Convergence to equilibrium for positive solutions of some mutation-selection model

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CONVERGENCE TO EQUILIBRIUM FOR POSITIVE SOLUTIONS OF SOME MUTATION-SELECTION MODEL

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ABSTRACT. In this paper we are interested in the long time behaviour of the positive solutions of the mutation selection model with Neumann Boundary condition:

$$\frac{\partial u(x,t)}{dt} = u\left[r(x) - \int_\Omega K(x,y)|u|^p(y)\,dy\right] + \nabla\cdot(A(x)\nabla u(x))\,, \qquad \text{in} \quad \mathbb{R}^+\times\Omega$$

where $\Omega\subset\mathbb{R}^N$ is a bounded smooth domain, $k(.,.)\in C(\bar\Omega\times C(\bar\Omega),\mathbb{R}), p\geq 1$ and $A(x)$ is a smooth elliptic matrix.

In a blind competition situation, i.e $K(x,y)=k(y)$, we show the existence of a unique positive steady state which is positively globally stable. That is, the positive steady state attracts all the possible trajectories initiated from any non negative initial datum. When K is a general positive kernel, we also present a necessary and sufficient condition for the existence of a positive steady states. We prove also some stability result on the dynamic of the equation when the competition kernel K is of the form $K(x,y)=k_0(y)+\epsilon k_1(x,y)$. That is, we prove that for sufficiently small ϵ there exists a unique steady state, which in addition is positively asymptotically stable. The proofs of the global stability of the steady state essentially rely on non-linear relative entropy identities and an orthogonal decomposition. These identities combined with the decomposition provide us some a priori estimates and differential inequalities essential to characterise the asymptotic behaviour of the solutions.

1. INTRODUCTION AND MAIN RESULTS

In this paper we are interested in the long time behaviour of the positive solutions of the nonlocal equation

$$\frac{\partial u(t,x)}{\partial t} = u(t,x)\left[r(x) - \int_\Omega K(x,y)|u(t,y)|^p\,dy\right] + \nabla\cdot(A(x)\nabla u(t,x)) \qquad \text{in} \quad \mathbb{R}^+\times\Omega \tag{1.1}$$

$$\frac{\partial u(t,x)}{\partial n} = 0, \quad \text{in} \quad \mathbb{R}^+\times\partial\Omega \tag{1.2}$$

$$u(0,x) = u_0(x) \tag{1.3}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded smooth domain, $r(x)\in C^{0,1}(\bar\Omega)$ is positive, $p\geq 1$, $K(.,.)\in C^{0,1}(\bar\Omega\times\bar\Omega)$ and $A(x)\in\mathcal{M}_{n\times n}(\mathbb{R})$ is a uniform smooth ($C^{1,\alpha}$) elliptic matrix.

Such type of nonlocal model has been introduced to capture the evolution of a population structured by a phenotypical trait [9, 10, 22, 32]. In this context $u(x,t)$ represents the density of a population at the phenotypical trait x at time t, which is submitted to two essential interactions: mutation and selection. Here, the mutation process, which acts as a diffusion operator on the traits space, is modelled by a classical diffusion operator whereas the selection process is modelled by the nonlocal term $u(t,x)\int_\Omega K(x,y)|u(t,y)|^p\,dy$. In the literature, the selection operator takes often the form $u(t,x)\int_\Omega K(x,y)|u(t,y)|\,dy$ [5, 9, 32]. A rigorous derivation of these equations from stochastic processes can be found in [17, 26].

To our knowledge, a large part of the analysis of the long time behaviour of solutions of (2.1) concerns either situations where no mutation occurs [4, 5, 9, 11, 12, 15, 20, 21, 28] or in the context of "adaptive dynamics", i.e. the evolution of the population is driven by small mutations, [9, 10, 14, 15, 16, 29] and references therein.

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In the latter case, the matrix $A(x) := \epsilon A_0(x)$ and some asymptotic regimes are studied when $\epsilon \to 0$. In this situation, an extensive work have been done in developing a constrained Hamilton-Jacobi approach in order to analyse the long time behaviour of positive solutions of this type of models see for instance [4, 5, 15, 16, 21].

Analysis of variants of (1.1) involving a nonlocal mutation process of the form $\epsilon \int_\Omega \mu(x,y)(u(t,y) - u(t,x))\, dy$ instead of an elliptic diffusion can be found [11, 12, 13, 34, 35]. For these variants, approaches based on semi-group theory have been developed to analyse the asymptotic behaviour and local stability of the positive stationary solution of (1.1) when $\epsilon \to 0$, see [11, 12, 13].

In all those works, the small mutation assumptions appears to be a key feature in the analysis. Our goal here is to analyse the long time behaviour of the solution to (1.1) – (1.3) in situations where no restriction on the mutation operator are imposed. In particular, we want to understand situations where the rate of mutations is not small compared to selection. This appears for example in some virus population where the rate of mutation per reproduction cycle is high [19, 24, 36, 38].

In what follows, we will always make the following assumptions on r, K

$$\begin{cases} A \in \mathcal{M}_{n\times n}(\mathbb{R}) \text{ is a smooth uniform elliptic matrix,} \\ r \in C^{0,1}(\Omega) \text{ is positive,} \\ \Omega \text{ is a bounded Lipschitz domain in } \mathbb{R}^N. \\ K \in C^{0,1}(\bar\Omega \times \bar\Omega), K > 0, \end{cases} \tag{1.4}$$

Under the above assumptions the existence of a positive solution to the Cauchy problem (1.1)–(1.3) is guarantee. Namely, we can easily prove

Theorem 1.1. *Assume A, r, K satisfy* (1.4) *and $p \geq 1$ then for all $u_0 \in L^p(\Omega)$ there exists a positive smooth solution u to* (1.1) – (1.3) *so that $u \in C([0,+\infty), L^p(\Omega)) \cap C^1((0,+\infty), C^{2,\alpha}(\Omega))$.*

The main problematic then remains to characterise the long time behaviour of these solutions. In this direction our first result concerns the situations of blind competition, that is when the kernel $K(x,y)$ is independent of x. In this context the equations (1.1) – (1.3) rewrite

$$\frac{\partial u}{\partial t}(t,x) = u(t,x)\left(r(x) - \int_\Omega k(y)|u(t,y)|^p\, dy\right) + \nabla\cdot(A(x)\nabla u(t,x)) \quad \text{in} \quad \mathbb{R}^+\times\Omega \tag{1.5}$$

$$\frac{\partial u}{\partial n}(t,x) = 0 \quad \text{in} \quad \mathbb{R}^+\times\partial\Omega \tag{1.6}$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega. \tag{1.7}$$

In this situation, we have

Theorem 1.2. *Assume A, r, k satisfy* (1.4) *and $p \geq 1$.Let λ_1 be the first eigenvalue of the operator $\nabla\cdot(A(x)\nabla) + r(x)$ with Neumann boundary condition and let ϕ_1 be a positive eigenfunction associated with λ_1, that is ϕ_1 satisfies*

$$\nabla\cdot(A(x)\nabla\phi_1) + r(x)\phi_1 = -\lambda_1\phi_1 \quad \textit{in} \quad \Omega, \tag{1.8}$$

$$\frac{\partial\phi_1}{\partial n}(x) = 0 \quad \textit{on} \quad \partial\Omega. \tag{1.9}$$

Then we have the following asymptotic behaviour for any positive smooth (at least C^2) solution $u(t,x)$ to (1.5) – (1.6)

- *if $\lambda_1 \geq 0$, there is no positive stationary solution and $u(t,x) \to 0$ as $t \to \infty$*
- *if $\lambda_1 < 0$, then*

$$u(t,x) \to \mu\phi_1$$

where $\mu = \left(\frac{-\lambda_1}{\int_\Omega k(y)|\phi_1|^p(y)\, dy}\right)^{\frac{1}{p}}$ and ϕ_1 has been normalized by $\|\phi_1\|_{L^2(\Omega)} = 1$.

Next we establish an optimal existence criteria for the positive stationary solution to (1.1)-(1.2). Namely, we prove

Theorem 1.3. *Assume A, r, K satisfy* (1.4) *and $p \geq 1$. Then there exists at least a positive smooth solution $\bar{u}$ of* (1.1) – (1.3) ***if and only if*** *$\lambda_1 < 0$, where λ_1 is defined in Theorem 1.2.*

Finally, we prove that the dynamic observed for blind selection kernel $K(x, y) = k(y)$ still holds for some perturbation of k. More precisely, let us consider a kernel $k_\epsilon(x, y) = k_0(y) + \epsilon k_1(x, y)$ with k_i satisfying the assumption (1.4), then we have the following

Theorem 1.4. *Assume A, r, K satisfy* (1.4) *and $p = 1$ or $p = 2$. Assume further that $K = k_\epsilon$ and let $u(t, x)$ be a positive smooth solution to* (1.1)–(1.2) *with $K = k_\epsilon$. Then we have the following asymptotic behaviour:*

- *if $\lambda_1 \geq 0$, there is no positive stationary solution and $u(t, x) \to 0$ as $t \to \infty$ uniformly.*
- *if $\lambda_1 < 0$, then there exists ϵ^* so that for all $\epsilon \leq \epsilon^*$ there exists a unique positive globally attractive equilibrium $\bar{u}_\epsilon$ to* (1.1)-(1.2) *i.e. for all $u_0 \geq_{\not\equiv} 0$, then we have for all $x \in \Omega$,*

$$\lim_{t\to\infty} u(t, x) \to \bar{u}_\epsilon(x).$$

1.1. **Comments.** Before going to the proofs of these results, we would like to make some comments. First, it comes directly from the proofs that the Theorems 1.2 and 1.3 can be generalised to more general selection process. In particular, Theorem 1.2 holds true if instead of considering a selection of the form $u \int_\Omega k(y)|u(t, y)|^p \, dy$, we consider a selection of the form $u\mathcal{R}(u)$ with $\mathcal{R} : dom(\mathcal{R}) \to \mathbb{R}^+$ a positive functional satisfying: $\exists p, q \geq 1$ and $c_p, \alpha_p, R_p, C_q, \alpha_q, R_q$ positive constants such that ,

$$\begin{aligned} \mathcal{R}(u) &> c_p \|u\|^{p\alpha_p}_{L^p(\Omega)} \quad \text{when} \quad \|u\|_{L^p(\Omega)} \geq R_p, \\ \mathcal{R}(u) &< C_q \|u\|^{q\alpha_q}_{L^q(\Omega)} \quad \text{when} \quad \|u\|_{L^q(\Omega)} \leq R_q. \end{aligned}$$

A simple example of such $\mathcal{R}$ is the functional $\mathcal{R}(u) := \|u\|^p_{L^p(\Omega)} \|u\|^q_{L^q(\Omega)}$.

Similarly, the optimal existence criteria Theorem 1.3 will hold as well for a selection process $u\mathcal{R}(x, u)$ such that

$$\mathcal{R}_1(\cdot) \leq \mathcal{R}(x, \cdot) \leq \mathcal{R}_2(\cdot),$$

where the $\mathcal{R}_i$ satisfy the above assumptions.

We also wanted to stress that the regularity on the coefficient is far from optimal and extension of our results for rougher coefficients r, k, A should hold true. In order to keep our analysis of the asymptotic behaviour as simple as possible, we deliberately impose some regularity on the considered coefficients. We believe that theses assumptions highlight the important point of the method we used without altering the pertinence of the results obtained.

We also want to emphasize that these results are strongly related to the eigenvalue problem obtained by linearising the equation (1.5) around the steady state 0 which is a common feature for classical reaction diffusion

$$\frac{\partial u}{\partial t} = \Delta u + f(x, u),$$

where f is a KPP type. However, the extension of Theorems 1.2, 1.3 to unbounded domains Ω is far from obvious considering the multiplicity of notion of generalised eigenvalue [7]. Moreover, in these situation the strict positivity of the kernel k seems to introduce a strong dichotomy for the properties of the stationary solutions and consequently the dynamics observed for evolution problem. Indeed, in this direction some progress have recently been made for the so called nonlocal Fisher-KPP equation :

$$\frac{\partial u}{\partial t} = \Delta u + u(1 - \phi \star u), \tag{1.10}$$

where ϕ is a non-negative kernel. When ϕ is a positive integrable function, the constant 1 is a positive solution. Moreover, for $\phi \in L^1 \cap C^1$ positive so that $x^2\phi \in L^1$, it is shown in [6] that

travelling semi-front exists for all speed $c \geq c^*$, i.e there exists (U, c), so that $U > 0$ and U satisfies

$$U_{xx} + cU_x + U(1 - \phi \star U) = 0,$$
$$\lim_{x \to +\infty} U = 0, \quad \liminf_{x \to -\infty} U > 0.$$

In particular when c is large or ϕ is sufficiently concentrated or has a positive Fourier transform, we have $\liminf_{x\to-\infty} U = \limsup_{x\to-\infty} U = 1$, see [1, 6, 25, 31]. On the contrary, from our analysis the positive solution of

$$\frac{\partial u}{\partial t} = \Delta u + u \left(1 - \int_{\mathbb{R}^n} u(t, y)\, dy\right), \tag{1.11}$$

converges uniformly to 0, which is actually the only non-negative stationary solution.

We mention also a recent related study [2] on a spatial demo-genetic model

$$\frac{\partial u}{\partial t}(t, x, y) = \Delta u(t, x, y) + u \left(r(x - By) - \int_{\mathbb{R}} u(t, x, y')\, dy'\right), \tag{1.12}$$

which can be viewed as an extension of (1.1) where a spatial local adaptation is taken into account. The interplay between the space variable x and the phenotypical trait variable y corresponding to local adaptation is modelled through the growth term $r(x - By)$ which is a function taking its maximum at 0. Generalisation of (1.12) have been studied in [3, 33]

The extension of Theorems 1.2, 1.3 and 1.4 for mutation-selection equations involving a mutation kernel such as

$$\frac{\partial u}{\partial t} = u \left(r(x) - \int_\Omega k_\epsilon(y)|u|^p(t, y)\, dy\right) + \int_\Omega M(x, y)[u(t, y) - u(t, x)]\, dy \quad \text{in} \quad \mathbb{R}^+ \times \Omega \tag{1.13}$$

is still a work in progress. However, although the technique and tools developed in this article are quite robust and can be applied in many situation, the lack of regularity of the positive solutions to (1.13) introduces some strong difficulty that cannot be easily overcome. Moreover, it has been proved by the author that such nonlocal problem can generates blow up phenomena, i.e. $u(x, t) \rightharpoonup \delta_{x_0} + g$ with δ_{x_0} the Dirac mass and g a singular L^1 function. This blow up phenomena is in accordance with a recent result showing that in some situation the only stationary solution to (1.13) are positive measure having a non-zero singular part [18]. The understanding of the long time behaviour of the positive solution to (1.13) require then the development of new analytical tools in order to analyse these blow-up phenomena.

This paper is organised as follows. The Section 2 is dedicated to the nonlinear relative entropies and some functional inequalities that we will frequently use along this article. Next, we prove in Section 3 the Theorem 1.2. Finally in Section 4 and 5 we prove the existence of positive steady states (Theorem 1.3) and the global stability (Theorem 1.4). A construction of a smooth positive solution to the Cauchy problem is made in the appendix.

2. Non-linear relative entropy identities and related functional inequality

In this section we first establish a general identity which can be assimilated to a nonlinear relative entropy principle. We consider a parabolic equation of the form

$$\frac{\partial u}{\partial t}(t, x) = u(t, x)(r(x) - \Psi(x, u)(t)) + \nabla \cdot (A(x)\nabla u(t, x)) \quad \text{in} \quad \mathbb{R}^+ \times \Omega, \tag{2.1}$$

$$\frac{\partial u}{\partial n}(t, x) = 0, \quad \text{in} \quad \mathbb{R}^+ \times \partial\Omega \tag{2.2}$$

where $\Psi(x, u)(t)$ denotes $\Psi(x, u)(t) := \int_\Omega K(x, y)|u|^p(t, y)\, dy$. Then for any solution of (2.1)–(2.2) we have

Theorem 2.1 (General Identity)**.** *Let H be a smooth (at least C^2) function. Let $\bar{u} > 0$ and u be two smooth solutions of* (2.1)–(2.2)*. Assume further that $\bar{u}$ is a stationary solution of* (2.1)–(2.2)*. Then we have*

$$\frac{d\mathcal{H}_{H,\bar{u}}[u](t)}{dt} = -\mathcal{D}(u) + \int_\Omega \bar{u}(x) H'\left(\frac{u}{\bar{u}}(t,x)\right)\Gamma(t,x)u(t,x)\,dx \tag{2.3}$$

where $\mathcal{H}_{H,\bar{u}}[u](t)$, $\mathcal{D}$ are the following quantity:

$$\Gamma(t,x) := \Psi(x,\bar{u}) - \Psi(x,u)$$
$$\mathcal{H}_{H,\bar{u}}[u](t) := \int_\Omega \bar{u}^2(x) H\left(\frac{u(x)}{\bar{u}(x)}\right)\,dx$$
$$\mathcal{D}(u) := \int_\Omega \bar{u}^2(x) H''\left(\frac{u(x)}{\bar{u}(x)}\right)\left(\nabla\left(\frac{u}{\bar{u}}\right)\right)^t A(x)\nabla\left(\frac{u}{\bar{u}}\right)\,dx$$

where $(\vec{a})^t$ denotes the transpose of a vector of $\mathbb{R}^N$.

Proof:

By (2.1), by defining $\Gamma(t,x) := \Psi(x,\bar{u}(x)) - \Psi(x,u(t,x))$ we have

$$\frac{\partial u}{\partial t} = (r(x) - \Psi(x,\bar{u})u + \nabla\cdot(A(x)\nabla u)) + \Gamma(t,x)u(x) \tag{2.4}$$

Using that $\bar{u}$ is also a stationary solution, we have for all x

$$(r(x) - \Psi(x,\bar{u}))\bar{u} = -\nabla\cdot(A(x)\nabla\bar{u}),$$

and we can rewrite the above equation as follows

$$\frac{\partial u(x)}{\partial t} = \nabla\cdot(A(x)\nabla u) - \frac{u}{\bar{u}}\nabla\cdot(A(x)\nabla\bar{u}) + \Gamma(t,x)u(x)$$

By multiplying the above equality by $\bar{u}(x)H'\left(\frac{u(x)}{\bar{u}(x)}\right)$ and by integrating over Ω we achieve

$$\begin{aligned}\int_\Omega \bar{u}(x)H'\left(\frac{u(x)}{\bar{u}(x)}\right)\frac{\partial u(x)}{\partial t}\,dx &= \int_\Omega \bar{u}(x)H'\left(\frac{u(x)}{\bar{u}(x)}\right)\Gamma(t,x)u(x)\,dx \\ &+ \int_\Omega H'\left(\frac{u(x)}{\bar{u}(x)}\right)\left[\bar{u}(x)\,\nabla\cdot(A(x)\nabla u) - u(x)\,\nabla\cdot(A(x)\nabla\bar{u}(x))\right]dx.\end{aligned} \tag{2.5}$$

By integrating by part the last term and rearranging the terms, it follows that

$$\begin{aligned}\int_\Omega \bar{u}(x)H'\left(\frac{u(x)}{\bar{u}(x)}\right)\frac{\partial u(x)}{\partial t}\,dx &= \int_\Omega \bar{u}(x)H'\left(\frac{u(x)}{\bar{u}(x)}\right)\Gamma(t,x)u(x)\,dx \\ &- \int_\Omega \bar{u}^2(x)H''\left(\frac{u(x)}{\bar{u}(x)}\right)\left(\nabla\left(\frac{u}{\bar{u}}\right)\right)^t A(x)\nabla\left(\frac{u}{\bar{u}}\right)\,dx.\end{aligned} \tag{2.6}$$

Hence, we have

$$\frac{d\mathcal{H}_{H,\bar{u}}[u](t)}{dt} = \int_\Omega \bar{u}(x)H'\left(\frac{u(x)}{\bar{u}(x)}\right)\Gamma(t,x)u(x)\,dx - \mathcal{D}(u).$$

□

Remark 2.2. We want to stress that if we replace $\bar{u}$ by any positive function $\widetilde{u}$ satisfying

$$\nabla\cdot(A(x)\nabla\widetilde{u}(x)) = -\widetilde{u}(x)\left(r(x) - \widetilde{\Psi}(x,\widetilde{u})(t)\right) \quad \text{in} \quad \Omega,$$
$$\frac{\partial\widetilde{u}}{\partial n}(x) = 0, \quad \text{in} \quad \partial\Omega$$

it will affect the equality in Theorem 2.1 only through the term Γ which will be transform into

$$\Gamma(t,x) = \widetilde{\Psi}(x,\widetilde{u}(x)) - \Psi(x,u(t,x)).$$

Remark 2.3. Under the extra assumption $\frac{u}{\bar{u}} \in L^\infty(\Omega)$, we remark that the formulas will holds as well if we consider homogeneous Dirichlet boundary conditions instead of Neumann boundary conditions. It is worth noticing that this extra condition is always satisfied in the Neumann case since for all positive stationary solution with homogeneous Neumann Boundary condition, we can show that $\inf_{\bar{\Omega}} \bar{u} > 0$.

Remark 2.4. We remark that the above formula do not require any particular assumption on the Ψ and as a consequence no particular assumption on the kernel K. Thus the formula holds as well for $K(x,y) = \delta_0$, which turns the equation (2.1) into a semi-linear PDE. In particular when $\Psi(x,u)$ is independent of u $i.ep = 0, K = \delta_0$ then the formula in Theorem 2.1 is known as the standard relative entropy principle for linear equations see [30].

Next we establish a useful functional inequality satisfied by vectors $h \in \bar{v}^\perp$ where $\bar{v}^\perp$ denotes the linear subspace of $H^1(\Omega)$:

$$\bar{v}^\perp := \left\{ h \in H^1(\Omega) \;\middle|\; \int_\Omega h\bar{v} = 0, \quad \bar{v}\nabla h \cdot n - h\nabla\bar{v}\cdot n = 0 \quad \text{on} \quad \partial\Omega \right\}$$

Lemma 2.5. *Let $\bar{v}$ be a smooth ($C^{1,\alpha}(\Omega)$) positive bounded function in Ω, so that $\inf_{\bar{\Omega}} \bar{v} > 0$. Then there exists $\rho_1 > 0$ so that for all $h \in \bar{v}^\perp$*

$$\rho_1 \|h\|^2_{L^2(\Omega)} \leq \int_\Omega \bar{v}^2 \left(\nabla\left(\frac{h}{\bar{v}}\right)\right)^t A(x) \nabla\left(\frac{h}{\bar{v}}\right).$$

Moreover $\rho_1 = \lambda_2$ where λ_2 is the second eigenvalue of the linear eigenvalue problem

$$\nabla\cdot\left(A(x)\bar{v}^2\nabla\left(\frac{h}{\bar{v}}\right)\right) = -\lambda h\bar{v} \quad \textit{in} \quad \Omega$$
$$\bar{v}\frac{\partial h}{\partial n} - h\frac{\partial \bar{v}}{\partial n} = 0 \quad \textit{in} \quad \partial\Omega$$

Proof :

Let $\mathcal{I}$ be the following functional in $H^1(\Omega)$,

$$\mathcal{I}(h) := \frac{1}{\|h\|_2^2} \int_\Omega \bar{v}^2 \left(\nabla\left(\frac{h}{\bar{v}}\right)\right)^t A(x) \nabla\left(\frac{h}{\bar{v}}\right). \tag{2.7}$$

Observe that from the homogeneity of the L^2 norm we have

$$\inf_{h\in\bar{v}^\perp, \|h\|_2=1} \mathcal{I}(h) = \inf_{h\in\bar{v}^\perp} \mathcal{I}(h), \tag{2.8}$$

and the first part of the Lemma is proved if we show that

$$\inf_{h\in\bar{v}^\perp, \|h\|_2=1} \mathcal{I}(h) > 0, \tag{2.9}$$

Let $d\mu$ denotes the positive measure $\bar{v}^2 dx$, then by construction $d\mu$ is absolutely continuous with respect to the Lebesgue measure and vice versa. So the Hilbert functional spaces $L^2_{d\mu}$ and $H^1_{d\mu}$ below are well defined :

$$L^2_{d\mu}(\Omega) := \left\{ u \;\middle|\; \int_\Omega u^2(x) d\mu(x) < +\infty \right\},$$
$$H^1_{d\mu}(\Omega) := \left\{ u \in L^2_{d\mu}(\Omega) \;\middle|\; \int_\Omega |\nabla u|^2(x) d\mu(x) < +\infty \right\}.$$

Moreover the Rellich-Kondrakov compact embedding $H^1_{d\mu}(\Omega) \hookrightarrow L^2_{d\mu}(\Omega)$ holds [27]. To obtain (2.9), we argue as follows. Let $(h_n)_{n\in\mathbb{N}}$ be a minimising sequence, by (2.9) we can take $(h_n)_{n\in\mathbb{N}}$ so that $h_n \in \bar{v}^\perp$, $\|h_n\|_2 = 1$ for all n. Let $g_n := \frac{h_n}{\bar{v}}$, then by straightforward computation, from (2.7) – (2.9), we see that $(g_n)_{n\in\mathbb{N}}$ is a minimising sequence of the functional

$$\mathcal{J}(g) := \frac{1}{\|g\|_{L^2_\mu(\Omega)}} \int_\Omega (\nabla(g))^t A(x) \nabla(g)\, d\mu,$$

satisfying for all n, $\|g_n\|_{L^2_\mu(\Omega)} = \|h_n\|_2 = 1$. Moreover, we have for all n, $\frac{\partial g_n}{\partial n} = 0$ on $\partial\Omega$ and

$$\int_\Omega g_n(x)\, d\mu(x) = \int_\Omega h_n(x)\bar{v}(x)\, dx = 0. \tag{2.10}$$

We can also easily verify that

$$\inf_{h\in\bar{v}^\perp, \|h\|_2=1} \mathcal{I}(h) = \inf_{g\in H^1_{d\mu}, \int_\Omega g\, d\mu=0} \mathcal{J}(g).$$

By construction the sequence $(g_n)_{n\in\mathbb{N}}$ is uniformly bounded in $H^1_{d\mu}(\Omega)$ and thanks to Rellich-Kondrakov compact embedding, there exists a subsequence $(g_{n_k})_{k\in N}$ which converges weakly in $H^1_{d\mu}(\Omega)$ and strongly in $L^2_{d\mu}(\Omega)$ to some $\widetilde{g} \in H^1_{d\mu}(\Omega)$. Moreover, $\widetilde{g}$ is a weak solution of

$$\nabla\cdot\left(A(x)\bar{v}^2\nabla\left(\widetilde{g}\right)\right) = -\lambda\widetilde{g}\bar{v}^2 \quad \text{in} \quad \Omega, \tag{2.11}$$

$$\frac{\partial\widetilde{g}}{\partial n} = 0 \tag{2.12}$$

for some $\lambda \in \mathbb{R}$. Furthermore $\widetilde{g}$ satisfies

$$\int_\Omega \widetilde{g}(x)\, d\mu(x) = 0. \tag{2.13}$$

Now assume that $\lambda = 0$, then the above equations (2.11)–(2.13) enforce $\widetilde{g} = 0$ leading to the contradiction $0 = \|\widetilde{g}\|_{L^2_\mu(\Omega)} = 1$. Therefore $\lambda \neq 0$ and (2.9) holds.

Now, since $A(x)$ and $\bar{v}$ are smooth and μ is absolutely continuous with respect to the Lebesgue measure, by standard elliptic regularity we have $\widetilde{g} \in C^{2,\alpha}(\Omega)$ for some α and the function $\widetilde{h} := \bar{v}\widetilde{g} \in C^2$ satisfies

$$\begin{aligned}
&\nabla\cdot\left(A(x)\bar{v}^2\nabla\left(\frac{\widetilde{h}}{\bar{v}}\right)\right) = -\lambda\widetilde{h}\bar{v} \quad \text{in} \quad \Omega,\\
&\int_\Omega \widetilde{h}\bar{v}\, dx = 0,\\
&\bar{v}\frac{\partial h}{\partial n} - h\frac{\partial\bar{v}}{\partial n} = 0 \quad \text{in} \quad \partial\Omega.
\end{aligned}$$

Now by dividing (2.11) by $\bar{v}^2$ we get the following eigenvalue problem

$$\begin{aligned}
&\frac{1}{\bar{v}^2}\nabla\cdot\left(A(x)\bar{v}^2\nabla g\right) = -\lambda g \quad \text{in} \quad \Omega,\\
&\frac{\partial g}{\partial n} = 0 \quad \text{in} \quad \partial\Omega,
\end{aligned}$$

From standard Theory [27] there exists a sequence $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ of eigenvalue of the above problem. Moreover there exists an orthonormal basis $\{\psi_k\}_{k=1}^\infty$ of L^2, so that ψ_k satisfies

$$\begin{aligned}
&\frac{1}{\bar{v}^2}\nabla\cdot\left(A(x)\bar{v}^2\nabla\psi_k\right) = -\lambda_k\psi_k \quad \text{in} \quad \Omega,\\
&\frac{\partial\psi_k}{\partial n} = 0 \quad \text{in} \quad \partial\Omega.
\end{aligned}$$

By setting $\phi_k := \frac{\psi_k}{\bar{v}}$, we can check that

$$\nabla\cdot\left(A(x)\bar{v}^2\nabla\left(\frac{\phi_k}{\bar{v}}\right)\right) = -\lambda_k\phi_k\bar{v} \quad \text{in} \quad \Omega, \tag{2.14}$$

$$\bar{v}\frac{\partial\phi_k}{\partial n} - \phi_k\frac{\partial\bar{v}}{\partial n} = 0 \quad \text{in} \quad \partial\Omega. \tag{2.15}$$

Here since $(0, \bar{v})$ is a solution to (2.14)–(2.15) and $\bar{v} > 0$, we see that $\phi_1 = \bar{v}$ and $\lambda_1 = 0$. So

$$\inf_{h\in\bar{v}^\perp, \|h\|_2=1} \mathcal{I}(h) = \lambda_2,$$

since the λ_i are ordered and $\phi_2 \in \bar{v}^\perp$.

□

3. THE BLIND COMPETITION CASE:

In this section we analyse the asymptotic behaviour of a positive smooth solution to (1.1)–(1.3) when the competition kernel $K(x,y)$ is independent of x, i.e $K(x,y) = k(y)$ with k satisfying (1.4). As we expressed in Theorem 1.2 that we recall below, in this situation the problem (1.5)–(1.6) has a unique positive stationary solution which attracts all the trajectories initiated from any nonnegative and non zero initial data. More precisely, we prove

Theorem 3.1. *Assume A, r, k satisfy* (1.4) *and $p \geq 1$.Let λ_1 be the first eigenvalue of the problem*

$$\nabla \cdot (A(x)\nabla\phi(x)) + r(x)\phi(x) = -\lambda\phi(x) \quad in \quad \Omega, \tag{3.1}$$

$$\frac{\partial\phi(x)}{\partial n} = 0 \quad on \quad \partial\Omega, \tag{3.2}$$

then we have the following asymptotic behaviour for any positive smooth solution $u(t,x)$ to (1.5)–(1.6)

- *if $\lambda_1 \geq 0$, there is no positive stationary solution and $u(t,x) \to 0$ as $t \to \infty$*
- *if $\lambda_1 < 0$, then*

$$u(t,x) \to \mu\phi_1$$

where $\mu = \left(\frac{-\lambda_1}{\int_\Omega k(y)|\phi_1(y)|^p\,dy}\right)^{\frac{1}{p}}$ and ϕ_1 is the positive eigenfunction associated to λ_1 normalized by $\|\phi_1\|_{L^2(\Omega)} = 1$.

In the sequel of this section to simplify the presentation we introduce the notation

$$\Psi(u) := \int_\Omega k(z)|u(y)|^p\,dy.$$

Before proving the Theorem, we start by establishing some useful Lemmas.

Lemma 3.2. *Assume $\lambda_1 < 0$, then there exists $\mu > 0$ so that $\mu\phi_1$ is a positive stationary solution of* (1.5).

Proof:

Let us normalised ϕ_1 by $\|\phi_1\|_{L^2(\Omega)} = 1$. Then, by plugging $\mu\phi_1$ in (1.5), we end up finding μ so that

$$\Psi(\mu\phi_1) = -\lambda_1.$$

Thus for $\mu = \left(\frac{-\lambda_1}{\int_\Omega k(y)|\phi_1(y)|^p\,dy}\right)^{\frac{1}{p}}$, $\mu\phi_1$ is a stationary solution of (1.5).

□ .

Next, we establish some useful identities. Namely, we show

Lemma 3.3. *Let $q \geq 1$ and H be the smooth convex function $H(s) : s \mapsto s^q$. Let $\bar{u}$ be a positive stationary solution of* (1.5)-(1.6)*, then a positive smooth solution $u(t,x)$ of* (1.5)–(**??**) *satisfies*

$$\frac{d\mathcal{H}_{q,\bar{u}}[u](t)}{dt} = -q(q-1)\int_\Omega \left(\frac{u(t,x)}{\bar{u}(x)}\right)^{q-2} \bar{u}^2 \left(\nabla\left(\frac{u(t,x)}{\bar{u}(x)}\right)\right)^t A(x)\nabla\left(\frac{u(t,x)}{\bar{u}(x)}\right) dx + q(\Psi(\bar{u}) - \Psi(u))\mathcal{H}_{q,\bar{u}}[u](t). \tag{3.3}$$

where $\mathcal{H}_{q,\bar{u}}[u](t) := \int_\Omega \bar{u}^2(x)\left(\frac{u(t,x)}{\bar{u}(x)}\right)^q dx$. Furthermore, the functional $\mathcal{F}(u) := \log\left(\frac{\mathcal{H}_{q,\bar{u}}[u](t)}{\left(\mathcal{H}_{1,\bar{u}}[u](t)\right)^q}\right)$ satisfies:

$$\frac{d}{dt}\mathcal{F}(u) = -\frac{q(q-1)}{\mathcal{H}_{q,\bar{u}}[u](t)}\int_\Omega \left(\frac{u(t,x)}{\bar{u}(x)}\right)^q \bar{u}^2 \left(\nabla\left(\frac{u(t,x)}{\bar{u}(x)}\right)\right)^t A(x)\nabla\left(\frac{u(t,x)}{\bar{u}(x)}\right) dx. \tag{3.4}$$

Remark 3.4. Note that in the particular case of $H(s) = s^2$, $\mathcal{H}_{2,\bar{u}}[u] = \|u\|_2^2$. So we get a Lyapunov functional involving the L^2 norm of u instead of a weighted L^q norm of u. Indeed, we have

$$\frac{\partial}{\partial t}\left(\log\left(\frac{\|u\|_2^2}{\left(\mathcal{H}_{1,\bar{u}}[u]\right)^2}\right)\right) = -\frac{2}{\|u\|_2^2}\int_\Omega \bar{u}^2 \left(\nabla\left(\frac{u(t,x)}{\bar{u}(x)}\right)\right)^t A(x)\nabla\left(\frac{u(t,x)}{\bar{u}(x)}\right) dx.$$

Proof:

The identity (3.3) is a straightforward consequence of Lemma 2.1. Indeed, for $H(s) := s^q$, by the Theorem 2.1 we have:

$$\frac{d\mathcal{H}_{_{q,\bar{u}}}[u](t)}{dt} = -\mathcal{D}(u) + \int_\Omega \bar{u}(x) H'\left(\frac{u(t,x)}{\bar{u}(x)}\right)\Gamma(x)u(x)\,dx$$

where Γ, $\mathcal{D}$ are the following quantity:

$$\Gamma(u(t)) := \Psi(\bar{u}) - \Psi(u)$$

$$\mathcal{D}(u) := \int_\Omega H''\left(\frac{u(t,x)}{\bar{u}(x)}\right)\bar{u}^2(x)\left(\nabla\left(\frac{u(t,x)}{\bar{u}(x)}\right)\right)^t A(x)\nabla\left(\frac{u(t,x)}{\bar{u}(x)}\right)\,dx$$

By observing that $\bar{u}(x)u(x)H'\left(\frac{u(t,x)}{\bar{u}(x)}\right) = q\mathcal{H}_{_{q,\bar{u}}}[u](t)$ and that Γ is independent of x, we see that

$$\frac{d\mathcal{H}_{_{q,\bar{u}}}[u](t)}{dt} = -\mathcal{D}(u) + q\Gamma\mathcal{H}_{_{q,\bar{u}}}[u](t),$$

and the formula (3.3) holds.

To obtain (3.4), we observe that by taking $q = 1$ in the formula (3.3) we get

$$\frac{d\mathcal{H}_{_{1,\bar{u}}}[u](t)}{dt} = \Gamma\mathcal{H}_{_{1,\bar{u}}}[u](t).$$

Since $\mathcal{H}_{_{1,\bar{u}}}[u](t) = \int_\Omega u(t,x)\bar{u}(x)\,dx > 0$ for all times we see that

$$\frac{d}{dt}\log(\mathcal{H}_{_{1,\bar{u}}}[u](t)) = (\Psi(\bar{u}) - \Psi(u)). \tag{3.5}$$

Similarly, since $\mathcal{H}_{_{q,\bar{u}}}[u](t) > 0$ for all times we have also

(3.6)

$$\frac{d}{dt}\log(\mathcal{H}_{_{q,\bar{u}}}[u](t)) = -\frac{q(q-1)}{\mathcal{H}_{_{q,\bar{u}}}[u](t)}\int_\Omega\left(\frac{u(t,x)}{\bar{u}(x)}\right)^{q-2}\bar{u}^2(x)\left(\nabla\left(\frac{u(t,x)}{\bar{u}(x)}\right)\right)^t A(x)\nabla\left(\frac{u(t,x)}{\bar{u}(x)}\right)\,dx + q(\Psi(\bar{u}) - \Psi(u)).$$

By combining (3.5) and (3.6) we end up with

$$\frac{d}{dt}\left(\log\left(\frac{\mathcal{H}_{_{q,\bar{u}}}[u](t)}{(\mathcal{H}_{_{1,\bar{u}}}[u](t))^q}\right)\right) = -\frac{q(q-1)}{\mathcal{H}_{_{q,\bar{u}}}[u](t)}\int_\Omega\left(\frac{u(t,x)}{\bar{u}(t,x)}\right)^{q-2}\bar{u}^2(x)\left(\nabla\left(\frac{u(t,x)}{\bar{u}(x)}\right)\right)^t A(x)\nabla\left(\frac{u(t,x)}{\bar{u}(x)}\right)\,dx.$$

□

As a straightforward application of this Lemma, we deduce the following *a priori* estimates on the solution of (1.5)–(1.7). Namely, we have

Lemma 3.5. *Let $u(t,x) \in C^1((0,+\infty), C^{2,\alpha}(\Omega))$ be a positive solution of* (1.5)-(1.6) *then for all $q \geq 1$ there exists a positive constant $c_q(q, u(x,1)) < C_q(q, u(x,1))$ so that for all $t \geq 1$*

$$c_q \leq \|u\|_{L^q(\Omega)} \leq C_q.$$

Proof:

Let us first show that for all $q \geq 1$ then there exists $C_q(q, u(x,1))$ so that for all $t \geq 1$

$$\|u\|_{L^q(\Omega)} \leq C_q. \tag{3.7}$$

First, let us obtain an upper bound for u when $q = 1$. By Lemma 3.3, we have

$$\frac{d\mathcal{H}_{_{1,\mu\phi_1}}[u](t)}{dt} = (\Psi(\mu\phi_1) - \Psi(u))\mathcal{H}_{_{1,\mu\phi_1}}[u](t),$$

where $\mu\phi_1$ is the stationary solution constructed in Lemma 3.2. By using the definition of Ψ and $\mathcal{H}_{_{1,\mu\phi_1}}[u](t)$, and Hölder's inequality , we have for some $c_0 > 0$

$$\frac{d\mathcal{H}_{_{1,\phi_1}}[u](t)}{dt} \leq \left[\lambda_1 - c_0\left(\int_\Omega |u(t,y)|\,dy\right)^p\right]\mathcal{H}_{_{1,\phi_1}}[u](t).$$

Since $\|u\|_{L^1(\Omega)} \sim \mathcal{H}_{1,\phi_1}[u](t)$, we get for some $\widetilde{c}_0$

$$\frac{d\mathcal{H}_{1,\phi_1}[u](t)}{dt} \leq \left[\lambda_1 - \widetilde{c}_0 \left(\mathcal{H}_{1,\phi_1}[u](t)\right)^p\right] \mathcal{H}_{1,\phi_1}[u](t).$$

So $\mathcal{H}_{1,\phi_1}[u](t)$ satisfies a logistic differential inequation, therefore there exists $C_1(u(x,1)) > 0$ so that for all $t \geq 1$,

$$\mathcal{H}_{1,\phi_1}[u](t) \leq C_1. \tag{3.8}$$

Now we can get an upper bounded for u for all $q \geq 1$. Indeed, let us assume that $q > 1$ then by a straightforward application of the Lemma 3.3 we have for all $q > 1$ and for all $t \geq 1$,

$$\mathcal{H}_{q,\mu\phi_1}[u](t) \leq \left(\mathcal{H}_{1,\mu\phi_1}[u](t)\right)^q \left(\frac{\mathcal{H}_{q,\mu\phi_1}[u](1)}{\left(\mathcal{H}_{1,\mu\phi_1}[u](1)\right)^q}\right).$$

By using the homogeneity of the norm $\mathcal{H}_{q,\mu\phi_1}[u]$ and (3.8) we see that for all $q > 1$ and for all $t \geq 1$,

$$\mathcal{H}_{q,\phi_1}[u](t) \leq \left(\mathcal{H}_{1,\phi_1}[u](t)\right)^q \left(\frac{\mathcal{H}_{q,\phi_1}[u](1)}{\left(\mathcal{H}_{1,\phi_1}[u](1)\right)^q}\right) \leq C_1^q \left(\frac{\mathcal{H}_{q,\phi_1}[u](1)}{\left(\mathcal{H}_{1,\phi_1}[u](1)\right)^q}\right).$$

Since for $q \geq 1$ $\|u\|_{L^q(\Omega)} \sim \mathcal{H}_{q,\phi_1}[u]$, (3.7) holds.

To prove the lower bound for u, by Hölder's inequality, it is enough to have a lower bound for $\|u\|_{L^1(\Omega)}$. Recall that $\mathcal{H}_{1,\mu\phi_1}[u](t)$ satisfies

$$\frac{d\mathcal{H}_{1,\mu\phi_1}[u](t)}{dt} = \left(\Psi(\mu\phi_1) - \int_\Omega k(y)|u(t,y)|^p\, dy\right) \mathcal{H}_{1,\mu\phi_1}[u](t).$$

Since (3.7) holds for all $q \geq 1$, by interpolation there exits positive constants C, α so that for all $t > 1$ $\|u\|_{L^p(\Omega)} \leq C\|u\|^\alpha_{L^1(\Omega)}$. Therefore $\mathcal{H}_{1,\mu\phi_1}[u](t)$ satisfies for all $t > 1$

$$\frac{d\mathcal{H}_{1,\mu\phi_1}[u](t)}{dt} \geq \left(\Psi(\mu\phi_1) - C^p|k|_\infty \mathcal{H}_{1,\mu\phi_1}[u]^{\alpha p}\right) \mathcal{H}_{1,\mu\phi_1}[u](t).$$

By using the logistic character of the above differential inequation, we deduce that $\mathcal{H}_{1,\mu\phi_1}[u](t) \geq c_1(u(x,1))$ for all $t > 1$.

□

We are now in position to prove the Theorem 1.2.

Proof of Theorem 1.2:

Let $u(t,x) \in C^1((0,+\infty), C^{2,\alpha}(\Omega))$ be a positive solution of (1.5)–(1.6). Assume first that $\lambda_1 < 0$. Since $u > 0$ then u is a sub-solution of

$$\frac{\partial v(t,x)}{\partial t} = \nabla \cdot (A(x)\nabla v(t,x)) + r(x)v(t,x) \quad \text{in} \quad \mathbb{R}^+ \times \Omega \tag{3.9}$$

$$\frac{\partial v(t,x)}{\partial n} = 0 \quad \text{in} \quad \mathbb{R}^+ \times \partial\Omega \tag{3.10}$$

$$v(x,0) = u(1,x) \quad \text{in} \quad \Omega. \tag{3.11}$$

Since $\lambda_1 > 0$ and $u(1,x) \in L^\infty$, for a large constant $Ce^{\lambda_1 t}\phi_1(x)$ is then a super-solution of (3.9)–(3.11) and by the parabolic maximum principle we have

$$u(x,t) \leq Ce^{\lambda_1 t}\phi_1(x) \to 0 \quad \text{as} \quad t \to \infty.$$

Now let us assume that $\lambda_1 = 0$. In this situation, by Lemma 3.3 and using Remark (2.2), we observe that for all $q \geq 1$ we have,

$$\frac{d\mathcal{H}_{q,\phi_1}[u](t)}{dt} = -q(q-1)\int_\Omega \left(\frac{u(t,x)}{\phi_1(x)}\right)^{q-2} \phi_1^2 \left(\nabla\left(\frac{u(t,x)}{\phi_1(x)}\right)\right)^t A(x)\nabla\left(\frac{u(t,x)}{\phi_1(x)}\right) dx - q\Psi(u))\mathcal{H}_{q,\phi_1}[u](t).$$

Therefore, since $\Psi(u)$ is non-negative, we get $\|\nabla u\|_2 \to 0$ and for all $q \geq 1$ $\|u\|_{L^q(\Omega)} \to 0$ as $t \to +\infty$. Since the coefficients of the parabolic equation are uniformly bounded, by a bootstrap argument using the Parabolic regularity, we get $\|u\|_\infty \to 0$ as $t \to \infty$.

Lastly, we assume $\lambda_1 < 0$ and let us denote $<,>$ the standard scalar product of $L^2(\Omega)$. Let $\bar{u}$ be the stationary solution of (1.5)– (1.6) constructed in Lemma 3.2 , i.e $\bar{u} := \mu\phi_1$. Since for all $t > 0$, the solution $u(t,x) \in L^\infty$, then we can decompose u the following way:

$$u(t,x) := \lambda(t)\bar{u}(x) + h(t,x)$$

with h so that $< \phi_1, h >= 0$.

Substituting u by this decomposition in (1.5) and using the equation satisfied by $\bar{u}$ it follows that

(3.12)

$$\lambda'(t)\bar{u}(x) + \frac{\partial h(t,x)}{\partial t} = (\lambda_1 - \Psi(u(t)))\lambda(t)\bar{u}(x) + (r(x) - \Psi(u))h(t,x) + \nabla\cdot(A(x)\nabla(h(t,x))).$$

By multiplying the above equation by h and integrating over Ω, it follows that

$$< \frac{\partial h(t)}{\partial t}, h >=< (r(x) - \Psi(u))h + \nabla\cdot(A(x)\nabla(h)), h > .$$

where we use that h is orthogonal to $\bar{u}$. Thus since $\mathcal{H}_{_{2,\bar{u}}}[h](t) := \|h(t)\|^2_{L^2(\Omega)}$, we have

$$< \frac{\partial h}{\partial t}, h >= \frac{1}{2}\frac{d\mathcal{H}_{_{2,\bar{u}}}[h](t)}{dt} =< (r - \Psi(u))h + \nabla\cdot(A\nabla(h)), h > .$$

By following the computation developed for the proof of Theorem 2.1 with $H(s) = s^2$, we see that

$$\frac{d\mathcal{H}_{_{2,\bar{u}}}[h](t)}{dt} = -\int_\Omega \bar{u}^2(x)\left(\nabla\left(\frac{h(t,x)}{\bar{u}(x)}\right)\right)^t A(x)\nabla\left(\frac{h(t,x)}{\bar{u}(x)}\right) + (\lambda_1 - \Psi(u(t))\mathcal{H}_{_{2,\bar{u}}}[h](t). \tag{3.13}$$

Since $\mathcal{H}_{_{2,\bar{u}}}[h](t) \geq 0$ for all times, let us analyse separately the two situations: $\mathcal{H}_{_{2,\bar{u}}}[h](t) > 0$ for all times t or there exists $t_0 \in \mathbb{R}$ so that $\mathcal{H}_{_{2,\bar{u}}}[h](t_0) = 0$. In the latter case, from the above equation we see that we must have $\mathcal{H}_{_{2,\bar{u}}}[h](t) = 0$ for all $t \geq t_0$ and so for all $t \geq t_0$, we must have $u(t) = \lambda(t)\bar{u}$ almost everywhere. Hence from (3.12) we are reduced to analyse the following ODE equation

$$\lambda'(t) = \lambda(t)(\lambda_1 - \widetilde{\Psi}(\lambda(t)))$$

where $\widetilde{\Psi}$ is the increasing locally Lipschitz function defined by $\widetilde{\Psi}(s) := s^p\int_\Omega k(y)\bar{u}(y)^p\,dx$.

Note that since by Lemma 3.5 we have

$$\lambda(t) < \bar{u}, \bar{u} >=< \bar{u}, u >= \mathcal{H}_{_{1,\bar{u}}}[u](t) \leq C_1, \tag{3.14}$$

we have $\lambda(t) \geq 0$ for all times t. The above ODE is of logistic type with non negative initial datum therefore by a standard argumentation we see that $\lambda(t)$ converges to $\bar{\lambda} > 0$ where $\bar{\lambda}$ is the unique solution of $\widetilde{\Psi}(\bar{\lambda}) = \lambda_1$. By construction we have $\widetilde{\Psi}(1) = \lambda_1$, so we deduce that $\bar{\lambda} = 1$. Hence, in this situation, u converges pointwise to $\bar{u}$ as time goes to infinity.

In the other situation, $\mathcal{H}_{_{2,\bar{u}}}[h](t) > 0$ for all t and we claim that

Claim 3.6. *$\mathcal{H}_{_{2,\bar{u}}}[h](t) \to 0$ as $t \to +\infty$.*

Assume the Claim holds true then we can conclude the proof by arguing as follows. From the decomposition $u(t,x) = \lambda(t)\bar{u}(x)+h(t,x)$, we can express the function $\mathcal{H}_{_{1,\bar{u}}}[u](t)$ by $\mathcal{H}_{_{1,\bar{u}}}[u](t) =< u, \bar{u} >= \lambda(t) < \bar{u}, \bar{u} >$. Therefore by using Theorem 3.3 we deduce that

$$\lambda'(t) = (\lambda_1 - \Psi[\lambda(t)\bar{u}(x) + h(t,x)])\,\lambda(t). \tag{3.15}$$

By using the definition of Ψ and the binomial expansion it follows that λ verifies the following ODE

$$\begin{aligned}\lambda'(t) &= (\lambda_1 - \widetilde{\Psi}(\lambda(t)))\lambda(t) + \lambda(t)(\Psi(\lambda(t)\bar{u}(x)) - \Psi(\lambda(t)\bar{u} + h(t)))\\ &= (\lambda_1 - \widetilde{\Psi}(\lambda(t)))\lambda(t) + \lambda(t)\left(\sum_{i=1}^{p}\binom{i}{p}\lambda^i(t)\int_\Omega \bar{u}^i h^{p-i}(t,x)\,dx\right),\end{aligned}$$

where $\binom{i}{p}$ denotes the binomial coefficient. Now by using $\|h(t)\|_2^2 = \mathcal{H}_{2,\bar{u}}[h](t) \to 0$ and Lemma 3.5, by interpolation we deduce that $\|h(t)\|_{L^q(\Omega)} \to 0$ for all $q \geq 1$. Therefore, since $\bar{u} \in L^\infty$ and by (3.14) λ is bounded, we have

$$\lim_{t\to\infty} \left(\sum_{i=1}^{p} \binom{i}{p} \lambda^i(t) \int_\Omega \bar{u}^i h^{p-i}(t,x)\, dx \right) = 0.$$

Thus λ satisfies

$$\lambda'(t) = (\lambda_1 - \widetilde{\Psi}(\lambda(t)))\lambda(t) + \lambda(t)o(1),$$

and as above we can conclude that $\lambda(t) \to 1$ and u converges to $\bar{u}$ almost everywhere.

□

Proof of Claim 3.6:

Since $\mathcal{H}_{2,\bar{u}}[h](t) > 0$ for all t, from (3.13) and by following the proof of Lemma 3.3 we see that

$$\frac{d}{dt} \log \left[\frac{\mathcal{H}_{2,\bar{u}}[h](t)}{\left(\mathcal{H}_{1,\bar{u}}[u](t)\right)^2} \right] = -\frac{1}{\mathcal{H}_{2,\bar{u}}[h](t)} \int_\Omega \bar{u}^2(x) \left(\nabla \left(\frac{h(t,x)}{\bar{u}(x)} \right) \right)^t A(x) \nabla \left(\frac{h(t,x)}{\bar{u}(x)} \right) dx. \tag{3.16}$$

Thus the function $\widetilde{F} := \log \left[\frac{\mathcal{H}_{2,\bar{u}}[h](t)}{\left(\mathcal{H}_{1,\bar{u}}[u](t)\right)^2} \right]$ is a decreasing smooth function.

First we observe that the claim is proved if there exists a sequence $(t_n)_{n\in\mathbb{N}}$ going to infinity so that $\mathcal{H}_{2,\bar{u}}[h](t_n) \to 0$. Indeed, assume such sequence exists and let $(s_k)_{k\in\mathbb{N}}$ be a sequence going to $+\infty$. Then there exists k_0 and a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ of $(t_n)_{n\in\mathbb{N}}$ so that for all $k \geq k_0$, we have $s_k \geq t_{n_k}$. Therefore from the monotonicity of $\widetilde{F}$ we have for all $k \geq k_0$

$$\log \left[\frac{\mathcal{H}_{2,\bar{u}}[h](s_k)}{\left(\mathcal{H}_{1,\bar{u}}[u](s_k)\right)^2} \right] \leq \log \left[\frac{\mathcal{H}_{2,\bar{u}}[h](t_{n_k})}{\left(\mathcal{H}_{1,\bar{u}}[u](t_{n_k})\right)^2} \right].$$

By letting k to infinity in the above inequality, we deduce that

$$\lim_{k\to\infty} \log \left[\frac{\mathcal{H}_{2,\bar{u}}[h](s_k)}{\left(\mathcal{H}_{1,\bar{u}}[u](s_k)\right)^2} \right] = -\infty,$$

which implies that $\mathcal{H}_{2,\bar{u}}[h](s_k) \to 0$, since by Lemma 3.5 $(\mathcal{H}_{1,\bar{u}}[u](t_k))_{k\in\mathbb{N}}$ is uniformly bounded. The sequence $(s_k)_{k\in\mathbb{N}}$ being chosen arbitrarily this implies that $\mathcal{H}_{2,\bar{u}}[h](t) \to 0$ as $t \to +\infty$.

Let us now prove that such sequence $(t_n)_{n\in\mathbb{N}}$ exists. Let us assume by contradiction that $\inf_{t\in\mathbb{R}^+} \mathcal{H}_{2,\bar{u}}[h](t) > 0$.

From the monotonicity and the smoothness of $\widetilde{F}$ we deduce that there is $c_0 \in \mathbb{R}$ so that

$$\widetilde{F}(h(t)) \to c_0 \quad \text{and} \quad \frac{d}{dt}\widetilde{F}(h(t)) \to 0 \quad \text{as} \quad t \to +\infty.$$

Thus by Lemma 3.5 and (3.16) it follows that

$$\lim_{t\to\infty} \int_\Omega \bar{u}^2(x) \left(\nabla \left(\frac{h(t,x)}{\bar{u}(x)} \right) \right)^t A(x) \nabla \left(\frac{h(t,x)}{\bar{u}(x)} \right) dx = 0. \tag{3.17}$$

Since for all t, $h(t) \in \bar{u}^\perp$, $\mathcal{H}_{2,\bar{u}}[h](t) = \|h(t)\|_2^2$ and $\bar{u} = \mu\phi_1 \in C^{2,\alpha}$ is strictly positive in $\bar{\Omega}$, by combining (3.17) and the Lemma 2.5 we get the contradiction

$$0 < \lim_{t\to\infty} \|h(t)\|_2^2 \leq \frac{1}{\rho_1} \lim_{t\to\infty} \int_\Omega \bar{u}^2(x) \left(\nabla \left(\frac{h(t,x)}{\bar{u}(x)} \right) \right)^t A(x) \nabla \left(\frac{h(t,x)}{\bar{u}(x)} \right) dx = 0.$$

□

4. The general competition case: Existence of positive stationary solution

In this section we investigate the existence of a positive stationary solution of (2.1) and prove Theorem 1.3. That is we look for positive solution of

$$\nabla \cdot (A(x)\nabla v) + v\,(r(x) - \Psi(x,v)) = 0 \quad \text{in} \quad \Omega, \tag{4.1}$$

$$\frac{\partial v}{\partial n}(x) = 0 \quad \text{in} \quad \partial\Omega, \tag{4.2}$$

where $\Psi(x,v) = \int_\Omega K(x,y)|v(y)|^p\,dy$. First observe that when $\lambda_1 \geq 0$, then there is no positive solution of (4.1)–(4.2). Indeed, by multiplying by ϕ_1 the equation (4.1) and integrating by parts it follows that

$$0 = -\lambda_1 \int_\Omega v(x)\phi_1(x)\,dx - \int_\Omega \Psi(x,v)v(x)\phi_1(x)\,dx,$$

which implies $\lambda_1 \int_\Omega v(x)\phi_1(x)\,dx = \int_\Omega \Psi(x,v)v(x)\phi_1(x)\,dx = 0$ since $\Psi(x,v), v$ and ϕ_1 are non negative. Thus $v = 0$ almost everywhere since $\phi_1 > 0$.

Let us then assume that $\lambda_1 < 0$. Let $k > 0$ so that the operator $\nabla \cdot (A(x)\nabla) + r(x) - k$ with Neumann boundary condition is invertible in $C^{0,\alpha}(\Omega)$ and a positive solution of (4.1)–(4.2) is a positive fixed point of the map T

$$\begin{array}{rcl} T : C^{0,\alpha}(\Omega) & \to & C^{0,\alpha}(\Omega) \\ v & \mapsto & Tv := (\nabla \cdot (A(x)\nabla) + r(x) - k)_n^{-1}[\Psi(x,v)v - kv] \end{array}.$$

To check that T has a positive fixed point we use a degree argument. Let $x_0 \in \Omega$ be fixed and let $K^s(x,y)$ be defined by

$$K^s(x,y) := sK(x,y) + (1-s)K(x_0,y).$$

Let us now consider the homotopy $H \in C([0,1] \times C^{0,\alpha}(\Omega), C^{0,\alpha}(\Omega))$ defined by

$$\begin{array}{rcl} H : [0,1] \times C^{0,\alpha}(\Omega) & \to & C^{0,\alpha}(\Omega) \\ (s,v) & \mapsto & H(s,v) := (\nabla \cdot (A(x)\nabla) + r(x) - k)_n^{-1}[\Psi_s(x,v)v - kv]. \end{array}\ ,$$

where $\Psi_s(x,v) := \int_\Omega K^s(x,y)|v|^p(y)\,dy$.

One can see that $H(1,.) = T$ and $H(0,.) = T_0$ where T_0 corresponds to the map

$$\begin{array}{rcl} T_0 : C^{0,\alpha}(\Omega) & \to & C^{0,\alpha}(\Omega) \\ v & \mapsto & T_0 v := (\Psi_0(v) - k)(\nabla \cdot (A(x)\nabla) + r(x) - k)_n^{-1} v. \end{array}$$

Note that there exists an unique positive fixed point to T_0 which can be constructed as in Section 3.

Before computing the degree of T_1, we obtain some a priori estimates on the fixed point of the map $H(\cdot,\cdot)$. That is some estimates on the positive solution to the equation

$$(\nabla \cdot (A(x)\nabla) + r(x) - k)_n^{-1}[\Psi_s(x,v)v - kv] = v$$

which rewrites:

$$\nabla \cdot (A(x)\nabla v) + r(x)v = \Psi_s(x,v)v \tag{4.3}$$

$$\partial_n v = 0 \quad \text{on } \partial\Omega \tag{4.4}$$

Lemma 4.1. *Let v be a continuous non negative solution of* (4.3)-(4.4)*. Then either $v \equiv 0$ or $v > 0$ and there exists $\bar{c}_1$ and $\bar{C}_1$ independent of s so that*

$$\bar{c}_1 \leq \int_\Omega |v|^p(x)\,dx \leq \bar{C}_1.$$

Proof:

The strict positivity of the solution v is a straightforward consequence of the strong maximum principle. Therefore either $v \equiv 0$ or $v > 0$. So let us assume that $v > 0$ and then by multiplying by v the equation (4.3) and integrating by parts we see that

$$\int_\Omega r(x)v^2(x)\,dx - \int_\Omega (\nabla v(x))^t\, A(x)\nabla v(x)\,dx = \int_\Omega \Psi^s(x,v)v(x)^2\,dx \geq K_{min}\int_\Omega |v(y)|^p\,dy \int_\Omega v^2(x)\,dx,$$

where $K_{min} := \min_{x,y\in\bar\Omega\times\bar\Omega} K(x,y)$. Therefore we get

$$\frac{\|r\|_\infty}{K_{min}} \geq \int_\Omega |v(y)|^p\,dy.$$

We also get

$$\int_\Omega r(x)v^2(x)\,dx - \int_\Omega (\nabla v(x))^t\, A(x)\nabla v(x)\,dx \leq K_{max}\int_\Omega |v(y)|^p\,dy \int_\Omega v^2(x)\,dx$$

with $K_{max} := \max_{x,y\in\bar\Omega\times\bar\Omega} K(x,y)$ which leads to

$$\frac{\lambda_1}{K_{max}} \leq \int_\Omega |v(y)|^p\,dy.$$

□

We are now in position to prove the existence of a positive solution to the equation (4.1) by means of the computation of the topological degree of $T - id$ on a well chosen set $\mathcal{O} \subset C^{0,\alpha}(\Omega)$. Let us choose positive constants c_2 and C_2 so that $c_2 < \bar c_1$ and $C_2 > \bar C_1$ where $\bar c_1$ and $\bar C_1$ are the constants obtained in Lemma 4.1. Let Ω be the following open set

$$\mathcal{O} := \left\{ v \in C^{0,\alpha}(\Omega), v \geq 0 \,|\, c_2 \leq \int_\Omega v^p(x)\,dx \leq C_2 \right\}$$

and let us compute $deg(T - Id, \mathcal{O}, 0)$. By Lemma 4.1 for all $s \in [0,1]$ $H(s,v) - v \neq 0$ on $\partial\mathcal{O}$. Therefore using that $H(.,.)$ is an homotopy, since T is a compact operator, we conclude that $deg(T - Id, \mathcal{O}, 0) = deg(H(1,.) - Id, \mathcal{O}, 0) = deg(H(0,.) - Id, \mathcal{O}, 0)$. By construction, from Section 3, one can check that $deg(H(0,.) - Id, \mathcal{O}, 0) \neq 0$ since the map T_0 has a unique positive non degenerated fixed point. Thus $deg(T - Id, \mathcal{O}, 0) \neq 0$ which shows that T has a fixed point in $\mathcal{O}$.

□

5. Stability of the dynamics, convergence to the equilibria

In this section we prove Theorem 1.4. That is to say, we analyse the stability under some perturbation of the dynamics established for (1.5)–(1.6) in Section 3. More precisely we investigate the global dynamics of solution of

$$\frac{\partial u(x,t)}{\partial t} = u\left[r(x) - \int_\Omega k_\epsilon(x,y)|u|^p(y)\,dy\right] + \nabla\cdot(A(x)\nabla u(t,x)) \quad \text{in} \quad \Omega\times\mathbb{R}^+, \tag{5.1}$$

$$\frac{\partial u}{\partial n}(t,x) = 0 \qquad \text{in} \quad \partial\Omega\times\mathbb{R}^{+,*}, \tag{5.2}$$

$$u(x,0) = u_0(x) \geq 0, \tag{5.3}$$

where $p = 1$ or 2 and $k_\epsilon(x,y) := k_0(y) + \epsilon k_1(x,y)$ with ϵ a small parameter. To obtain the asymptotic behaviour in this case, we follow the strategy developed in Section 3. Namely, we start by showing some *a priori* estimates on the solution $u(t,x)$, then we analyse the convergence by means of some differential inequalities. For convenience, we dedicate a subsection to each essential part of the proof.

5.1. **A priori estimate.**
We start by establishing some useful differential inequalities. Namely we show that

Lemma 5.1. *Assume that A, r, k_i satisfies* (1.4) *and let ϕ_1 be the positive eigenfunction associated to $\lambda_1(\nabla\cdot(A(x)\nabla)) + r(x)$ with Neumann boundary condition. Let $q \geq 1$ and H be the smooth convex function $H(s) : s \mapsto s^q$. Then there exists ϵ_0 so that for all $\epsilon \leq \epsilon_0$ and for all positive solution $u \in C^1((0,\infty), C^{2,\alpha}(\Omega))$ of* (5.1)–(5.2)*, we have for $t > 0$*

$$\frac{d\mathcal{H}_{q,\phi_1}[u](t)}{dt} \leq -\mathcal{D}_{q,\phi_1}[u](t) + q(-\lambda_1 - \alpha_{\epsilon,-}(u))\mathcal{H}_{q,\phi_1}[u](t)$$
$$\frac{d\mathcal{H}_{q,\phi_1}[u](t)}{dt} \geq -\mathcal{D}_{q,\phi_1}[u](t) + q(-\lambda_1 - \alpha_{\epsilon,+}(u))\mathcal{H}_{q,\phi_1}[u](t)$$

where

$$\mathcal{D}_{q,\phi_1}[u](t) := q(q-1)\int_\Omega \left(\frac{u(t,x)}{\phi_1(x)}\right)^{q-2}\phi_1^2(x)\left(\nabla\left(\frac{u(t,x)}{\phi_1(x)}\right)\right)^t A(x)\nabla\left(\frac{u(t,x)}{\phi_1(x)}\right)dx$$
$$\mathcal{H}_{q,\phi_1}[u] := \int_\Omega \left(\frac{u(t,x)}{\phi_1(x)}\right)^q \phi_1^2(x)\,dx$$
$$\alpha_{\epsilon,\pm}(u) := \int_\Omega (k_0(y) \pm \epsilon\|k_1\|_\infty)\,|u(t,y)|^p\,dy$$

Proof:

Observe that since u is positive, from (5.1) it follows that

$$\frac{\partial u}{\partial t}(t,x) \leq [r(x) - \alpha_{-,\epsilon}(u)]u(t,x) + \nabla\cdot(A(x)\nabla u(t,x))\,,$$
$$\frac{\partial u}{\partial t}(t,x) \geq [r(x) - \alpha_{+,\epsilon}(u)]u(t,x) + \nabla\cdot(A(x)\nabla u(t,x))\,.$$

Let $\bar\omega_\epsilon^+$ and $\bar\omega_\epsilon^-$ be the stationary solutions of the corresponding equations with homogeneous Neumann boundary condition:

$$\frac{\partial\omega_\epsilon^-(t,x)}{\partial t} = [r(x) - \alpha_{-,\epsilon}(\omega_\epsilon^-)]\omega_\epsilon^-(t,x) + \nabla\cdot\left(A(x)\nabla\omega_\epsilon^-(t,x)\right),$$
$$\frac{\partial\omega_\epsilon^+(t,x)}{\partial t} = [r(x) - \alpha_{+,\epsilon}(\omega_\epsilon^+)]\omega_\epsilon^+(t,x) + \nabla\cdot\left(A(x)\nabla\omega_\epsilon^+(t,x)\right).$$

Let ϵ small enough, says $\epsilon \leq \frac{k_{0,min}}{2\|k_1\|_\infty}$, then by construction $\bar\omega_\epsilon^\pm$ exists and we have $\bar\omega_\epsilon^\pm = \mu_\epsilon^\pm\phi_1$. Now by arguing as in the proof of Theorem 2.1, we obtain

$$\frac{d\mathcal{H}^-_{H,\bar\omega_\epsilon^-}[u](t)}{dt} \leq -\mathcal{D}_{H,\bar\omega_\epsilon^-}[u](t) + q[-\lambda_1 - \alpha_{\epsilon,-}(u)]\mathcal{H}^-_{H,\bar\omega_\epsilon^-}[u](t),$$
$$\frac{d\mathcal{H}^+_{H,\bar\omega_\epsilon^+}[u](t)}{dt} \geq -\mathcal{D}_{H,\bar\omega_\epsilon^+}[u](t) + q[-\lambda_1 - \alpha_{\epsilon,+}(u)]\mathcal{H}^+_{H,\bar\omega_\epsilon^+}[u](t).$$

where

$$\mathcal{H}^\pm_{H,\bar\omega_\epsilon^\pm}[u](t) := \int_\Omega (\bar\omega_\epsilon^\pm)^2(x)H\left(\frac{u(t,x)}{\bar\omega_\epsilon^\pm(x)}\right)dx,$$
$$\mathcal{D}_{H,\bar\omega_\epsilon^\pm}[u](t) := \int_\Omega H''\left(\frac{u(t,x)}{\bar\omega_\epsilon^\pm(x)}\right)(\bar\omega_\epsilon^\pm(x))^2\left(\nabla\left(\frac{u(t,x)}{\bar\omega_\epsilon^\pm(x)}\right)\right)^t A(x)\nabla\left(\frac{u(t,x)}{\bar\omega_\epsilon^\pm(x)}\right)dx.$$

By using that $\bar\omega_\epsilon^\pm = \mu_\epsilon^\pm\phi_1$, the definition of H and the homogeneity of $\mathcal{H}_{q,\mu_\epsilon^\pm\phi_1}[u]$, we deduce that

$$\frac{d\mathcal{H}_{q,\phi_1}[u](t)}{dt} \leq -\mathcal{D}_{q,\phi_1}[u](t) + q[-\lambda_1 - \alpha_{\epsilon,-}(u)]\mathcal{H}_{q,\phi_1}[u](t),$$
$$\frac{d\mathcal{H}_{q,\phi_1}[u](t)}{dt} \geq -\mathcal{D}_{q,\phi_1}[u](t) + q[-\lambda_1 - \alpha_{\epsilon,+}(u)]\mathcal{H}_{q,\phi_1}[u](t).$$

□

Next, we derive some *a priori* estimates for the solutions $u \in C^1((0,\infty), C^{2,\alpha}(\Omega))$of (5.1)–(5.2).

Lemma 5.2. *Assume that A, r, k_i satisfies* (1.4)*. Then there exists ϵ_1 so that we have :*

(i) *For all $q' \geq 1$ there exists $\bar{c}_{q'} < \bar{C}_{q'}$ so that for all $\epsilon \leq \epsilon_1$ and for all positive continuous stationary solution $\bar{u}_\epsilon$ to* (5.1)–(5.2)
$$\bar{c}_{q'} \leq \|\bar{u}_\epsilon\|_{L^{q'}(\Omega)} < \bar{C}_{q'}.$$

(ii) *There exists $0 < \bar{c}_\infty < \bar{C}_\infty$, so that for all $\epsilon \leq \epsilon_1$ and for all continuous stationary solution $\bar{u}_\epsilon$ to* (5.1)–(5.2)
$$\bar{c}_\infty \leq \bar{u}_\epsilon \leq \bar{C}_\infty.$$

(iii) *For all $1 \leq q' \leq p$, there exists $0 < C_{q'}$, so that for all $\epsilon \leq \epsilon_1$ and for all $u_\epsilon \in C^1((0,\infty), C^{2,\alpha}(\Omega))$ positive solution to* (5.1)–(5.2) *there exists $\bar{t}$ so that for all $t \geq \bar{t}$*
$$\|u_\epsilon(t)\|_{L^{q'}(\Omega)} \leq C_{q'}.$$

(iv) *For $p = 1$ or $p = 2$ there exists a positive constant c_1 , so that for all $\epsilon \leq \epsilon_1$ and for all $u_\epsilon \in C^1((0,\infty), C^{2,\alpha}(\Omega))$ positive solution to* (5.1)–(5.2) *there exists $\bar{t}$ so that for all $t \geq \bar{t}$*
$$\|u_\epsilon(t)\|_{L^1(\Omega)} \geq c_1.$$

Proof:

Let us first observe that (ii) is a straightforward consequence of (i) since $\bar{u}_\epsilon$ satisfies an elliptic equation with uniformly bounded continuous coefficient with respect to ϵ and $\bar{u}_\epsilon$. To prove (i), we first show the estimates for $q' = p$. First let us observe that by replacing u_ϵ by $\bar{u}_\epsilon$ and taking $q = 1$ in the formulas of Lemma 5.1, we get for $\epsilon \leq \epsilon_0$
$$0 \leq [-\lambda_1 - \alpha_{\epsilon,-}(\bar{u}_\epsilon)]\mathcal{H}_{1,\phi_1}[\bar{u}_\epsilon],$$
$$0 \geq (-\lambda_1 - \alpha_{\epsilon,+}(\bar{u}_\epsilon))\mathcal{H}_{1,\phi_1}[\bar{u}_\epsilon].$$

From the latter inequalities, by using the positivity of $\bar{u}_\epsilon$ and ϕ_1 it follows that
$$-\lambda_1 \geq \int_\Omega (k_0(y) - \sigma)\bar{u}_\epsilon^p(y)\,dy \geq \inf_{x\in\Omega}(k_0(x) - \sigma)\|\bar{u}_\epsilon\|^p_{L^p(\Omega)},$$
$$-\lambda_1 \leq \int_\Omega (k_0(y) + \sigma)\bar{u}_\epsilon^p(y)\,dy \leq \sup_{x\in\Omega}(k_0(x) + \sigma)\|\bar{u}_\epsilon\|^p_{L^p(\Omega)},$$

where $\sigma := \epsilon\|k_1\|_\infty$. Let $\kappa_0 := \frac{\inf_{x\in\Omega} k_0(x)}{2}$ and choose ϵ small enough, says so that $\epsilon < \frac{\kappa_0}{\|k_1\|_\infty} =: \epsilon'$, we achieve for all $\epsilon \leq \epsilon'$ and all stationary solution $\bar{u}_\epsilon$

$$\left(\frac{-\lambda_1}{\|k_0\|_\infty + \epsilon_1\|k_1\|_\infty)}\right)^{\frac{1}{p}} =: \bar{c}_p \leq \|\bar{u}_\epsilon\|_{L^p(\Omega)} \leq \bar{C}_p := \left(\frac{-\lambda_1}{\kappa_0}\right)^{\frac{1}{p}}. \tag{5.4}$$

Now recall that $\bar{u}_\epsilon$ satisfies the elliptic equation
$$\nabla\cdot(A(x)\nabla\bar{u}_\epsilon(x)) + \left(r(x) - \int_\Omega k_\epsilon(x,y)\bar{u}_\epsilon^p(y)\,dy\right)\bar{u}_\epsilon(x) = 0 \quad \text{in} \quad \Omega,$$
$$\frac{\partial\bar{u}_\epsilon(x)}{\partial n} = 0 \quad \text{in} \quad \partial\Omega.$$

From (5.4), the coefficients of this linear equation are uniformly bounded in L^∞ with respect to $\epsilon \in [0,\epsilon']$. So by using the elliptic regularity and Sobolev's embedding [8], we can show that for all $q \geq 1$ there exists $C > 0$ so that
$$\|\bar{u}_\epsilon\|_{W^{2,q}(\Omega)} \leq C,$$
with C independent of ϵ and $\bar{u}_\epsilon$. Thus there exists $C_\infty > 0$ independent of $\bar{u}_\epsilon$, so that
$$\|\bar{u}_\epsilon\|_\infty \leq C_\infty. \tag{5.5}$$

To obtain the desired uniform lower bound $\bar{c}_q$, a standard interpolation argument can be used [8] combining (5.4) and (5.5).

Let us now prove (iii). Let $\kappa_1 := \|k_0\|_\infty + \epsilon_1\|k_1\|_\infty$ and $\kappa_0 := \frac{\inf_{x\in\Omega} k_0(x)}{2}$ then by Lemma 5.1, since $\epsilon \le \epsilon'$ we get for all $q \ge 1$ and all $t > 0$

$$\frac{d\mathcal{H}_{q,\phi_1}[u_\epsilon](t)}{dt} \le -q(q-1)\int_\Omega \left(\frac{u_\epsilon(t,x)}{\phi_1(x)}\right)^{q-2}\phi_1^2(x)\left(\nabla\left(\frac{u(t,x)}{\phi_1(x)}\right)\right)^t A(x)\nabla\left(\frac{u_\epsilon(t,x)}{\phi_1(x)}\right)dx + q[-\lambda_1 - \kappa_0\|u_\epsilon\|^p_{L^p(\Omega)}]\mathcal{H}_{q,\phi_1}[u_\epsilon](t).$$

Since $\mathcal{H}_1(u) \sim \|u\|_{L^1(\Omega)}$, by Hölder's inequality and by choosing $q = 1$ in the above inequality, it follows that

$$\frac{d\mathcal{H}_{1,\phi_1}[u_\epsilon](t)}{dt} \le [-\lambda_1 - \widetilde{\kappa}_0\mathcal{H}_{1,\phi_1}[u_\epsilon](t)^p]\mathcal{H}_{1,\phi_1}[u_\epsilon](t) \quad \text{in} \quad (0,\infty). \tag{5.6}$$

Using the logistic character of the above equation, there exists t_1 so that $\mathcal{H}_{1,\phi_1}[u_\epsilon](t) \le \frac{-2\lambda_1}{\widetilde{\kappa}_0}$ for all $t \ge t_1$. A similar argument can be done for $q = p$, thus $\mathcal{H}_{p,\phi_1}[u_\epsilon](t) \le C_p$ for all $t \ge t_p$ and by interpolation we get for all $1 \le q \le p$

$$\|u_\epsilon\|_{L^q(\Omega)} \le C_q \quad \text{for all} \quad t \ge t' := \sup\{t_1, t_p\}. \tag{5.7}$$

To obtain the lower bound (iv), it is enough to get an uniform lower bound for $\mathcal{H}_{1,\phi_1}[u_\epsilon](t)$. By Lemma 5.1 we have

$$\frac{d\mathcal{H}_{1,\phi_1}[u_\epsilon](t)}{dt} \ge \left(-\lambda_1 - \|k\|_\infty\int_\Omega u_\epsilon^p(y)\,dy\right)\mathcal{H}_{1,\phi_1}[u_\epsilon](t). \tag{5.8}$$

Case 1: $p = 1$**.** In this situation, since $\mathcal{H}_{1,\phi_1}[u_\epsilon](t) \sim \|u_\epsilon\|_{L^1(\Omega)}$, we deduce that

$$\frac{d\mathcal{H}_{1,\phi_1}[u_\epsilon](t)}{dt} \ge (-\lambda_1 - \kappa_1\mathcal{H}_{1,\phi_1}[u_\epsilon](t))\mathcal{H}_{1,\phi_1}[u_\epsilon](t),$$

for some $\kappa_1 > 0$. Hence, there exists $\bar{t}$ so that $\mathcal{H}_{1,\phi_1}[u_\epsilon](t) \ge \frac{-\lambda_1}{2\kappa_1}$ for all $t > \bar{t}$.

Case 2: $p = 2$**.** In this situation, let us rewrite $u_\epsilon(x,t) := \mu_\epsilon(t)\phi_1(x) + g_\epsilon(t,x)$ with $g(t,x) \perp \phi_1$ in $L^2(\Omega)$. Equipped with this decomposition, we have

$$\mathcal{H}_{1,\phi_1}[u](t) = \mu_\epsilon(t) \tag{5.9}$$

$$\|u(t)\|_2^2 = \mathcal{H}_{2,\phi_1}[u](t) = \mu_\epsilon^2(t) + \|g_\epsilon(t)\|_2^2 \tag{5.10}$$

$$\frac{d\mathcal{H}_{2,\phi_1}[g_\epsilon](t)}{dt} = \frac{d\mathcal{H}_{2,\phi_1}[u_\epsilon](t)}{dt} - 2\mu_\epsilon(t)\mu_\epsilon'(t) \tag{5.11}$$

So from (5.8), we get

$$\mu_\epsilon'(t) \ge \left(-\lambda_1 - \|k_\epsilon\|_\infty\mu_\epsilon^2(t) - \|k_\epsilon\|_\infty\|g_\epsilon(t)\|_2^2\right)\mu_\epsilon(t). \tag{5.12}$$

Now by combining (5.9), (5.11) and Lemma 5.1 we see that

$$\frac{d\mathcal{H}_{2,\phi_1}[g_\epsilon](t)}{dt} \le -2\int_\Omega \phi_1^2(x)\left(\nabla\left(\frac{g_\epsilon(t,x)}{\phi_1(x)}\right)\right)^t A(x)\nabla\left(\frac{g_\epsilon(t,x)}{\phi_1(x)}\right)dx + \frac{d\log\mu_\epsilon^2(t)}{dt}\mathcal{H}_{2,\phi_1}[g_\epsilon](t) + 2[\alpha_{\epsilon,+}(u_\epsilon) - \alpha_{\epsilon,-}(u_\epsilon)]\left(\mu_\epsilon^2(t) + \mathcal{H}_{2,\phi_1}[g_\epsilon](t)\right) \tag{5.13}$$

By Lemma 2.5 and using (5.7) it follows that for $t \ge t'$

$$\frac{d\mathcal{H}_{2,\phi_1}[g_\epsilon](t)}{dt} - \frac{d\log\mu_\epsilon^2(t)}{dt}\mathcal{H}_{2,\phi_1}[g_\epsilon](t) \le -(2\rho_1(\phi_1) - 4\epsilon\|k_1\|_\infty C_2)\mathcal{H}_{2,\phi_1}[g_\epsilon](t) + 4\epsilon\|k_1\|_\infty C_2C_1^2. \tag{5.14}$$

Let $\Sigma := \{t \ge t' \,|\, \mathcal{H}_{2,\phi_1}[g_\epsilon](t) > 0\}$, then we have for all $t \in \Sigma$

$$\frac{d}{dt}\left(\log\left[\frac{\mathcal{H}_{2,\phi_1}[g_\epsilon](t)}{\mu_\epsilon^2(t)}\right]\right) \le -(2\rho_1(\phi_1) - 4\epsilon\|k_1\|_\infty C_2) + \frac{4\epsilon\|k_1\|_\infty C_2C_1^2}{\mathcal{H}_{2,\phi_1}[g_\epsilon](t)}. \tag{5.15}$$

By choosing ϵ small enough, say $\epsilon \leq \epsilon" := \min\left\{\epsilon', \frac{\rho_1(\phi_1)}{4C_2\|k_1\|_\infty}\right\}$, and by letting $\delta := 4\|k_1\|_\infty C_2 C_1^2$, by (5.15) we achieve for all $t \in \Sigma$

$$\frac{d}{dt}\left(\log\left[\frac{\mathcal{H}_{2,\phi_1}[g_\epsilon](t)}{\mu_\epsilon^2(t)}\right]\right) \leq -\rho_1(\phi_1) + \frac{\epsilon\delta}{\mathcal{H}_{2,\phi_1}[g_\epsilon](t)}. \tag{5.16}$$

To obtain the lower bound, the proof follows now three steps:

Step One. We claim that

Claim 5.3. *For all $\epsilon \leq \epsilon^{''}$, there exists $t_0 > t'$ so that*

$$\mathcal{H}_{2,\phi_1}[g_\epsilon](t_0) < \frac{2\delta\epsilon}{\rho_1(\phi_1)}.$$

Proof:

Assume by contradiction that for all $t \geq t'$ we have

$$\mathcal{H}_{2,\phi_1}[g_\epsilon](t_0) \geq \frac{2\delta\epsilon}{\rho_1(\phi_1)}.$$

Therefore it follows from (5.16) that for all $t > t'$

$$\frac{d}{dt}\left(\log\left[\frac{\mathcal{H}_{2,\phi_1}[g_\epsilon](t)}{\mu_\epsilon^2(t)}\right]\right) \leq -\frac{\rho_1(\phi_1)}{2}. \tag{5.17}$$

Thus $F(t) := \log\left[\frac{\mathcal{H}_{2,\phi_1}[g_\epsilon](t)}{\mu_\epsilon^2(t)}\right]$ is a decreasing function which by assumption is bounded from below for all $t \geq t'$. Therefore F converges as t tends to $+\infty$ and $\frac{dF}{dt} \to 0$. Hence for t large enough, we get the contradiction

$$-\frac{\rho_1(\phi_1)}{4} \leq \frac{d}{dt}\left(\log\left[\frac{\mathcal{H}_{2,\phi_1}[g_\epsilon](t)}{\mu_\epsilon^2(t)}\right]\right) \leq -\frac{\rho_1(\phi_1)}{2}.$$

□

Step Two. Let ϵ_1 and $\gamma(t_0)$ be the following quantities

$$\epsilon_1 := \min\left\{\epsilon", \frac{-\lambda_1\rho_1(\phi_1)}{8\|k_\epsilon\|_\infty\delta}\right\},$$
$$\gamma(t_0) := \min\left\{\mu_\epsilon(t_0), \sqrt{\frac{-\lambda_1}{2\|k_\epsilon\|_\infty}}\right\}$$

and let Q be the real map

$$\begin{array}{ccc} \mathbb{R}^+ & \to & \mathbb{R}^+ \\ x & \mapsto & A\frac{Bx}{Bx+C} \end{array}$$

where $A := \frac{-\lambda_1}{2\|k\|_\infty}$, $B := \rho_1(\phi_1)$ and $C := 2\epsilon\delta$. We claim that

Claim 5.4. *For all $\epsilon \leq \epsilon_1$ we have*

(i) *For all $t \geq t_0$,*

$$\mu_\epsilon^2(t) \geq \gamma^2(t_0).$$

(ii) *There exists $t_1' \geq t_0$ so that for all $t > t_1'$*

$$\mu_\epsilon^2(t) \geq Q(\gamma^2(t_0)).$$

Proof:

Let us denote $\Sigma^{\pm}$ and Σ_0 the following sets

$$\Sigma^+ := \left\{ t \geq t_0 \,|\, \mathcal{H}_{2,\phi_1}[g_\epsilon](t) > \frac{2\delta\epsilon}{\rho_1(\phi_1)} \right\},$$
$$\Sigma^- := \left\{ t \geq t_0 \,|\, \mathcal{H}_{2,\phi_1}[g_\epsilon](t) \leq \frac{2\delta\epsilon}{\rho_1(\phi_1)} \right\},$$
$$\Sigma_0 := \left\{ t \geq t_0 \,|\, \mu_\epsilon(t) \geq \min\left\{ \mu_\epsilon(t_0), \sqrt{\frac{-\lambda_1}{2\|k_\epsilon\|_\infty}} \right\} \right\}.$$

By construction $[t_0, +\infty) = \Sigma^+ \cup \Sigma^-, t_0 \in \Sigma^-$ and for all $\epsilon \leq \epsilon_1$ we have

$$-\lambda_1 - \|k_\epsilon\|_\infty \frac{2\epsilon\delta}{\rho_1(\phi_1)} \geq -\frac{-\lambda_1}{2}.$$

Let us now prove (i). Let $\widetilde{t_0}$ be the following time

$$\widetilde{t_0} := \sup\{t \geq t_0 \,|\, [t_0, t] \subset \Sigma^-\}.$$

By continuity of $\mathcal{H}_{2,\phi_1}[g_\epsilon](t)$, it follows from $\mathcal{H}_{2,\phi_1}[g_\epsilon](t_0) < \frac{2\delta\epsilon}{\rho_1(\phi_1)}$ that $\widetilde{t_0} > t_0$. Moreover we deduce from (5.12) that μ_ϵ satisfies on $(t_0, \widetilde{t_0})$:

$$\mu_\epsilon'(t) \geq \left(-\frac{\lambda_1}{2} - \|k_\epsilon\|_\infty \mu_\epsilon^2(t) \right) \mu_\epsilon(t). \tag{5.18}$$

Therefore $\mu_\epsilon(t) \geq \min\left\{ \mu_\epsilon(t_0), \sqrt{\frac{-\lambda_1}{2\|k_\epsilon\|_\infty}} \right\}$ for $t \in [t_0, \widetilde{t_0})$ which enforces $(t_0, \widetilde{t_0}) \subset \Sigma_0$. Let t^* be the following quantity

$$t^* := \sup\{t \geq t_0 \,|\, (t_0, t) \subset \Sigma_0\}.$$

From above $(t_0, \widetilde{t_1}) \subset \Sigma_0$, so we have $t^* \in (t_0, +\infty]$. We will show that $t^* = +\infty$. If not, $t^* < +\infty$ and from the above arguments we can see that $\mathcal{H}_{2,\phi_1}[g_\epsilon](t^*) \geq \frac{2\delta\epsilon}{\rho_1(\phi_1)}$. By definition of t^*, we have the following dichotomy since $[t_0, +\infty) = \Sigma^+ \cup \Sigma^-$:

- $t^* \in \Sigma^-$ and there exists $t^* < t^{*,+} \in \Sigma^+$ so that $(t^*, t^{*,+}) \subset \Sigma^+$
- $t^* \in \Sigma^+$ and there exists $t^{*,-} < t^* < t^{*,+}$ so that $t^{*,-} \in \Sigma_0 \cap \Sigma^-, t^{*,+} \in \Sigma^+$ and $(t^{*,-}, t^{*,+}] \subset \Sigma^+$

In both cases we see from (5.17) that on $(t^{*,-}, t^{*,+}]$ the function $F(t) = \log\left[\frac{\mathcal{H}_{2,\phi_1}[g_\epsilon](t)}{\mu_\epsilon^2(t)}\right]$ is decreasing and we have for all $t \in (t^{*,-}, t^{*,+}] F(t) < F(t^{*,-})$ which leads to

$$\mu_\epsilon^2(t^{*,-}) \leq \mu_\epsilon^2(t) \frac{\mathcal{H}_{2,\phi_1}[g_\epsilon](t^{*,-})}{\mathcal{H}_{2,\phi_1}[g_\epsilon](t)}.$$

Thus we get for all $t \in (t^{*,-}, t^{*,+}]$

$$\gamma(t_0) \leq \mu_\epsilon(t),$$

since $t^{*,-} \in \Sigma^- \cap \Sigma_0$ and $t \in \Sigma^+$. As a consequence we have $t^* < t^{*,+} \in \Sigma_0$, which contradicts the definition of t^*.

Hence $t^* = \infty$ and

$$\mu_\epsilon(t) \geq \min\left\{ \mu_\epsilon(t_0), \sqrt{\frac{-\lambda_1}{2\|k_\epsilon\|_\infty}} \right\} \quad \text{for all} \quad t \geq t_0. \tag{5.19}$$

Let us now prove (ii). By arguing on each connected component of Σ^+, since by (5.16) $F(t) = \log\left[\frac{\mathcal{H}_{2,\phi_1}[g_\epsilon](t)}{\mu_\epsilon^2(t)}\right]$ is a decreasing function one has for all $t \in \Sigma^+$

$$\mathcal{H}_{2,\phi_1}[g_\epsilon](t) \leq \frac{\mu_\epsilon^2(t)}{\gamma^2(t_0)} \frac{2\epsilon\delta}{\rho_1(\phi_1)}.$$

By construction, from (5.19) we also have for all $t \in \Sigma^-$

$$\mathcal{H}_{2,\phi_1}[g_\epsilon](t) \leq \frac{\mu_\epsilon^2(t)}{\gamma^2(t_0)} \frac{2\epsilon\delta}{\rho_1(\phi_1)}.$$

Therefore for all $t \geq t_0$ we get

$$\mathcal{H}_{2,\phi_1}[g_\epsilon](t) \leq \frac{\mu_\epsilon^2(t)}{\gamma^2(t_0)} \frac{2\epsilon\delta}{\rho_1(\phi_1)}. \tag{5.20}$$

Now by combining (5.20) with (5.12) it follows that for all $t \geq t_0$, $\mu_\epsilon(t)$ satisfies

$$\mu_\epsilon'(t) \geq \left[-\lambda_1 - \|k\|_\infty \mu_\epsilon^2(t) \left(1 + \frac{2\epsilon\delta}{\rho_1(\phi_1)\gamma^2(t_0)} \right) \right] \mu_\epsilon(t).$$

Hence, by using the logistic character of the above equation we have for some t_1' for all $t \geq t_1'$

$$\mu_\epsilon^2(t) \geq \frac{-\lambda_1}{2\|k\|_\infty} \frac{\rho_1(\phi_1)\gamma^2(t_0)}{\gamma^2(t_0)\rho_1(\phi_1) + 2\epsilon\delta} = Q(\gamma^2(t_0)). \tag{5.21}$$

□

Step Three. Finally we claim that

Claim 5.5. *There exists $\bar{t}$ so that for all $t \geq \bar{t}$*

$$\mu_\epsilon^2(t) \geq \frac{-\lambda_1}{8\|k\|_\infty}.$$

Proof:

By an elementary analysis, one can check that the map $Q(x) = A\frac{Bx}{Bx+C}$ is monotone increasing and has a unique positive fixed point $x_0 = \frac{AB-C}{B} = \frac{-\lambda_1}{2\|k\|_\infty} - \frac{2\epsilon\delta}{\rho_1(\phi_1)} \geq \frac{-\lambda_1}{4\|k\|_\infty} > 0$. We can also check that the iterated map $Q^{n+1}(x) := Q(Q^n(x))$ satisfies for any $x^* \in (0, +\infty)$

$$\lim_{n\to\infty} Q^n(x^*) = x_0. \tag{5.22}$$

Now recall that by the previous step, we have for all $t \geq t_1'$,

$$\mu_\epsilon^2(t) \geq Q(\gamma^2(t_0)) = Q\left(\min\left\{ \mu_\epsilon^2(t_0), \frac{-\lambda_1}{2\|k\|_\infty} \right\} \right).$$

Since Q is monotone increasing and $\frac{-\lambda_1}{2\|k\|_\infty} > x_0$ we deduce from (5.21) that for all $t \geq t_1'$

$$\mu_\epsilon^2(t) \geq \min\left\{ x_0, Q(\mu_\epsilon^2(t_0)) \right\}. \tag{5.23}$$

By using now step one with t_1' instead of t', it follows that there exists $t_1 \geq t_1'$ so that $\mathcal{H}_{2,\phi_1}[g_\epsilon](t_1) < \frac{2\delta\epsilon}{\rho_1(\phi_1)}$. We can then replace t_0 by t_1 in Step two, to obtain the existence of $t_2' > t_1$ so that for all $t \geq t_2'$ we have

$$\mu_\epsilon^2(t) \geq Q(\gamma(t_1)^2) = Q\left(\min\left\{ \mu_\epsilon^2(t_1), \frac{-\lambda_1}{2\|k\|_\infty} \right\} \right),$$

which by using the monotonicity of Q, $\frac{-\lambda_1}{2\|k\|_\infty} > x_0$ and (5.23) leads to

$$\mu_\epsilon^2(t) \geq \min\left\{ x_0, Q\left[\min\left\{ x_0, Q(\mu_\epsilon^2(t_0) \right\} \right] \right\} \tag{5.24}$$

for all $t \geq t_2'$.

Since x_0 is a fixed point of Q, it follows from (5.24) that for all $t \geq t_2'$

$$\mu_\epsilon^2(t) \geq \min\left\{ x_0, Q[Q(\mu_\epsilon^2(t_0))] \right\} = \min\left\{ x_0, Q^2(\mu_\epsilon^2(t_0)) \right\}. \tag{5.25}$$

By arguing inductively, we can then construct an increasing sequence $(t_n')_{n\in\mathbb{N}_0}$ so that for all n and for all $t \geq t_n'$ we have

$$\mu_\epsilon^2(t) \geq \min\left\{ x_0, Q^n(\mu_\epsilon^2(t_0)) \right\}. \tag{5.26}$$

Since $\mu^2_\epsilon(t_0) > 0$, by (5.22) there exists n_0 so that $Q^n(\mu^0_\epsilon(t_0)) \geq \frac{x_0}{4} = \frac{-\lambda_1}{8\|k\|_\infty}$. Hence, by (5.26) we have for all $t \geq t'_{n_0}$

$$\mu^2_\epsilon(t) \geq \frac{-\lambda_1}{8\|k\|_\infty}.$$

□

Finally, we establish an estimate on $\rho_1(\bar{u}_\epsilon)$ where $\rho_1(\bar{u}_\epsilon)$ is the constant defined in Lemma 2.5 for the positive vector $\bar{u}_\epsilon$. Namely, we show that

Lemma 5.6. *There exists $\bar{\rho} > 0$, so that for all $\epsilon \in [0, \epsilon_1)$ and for all positive stationary solution $\bar{u}_\epsilon$ of* (5.1)–(5.2)*, we have*

$$\rho(\bar{u}_\epsilon) \geq \bar{\rho}$$

Proof:

From the proof of Lemma 2.5, if we let $d\mu_\epsilon$, $L^2_{\mu_\epsilon}$ and $H^1_{\mu_\epsilon}$ be respectively the positive measure $d\mu_\epsilon = \bar{u}^2_\epsilon dx$, the following functional space:

$$L^2_{\mu_\epsilon}(\Omega) := \left\{ u \;\middle|\; \int_\Omega u^2(x) d\mu_\epsilon(x) < +\infty \right\}$$
$$H^1_{\mu_\epsilon}(\Omega) := \left\{ u \in L^2_{\mu_\epsilon} \;\middle|\; \int_\Omega |\nabla u|^2(x) d\mu_\epsilon(x) < +\infty \right\}$$

we have

$$0 < \rho(\bar{u}_\epsilon) = \inf_{g \in H^1_{d\mu_\epsilon}, \int_\Omega g \, d\mu_\epsilon = 0} \mathcal{J}(g)$$

with $\mathcal{J}$ the functional

$$\mathcal{J}(g) := \frac{1}{\|g\|_{L^2_{\mu_\epsilon}(\Omega)}} \int_\Omega (\nabla(g))^t \, A(x) \nabla(g) \, d\mu_\epsilon.$$

Let

$$\nu := \inf_{d\mu_\epsilon = \bar{u}^2_\epsilon dx} \rho(\bar{u}_\epsilon),$$

where $\epsilon \in [0, \epsilon_1]$ and $\bar{u}_\epsilon$ is any stationary solution of (5.1)–(5.2), then we have

$$\rho(\bar{u}_\epsilon) \geq \nu \geq 0.$$

We claim that $\nu > 0$. Indeed, if not then there exists a sequence of positive measure $\bar{u}^2_n dx$ so that

$$\lim_{n \to \infty} \rho(\bar{u}_n) = 0.$$

Since $0 \leq \epsilon \leq \epsilon_1$, by Lemma 5.2 the sequence $(\bar{u}_n)_{n \in \mathbb{N}}$ is uniformly bounded in $W^{2,q}(\Omega)$ for all $q \geq 1$. Therefore by the Rellich-Kondrakov Theorem, there exists a subsequence $(\bar{u}_{n_k})_{k \in \mathbb{N}}$ which converges to $\widetilde{u}$ a non-negative solution of (5.1)–(5.2) for some $\bar{\epsilon}$. By Lemma 5.2, we see also that $\widetilde{u}$ is non trivial and positive. Thus by applying Lemma 2.5 with $\widetilde{u}$ we get the contradiction

$$0 < \rho(\widetilde{u}) = 0.$$

□

5.2. **Asymptotic Behaviour.**

We are now in position to obtain the asymptotic behaviour of the solution $u_\epsilon(t, x)$ as t goes to $+\infty$ for $\epsilon \in [0, \epsilon^*]$, where ϵ^* is to be determined later on.

Let us first introduce some practical notation:

$$\Psi_0(v) := \int_\Omega k_0(y)|v(y)|^p \, dy, \;\; \Psi_1(x, v) := \int_\Omega k_1(x, y)|v(y)|^p \, dy, \;\; \Psi_\epsilon(x, v) := \Psi_0(v) + \epsilon \Psi_1(x, v)$$
$$\widetilde{\Psi}_\epsilon(v) := \int_\Omega \Psi_\epsilon(x, v) v^2(x) \, dx.$$

When $\lambda_1 \leq 0$, then the proof of Section 3 holds as well for solution of (5.1) – (5.3) and $u(t, x) \to 0$ as $t \to 0$. So let us assume $\lambda_1 < 0$ and let us denote $<, >$ the standard scalar product of $L^2(\Omega)$.

Let $\bar{u}_\epsilon$ be a positive stationary solution of (5.1)– (5.2). Such solution exists from Section 4. Since for all $t > 0$ the solution $u_\epsilon(t,x) \in L^2$, we can decompose u_ϵ as follows:

$$u_\epsilon(t,x) := \lambda_\epsilon(t)\bar{u}_\epsilon + h_\epsilon(t,x)$$

with h_ϵ so that $< \bar{u}_\epsilon, h_\epsilon >= 0$.

From this decomposition and by using Theorem 2.1 we get:

$$\lambda_\epsilon(t) < \bar{u}_\epsilon, \bar{u}_\epsilon >= \mathcal{H}_{1,\bar{u}_\epsilon}[u_\epsilon](t), \tag{5.27}$$

$$\frac{d\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}{dt} = \frac{d\mathcal{H}_{2,\bar{u}_\epsilon}[u_\epsilon](t)}{dt} - 2\lambda\lambda' < \bar{u}_\epsilon, \bar{u}_\epsilon > \tag{5.28}$$

$$\lambda'_\epsilon(t) < \bar{u}_\epsilon, \bar{u}_\epsilon >= \int_\Omega (\Psi_\epsilon(x,\bar{u}_\epsilon) - \Psi_\epsilon(x,u_\epsilon))\bar{u}_\epsilon(x)u_\epsilon(x,t)\,dx. \tag{5.29}$$

By Lemma 5.2 and (5.27), we can check that when $\epsilon \leq \epsilon_1$ there exists positives constants $c_1, C_1, \bar{c}_2, \bar{C}_2$ independent of ϵ such that for any positive smooth solutions u_ϵ to (5.1)–(5.2) there exists $\bar{t}(u_\epsilon)$ so that

$$\hat{c} := \frac{c_1}{\bar{C}_2} \leq \lambda_\epsilon(t) \leq \frac{C_1}{\bar{c}_2} =: \hat{C} \quad \text{for all} \quad t > \bar{t}. \tag{5.30}$$

From the decomposition, by using (5.30) and Lemma 5.2 we can also check that h_ϵ is smooth (i.e $C^{2,\alpha}(\Omega)$) and therefore belongs to $L^2(\Omega)$ for all times.

By plugging the decomposition of u_ϵ in (5.29) and using the definition of Ψ_ϵ, we can check that

$$\lambda'_\epsilon(t) = \frac{\widetilde{\Psi}_\epsilon(\bar{u}_\epsilon)\lambda_\epsilon(t)}{\|\bar{u}_\epsilon\|^2_{L^2(\Omega)}}(1-\lambda^p_\epsilon(t)) + \mathcal{R}_1(t) + \mathcal{R}_2(t) \tag{5.31}$$

where $\mathcal{R}_i$ are the following quantity:

$$\mathcal{R}_1(t) := \frac{1}{\|\bar{u}_\epsilon\|^2_{L^2(\Omega)}} \int_\Omega [\Psi_\epsilon(x,\bar{u}_\epsilon) - \Psi_\epsilon(x,u_\epsilon)]\bar{u}_\epsilon(x)h_\epsilon(t,x)\,dx \tag{5.32}$$

$$\mathcal{R}_2(t) := \frac{\lambda_\epsilon(t)}{\|\bar{u}_\epsilon\|^2_{L^2(\Omega)}} \int_\Omega \left(\sum_{k=1}^{p} \binom{k}{p} \lambda^{p-k}_\epsilon(t) \int_\Omega k_\epsilon(x,y)\bar{u}^{p-k}_\epsilon(y)h^k_\epsilon(t,y)\,dy \right) \bar{u}_\epsilon(x)^2\,dx \tag{5.33}$$

Next, we show that

Lemma 5.7. *Let $p = 1$ or $p = 2$ then there exists $\epsilon^* \leq \min\{\epsilon_0, \epsilon_1\}$, so that for all $\epsilon \leq \epsilon^*$ then any positive smooth solution u_ϵ of* (5.1)–(5.2) *satisfies*

$$\lim_{t\to\infty} \mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)) = 0.$$

Assume the lemma holds true, then we can conclude the proof of Theorem 1.4 by arguing as follows. By combining Lemma 5.2, Lemma 5.7 and by using Hölder's inequality, since $p = 1$ or 2 we see that $\mathcal{R}_i(t) \to 0$ as $t \to +\infty$. Thus $\lambda_\epsilon(t)$ satisfies

$$\lambda'_\epsilon(t) = \frac{\widetilde{\Psi}_\epsilon(\bar{u}_\epsilon)\lambda_\epsilon(t)}{\|\bar{u}_\epsilon\|^2_{L^2(\Omega)}}(1 + o(1) - \lambda^p_\epsilon(t)), \tag{5.34}$$

The above ODE is of logistic type with a perturbation $o(1) \to 0$ with a non negative initial datum. Therefore, when $\epsilon \leq \epsilon^*$ $\lambda_\epsilon(t)$ converges to 1 and we conclude that when $\epsilon \leq \epsilon^*$ then any positive solution u_ϵ to (5.1)–(5.2) converges to $\bar{u}_\epsilon$ almost everywhere.

□

Let us now turn our attention to the proof of the Lemma 5.7.

Proof of Lemma 5.7:

First, let us denote $\Gamma(t,x) := \Psi_\epsilon(x,\bar{u}_\epsilon) - \Psi_\epsilon(x,u_\epsilon)$. By (5.28) (5.29) and by using Theorem 2.1 we achieve

$$\frac{d\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}{dt} = -2\int_\Omega \bar{u}^2_\epsilon \left(\nabla\left(\frac{h_\epsilon(t,x)}{\bar{u}_\epsilon(x)}\right)\right)^t A(x)\nabla\left(\frac{h_\epsilon}{\bar{u}_\epsilon}\right) + 2\int_\Omega \Gamma(t,x)h_\epsilon(x)u_\epsilon(x)\,dx.$$

Therefore using the definition of Ψ_ϵ and that $\bar{u}_\epsilon \perp h_\epsilon$ we have

$$\frac{d\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}{dt} = -2\int_\Omega \bar{u}_\epsilon^2 \left(\nabla\left(\frac{h_\epsilon(t,x)}{\bar{u}_\epsilon(x)}\right)\right)^t A(x)\nabla\left(\frac{h_\epsilon}{\bar{u}_\epsilon}\right) + 2(\Psi_0(\bar{u}_\epsilon)-\Psi_0(u_\epsilon))\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)$$
$$+ 2\epsilon\int_\Omega (\Psi_1(x,\bar{u}_\epsilon)-\Psi_1(x,u_\epsilon))h_\epsilon^2(x)\,dx + 2\epsilon\lambda_\epsilon\int_\Omega (\Psi_1(x,\bar{u}_\epsilon)-\Psi_1(x,u_\epsilon))h_\epsilon(x)\bar{u}_\epsilon\,dx.$$

Let $\epsilon \le \min\{\epsilon_1,\epsilon_2\}$, by Lemma 5.2 any stationary solution $\bar{u}_\epsilon$ to (5.1)–(5.2) is bounded in $L^p(\Omega)$ and for any positive solution u_ϵ to (5.1)–(5.2) there exists $\bar{t}(u_\epsilon)$ so that for all times $t \ge \bar{t}$,

$$c_p \le \|u_\epsilon\|_{L^p(\Omega)} < C_p.$$

So for all times $t \ge \bar{t}$ we have

$$|\Psi_1(x,\bar{u}_\epsilon)-\Psi_1(x,u_\epsilon)| \le 2\|k_1\|_\infty \sup\{C_p,\bar{C}_p\} =: \kappa_1,$$

which implies that for $t \ge \bar{t}$

(5.35)
$$\frac{d\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}{dt} \le -2\int_\Omega \bar{u}_\epsilon^2(x)\left(\nabla\left(\frac{h_\epsilon(t,x)}{\bar{u}_\epsilon(x)}\right)\right)^t A(x)\nabla\left(\frac{h_\epsilon}{\bar{u}_\epsilon}\right)dx + 2(\Psi_0(\bar{u}_\epsilon)-\Psi_0(u_\epsilon))\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)$$
$$+ 2\epsilon\kappa_1\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t) + 2\epsilon\lambda_\epsilon\int_\Omega (\Psi_1(x,\bar{u}_\epsilon)-\Psi_1(x,u_\epsilon))h_\epsilon(x)\bar{u}_\epsilon(x)\,dx.$$

By (5.27) (5.29), using the definition of Ψ_ϵ we also have

$$\frac{d}{dt}\mathcal{H}_{1,\bar{u}_\epsilon}[u_\epsilon](t) = (\Psi_0(\bar{u}_\epsilon)-\Psi_0(u_\epsilon))\mathcal{H}_{1,\bar{u}_\epsilon}[u_\epsilon](t) + \epsilon\int_\Omega (\Psi_1(x,\bar{u}_\epsilon)-\Psi_1(x,u_\epsilon))u_\epsilon(x)\bar{u}_\epsilon(x)\,dx.$$
$$\ge (\Psi_0(\bar{u}_\epsilon)-\epsilon\kappa_1-\Psi_0(u_\epsilon))\mathcal{H}_{1,\bar{u}_\epsilon}[u_\epsilon](t).$$

Since $\mathcal{H}_{1,\bar{u}_\epsilon}[u_\epsilon] > 0$ for all $t > 0$, we have

$$\frac{d\log(\mathcal{H}_{1,\bar{u}_\epsilon}[u_\epsilon])}{dt}(t) \ge (\Psi_0(\bar{u}_\epsilon)-\epsilon\kappa_1-\Psi_0(u_\epsilon)),$$

which combined with (5.35) implies that for $t \ge \bar{t}$

$$\frac{d\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}{dt} \le -2\int_\Omega \bar{u}_\epsilon^2(x)\left(\nabla\left(\frac{h_\epsilon(t,x)}{\bar{u}_\epsilon(x)}\right)\right)^t A(x)\nabla\left(\frac{h_\epsilon}{\bar{u}_\epsilon}\right)dx + \left(\frac{d\log\left(\mathcal{H}_{1,\bar{u}_\epsilon}[u_\epsilon]\right)^2}{dt}(t)\right)\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)$$
$$+ 4\epsilon\kappa_1\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t) + 2\epsilon\lambda_\epsilon(t)\int_\Omega \Gamma_1(t,x)h_\epsilon(x)\bar{u}_\epsilon(x)\,dx.$$

where $\Gamma_1(t,x) := \Psi_1(x,\bar{u}_\epsilon)-\Psi_1(x,u_\epsilon)$.

Since $\epsilon \le \epsilon_1$, by Lemma 5.6, and by rearranging the terms in the above inequality we get for $t \ge \bar{t}$

$$\frac{d\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}{dt} - \mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)\frac{d\log\left(\mathcal{H}_{1,\bar{u}_\epsilon}[u_\epsilon]\right)^2}{dt}(t) \le (-\bar{\rho}+4\epsilon\kappa_1)\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)$$
$$+ 2\epsilon\lambda_\epsilon(t)\int_\Omega \Gamma_1(t,x)h_\epsilon(x)\bar{u}_\epsilon(x)\,dx. \tag{5.36}$$

Now, we estimate the last term of the above inequality.

Case $p = 1$**.** In this situation, by using the definition of Γ_1 and the Cauchy-Schwartz inequality we have

$$|\Gamma_1(t,x)| \le |1-\lambda_\epsilon|\|\bar{u}_\epsilon\|_2 \sup_{x\in\Omega}\sqrt{\int_\Omega k_1(x,y)^2\,dy} + \|h_\epsilon\|_2 \sup_{x\in\Omega}\sqrt{\int_\Omega k_1(x,y)^2\,dy}.$$

Since $\|v\|_2 = \sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[v]}$, by the Cauchy-Schwartz inequality we achieve for $t \geq bart$

$$\int_\Omega \Gamma_1(t,x)h_\epsilon(x)\bar{u}_\epsilon(x) \leq \kappa\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[\bar{u}_\epsilon](t)}\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}\left[|1-\lambda(t)|\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[\bar{u}_\epsilon]} + \sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}\right],$$
$$\leq \kappa\bar{C}_2\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}\left[|1-\lambda(t)|\bar{C}_2 + \sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}\right],$$

where $\kappa := \sup_{x\in\Omega}\sqrt{\int_\Omega k_1(x,y)^2\,dy}$.

Case $p = 2$**.** In this situation, as above by using the definition of Γ_1 and the Cauchy-Schwartz inequality, we see that

$$|\Gamma_1(t,x)| \leq |1-\lambda_\epsilon^2|\|k_1\|_\infty\|\bar{u}_\epsilon\|_2^2 + 2\lambda_\epsilon\|k_1\|_\infty\|\bar{u}_\epsilon\|_2\|h_\epsilon\|_2 + \|k_1\|_\infty\|h_\epsilon(t)\|_2^2.$$

So we get for $t \geq \bar{t}$

$$\int_\Omega \Gamma_1(x)h_\epsilon(x)\bar{u}_\epsilon(x) \leq \kappa\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[\bar{u}_\epsilon](t)}\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}\left[|1-\lambda_\epsilon^2|\|\bar{u}_\epsilon\|_2^2 + 2\lambda_\epsilon\|\bar{u}_\epsilon\|_2\|h_\epsilon\|_2 + \|h_\epsilon(t)\|_2^2\right],$$
$$\leq \kappa\bar{C}_2\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}\left[|1-\lambda^2(t)|\bar{C}_2^2 + (\widetilde{C}_2+\hat{C})\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}\right],$$

where $\kappa = \|k_1\|_\infty$.

In both case, we can see that there exists κ_2 and κ_3 independent of ϵ, $\bar{u}_\epsilon$ and u_ϵ so that we have for $t \geq \bar{t}$.

$$\int_\Omega \Gamma_1(x)h_\epsilon(x)\bar{u}_\epsilon(x)\,dx \leq \kappa_2\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}\left[|1-\lambda^p(t)| + \kappa_3\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}\right]. \tag{5.37}$$

By combining (5.37) and (5.36), we achieve for $t \geq \bar{t}$

$$\frac{d\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}{dt} - \mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)\frac{d\log\left(\mathcal{H}_{1,\bar{u}_\epsilon}[u_\epsilon]\right)^2}{dt}(t) \leq (-\bar{\rho}+\epsilon\kappa_5)\,\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)$$
$$+ \epsilon\kappa_4|1-\lambda^p(t)|\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}, \tag{5.38}$$

where $\kappa_4 := 2\hat{C}\kappa_2$ and $\kappa_5 := 2\hat{C}\kappa_2\kappa_3 + 4\kappa_1$ are positive constants independent of ϵ, u_ϵ and $\bar{u}_\epsilon$.

The proof now will follow several steps:

Step One: Since $\epsilon \leq \epsilon_1$ by (5.30) we have $|1-\lambda^p(t)| \leq \kappa_6$ for all $t > \bar{t}$, with κ_6 a universal constant independent of ϵ. We claim that

Claim 5.8. *Let $\epsilon \leq \epsilon_3 := \min(\epsilon_1, \epsilon_2 := \frac{\bar{\rho}}{2\kappa_5})$, then for all u_ϵ positive solution to* (5.1)–(5.2) *there exists $\bar{t}' \geq \bar{t}$ so that for all $t \geq \bar{t}'$ we have*

$$\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)} \leq 2\epsilon\left(\frac{\hat{C}}{\hat{c}}\right)\frac{2\kappa_4\kappa_6}{\bar{\rho}}.$$

Proof:

Indeed for $\epsilon \leq \epsilon_3$ by (5.38) for $t \geq \bar{t}$ we have

$$\frac{d\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}{dt} - \mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)\frac{d}{dt}\log\left(\mathcal{H}_{1,\bar{u}_\epsilon}[u_\epsilon](t)\right)^2 \leq -\frac{\bar{\rho}}{2}\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t) + \epsilon\kappa_4\kappa_6\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}. \tag{5.39}$$

From the above differential inequality we can check that there exists $t_0' > \bar{t}$ so that

$$\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t_0)} \leq \frac{\epsilon 4\kappa_4\kappa_6}{\bar{\rho}}.$$

If not, then $\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)} > \frac{\epsilon 4\kappa_4\kappa_6}{\bar{\rho}}$ for all $t > \bar{t}$ and by dividing (5.39) by $\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}$ and by rearranging the terms, we get the inequality

$$\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}\frac{d}{dt}\log\left(\frac{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}{\mathcal{H}_{1,\bar{u}_\epsilon}[u_\epsilon](t)^2}\right) \leq -\frac{\bar{\rho}}{2}\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)} + \epsilon\kappa_4\kappa_6 < -\epsilon\kappa_4\kappa_6 \qquad \forall t \geq \bar{t}. \tag{5.40}$$

Thus $F(t) := \log\left(\frac{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}{\mathcal{H}_{1,\bar{u}_\epsilon}[u_\epsilon](t)^2}\right)$ is a decreasing function which is bounded from below since $\lambda_\epsilon \leq \hat{C}$. Moreover $\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)} > \frac{\epsilon 4\kappa_4\kappa_6}{\bar{\rho}}$ for all $t > \bar{t}$. Therefore F converges as t tends to $+\infty$ and $\frac{dF}{dt} \to 0$. Thus for t large enough, we get the contradiction

$$-\frac{\epsilon\kappa_4\kappa_6}{2\bar{\rho}} \leq \sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}\frac{d}{dt}\log\left(\frac{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}{\mathcal{H}_{1,\bar{u}_\epsilon}[u_\epsilon](t)^2}\right) \leq -\frac{\epsilon\kappa_4\kappa_6}{\bar{\rho}}.$$

Let Σ be the set $\Sigma := \left\{t > t_0' | \sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)} > \frac{\epsilon 4\kappa_4\kappa_6}{\bar{\rho}}\right\}$. Assume that Σ is non empty otherwise the claim is proved since $\frac{\hat{C}}{\hat{c}} > 1$. Let us denote $t^* := \inf\Sigma$. By construction, since h_ϵ is continuous we have $\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t^*)} = \frac{\epsilon 4\kappa_4\kappa_6}{\bar{\rho}}$.

Again, by dividing(5.39) by $\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}$ and rearranging the terms, we get for all $t \in \Sigma$

$$\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}\frac{d}{dt}\log\left(\frac{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}{\mathcal{H}_{1,\bar{u}_\epsilon}[u_\epsilon](t)^2}\right) \leq -\frac{\bar{\rho}}{4}\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)} + \epsilon\kappa_4\kappa_6 \leq 0. \tag{5.41}$$

Thus $\log\left(\frac{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}{\mathcal{H}_{1,\bar{u}_\epsilon}[u_\epsilon](t)^2}\right)$ is a decreasing function of t for all $t \in \Sigma$. By arguing on each connected component of Σ and by using Lemma 5.2 we can check that for $t \geq t^*$ we have

$$\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)} \leq \frac{\hat{C}}{\hat{c}}\frac{\epsilon 4\kappa_4\kappa_6}{\bar{\rho}}.$$

Hence, since $\frac{\hat{C}}{\hat{c}} > 1$ we get for all $t \geq t_0$,

$$\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)} \leq \frac{\hat{C}}{\hat{c}}\frac{\epsilon 4\kappa_4\kappa_6}{\bar{\rho}}.$$

□

Step Two: Recall that $\lambda_\epsilon(t)$ satisfies

$$\lambda_\epsilon'(t) = \frac{\widetilde{\Psi}_\epsilon(\bar{u}_\epsilon)\lambda_\epsilon(t)}{\|\bar{u}_\epsilon\|^2_{L^2(\Omega)}}(1-\lambda_\epsilon^p(t)) + \mathcal{R}_1(t) + \mathcal{R}_2(t) \tag{5.42}$$

where $\mathcal{R}_i$ are the following quantity:

$$\mathcal{R}_1(t) := \frac{1}{\|\bar{u}_\epsilon\|^2_{L^2(\Omega)}}\int_\Omega [\Psi_\epsilon(x,\bar{u}_\epsilon) - \Psi_\epsilon(x,u_\epsilon)]\bar{u}_\epsilon(x)h(t,x)\,dx \tag{5.43}$$

$$\mathcal{R}_2(t) := \frac{\lambda_\epsilon(t)}{\|\bar{u}_\epsilon\|^2_{L^2(\Omega)}}\int_\Omega\left(\sum_{k=1}^p\binom{k}{p}\lambda_\epsilon^{p-k}(t)\int_\Omega k_\epsilon(x,y)\bar{u}_\epsilon^{p-k}(y)h^k(t,y)\,dy\right)\bar{u}_\epsilon(x)^2\,dx \tag{5.44}$$

Since $p = 1$ or $p = 2$ then by Lemma 5.2 and Hölder's inequality, we can see that there exists κ_7 independent of $\epsilon, \bar{u}_\epsilon, u_\epsilon$ so that for all $t \geq \bar{t}$

$$|\mathcal{R}_1(t) + \mathcal{R}_2(t)| \leq \kappa_7\frac{\widetilde{\Psi}(\bar{u}_\epsilon)\lambda_\epsilon(t)}{\|\bar{u}_\epsilon\|^2_{L^2(\Omega)}}\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}. \tag{5.45}$$

Next, we define some constant quantities:

$$\delta_0 := \frac{\hat{C}}{\hat{c}}\frac{4\kappa_4\kappa_6}{\bar{\rho}}, \tag{5.46}$$

$$\epsilon^* := \min\left\{\epsilon_3, \frac{\bar{\rho}\hat{c}}{16\kappa_4\kappa_7\hat{C}}, \frac{1}{4\kappa_7\delta_0}\right\}, \tag{5.47}$$

By the previous step, we see that for $\epsilon \leq \epsilon^*$ we have for any positive solution u_ϵ to (5.1)–(5.2) there exists $\bar{t}'$ so that for all $t \geq \bar{t}'$

$$\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)} \leq \epsilon\delta_0.$$

We claim that

Claim 5.9. *For $\epsilon \leq \epsilon^*$, there exists $t_{\epsilon\delta_0} \geq \bar{t}'$ such that for all $t \geq t_{\epsilon\delta_0}$*

$$\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)} \leq \frac{\epsilon\delta_0}{2}.$$

Proof:

First, we can check that for $\epsilon \leq \epsilon^*$ there exists t^* so that for all $t \geq t^*$

$$|1 - \lambda_\epsilon^p(t)| \leq 2\epsilon\delta_0\kappa_7.$$

Let $\lambda_{\pm\epsilon\delta_0\kappa_7} \in C^1((\bar{t}', \infty), \mathbb{R}^+)$ be the solution of the ODE

$$\lambda'_{\pm\epsilon\delta_0\kappa_7}(t) = \frac{\widetilde{\Psi}_\epsilon(\bar{u}_\epsilon)\lambda_{\pm\epsilon\delta_0\kappa_7}}{\|\bar{u}_\epsilon\|^2_{L^2(\Omega)}}(1 \pm \epsilon\delta_0\kappa_7 - \lambda^p_{\pm\epsilon\delta_0\kappa_7}(t)), \quad \lambda_{\pm\epsilon\delta_0\kappa_7}(\bar{t}') = \lambda_\epsilon(\bar{t}'). \tag{5.48}$$

Since the above equation is of logistic type and $\lambda_{\pm\epsilon\delta_0\kappa_7}(\bar{t}') > 0$, $\lambda_{\pm\epsilon\delta_0\kappa_7}(t) \to \bar{\lambda}_\pm$ as $t \to \infty$ where $\bar{\lambda}_\pm$ is the solution of the algebraic equation $1 \pm \epsilon\delta_0\kappa_7 - \bar{\lambda}^p_\pm = 0$.

By (5.42) and (5.45), we can check that λ_ϵ satisfies for $t \geq \bar{t}'$

$$\lambda'_\epsilon(t) \geq \frac{\widetilde{\Psi}_\epsilon(\bar{u}_\epsilon)\lambda_\epsilon(t)}{\|\bar{u}_\epsilon\|^2_{L^2(\Omega)}}(1 - \epsilon\delta_0\kappa_7 - \lambda^p_\epsilon(t)), \tag{5.49}$$

$$\lambda'_\epsilon(t) \leq \frac{\widetilde{\Psi}_\epsilon(\bar{u}_\epsilon)\lambda_\epsilon(t)}{\|\bar{u}_\epsilon\|^2_{L^2(\Omega)}}(1 + \epsilon\delta_0\kappa_7 - \lambda^p_\epsilon(t)). \tag{5.50}$$

By the comparison principle, from (5.48) (5.49) and (5.50) we get $\lambda_{-\epsilon\delta_0\kappa_7}(t) \leq \lambda_\epsilon(t) \leq \lambda_{+\epsilon\delta_0\kappa_7}(t)$ for all $t \geq \bar{t}'$. Thanks to the convergence of $\lambda_{\pm\epsilon\delta_0\kappa_7}(t)$ to $\bar{\lambda}_{\pm\epsilon\delta_0\kappa_7}$ and the monotone behaviour of $\bar{\lambda}_{\pm\epsilon\delta_0\kappa_7}$ with respect to ϵ we get

$$\bar{\lambda}_{-2\epsilon\delta_0\kappa_7} \leq \lambda_\epsilon(t) \leq \bar{\lambda}_{+2\epsilon\delta_0\kappa_7} \quad \text{for} \quad t \geq t^*,$$

for some $t^* \geq \bar{t}'$. Therefore, for $t \geq t^*$ we have

$$|1 - \lambda^p_\epsilon(t)| \leq 2\epsilon\delta_0\kappa_7.$$

From the latter estimate, since $\epsilon \leq \epsilon_3$ we deduce from (5.38) that for $t \geq t^*$

$$\frac{d\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}{dt} - \mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)\frac{d}{dt}\log\left(\mathcal{H}_{1,\bar{u}_\epsilon}[u_\epsilon](t)\right)^2 \leq -\frac{\bar{\rho}}{2}\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t) + 2\epsilon^2\kappa_4\kappa_7\delta_0\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)}.$$

By following the argumentation of Step one, we can show that there exists $t_{\epsilon\delta_0} \geq t^*$ such that for $t \geq t_{\epsilon\delta_0}$ we have

$$\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)} \leq \frac{8\epsilon\hat{C}\kappa_4\kappa_7}{\hat{c}\bar{\rho}}\epsilon\delta_0,$$

which thanks to $\epsilon \leq \frac{\bar{\rho}\hat{c}}{16\kappa_4\kappa_7\hat{C}}$ leads to

$$\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)} \leq \frac{\epsilon\delta_0}{2}.$$

□

Step Three: Since for all $t \geq t_{\epsilon\delta_0}$,

$$\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)} \leq \frac{\epsilon\delta_0}{2},$$

by arguing as in the proof of Claim 5.9, we see that there exists $t_{\epsilon\frac{\delta_0}{2}}$ so that for all $t \geq t_{\epsilon\frac{\delta_0}{2}}$

$$\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)} \leq \frac{\epsilon\delta_0}{4}.$$

By reproducing inductively the above argumentation, we can construct a sequence $(t_n)_{n\in\mathbb{N}}$ so that for all $t \geq t_n$ we have

$$\sqrt{\mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t)} \leq \frac{\epsilon\delta_0}{2^n}.$$

Hence, when $\epsilon \leq \epsilon^*$ we deduce that

$$\lim_{t\to\infty} \mathcal{H}_{2,\bar{u}_\epsilon}[h_\epsilon](t) \to 0.$$

□

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APPENDIX A. EXISTENCE OF A POSITIVE SOLUTION

In this appendix, we present a construction of a smooth positive solution of (1.1) The construction is rather simple and follows some of the ideas used in [13]. First, let $p \geq 1$ be fixed and let us regularised u_0 by a smooth mollifier ρ_ϵ and consider the solution of (1.1)–(1.3) with initial condition $u_{\epsilon,0} := \rho_\epsilon \star u_0$ instead of u_0.

Now we introduce the following sequence of function $(u_n(x,t))_{n\in\mathbb{N}}$ where u_n is defined recursively by $u_0(x,t) = u_0(x)$ and for $n \geq 0$, u_{n+1} is the solution of

$$\frac{\partial u_{n+1}}{\partial t} = \nabla\cdot(A(x)\nabla u_{n+1}) + u_{n+1}\left(r(x) - \int_\Omega K(x,y)|u_n|^p(t,y)\,dy\right) \quad \text{in} \quad \mathbb{R}^+\times\Omega \tag{A.1}$$

$$\frac{\partial u_{n+1}}{\partial n}(t,x) = 0 \quad \text{in} \quad \mathbb{R}^+\times\partial\Omega \tag{A.2}$$

$$u_{n+1}(x,0) = u_{\epsilon,0}(x) \quad \text{in} \quad \Omega. \tag{A.3}$$

Since by assumption $u_{\epsilon,0} \in C^\infty(\Omega)$, $(u_n)_{n\in\mathbb{N}}$ is well defined from the standard parabolic theory see [8, 23]. Moreover since $u_{\epsilon,0} \geq 0$ and 0 is a sub-solution of the problem (A.1)– (A.3) for each n, by the parabolic strong maximum principle we deduce that $u_n(x,t) > 0$ for all n, x and $t > 0$.

Now since $u_n$and K are non-negative functions, for all $n \geq 0$, u_{n+1} is a subsolution of the linear problem:

$$\frac{\partial v}{\partial t} = \nabla\cdot(A(x)\nabla v) + r(x)v \quad \text{in} \quad \mathbb{R}^+\times\Omega \tag{A.4}$$

$$\frac{\partial v}{\partial n}(t,x) = 0 \quad \text{in} \quad \mathbb{R}^+\times\partial\Omega \tag{A.5}$$

$$v(x,0) = u_0^\epsilon(x) \quad \text{in} \quad \Omega. \tag{A.6}$$

and by the parabolic maximum principle, we have $u_n \leq v \leq \|u_0^\epsilon\|_\infty e^{\|r\|_\infty t}$ in $\mathbb{R}^+\times\Omega$ for all n. Therefore from the standard Schauder parabolic *a priori* estimates, we deduce that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $C^{1,\alpha}((0,T),C^{2,\beta}(\Omega))$ for each $T > 0$. Thus by diagonal extraction, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ which converges to a solution $u(x,t) \geq 0$ of (1.1)–(1.3) with initial condition $u_{\epsilon,0}$.

Let us now take the limit $\epsilon \to 0$. By multiplying (1.1) by ϕ_1 and integrate it over Ω we have

$$\frac{d}{dt}\left(\int_\Omega u_\epsilon(t,x)\phi_1(x)\,dx\right) = -\lambda_1\int_\Omega u_\epsilon\phi_1 - \int_{\Omega\times\Omega} K(x,y)\phi_1(x)u_\epsilon(t,x)u_\epsilon^p(t,y)\,dydx.$$

Since u^ϵ, ϕ_1 and $K(x,y)$ are positives in $\bar{\Omega}$ it follows that

$$\frac{d}{dt}\left(\int_\Omega u_\epsilon(t,x)\phi_1(x)\,dx\right) \leq \int_\Omega u_\epsilon\phi_1\left(-\lambda_1 - C_0\int_\Omega u_\epsilon\phi_1\right),$$

for some positive constant C_0 which depends only on ϕ_1 and K. Thanks to the logistic character of the above inequality, we deduce that $\|u_\epsilon\|_{L^1(\Omega)}$ is bounded uniformly in time independently

of ϵ. By using Theorem 2.1 and Remark 2.2 with $H(s) : s \mapsto s^p$ and ϕ_1, it follows that

$$\frac{d\mathcal{H}_{p,\phi_1}[u_\epsilon](t)}{dt} \leq -p(p-1)\int_\Omega \left(\frac{u_\epsilon(t,x)}{\phi_1(x)}\right)^{p-2} \phi_1^2 \left(\nabla\left(\frac{u_\epsilon(t,x)}{\phi_1(x)}\right)\right)^t A(x)\nabla\left(\frac{u_\epsilon(t,x)}{\phi_1(x)}\right) dx$$
$$+ p\int_\Omega \phi_1^2(x)\left(\frac{u_\epsilon}{\phi_1}(t,x)\right)^p \left[-\lambda_1 - \int_\Omega K(x,y)u_\epsilon^p(t,y)\,dy\right] dx.$$

As above since u^ϵ, ϕ_1 and $K(x,y)$ are positives in $\bar{\Omega}$ it follows that

$$\frac{d\mathcal{H}_{p,\phi_1}[u_\epsilon](t)}{dt} \leq C_1\mathcal{H}_{p,\phi_1}[u_\epsilon](t)\left[-\lambda_1 - C_2\mathcal{H}_{p,\phi_1}[u_\epsilon](t)\right],$$

for some positive constants C_1 and C_2 which depends only on ϕ_1 and K. Thus $\|u_\epsilon\|_{L^p(\Omega)}$ is bounded uniformly with respect to ϵ. Since the coefficient of the parabolic PDE are bounded in L^∞ independently of ϵ, by standard parabolic L^p estimates [37], it follows that for all $T > 0$, u_ϵ is bounded independently of ϵ in $W^{1,2,1}((0,T)\times\Omega) \cap W_0^{1,1,1}(0,T)\times\Omega)$, where for $p \geq 1$ $W^{1,2,p}$ and $W_0^{1,1,p}$ denote the Sobolev space

$$W^{1,2,p} := \{u \in L^p((0,T)\times\Omega) \,|\, \partial_t u, \nabla u, \partial_{ij} u \in L^p((0,T)\times\Omega)\},$$
$$W_0^{1,1,p} := \{u \in L^p((0,T)\times\Omega), \partial_n u = 0 \text{ on } \partial\Omega \,|\, \partial_t u, \nabla u \in L^p((0,T)\times\Omega)\}.$$

By a standard bootstrap argument using the Parabolic regularity, we see that for each $T > 0$, (u_ϵ) is bounded in $C^{1,\alpha}((0,T), C^{2,\beta}(\Omega))$ independently of ϵ. Thus by diagonal extraction, there exists a subsequence $(u_{\epsilon_{n_k}})_{k\in\mathbb{N}}$ which converges to a smooth solution $u(x,t) \geq 0$ of (1.1)–(1.3) with initial condition u_0.

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