma|_{H_T^s}^2 |\tau|_{H_T^s}^2. \end{aligned}$$

□

Proposition 4.9. *Let $\sigma(x, \theta)$ and $\tau(x, \theta)$ belong to Γ_T^s , $s > \frac{d}{2} + 3$ and let $(\sigma \tau)_p^*$ denote the unique primitive of $(\sigma \tau)_p$ that decays to zero as $|\theta| \rightarrow \infty$. Then*

$$|(\sigma \tau)_p^*|_{H_T^s} \leq C \frac{|\sigma|_{H_T^s} |\tau|_{H_T^s}}{p}.$$

Proof. Since Γ_T^s is a Banach algebra, Proposition 4.7 implies $(\sigma \tau)_p^* \in \Gamma_T^s$ with moment zero and

$$|(\sigma \tau)_p^*|_{H_T^s} \leq C \frac{|(\sigma \tau)_p|_{H_T^s}}{p}.$$

Since H_T^s is a Banach algebra, the result now follows from Proposition 4.6(a). □

4.2 Proof of Theorem 1.14

Choice of a and \tilde{a} . The conditions on the boundary datum $G(x', \theta_0)$ are different in Theorems 3.7 and 2.14. We need to choose a , \tilde{a} , and $G(x', \theta_0)$ so that both Theorems apply simultaneously. We also need a large enough so that we can apply Proposition 3.6 in the step (4.18) of the error analysis below. These conditions are met if we take a and \tilde{a} as in (1.21), and choose $G \in b\Gamma_T^{\tilde{a}}$ such that $\langle G \rangle_{b\Gamma_T^{a+2}}$ is small enough. Applying Theorems 3.7 and 2.14, we now have for $0 < \varepsilon \leq \varepsilon_0$ an exact solution $U_\varepsilon(x, \theta_0) \in E_T^{a-1}$ to the singular system (1.23) and profiles $\mathcal{U}^0 \in \tilde{\Gamma}_T^{a+1}$, $\mathcal{U}^1 \in \tilde{\Gamma}_T^{a-2}$ satisfying the profile equations (1.35) and (1.36).

Approximation. Fix $0 < p \leq 1$, we use Proposition 4.5 to choose moment zero approximations \mathcal{U}_p^0 and \mathcal{U}_p^1 such that

$$(4.6) \quad \begin{aligned} |\mathcal{U}^0 - \mathcal{U}_p^0|_{\mathcal{E}_T^{a-1}} &\leq C \sqrt{p}, & |\partial_{x_d} \mathcal{U}^0 - \partial_{x_d} \mathcal{U}_p^0|_{\mathcal{E}_T^{a-2}} &\leq C \sqrt{p}, \\ |\mathcal{U}^1 - \mathcal{U}_p^1|_{\mathcal{E}_T^{a-4}} &\leq C \sqrt{p}, & |\partial_{x_d} \mathcal{U}^1 - \partial_{x_d} \mathcal{U}_p^1|_{\mathcal{E}_T^{a-5}} &\leq C \sqrt{p}. \end{aligned}$$

Having made these choices, we can now state the main result of this section, which yields the final convergence result of Theorem 1.14 as an immediate corollary.

Theorem 4.10. *We make the same Assumptions as in Theorem 1.14 and let a and \tilde{a} be chosen as in (1.21). Consider the leading order approximate solution to the singular semilinear system (1.23) given by*

$$\mathcal{U}_\varepsilon^0(x, \theta_0) := \mathcal{U}^0\left(x, \theta_0, \frac{x_d}{\varepsilon}\right),$$

and let $U_\varepsilon(x, \theta_0) \in E_T^{a-1}$ be the exact solution to (1.23) just obtained. Then

$$|U_\varepsilon(x, \theta_0) - \mathcal{U}_\varepsilon^0(x, \theta_0)|_{E_T^{a-6}} \leq C \varepsilon^{\frac{1}{2M_1+3}}, \quad \text{where } M_1 = \left\lfloor \frac{d}{2} + 3 \right\rfloor.$$

Proof. We shall fill in the sketch provided in section 1.5. The approximate solution to the singular system defined by $(\mathcal{U}^0 + \varepsilon \mathcal{U}^1)(x, \theta_0, \frac{x_d}{\varepsilon})$ is too crude for the error analysis, so we must construct an additional corrector.

1.) As a first try we could use Proposition 1.19 to construct $\tilde{\mathcal{U}}_p^2(x, \theta_0, \xi_d)$ satisfying

$$(4.7) \quad \tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d}) \tilde{\mathcal{U}}_p^2 = -(I - \mathbf{E}) \left(\tilde{L}(\partial) \mathcal{U}_p^1 + D(0) \mathcal{U}_p^1 + (dD(0) \mathcal{U}_p^0) \mathcal{U}_p^0 \right) := -(I - \mathbf{E}) \mathcal{G}.$$

This choice of $\tilde{\mathcal{U}}_p^2(x, \theta_0, \frac{x_d}{\varepsilon})$ turns out not to lie in any E_T^r space, and is thus too large to be useful in the error analysis. Observe however that \mathcal{G} is of type \mathcal{F} .

To remedy this problem we replace $(I - \mathbf{E}) \mathcal{G}$ by a modification $[(I - \mathbf{E}) \mathcal{G}]_{mod}$ defined as follows. First, using the earlier formulas for $\mathbf{E} \mathcal{U}^1$ and $(I - \mathbf{E}) \mathcal{U}^1$ and Remark 1.17, we have

$$(4.8) \quad \begin{aligned} -(I - \mathbf{E}) \mathcal{G} &= \sum_{i=1}^3 \left(- \sum_{k \neq i} V_k^i \tau_{k,p} - \sum_{k \neq i} X_{\phi_i} b_{k,p} - \sum_{k \neq i, l \neq k, l \neq i} V_k^i b_{l,p} + \sum_{k \neq i} e_k^i \tau_{k,p} \right) r_i \\ &+ \sum_{i=1}^3 \left(\sum_{k \neq i} c_k^i \sigma_{k,p}^2 + \sum_{l \neq m} d_{l,m}^i \sigma_{l,p} \sigma_{m,p} \right) r_i =: A + B, \end{aligned}$$

where B is the second sum over i in (4.8), $\sigma_{q,p} = \sigma_{q,p}(x, \theta_0 + \omega_q \xi_d)$, and $\tau_{r,p} = \tau_{r,p}(x, \theta_0 + \omega_r \xi_d)$. The problem is caused by the self-interaction terms given by the sum over $k \neq i$ in B , so we define

$$(4.9) \quad -[(I - \mathbf{E}) \mathcal{G}]_{mod} = A + \sum_{i=1}^3 \left(\sum_{k \neq i} c_k^i (\sigma_{k,p}^2)_p + \sum_{l \neq m} d_{l,m}^i \sigma_{l,p} \sigma_{m,p} \right) r_i,$$

and we set

$$(4.10) \quad \mathcal{U}_p^2 := -\mathbf{R}_\infty [(I - \mathbf{E}) \mathcal{G}]_{mod}.$$

Instead of (4.7) we have

$$(4.11) \quad \tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d}) \mathcal{U}_p^2 = -[(I - \mathbf{E})\mathcal{G}]_{mod}.$$

For later use we set

$$D(x, \theta_0, \xi_d) := (I - \mathbf{E})\mathcal{G} - [(I - \mathbf{E})\mathcal{G}]_{mod}$$

and estimate

$$|D(x, \theta_0, \frac{x_d}{\varepsilon})|_{E_T^a} \leq C \sqrt{p}.$$

Indeed, using Propositions 4.8 and 4.6(a) we have

$$\left| (\sigma_{k,p}^2 - (\sigma_{k,p}^2)_p) \left(x, \theta_0 + \omega_k \frac{x_d}{\varepsilon} \right) \right|_{E_T^a} \leq |(\sigma_{k,p}^2 - (\sigma_{k,p}^2)_p)|_{H_T^{a+1}} \leq |\sigma_k|_{H_T^{a+1}}^2 \sqrt{p}.$$

We then define the corrected approximate solution

$$\mathcal{U}_\varepsilon(x, \theta_0) := \mathcal{U}^0(x, \theta_0, \frac{x_d}{\varepsilon}) + \varepsilon \mathcal{U}^1(x, \theta_0, \frac{x_d}{\varepsilon}) + \varepsilon^2 \mathcal{U}_p^2(x, \theta_0, \frac{x_d}{\varepsilon}).$$

Since $\mathcal{U}^1 \in \tilde{\Gamma}_T^{a-2}$, Proposition 4.3 implies $\mathcal{U}_\varepsilon^1 \in E_T^{a-3}$.

2.) Estimate of $|\mathcal{U}_{p,\varepsilon}^2|_{E_T^{a-4}}$. For A as in (4.8) let us write $A = \sum_{i=1}^3 a_i(x, \theta_0, \xi_d) r_i$. By (4.10), (4.9) and the formula (1.32) for \mathbf{R}_∞ , for $i = 1, 2, 3$ we must estimate $|b_i(x, \theta_0, \frac{x_d}{\varepsilon})|_{E_T^{s-2}}$, where $b_i(x, \theta_0, \xi_d)$ reads as follows:

$$(4.12) \quad \begin{aligned} & \sum_{k \neq i} c_k^i \int_{\xi_d}^{+\infty} (\sigma_{k,p}^2)_p \left(x, \theta_0 + \omega_i \xi_d + s(\omega_k - \omega_i) \right) ds \\ & + \sum_{m \neq i} d_{i,m}^i \int_{\xi_d}^{+\infty} \sigma_{i,p}(x, \theta_0 + \omega_i \xi_d) \sigma_{m,p}(x, \theta_0 + \omega_i \xi_d + s(\omega_m - \omega_i)) ds \\ & + \sum_{l \neq i} d_{l,i}^i \int_{\xi_d}^{+\infty} \sigma_{l,p}(x, \theta_0 + \omega_i \xi_d + s(\omega_l - \omega_i)) \sigma_{i,p}(x, \theta_0 + \omega_i \xi_d) ds \\ & + \sum_{l \neq m, l \neq i, m \neq i} d_{l,m}^i \int_{\xi_d}^{+\infty} \sigma_{l,p}(x, \theta_0 + \omega_i \xi_d + s(\omega_l - \omega_i)) \sigma_{m,p}(x, \theta_0 + \omega_i \xi_d + s(\omega_m - \omega_i)) ds \\ & + \int_{\xi_d}^{+\infty} a_i(x, \theta_0 + \omega_i(\xi_d - s), s) ds = \sum_{r=1}^5 b_{i,r}(x, \theta_0, \xi_d), \end{aligned}$$

where $b_{i,r}$, $r = 1, \dots, 5$ are defined by the respective lines of (4.12). Since $\mathcal{U}^0 \in \tilde{\Gamma}_T^{a+1}$, using Corollary A.3 we find

$$\left| b_{i,1} \left(x, \theta_0, \frac{x_d}{\varepsilon} \right) \right|_{E_T^a} \leq C \sum_{k \neq i} \frac{|\sigma_k|_{H_T^{a+1}}^2}{p} \leq C/p.$$

Similarly, from Proposition A.4 we get

$$\left| b_{i,2} \left(x, \theta_0, \frac{x_d}{\varepsilon} \right) \right|_{E_T^a} \leq C/p; \quad \left| b_{i,3} \left(x, \theta_0, \frac{x_d}{\varepsilon} \right) \right|_{E_T^a} \leq C/p.$$

Proposition A.1 on transversal interactions implies

$$\left| b_{i,4} \left(x, \theta_0, \frac{x_d}{\varepsilon} \right) \right|_{E_T^a} \leq \frac{C}{p^{M_1}},$$

where we have used Proposition 4.6(b) to estimate $|\sigma_{m,p}|_{\Gamma_T^{M_1}} \leq \frac{C}{p^{M_1}} |\sigma_m|_{\Gamma_T^{M_1}}$.

Since $A \in \tilde{\Gamma}_T^{a-3}$, Proposition A.5 implies $|b_{i,5}(x, \theta_0, \frac{x_d}{\varepsilon})|_{E_T^{a-4}} \leq C/p$, so adding up we obtain

$$|\mathcal{U}_{p,\varepsilon}^2|_{E_T^{a-4}} \leq \frac{C}{p^t}.$$

To estimate $(\partial_{x_d} \mathcal{U}_p^2)_\varepsilon$ we differentiate (4.12) and estimate as above to find

$$|(\partial_{x_d} \mathcal{U}_p^2)_\varepsilon|_{E_T^{a-5}} \leq \frac{C}{p^{t+1}}.$$

3.) Next we study the error upon substituting the corrected approximate solution \mathcal{U}_ε into the singular system (1.23). Letting \mathbb{L}_ε denote the operator defined by the left side of (1.23), we compute

$$(4.13) \quad \mathbb{L}_\varepsilon(\mathcal{U}_\varepsilon) = \varepsilon \left[(\tilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d}) \mathcal{U}_p^2) \Big|_{\xi_d = \frac{x_d}{\varepsilon}} + \left(\tilde{L}(\partial) \mathcal{U}^1 + D(0) \mathcal{U}^1 + (dD(0) \mathcal{U}^0) \mathcal{U}^0 \right) \Big|_{\xi_d = \frac{x_d}{\varepsilon}} \right] + O(\varepsilon^2).$$

Here the profile equations (1.25) imply that the terms of order ε^{-1} and ε^0 vanish. Using (4.11) we can rewrite the coefficient of ε in (4.13) as

$$\begin{aligned} & \left[\tilde{L}(\partial) (\mathcal{U}^1 - \mathcal{U}_p^1) + D(0) (\mathcal{U}^1 - \mathcal{U}_p^1) + (dD(0) \mathcal{U}^0) \mathcal{U}^0 - (dD(0) \mathcal{U}_p^0) \mathcal{U}_p^0 \right] \Big|_{\xi_d = \frac{x_d}{\varepsilon}} \\ & + \left[\mathbf{E} \left(\tilde{L}(\partial) \mathcal{U}_p^1 + D(0) \mathcal{U}_p^1 + (dD(0) \mathcal{U}_p^0) \mathcal{U}_p^0 \right) \right] \Big|_{\xi_d = \frac{x_d}{\varepsilon}} + D(x, \theta_0, \frac{x_d}{\varepsilon}) := A + B + D(x, \theta_0, \frac{x_d}{\varepsilon}). \end{aligned}$$

Using (4.6), Proposition 4.3, and the fact that E_T^s is a Banach algebra for $s \geq [(d+1)/2]$, we see that

$$|A|_{E^{a-5}(\Omega_T)} < C\sqrt{p}.$$

To estimate B , let

$$F = \tilde{L}(\partial) \mathcal{U}^1 + D(0) \mathcal{U}^1 + (dD(0) \mathcal{U}^0) \mathcal{U}^0 \text{ and } F_p = \tilde{L}(\partial) \mathcal{U}_p^1 + D(0) \mathcal{U}_p^1 + (dD(0) \mathcal{U}_p^0) \mathcal{U}_p^0.$$

The profile equation (1.36)(b) implies $\mathbf{E}F = 0$. Since F and F_p are of type \mathcal{F} , we can use the explicit formula for the action of \mathbf{E} in Remark 1.17 and (4.6) to obtain

$$|B|_{E_T^{a-5}} = \left| (\mathbf{E}F_p) \Big|_{\xi_d = \frac{x_d}{\varepsilon}} \right|_{E_T^{a-5}} = \left| (\mathbf{E}(F - F_p)) \Big|_{\xi_d = \frac{x_d}{\varepsilon}} \right|_{E_T^{a-5}} \leq C\sqrt{p}.$$

4.) The $O(\varepsilon^2)$ terms in (4.13) consist of

$$(4.14) \quad \left| \varepsilon^2 (L(\partial) \mathcal{U}_p^2(x, \theta_0, \xi_d)) \Big|_{\xi_d = \frac{x_d}{\varepsilon}} \right|_{E_T^{a-5}} \leq C \frac{\varepsilon^2}{p^{t+1}},$$

as well as terms coming from the Taylor expansion of $D(\varepsilon\mathcal{U}_\varepsilon)\mathcal{U}_\varepsilon$ like $\varepsilon^2 (\mathrm{d}D(0)\mathcal{U}^0)\mathcal{U}^1|_{\xi_d=\frac{x_d}{\varepsilon}}$, all of which satisfy better estimates than (4.14).

Setting $R_\varepsilon(x, \theta_0) := \mathbb{L}_\varepsilon(\mathcal{U}_\varepsilon)$, we have shown

$$(4.15) \quad |R_\varepsilon|_{E_T^{a-5}} \leq C \varepsilon \left(\sqrt{p} + \frac{\varepsilon}{p^{t+1}} \right).$$

5.) The boundary profile equations (1.26), and the fact that the traces of $\mathcal{U}_\varepsilon^0$ and $\mathcal{U}_\varepsilon^1$ lie in $H^a(b\Omega_T)$ and $H^{a-3}(b\Omega_T)$, respectively, imply

$$(4.16) \quad \langle r_\varepsilon(x', \theta_0) \rangle_{H^{a-4}(b\Omega_T)} \leq C \frac{\varepsilon^2}{p^t}, \quad \text{where } r_\varepsilon := \psi(\varepsilon\mathcal{U}_\varepsilon)\mathcal{U}_\varepsilon - \varepsilon G(x', \theta_0).$$

Indeed, these $O(\varepsilon^2)$ terms include

$$\langle \varepsilon^2 B\mathcal{U}_{p,\varepsilon}^2(x', 0, \theta_0) \rangle_{H^{a-4}(b\Omega_T)} \leq C \frac{\varepsilon^2}{p^t},$$

and other terms satisfying the same estimate coming from the Taylor expansion of $\psi(\varepsilon\mathcal{U}_\varepsilon)\mathcal{U}_\varepsilon$.

6.) Next we consider the singular problem satisfied by the difference $W_\varepsilon := U_\varepsilon - \mathcal{U}_\varepsilon$:

$$(4.17) \quad \begin{aligned} \partial_d W_\varepsilon + \mathbb{A} \left(\partial_{x'} + \frac{\beta \partial_{\theta_0}}{\varepsilon} \right) W_\varepsilon + D_2(\varepsilon U_\varepsilon, \varepsilon \mathcal{U}_\varepsilon) W_\varepsilon &= -R_\varepsilon \\ \psi_2(\varepsilon U_\varepsilon, \varepsilon \mathcal{U}_\varepsilon) W_\varepsilon &= -r_\varepsilon \text{ on } x_d = 0 \\ W_\varepsilon &= 0 \text{ in } t < 0, \end{aligned}$$

where

$$\begin{aligned} D_2(\varepsilon U_\varepsilon, \varepsilon \mathcal{U}_\varepsilon) W_\varepsilon &:= D(\varepsilon U_\varepsilon)U_\varepsilon - D(\varepsilon \mathcal{U}_\varepsilon)\mathcal{U}_\varepsilon \\ &= D(\varepsilon U_\varepsilon)W_\varepsilon + \left(\int_0^1 \mathrm{d}D(\varepsilon \mathcal{U}_\varepsilon + s\varepsilon(U_\varepsilon - \mathcal{U}_\varepsilon)) W_\varepsilon \mathrm{d}s \right) (\varepsilon \mathcal{U}_\varepsilon), \end{aligned}$$

and $\psi_2(\varepsilon U_\varepsilon, \varepsilon \mathcal{U}_\varepsilon)W_\varepsilon$ is defined similarly. Since $U_\varepsilon \in E_T^{a-1}$ and $\mathcal{U}_\varepsilon \in E_T^{a-4}$, a short computation shows

$$\psi_2(\varepsilon U_\varepsilon, \varepsilon \mathcal{U}_\varepsilon)W_\varepsilon = \psi(\varepsilon U_\varepsilon)W_\varepsilon + (\mathrm{d}\psi(\varepsilon U)W_\varepsilon)\varepsilon \mathcal{U}_\varepsilon + O\left(\frac{\varepsilon^2}{p^{M_1}}\right) = \mathcal{B}(\varepsilon U, \varepsilon \mathcal{U})W_\varepsilon + O\left(\frac{\varepsilon^2}{p^{M_1}}\right),$$

where the error term is measured in $H^{a-4}(b\Omega_T)$ and \mathcal{B} is defined in (1.13). Similarly,

$$D_2(\varepsilon U_\varepsilon, \varepsilon \mathcal{U}_\varepsilon)W_\varepsilon = \mathcal{D}(\varepsilon U, \varepsilon \mathcal{U})W_\varepsilon + O\left(\frac{\varepsilon^2}{p^{M_1}}\right) \text{ in } E_T^{a-4}.$$

Thus, using (4.15) and (4.16) we find

$$\begin{aligned} \partial_d W_\varepsilon + \mathbb{A} \left(\partial_{x'} + \frac{\beta \partial_{\theta_0}}{\varepsilon} \right) W_\varepsilon + \mathcal{D}(\varepsilon U_\varepsilon, \varepsilon \mathcal{U}_\varepsilon) W_\varepsilon &= \varepsilon O\left(\sqrt{p} + \frac{\varepsilon}{p^{M_1+1}}\right) \text{ in } E_T^{a-5} \\ \mathcal{B}(\varepsilon U_\varepsilon, \varepsilon \mathcal{U}_\varepsilon) W_\varepsilon|_{x_d=0} &= O\left(\frac{\varepsilon^2}{p^{M_1}}\right) \text{ in } H^{a-4}(b\Omega_T) \\ W_\varepsilon &= 0 \text{ in } t < 0. \end{aligned}$$

Provided $\langle G \rangle_{\mathfrak{H}_T^{\alpha+2}}$ is small enough, we can apply the estimate of Proposition 3.6 to obtain

$$(4.18) \quad |W_\varepsilon|_{E_T^{\alpha-6}} \leq C \left(\sqrt{p} + \frac{\varepsilon}{p^{M_1+1}} \right).$$

Setting $p = \varepsilon^b$ and $\sqrt{p} = \varepsilon/p^{M_1+1}$, we find $b = \frac{2}{2M_1+3}$. Thus, $|W_\varepsilon|_{E_T^{\alpha-6}} \leq C \varepsilon^{\frac{1}{2M_1+3}}$, so (4.18) implies

$$|U_\varepsilon - \mathcal{U}_\varepsilon^0|_{E_T^{\alpha-6}} \leq C \varepsilon^{\frac{1}{2M_1+3}}.$$

□

5 Extension to the general $N \times N$ system

It is only the discussion of profiles and the error analysis that needs to be extended to the general case. For each $m \in \{1, \dots, M\}$ we let

$$\ell_{m,k}, \quad k = 1, \dots, \nu_{k_m}$$

denote a basis of real vectors for the left eigenspace of the real matrix $i\mathcal{A}(\beta)$ associated to the real eigenvalue $-\omega_m$ and chosen to satisfy

$$\ell_{m,k} \cdot r_{m',k'} = \begin{cases} 1, & \text{if } m = m' \text{ and } k = k', \\ 0, & \text{otherwise.} \end{cases}$$

For $v \in \mathbb{C}^N$ we set

$$P_{m,k}v := (\ell_{m,k} \cdot v) r_{m,k}, \quad (\text{no complex conjugation here}).$$

Functions of type \mathcal{F} (see Definition 1.16) have the form

$$F(x, \theta_0, \xi_d) = \sum_{m=1}^M \sum_{k=1}^{\nu_{k_m}} F_{m,k}(x, \theta_0, \xi_d) r_{m,k},$$

where

$$\begin{aligned} F_{m,k} &= \sum_{m'} f_{m'}^{m,k}(x, \theta_0 + \omega_{m'} \xi_d) \\ &\quad + \sum_{m',k',m'',k''} g_{m',k',m'',k''}^{m,k}(x, \theta_0 + \omega_{m'} \xi_d) h_{m',k',m'',k''}^{m,k}(x, \theta_0 + \omega_{m''} \xi_d). \end{aligned}$$

The operator \mathbf{E} is given by

$$\mathbf{E}F := \sum_{m,k} \left(\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T F_{m,k}(x, \theta_0 + \omega_m(\xi_d - s), s) ds \right) r_{m,k},$$

and we leave it to the reader to formulate the analogue of Remark 1.17. On functions of type \mathcal{F} such that $\mathbf{E}F = 0$ the action of the operator \mathbf{R}_∞ is given by

$$\mathbf{R}_\infty F := - \sum_{m,k} \left(\int_{\xi_d}^{+\infty} F_{m,k}(x, \theta_0 + \omega_m(\xi_d - s), s) \, ds \right) r_{m,k}.$$

The general form of the profile equations (1.35), (1.36) still applies. With

$$W(x, \theta_0, \xi_d) = \sum_{m,k} w_{m,k}(x, \theta_0, \xi_d) r_{m,k},$$

the decomposition of Proposition 2.1 now has the form

$$(5.1) \quad \tilde{L}(\partial)W = \sum_{m,k} (X_{\phi_m} w_{m,k}) r_{m,k} + \sum_{m,k} \left(\sum_{m' \neq m, k'} V_{m',k'}^{m,k} w_{m',k'} \right) r_{m,k},$$

where $V_{m',k'}^{m,k}$ is the tangential vector field

$$V_{m',k'}^{m,k} = \sum_{j=0}^{d-1} (\ell_{m,k} A_j r_{m',k'}) \partial_{x_j}.$$

Recalling that $\{1, \dots, M\} = \mathcal{O} \cup \mathcal{I}$, where \mathcal{O} and \mathcal{I} contain the indices corresponding to outgoing and incoming phases, we have the expressions in (1.19) for \mathcal{U}^0 and \mathcal{U}^1 in the general case. As in (1.10) we have

$$\ker B \cap \mathbb{E}^s(\beta) = \text{Span}\{e\} = \sum_{m \in \mathcal{I}} e_m = \sum_{m \in \mathcal{I}} \sum_{k=1}^{\nu_{k_m}} e_{m,k},$$

where $e_{m,k} := P_{m,k} e$. Using (5.1) and arguing as in Propositions 2.2 and 2.3, we find that the key subsystem (2.11) now takes the form

$$(5.2) \quad \begin{aligned} X_{\phi_m} \sigma_{m,k} + e_{m,k} \sigma_{m,k} &= 0, \quad m \in \mathcal{I}, \quad k = 1, \dots, \nu_{k_m}, \\ X_{Lop} a + c a + d \partial_{\theta_0}(a^2) &= -b \cdot \partial_{\theta_0} G \text{ on } \{x_d = 0, \xi_d = 0\}, \end{aligned}$$

where

$$\sigma_{m,k}(x', 0, \theta_0) r_{m,k} = a(x', \theta_0) e_{m,k},$$

and all unknowns are zero in $t < 0$. The key subsystem and then the full profile system are solved as before (using a Picard iteration for a and integration along the characteristics for the $\sigma_{m,k}$'s), and Theorem 2.14 on the existence and regularity of \mathcal{U}^0 and \mathcal{U}^1 still holds as stated.

The error analysis in the proof of Theorem 4.10 goes through with the obvious minor changes. For example, the troublesome self-interaction terms $c_k^i \sigma_{k,p}^2$ in the sum that is part of B in (4.8) are now replaced by terms of the form $c_{m,k,k'}^i \sigma_{m,k,p} \sigma_{m,k',p}$, $m \neq i$, where the index p as before denotes a moment-zero approximation. These terms are handled just as before by introducing $-[(I - \mathbf{E})\mathcal{G}]_{mod}$, see (4.9), in which they are replaced by $c_{m,k,k'}^i (\sigma_{m,k,p} \sigma_{m,k',p})_p$. The contribution of these terms to \mathcal{U}_p^2 is estimated as before using Corollary A.3. The statement of Theorem 4.10 applies unchanged in the general case.

A Estimates of interaction integrals

Pulses do not interact to produce resonances that affect the leading order profiles as in the wavetrain case. However, interaction integrals must be estimated carefully in the error analysis of Section 4. All results below are proved in our earlier paper [CW13] so we shall not reproduce the proofs here.

The following propositions are used in the error analysis for estimating integrals appearing in the definition of $\mathcal{U}_{p,\varepsilon}^2$ (step **2** of Theorem 4.10). In particular, we must estimate primitives of products of pulses. In some of the estimates below we must introduce moment-zero approximations to avoid errors that would fail to lie in any E_T^r space, which are thus too large to be useful in the error analysis. We begin with an estimate of “transversal interactions”, see Proposition 4.10 in [CW13].

Proposition A.1. *Let t be the smallest integer greater than $\frac{d}{2} + 3$, and let $s \geq 0$. Let $\sigma_1(x, \theta)$, $\sigma_2(x, \theta)$ belong to $\Gamma_T^t \cap H_T^{s+1}$, and define*

$$u(x, \theta_0, \xi_d) := - \int_{\xi_d}^{+\infty} \sigma_1(x, \theta_0 + \omega \xi_d + \alpha s) \sigma_2(x, \theta_0 + \omega \xi_d + s) ds,$$

where ω, α are real and $\alpha \notin \{0, 1\}$. With $u_\varepsilon(x, \theta_0) := u(x, \theta_0, \frac{x_d}{\varepsilon})$, we have

$$|u_\varepsilon|_{E_T^s} \leq C (|\sigma_1|_{H_T^{s+1}} |\sigma_2|_{\Gamma_T^t} + |\sigma_2|_{H_T^{s+1}} |\sigma_1|_{\Gamma_T^t}),$$

uniformly for $\varepsilon \in (0, 1]$.

The previous estimate of transversal interactions did not require the use of moment-zero approximations. However, nontransversal interactions of pulses can produce errors that do not lie in any E_T^r space, and are thus too big to be helpful in the error analysis. Thus, we are forced to use a moment-zero approximation in the next two propositions (see corresponding Proposition 4.11, Corollary 4.12 and Proposition 4.14 in [CW13]).

Proposition A.2. *Let $\sigma(x, \theta)$ and $\tau(x, \theta)$ belong to Γ_T^s , $s > \frac{d}{2} + 3$. For $\alpha, \omega \in \mathbb{R}$, $\alpha \neq 0$ set*

$$f(x, \theta_0, \xi_d) := - \int_{\xi_d}^{+\infty} (\sigma \tau)_p(x, \theta_0 + \omega \xi_d + \alpha s) ds.$$

Then

$$\left| f\left(x, \theta_0, \frac{x_d}{\varepsilon}\right) \right|_{E_T^{s-1}} \leq C \frac{|\sigma|_{H_T^s} |\tau|_{H_T^s}}{p}.$$

Corollary A.3. *Let $\sigma(x, \theta)$, $\tau(x, \theta)$, and ω, α be as in Proposition A.2 and set*

$$g(x, \theta_0, \xi_d) := - \int_{\xi_d}^{+\infty} (\sigma_p \tau_p)_p(x, \theta_0 + \omega \xi_d + \alpha s) ds.$$

Then

$$\left| g\left(x, \theta_0, \frac{x_d}{\varepsilon}\right) \right|_{E_T^{s-1}} \leq C \frac{|\sigma|_{H_T^s} |\tau|_{H_T^s}}{p}.$$

Proposition A.4. For $s > \frac{d}{2} + 3$ let $\sigma(x, \theta) \in H_T^s$, $\tau(x, \theta) \in \Gamma_T^s$. With $\omega, \alpha \in \mathbb{R}$, $\alpha \neq 0$ set

$$j(x, \theta_0, \xi_d) := -\sigma(x, \theta_0 + \omega \xi_d) \int_{\xi_d}^{+\infty} \tau_p(x, \theta_0 + \omega \xi_d + \alpha s) ds.$$

Then

$$\left| j(x, \theta_0, \frac{x_d}{\varepsilon}) \right|_{E_T^{s-2}} \leq C \frac{|\sigma|_{H_T^s} |\tau|_{H_T^{s-1}}}{p}.$$

The next Proposition requires moment zero approximations because forcing by a noninteracting pulse can also produce errors that fail to lie in any E_T^r space, see Proposition 4.15 in [CW13].

Proposition A.5. For $s > \frac{d}{2} + 3$ and $\omega, \alpha \in \mathbb{R}$, $\alpha \neq 0$, let $\sigma \in \Gamma_T^s$ and set

$$k(x, \theta_0, \xi_d) = - \int_{\xi_d}^{+\infty} \sigma_p(x, \theta_0 + \omega \xi_d + \alpha s) ds.$$

Then

$$\left| k(x, \theta_0, \frac{x_d}{\varepsilon}) \right|_{E_T^{s-1}} \leq C \frac{|\sigma|_{H_T^s}}{p}.$$

B Singular pseudodifferential calculus for pulses

Here we summarize the parts of the singular pulse calculus constructed in [CGW12] that are needed in this article. First we define the singular Sobolev spaces used to describe mapping properties.

The variable in \mathbb{R}^{d+1} is denoted (x, θ) , $x \in \mathbb{R}^d$, $\theta \in \mathbb{R}$, and the associated frequency is denoted (ξ, k) . In this new context, the singular Sobolev spaces are defined as follows. We consider a fixed vector $\beta \in \mathbb{R}^d \setminus \{0\}$. Then for $s \in \mathbb{R}$ and $\varepsilon \in]0, 1]$, the anisotropic Sobolev space $H^{s, \varepsilon}(\mathbb{R}^{d+1})$ is defined by

$$H^{s, \varepsilon}(\mathbb{R}^{d+1}) := \left\{ u \in \mathcal{S}'(\mathbb{R}^{d+1}) / \widehat{u} \in L_{\text{loc}}^2(\mathbb{R}^{d+1}) \right. \\ \left. \text{and } \int_{\mathbb{R}^{d+1}} \left(1 + \left| \xi + \frac{k \beta}{\varepsilon} \right|^2 \right)^s |\widehat{u}(\xi, k)|^2 d\xi dk < +\infty \right\}.$$

Here \widehat{u} denotes the Fourier transform of u on \mathbb{R}^{d+1} . The space $H^{s, \varepsilon}(\mathbb{R}^{d+1})$ is equipped with the family of norms

$$\forall \gamma \geq 1, \quad \forall u \in H^{s, \varepsilon}(\mathbb{R}^{d+1}), \quad \|u\|_{H^{s, \varepsilon, \gamma}}^2 := \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} \left(\gamma^2 + \left| \xi + \frac{k \beta}{\varepsilon} \right|^2 \right)^s |\widehat{u}(\xi, k)|^2 d\xi dk.$$

When m is an integer, the space $H^{m, \varepsilon}(\mathbb{R}^{d+1})$ coincides with the space of functions $u \in L^2(\mathbb{R}^{d+1})$ such that the derivatives, in the sense of distributions,

$$\left(\partial_{x_1} + \frac{\beta_1}{\varepsilon} \partial_\theta \right)^{\alpha_1} \dots \left(\partial_{x_d} + \frac{\beta_d}{\varepsilon} \partial_\theta \right)^{\alpha_d} u, \quad \alpha_1 + \dots + \alpha_d \leq m,$$

belong to $L^2(\mathbb{R}^{d+1})$. In the definition of the norm $\|\cdot\|_{H^{m, \varepsilon, \gamma}}$, one power of γ counts as much as one derivative.

B.1 Symbols

Our singular symbols are built from the following sets of classical symbols.

Definition B.1. Let $\mathcal{O} \subset \mathbb{R}^N$ be an open subset that contains the origin. For $m \in \mathbb{R}$ we let $\mathbf{S}^m(\mathcal{O})$ denote the class of all functions $\sigma : \mathcal{O} \times \mathbb{R}^d \times [1, \infty) \rightarrow \mathbb{C}^{N \times N}$, $N \geq 1$, such that σ is C^∞ on $\mathcal{O} \times \mathbb{R}^d$ and for all compact sets $K \subset \mathcal{O}$:

$$\sup_{v \in K} \sup_{\xi' \in \mathbb{R}^d} \sup_{\gamma \geq 1} (\gamma^2 + |\xi|^2)^{-(m-|\nu|)/2} |\partial_v^\alpha \partial_{\xi'}^\nu \sigma(v, \xi, \gamma)| \leq C_{\alpha, \nu, K}.$$

Let $\mathcal{C}_b^k(\mathbb{R}^{d+1})$, $k \in \mathbb{N}$, denote the space of continuous and bounded functions on \mathbb{R}^{d+1} , whose derivatives up to order k are continuous and bounded. Let us first define the singular symbols.

Definition B.2 (Singular symbols). Fix $\beta \in \mathbb{R}^d \setminus 0$, let $m \in \mathbb{R}$, and let $n \in \mathbb{N}$. Then we let S_n^m denote the set of families of functions $(a_{\varepsilon, \gamma})_{\varepsilon \in]0, 1], \gamma \geq 1}$ that are constructed as follows:

$$(B.1) \quad \forall (x, \theta, \xi, k) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}, \quad a_{\varepsilon, \gamma}(x, \theta, \xi, k) = \sigma \left(\varepsilon V(x, \theta), \xi + \frac{k \beta}{\varepsilon}, \gamma \right),$$

where $\sigma \in \mathbf{S}^m(\mathcal{O})$, V belongs to the space $\mathcal{C}_b^n(\mathbb{R}^{d+1})$ and where furthermore V takes its values in a convex compact subset K of \mathcal{O} that contains the origin (for instance K can be a closed ball centered round the origin).

All results below extend to the case where in place of a function V that is independent of ε , the representation (B.1) is considered with a function V_ε that is indexed by ε , provided that we assume that all functions $\varepsilon V_\varepsilon$ take values in a fixed convex compact subset K of \mathcal{O} that contains the origin, and $(V_\varepsilon)_{\varepsilon \in (0, 1]}$ is a bounded family of $\mathcal{C}_b^n(\mathbb{R}^{d+1})$. Such singular symbols with a function V_ε are exactly the kind of symbols that we manipulated in the construction of exact solutions to the singular system (1.23).

B.2 Definition of operators and action on Sobolev spaces

To each symbol $a = (a_{\varepsilon, \gamma})_{\varepsilon \in]0, 1], \gamma \geq 1} \in S_n^m$ given by the formula (B.1), we associate a singular pseudodifferential operator $\text{Op}^{\varepsilon, \gamma}(a)$, with $\varepsilon \in]0, 1]$ and $\gamma \geq 1$, whose action on a function $u \in \mathcal{S}(\mathbb{R}^{d+1}; \mathbb{C}^N)$ is defined by

$$(B.2) \quad \text{Op}^{\varepsilon, \gamma}(a) u(x, \theta) := \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} e^{i(\xi \cdot x + k \theta)} \sigma \left(\varepsilon V(x, \theta), \xi + \frac{k \beta}{\varepsilon}, \gamma \right) \widehat{u}(\xi, k) \, d\xi \, dk.$$

Let us briefly note that for the Fourier multiplier $\sigma(v, \xi, \gamma) = i \xi_1$, the corresponding singular operator is $\partial_{x_1} + (\beta_1/\varepsilon) \partial_\theta$. We now describe the action of singular pseudodifferential operators on Sobolev spaces.

Proposition B.3. Let $n \geq d + 1$, and let $a \in S_n^m$ with $m \leq 0$. Then $\text{Op}^{\varepsilon, \gamma}(a)$ in (B.2) defines a bounded operator on $L^2(\mathbb{R}^{d+1})$: there exists a constant $C > 0$, that only depends on σ and V in the representation (B.1), such that for all $\varepsilon \in]0, 1]$ and for all $\gamma \geq 1$, there holds

$$\forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \|\text{Op}^{\varepsilon, \gamma}(a) u\|_0 \leq \frac{C}{\gamma^{|m|}} \|u\|_0.$$

The constant C in Proposition B.3 depends uniformly on the compact set in which V takes its values and on the norm of V in \mathcal{C}_b^{d+1} . For operators defined by symbols of order $m > 0$ we have:

Proposition B.4. *Let $n \geq d + 1$, and let $a \in S_n^m$ with $m > 0$. Then $\text{Op}^{\varepsilon, \gamma}(a)$ in (B.2) defines a bounded operator from $H^{m, \varepsilon}(\mathbb{R}^{d+1})$ to $L^2(\mathbb{R}^{d+1})$: there exists a constant $C > 0$, that only depends on σ and V in the representation (B.1), such that for all $\varepsilon \in]0, 1]$ and for all $\gamma \geq 1$, there holds*

$$\forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \|\text{Op}^{\varepsilon, \gamma}(a) u\|_0 \leq C \|u\|_{H^{m, \varepsilon, \gamma}}.$$

The next proposition describes the smoothing effect of operators of order -1 .

Proposition B.5. *Let $n \geq d + 2$, and let $a \in S_n^{-1}$. Then $\text{Op}^{\varepsilon, \gamma}(a)$ in (B.2) defines a bounded operator from $L^2(\mathbb{R}^{d+1})$ to $H^{1, \varepsilon}(\mathbb{R}^{d+1})$: there exists a constant $C > 0$, that only depends on σ and V in the representation (B.1), such that for all $\varepsilon \in]0, 1]$ and for all $\gamma \geq 1$, there holds*

$$\forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \|\text{Op}^{\varepsilon, \gamma}(a) u\|_{H^{1, \varepsilon, \gamma}} \leq C \|u\|_0.$$

Remark B.6. In applications of the pulse calculus, we verify the hypothesis that for V as in (B.1), $V \in \mathcal{C}_b^n(\mathbb{R}^{d+1})$, by showing $V \in H^s(\mathbb{R}^{d+1})$ for some $s > \frac{d+1}{2} + n$.

B.3 Adjoints and products

For proofs of the following results we refer to [CGW12]. The two first results deal with adjoints of singular pseudodifferential operators while the last two deal with products.

Proposition B.7. *Let $a = \sigma(\varepsilon V, \xi + \frac{k\beta}{\varepsilon}, \gamma) \in S_n^0$, $n \geq 2(d + 1)$, where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$, and let a^* denote the conjugate transpose of the symbol a . Then $\text{Op}^{\varepsilon, \gamma}(a)$ and $\text{Op}^{\varepsilon, \gamma}(a^*)$ act boundedly on L^2 and there exists a constant $C \geq 0$ such that for all $\varepsilon \in]0, 1]$ and for all $\gamma \geq 1$, there holds*

$$\forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \|\text{Op}^{\varepsilon, \gamma}(a)^* u - \text{Op}^{\varepsilon, \gamma}(a^*) u\|_0 \leq \frac{C}{\gamma} \|u\|_0.$$

If $n \geq 3d + 3$, then for another constant C , there holds

$$\forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \|\text{Op}^{\varepsilon, \gamma}(a)^* u - \text{Op}^{\varepsilon, \gamma}(a^*) u\|_{H^{1, \varepsilon, \gamma}} \leq C \|u\|_0,$$

uniformly in ε and γ .

Proposition B.8. *Let $a = \sigma(\varepsilon V, \xi + \frac{k\beta}{\varepsilon}, \gamma) \in S_n^1$, $n \geq 3d + 4$, where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$, and let a^* denote the conjugate transpose of the symbol a . Then $\text{Op}^{\varepsilon, \gamma}(a)$ and $\text{Op}^{\varepsilon, \gamma}(a^*)$ map $H^{1, \varepsilon}$ into L^2 and there exists a family of operators $R^{\varepsilon, \gamma}$ that satisfies*

- there exists a constant $C \geq 0$ such that for all $\varepsilon \in]0, 1]$ and for all $\gamma \geq 1$, there holds

$$\forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \|R^{\varepsilon, \gamma} u\|_0 \leq C \|u\|_0,$$

- the following duality property holds

$$\forall u, v \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \langle \text{Op}^{\varepsilon, \gamma}(a) u, v \rangle_{L^2} - \langle u, \text{Op}^{\varepsilon, \gamma}(a^*) v \rangle_{L^2} = \langle R^{\varepsilon, \gamma} u, v \rangle_{L^2}.$$

In particular, the adjoint $\text{Op}^{\varepsilon, \gamma}(a)^*$ for the L^2 scalar product maps $H^{1, \varepsilon}$ into L^2 .

Proposition B.9. (a) Let $a, b \in S_n^0$, $n \geq 2(d+1)$, and suppose $b = \sigma(\varepsilon V, \xi + \frac{k\beta}{\varepsilon}, \gamma)$ where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$. Then there exists a constant $C \geq 0$ such that for all $\varepsilon \in]0, 1]$ and for all $\gamma \geq 1$, there holds

$$\forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \|\text{Op}^{\varepsilon, \gamma}(a) \text{Op}^{\varepsilon, \gamma}(b) u - \text{Op}^{\varepsilon, \gamma}(ab) u\|_0 \leq \frac{C}{\gamma} \|u\|_0.$$

If $n \geq 3d+3$, then for another constant C , there holds

$$\forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \|\text{Op}^{\varepsilon, \gamma}(a) \text{Op}^{\varepsilon, \gamma}(b) u - \text{Op}^{\varepsilon, \gamma}(ab) u\|_{H^{1, \varepsilon, \gamma}} \leq C \|u\|_0,$$

uniformly in ε and γ .

(b) Let $a \in S_n^1, b \in S_n^0$ or $a \in S_n^0, b \in S_n^1$, $n \geq 3d+4$, and in each case suppose $b = \sigma(\varepsilon V, \xi + \frac{k\beta}{\varepsilon}, \gamma)$ where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$. Then there exists a constant $C \geq 0$ such that for all $\varepsilon \in]0, 1]$ and for all $\gamma \geq 1$, there holds

$$\forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \|\text{Op}^{\varepsilon, \gamma}(a) \text{Op}^{\varepsilon, \gamma}(b) u - \text{Op}^{\varepsilon, \gamma}(ab) u\|_0 \leq C \|u\|_0.$$

Proposition B.10. Let $a \in S_n^{-1}, b \in S_n^1$, $n \geq 3d+4$, and suppose $b = \sigma(\varepsilon V, \xi + \frac{k\beta}{\varepsilon}, \gamma)$ where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$. Then $\text{Op}^{\varepsilon, \gamma}(a) \text{Op}^{\varepsilon, \gamma}(b)$ defines a bounded operator on $H^{1, \varepsilon}$ and there exists a constant $C \geq 0$ such that for all $\varepsilon \in]0, 1]$ and for all $\gamma \geq 1$, there holds

$$\forall u \in \mathcal{S}(\mathbb{R}^{d+1}), \quad \|\text{Op}^{\varepsilon, \gamma}(a) \text{Op}^{\varepsilon, \gamma}(b) u - \text{Op}^{\varepsilon, \gamma}(ab) u\|_{H^{1, \varepsilon, \gamma}} \leq C \|u\|_0.$$

Our final result is Gårding's inequality.

Theorem B.11. Let $\sigma \in \mathbf{S}^0$ satisfy $\text{Re } \sigma(v, \xi, \gamma) \geq C_K > 0$ for all v in a compact subset K of \mathcal{O} . Let now $a \in S_0^n$, $n \geq 2d+2$ be given by (B.1), where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$ and is valued in a convex compact subset K . Then for all $\delta > 0$, there exists γ_0 which depends uniformly on V , the constant C_K and δ , such that for all $\gamma \geq \gamma_0$ and all $u \in \mathcal{S}(\mathbb{R}^{d+1})$, there holds

$$\text{Re } \langle \text{Op}^{\varepsilon, \gamma}(a) u; u \rangle_{L^2} \geq (C_K - \delta) \|u\|_0^2.$$

B.4 Extended calculus

In the proof of Corollary 3.4 we use a slight extension of the singular calculus. For given parameters $0 < \delta_1 < \delta_2 < 1$, we choose a cutoff $\chi^e(\xi', \frac{k\beta}{\varepsilon}, \gamma)$ such that

$$\begin{aligned} 0 &\leq \chi^e \leq 1, \\ \chi^e \left(\xi', \frac{k\beta}{\varepsilon}, \gamma \right) &= 1 \text{ on } \left\{ (\gamma^2 + |\xi'|^2)^{1/2} \leq \delta_1 \left| \frac{k\beta}{\varepsilon} \right| \right\}, \\ \text{supp } \chi^e &\subset \left\{ (\gamma^2 + |\xi'|^2)^{1/2} \leq \delta_2 \left| \frac{k\beta}{\varepsilon} \right| \right\}, \end{aligned}$$

and define a corresponding Fourier multiplier χ_D in the extended calculus by the formula (B.2) with $\chi^e(\xi', \frac{k\beta}{\varepsilon}, \gamma)$ in place of $\sigma(\varepsilon V, X, \gamma)$. Composition laws involving such operators are proved in [CGW12], but here we need only the fact that part (a) of Proposition B.9 holds when either a or b is replaced by an extended cutoff χ^e .

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