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► To cite this version:

Loïc Teyssier. Analyticity in spaces of convergent power series and applications. 2013. hal-00855142v2

HAL Id: hal-00855142

<https://hal.science/hal-00855142v2>

Preprint submitted on 3 Oct 2014 (v2), last revised 6 May 2015 (v5)

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ANALYTICITY IN SPACES OF CONVERGENT POWER SERIES AND APPLICATIONS[†]

[†]Preprint version: October 2014

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 This work has been partially supported by French national grant ANR-13-JS01-0002-01.

ABSTRACT. We study the analytic structure of the space of germs of an analytic function at the origin of \mathbb{C}^m , namely the space $\mathbb{C}\{z\}$ where $z = (z_1, \dots, z_m)$, equipped with a convenient locally convex topology. We are particularly interested in studying the properties of analytic sets of $\mathbb{C}\{z\}$ as defined by the vanishing loci of analytic maps. While we notice that $\mathbb{C}\{z\}$ is not Baire we also prove it enjoys the analytic Baire property: the countable union of proper analytic sets of $\mathbb{C}\{z\}$ has empty interior. This property underlies a quite natural notion of a generic property of $\mathbb{C}\{z\}$, for which we prove some dynamics-related theorems. We also initiate a program to tackle the task of characterizing glocal objects in some situations.

Infinite-dimensional holomorphy, complex dynamical systems, holomorphic solutions of differential equations, Liouvillian integrability of foliations
 [2000][MSC2010] 46G20, 58B12, 34M99, 37F75

1. INTRODUCTION

This article purports to provide a context in which the following results make sense:

Corollary A. *Fix $m \in \mathbb{N}$. The generic finitely generated subgroup $G < \text{Diff}(\mathbb{C}^m, 0)$ of germs of biholomorphisms fixing $0 \in \mathbb{C}^m$, identified with a tuple of generators $(\Delta_1, \dots, \Delta_n)$, is free. Besides the set of non-solvable subgroups of $\text{Diff}(\mathbb{C}, 0)$ generated by two elements is Zariski-full, in the sense that it contains (and, as it turns out, is equal to) an open set defined as the complement of a proper analytic subset of $\text{Diff}(\mathbb{C}, 0)^n$. The latter result persists in the case of n -generated subgroups of $\text{Diff}(\mathbb{C}, 0)$ everyone of which is tangent to the identity.*

Corollary B. *The generic germ of a holomorphic function $f \in \mathbb{C}\{z\}$ near the origin of \mathbb{C} is not solution of any ordinary differential equation $f^{(k+1)}(z) = P(z, f(z), \dots, f^{(k)}(z))$, where P is an elementary function of all its variables, differentiable at $(0, f(0), \dots, f^{(k)}(0))$, and $k \in \mathbb{N}$.*

Corollary C. *The set of coprime $P, Q \in \mathbb{C}\{x, y\}$, with zero first-jet, such that*

$$y' = \frac{P(x, y)}{Q(x, y)}$$

is not solvable in «closed form» constitutes a Zariski-full set.

The main concern of the article is proposing a framework in which the analytic properties of $\mathbb{C}\{z\}$ can easily be manipulated. For that reason the above results should be understood as showcase consequences of more general theorems stated below. Other consequences in the realm of dynamics will also be detailed in due time, for there lies the original motivation of this work. In the meantime the objects involved above must be outlined, and we postpone more formal definitions to the body of the article.

1.1. Statement of the main results.

Roughly speaking a property \mathcal{P} expressed on the set of germs of holomorphic functions is *generic*¹ if the subset where \mathcal{P} does not hold is contained in countably many (proper) analytic sets. This concept supports a genericity *à la* Baire embodied in a metrizable, locally convex topology on $\mathbb{C}\{z\}$, where $z = (z_1, \dots, z_m)$. The topology is induced by a convenient family of norms. Such spaces will be referred to as *normally convex* spaces. Special norms will be of interest, those defined by

$$(\star) \quad \left\| \sum_{\mathbf{n} \in \mathbb{N}^m} f_{\mathbf{n}} z^{\mathbf{n}} \right\|_a := \sum_{\mathbf{n} \in \mathbb{N}^m} a_{\mathbf{n}} |f_{\mathbf{n}}|,$$

where $a = (a_{\mathbf{n}})_{\mathbf{n} \in \mathbb{N}^m}$ is a multi-sequence of positive real numbers satisfying the additional growth condition

$$\lim_{|\mathbf{n}| \rightarrow \infty} a_{\mathbf{n}}^{1/|\mathbf{n}|} = 0,$$

which ensures the convergence of the series (\star) . This particular choice of a topology on $\mathbb{C}\{z\}$, instead of the «usual» ones, is motivated by the theory of analyticity between locally convex spaces, developed in its modern form during the 70's by various mathematicians (S. DINEEN, L. NACHBIN and J. SEBASTIÃO E SILVA among others) in the wake of the works of J.-P. RAMIS for Banach analytic spaces. A brief history of the theory of infinite-dimensional holomorphy in locally convex space is conducted in [Din81, p101-104]. In that setting a map $\Lambda : E \rightarrow F$ is *analytic* if it is the sum of a «convergent power series» whose term of «homogeneous degree» $p \in \mathbb{N}$ is some p -linear, continuous mapping².

To keep matters brief we wish to manipulate relations like

$$f \circ g = g \circ f$$

or, for the sake of example,

$$(f'')^2 = 1 + \exp f' + f \times f^{(3)},$$

as analytic relations on the corresponding space of holomorphic germs. While this is true for the inductive topology on $\mathbb{C}\{z\}$, it is not too difficult to prove that the standard differentiation $f \mapsto \frac{\partial f}{\partial z}$ or the composition mapping

$$(f, g) \mapsto f \circ g$$

¹An element satisfying a generic property will also be called «generic» for convenience, as in the three corollaries stated at the very beginning of the introduction.

²We refer to the books by J.-A. BARROSO [Bar85] and P. MAZET [Maz84] for a presentation of the different (equivalent) types of analyticity and related analytic sets, of which we give a short summary in Section 3.

are analytic if one endows $\mathbb{C}\{\mathbf{z}\}$ with a convenient sub-family of norms (\star). We would like to point out that not only is the continuity of the linear left-composition operator $g^* : f \mapsto f \circ g$ ascertained, provided $g(0) = 0$, but so is the analyticity of the right-composition mapping $f_* : g \mapsto f \circ g$, for given f . The Taylor expansion

$$f \circ g = \sum_{p=0}^{\infty} \frac{f^{(p)}(g(0))}{p!} (g - g(0))^p$$

is indeed convergent on the open set

$$\{g \in \mathbb{C}\{\mathbf{z}\} : |g(0)| < \mathcal{R}(f)\}$$

(see Proposition 3.20) where $\mathcal{R}(f)$ stands for the radius of convergence of the Taylor series of f at 0. It should be noticed straight away that \mathcal{R} is lower semi-continuous but cannot be positively lower-bounded on any domain of $\mathbb{C}\{\mathbf{z}\}$, which constitutes of course a source of trouble and, as a by-product, reveals that the composition mapping $(f, g) \mapsto f \circ g$ cannot be defined on any domain of $\mathbb{C}\{\mathbf{z}\} \times \mathbb{C}\{\mathbf{z}\}$ (for given g with $g(0) \neq 0$ there will always exist f not defined at $g(0)$).

Once we are granted the notion of analyticity we can speak of *analytic sets*, closed subsets locally defined by the vanishing of a collection of analytic functions. As we recall later the space $\mathbb{C}\{\mathbf{z}\}$ is not Baire and at first glance the notion of G_δ -genericity must arguably be discarded. Yet it is possible to salvage this concept by hardening the rules Baire's closed sets are required to play by:

Theorem A. *The space $\mathbb{C}\{\mathbf{z}\}$ enjoys the analytical Baire property : at most countable unions of proper analytic sets of $\mathbb{C}\{\mathbf{z}\}$ have empty interior.*

This theorem bolsters the well-foundedness of the concept of genericity in $\mathbb{C}\{\mathbf{z}\}$ as introduced above. It has been originally proved in a weaker form in [GT10] while encompassing issues of glocality, which is a subject we come back to afterward. In fact we can prove a stronger refinement of the main result of the cited reference:

Theorem B. *Let $\Lambda : U \subset E \rightarrow \mathbb{C}\{\mathbf{w}\}$ be a map analytic on a relative open set in a linear subspace $E < \mathbb{C}\{\mathbf{z}\}$ of at most countable dimension. Then the range of Λ is analytically meager: $\Lambda(U)$ is contained in a countable union of proper analytic sets of $\mathbb{C}\{\mathbf{w}\}$.*

Such a result is typically useful in conjunction with Theorem A to derive existential properties by proving, say, that the image of the set of polynomials by a given analytic map cannot cover a domain of $\mathbb{C}\{\mathbf{w}\}$ (in particular it cannot be locally onto). That was the targeted objective of [GT10] through the use of a different concept of analyticity (the so-called here *quasi-strong analyticity*) and analytic sets, which we also come back to later. The topology on $\mathbb{C}\{\mathbf{z}\}$ was there induced by the norm $\|\bullet\|_{1/n!}$. This topology, though, is too coarse to be used in Corollaries A, B and C. Instead one can consider the metrizable topology induced by the collection $(\|\bullet\|_{n!^{-1/k}})_{k \in \mathbb{N}_{>0}}$, which we name *factorial topology*. Other choices, called *useful topologies*, are possible to carry out most proofs; roughly speaking any collection of norms making multiplication, composition and differentiation continuous can do. Notice that no finite collection can be useful: the presence of normally convex spaces equipped with an infinite collection of norms (instead

of *e.g.* normed space) is a necessity when dealing with the composition or the differentiation. The major drawback is that no differential geometry (*e.g.* Fixed-Point Theorem³) exists in such a general framework, although this is precisely what everyone would like to use in conjunction with the Fréchet calculus on first-order derivatives. In order to make a constructive use of the latter we need some rudiments of analytical geometry in $\mathbb{C}\{z\}$, which is unfortunately not available for now.

1.2. Deduction of the Corollaries from the Theorems.

1.2.1. Corollary A.

Any algebraic relation between n generators $(\Delta_1, \dots, \Delta_n)$ of G writes

$$\bigcirc_{\ell=1}^k \Delta_{j_\ell}^{\circ n_\ell} - \text{Id} = 0$$

for a given depth $k \in \mathbb{N}_{>0}$ and a collection of couples $(j_\ell, n_\ell) \in \mathbb{N}_{\leq n} \times \mathbb{Z}_{\neq 0}$. Each one of these countably many choices defines an analytic set⁴ of $\text{Diff}(\mathbb{C}^m, 0)^n$, which is proper if the relation does not boil down to the trivial relation $\Delta \circ \Delta^{\circ-1} - \text{Id} = 0$ (after recursive simplification of said trivial relations). Abelian groups form the analytic set defined by the (non-trivial) relations

$$[\Delta_j, \Delta_k] - \text{Id} = 0, \quad j \neq k,$$

where as usual $[f, g] := f^{\circ-1} \circ g^{\circ-1} \circ f \circ g$. In the unidimensional case D. CERVEAU and R. MOUSSU on the one hand, Y. IL'YASHENKO and A. SHERBAKOV on the other hand, have studied the structure of the finitely-generated subgroups of $\text{Diff}(\mathbb{C}, 0)$, proving results later refined by F. LORAY [Lor94] which are as follows.

- ⊗ A finitely generated subgroup G of $\text{Diff}(\mathbb{C}, 0)$ is solvable if, and only if, it is meta-Abelian (*i.e.* its second derived group $[[G, G], [G, G]]$ is trivial).
- ⊗ $G := \langle f, g \rangle$ is solvable if, and only if,

$$[f, [f, g^{\circ 2}]] - \text{Id} = 0.$$

This again is an analytic relation and the complement in $\text{Diff}(\mathbb{C}, 0)^2$ of the corresponding analytic set is a Zariski-full open set.

- ⊗ If all the generators of G are tangent to the identity then G is solvable if, and only if, it is Abelian.

All these points put together prove the corollary.

1.2.2. Corollary B.

Corollary B is a consequence of Theorem C below. In fact we prove a stronger result: being given $k \in \mathbb{N}$ and \mathcal{E}_k an analytically meager subset of $\mathbb{C}\{z, \delta_0, \dots, \delta_k\}$, the set \mathcal{S} of germs f in $\mathbb{C}\{z\}$ that are solutions of some differential equation

$$P(z, f(z) - f(0), \dots, f^{(k)}(z) - f^{(k)}(0)) = 0,$$

³We explain in the course of the article why Nash-Moser theorem cannot be applied in the generality of the spaces under study.

⁴Since each diffeomorphism Δ_j fixes 0 the composition mappings $(\Delta_j, \Delta_k) \mapsto \Delta_j \circ \Delta_k$ are analytic on the whole $\text{Diff}(\mathbb{C}^m, 0)^2$, as stated in Section 3.2. We also refer to this section for the definition of analytic mappings from and into the non-linear space $\text{Diff}(\mathbb{C}^m, 0)$.

with

$$P \in \mathcal{E}_k^* := \mathcal{E}_k \setminus \left\{ Q : \frac{\partial Q}{\partial \delta_k}(\mathbf{0}) = 0 \right\},$$

is analytically meager. We explain at the end of this paragraph which meaning we place on elementary functions in the variable $\mathbf{z} = (z_1, \dots, z_m)$, and why the trace of their set on $\mathbb{C}\{\mathbf{z}\}$ is analytically meager.

For the sake of concision set

$$F_k^f(\mathbf{z}) := \left(\mathbf{z}, f(\mathbf{z}) - f(\mathbf{0}), \dots, f^{(k)}(\mathbf{z}) - f^{(k)}(\mathbf{0}) \right).$$

The natural approach is to consider the vanishing locus Ω of the analytic map

$$\begin{aligned} \mathbb{C}\{\mathbf{z}, \delta_0, \dots, \delta_k\}^* \times \mathbb{C}\{\mathbf{z}\} &\longrightarrow \mathbb{C}\{\mathbf{z}\} \\ (P, f) &\longmapsto P \circ F_k^f \end{aligned}$$

(this map is analytic if both source and range spaces are given a useful topology). The set of those germs satisfying at least one differential equation of the requested type is therefore the sub-analytic set given by the canonical projection of $\Omega \cap (\mathcal{E}_k^* \times \mathbb{C}\{\mathbf{z}\})$ on the second factor $\mathbb{C}\{\mathbf{z}\}$. To guarantee that this projection has empty interior we provide a parameterized covering by the space of equations and initial conditions, using:

Theorem C. Fix $m \in \mathbb{N}$ and consider the space \mathbf{VF} of germs at $\mathbf{0} \in \mathbb{C}^m$ of a holomorphic vector field, identified with $\mathbb{C}\{\mathbf{z}\}^m$, endowed with the factorial topology. For $X \in \mathbf{VF}$ we name Φ_X the flow of X , that is the unique germ of a holomorphic mapping near $(\mathbf{0}, 0)$

$$\begin{aligned} \Phi_X : \mathbb{C}^m \times \mathbb{C} &\longrightarrow \mathbb{C}^m \\ (\mathbf{p}, t) &\longmapsto \Phi_X^t(\mathbf{p}) \end{aligned}$$

that is solution of the differential system

$$\begin{cases} \dot{\mathbf{z}}(\mathbf{p}, t) &= X(\mathbf{z}(\mathbf{p}, t)) \\ \mathbf{z}(\mathbf{p}, 0) &= \mathbf{p}. \end{cases}$$

Then the «flow mapping»

$$\begin{aligned} \mathbf{VF} &\longrightarrow \mathbb{C}\{\mathbf{z}, t\}^m \\ X &\longmapsto \Phi_X, \end{aligned}$$

where the target space is also given the factorial topology, is analytic.

Remark 1.1. The flow mapping admit the analytic section

$$\begin{aligned} \mathbb{C}\{\mathbf{z}, t\} &\longrightarrow \mathbb{C}\{\mathbf{z}\} \\ \phi &\longmapsto \frac{\partial \phi}{\partial t}(\mathbf{z}, 0), \end{aligned}$$

so that proper analytic sets $\Omega \subset \mathbb{C}\{\mathbf{z}\}$ are mapped to proper analytic sets $\Phi_\Omega \subset \mathbb{C}\{\mathbf{z}, t\}$.

Take now $f \in \mathbb{C}\{z\}$ and $P \in \mathbb{C}\{z, \delta_0, \dots, \delta_k\}^*$. Using the usual trick of differentiating once more the equation we obtain $\frac{dP \circ F_k^f}{dz}(z) = 0$, which we rewrite $\hat{P} \circ F_{k+1}^f = 0$ with:

$$\hat{P}(z, \delta_0, \dots, \delta_{k+1}) := \frac{\partial P}{\partial z}(z, \delta_0, \dots, \delta_k) + \sum_{j=0}^k \frac{\partial P}{\partial \delta_j}(z, \delta_0, \dots, \delta_k) \delta_{j+1}.$$

From this we deduce an explicit non-trivial $(k+1)^{\text{th}}$ -order differential equation, whose solutions are obtained through the flow of the companion vector field

$$\mathfrak{X}(P) = \frac{\partial}{\partial z} + \sum_{j=0}^{k-1} \delta_{j+1} \frac{\partial}{\partial \delta_j} - \frac{\frac{\partial P}{\partial z} + \sum_{j=0}^{k-1} \frac{\partial P}{\partial \delta_j} \delta_{j+1}}{\frac{\partial P}{\partial \delta_k}} \frac{\partial}{\partial \delta_k},$$

which is holomorphic for $\frac{\partial P}{\partial \delta_k}(\mathbf{0}) \neq 0$. Obviously $P \in \mathbb{C}\{z, \delta_0, \dots, \delta_k\}^* \mapsto \mathfrak{X}(P)$ is analytic for any useful topology. Consider the map

$$\begin{aligned} \Phi_k : \mathbb{C}[z]_{\leq k} \times \mathbb{C}\{z, \delta_0, \dots, \delta_k\}^* &\longrightarrow \mathbb{C}\{z\} \\ (J, P) &\longmapsto \left(z \mapsto J(z) + \Pi \circ \Phi_{\mathfrak{X}(P)}^z(\mathbf{0}) \right) \end{aligned}$$

where Π denotes the natural projection $(z, \delta_0, \dots, \delta_k) \mapsto \delta_0$ and $\mathbb{C}[z]_{\leq k}$ the vector space of complex polynomials of degree at most k . Theorem C asserts the analyticity of the map if the space $\mathbb{C}\{z\}$ is equipped with the factorial topology. By construction, if f is given satisfying an equation $P \in \mathbb{C}\{z, \delta_0, \dots, \delta_k\}^*$ then $f = \Phi_k(J, P)$ where J is the k^{th} -jet of f at 0 (the canonical projection of $f \in \mathbb{C}\{z\}$ on $\mathbb{C}[z]_{\leq k}$). The set \mathcal{S} must therefore be included in the countable union of sets $\left(\Phi_k \left(\mathbb{C}[z]_{\leq k} \times \mathcal{E}_k^* \right) \right)_{k \in \mathbb{N}}$, each one of which is analytically meager, accounting for Remark 1.1, finally proving the sought result.

Let us conclude this paragraph by proving our claim regarding elementary functions. Denote by $\text{DiffAlg}_{\mathbb{K}}(x)$ the abstract set of all solutions of at least one autonomous, polynomial differential equations of finite order in the variable x with coefficients in a subfield $\mathbb{K} \leq \mathbb{C}$. By $\text{DiffAlg}_{\mathbb{K}}(x) \cap \mathbb{C}\{x\}$ we mean the concrete representation of those elements of $\text{DiffAlg}_{\mathbb{K}}(x)$ defining actual germs of analytic functions at 0. (Although not completely relevant here, this set is analytically meager for the factorial topology on $\mathbb{C}\{x\}$ thanks to Theorem B.)

Define the increasing union of fields

$$\mathbb{E}(\mathbf{z}) := \bigcup_{n \in \mathbb{N}} \mathbb{E}_n(\mathbf{z})$$

where

$$\begin{aligned} \mathbb{E}_0(\mathbf{z}) &:= \mathbb{C}(\mathbf{z}) \\ (1.1) \quad \mathbb{E}_{n+1}(\mathbf{z}) &:= \mathbb{C}(\mathbf{z}, \text{DiffAlg}_{\mathbb{C}}(x) \circ \mathbb{E}_n(\mathbf{z})). \end{aligned}$$

Hence $\mathbb{E}_{n+1}(\mathbf{z})$ is the field generated by all variables z_j and all compositions of an element of $\text{DiffAlg}(x)$ with an element of $\mathbb{E}_n(\mathbf{z})$. Notice that $\mathbb{E}(\mathbf{z})$ contains also most special functions, with the notable exception of Euler's Gamma function (Hölder's theorem, see also [Moo96]). The field $\text{Elem}(\mathbf{z})$ obtained by replacing $\text{DiffAlg}_{\mathbb{C}}(x)$ with $\{\exp x, \log x, \arcsin x\}$ in (1.1), is the field of *elementary functions* in \mathbf{z} , which is naturally a subfield of $\mathbb{E}(\mathbf{z})$.

E. H. MOORE proved the following result :

Lemma ($m = 1$, [Moo96, p54-55]). $\mathbb{E}(z) = \text{DiffAlg}_{\mathbb{C}}(z)$. Moreover this field is stable by differentiation and compositional inversion.

Pick $P \in \mathbb{C}[\delta_0, \dots, \delta_k]$ and $f \in \mathbb{E}(x)$ such that $P(f, f', \dots, f^{(k)}) = 0$ (for short, let us say that f is solution of $P = 0$). We may write $\delta = (\delta_0, \dots, \delta_k)$, $\delta^f := (f, f', \dots, f^{(k)})$ and $P(\delta) = \sum_{|\alpha| \leq d} \lambda_\alpha \delta^\alpha$ with $\lambda_\alpha \in \mathbb{C}$ and $\alpha \in \mathbb{N}^{k+1}$. By isolating a single variable λ_β appearing in P , that is

$$\lambda_\beta = - \frac{\sum_{\alpha \neq \beta, |\alpha| \leq d} \lambda_\alpha (\delta^f)^\alpha}{(\delta^f)^\beta},$$

and differentiating with respect to x we obtain that f is solution of another polynomial, autonomous differential equation of order $k + 1$ which does not depend on λ_β and in which the coefficients $(\lambda_\alpha)_{\alpha \neq \beta}$ continue to appear linearly. By induction we actually prove that there exists $\tilde{P} \in \mathbb{Q}[\delta_0, \dots, \delta_{k'}]$ for some $k' \geq k$ such that f is solution of $\tilde{P} = 0$, so that

$$\text{DiffAlg}_{\mathbb{Q}}(x) = \text{DiffAlg}_{\mathbb{C}}(x).$$

For $\mathcal{E} := \mathbb{Q}[(\delta_k)_{k \in \mathbb{N}}]$ is countable, Moore's lemma asserts that the set $\mathbb{E}(z) \cap \mathbb{C}\{z\}$ is analytically meager, settling the case $m = 1$.

Let us prove now that $\mathbb{E}(z) \cap \mathbb{C}\{z\}$ is analytically meager when $m > 1$. Fix an enumeration $\{Q_n : n \in \mathbb{N}\}$ of \mathcal{E} . For $f \in \mathbb{E}(z) \cap \mathbb{C}\{z\}$ and for given $\tilde{z} = (z_2, \dots, z_m)$ small enough consider the partial function defined by $\tilde{f}_{\tilde{z}}(z) := f(z, z_2, \dots, z_m)$, which belongs to $\mathbb{E}(z)$. According to the lemma there exists $k = k(\tilde{z}) \in \mathbb{N}$ and a polynomial $P_{\tilde{z}} \in \mathbb{Q}[\delta_0, \dots, \delta_k]$ such that $\tilde{f}_{\tilde{z}}$ is solution of $P_{\tilde{z}} = 0$. We can split a small domain $\Omega \subset \mathbb{C}^{m-1}$ containing $\mathbf{0}$ as into countably many sets Ω_n , consisting precisely of those points \tilde{z} for which $P_{\tilde{z}}$ is the n^{th} element $Q_n \in \mathbb{Q}[\delta_0, \dots, \delta_{k_n}]$ of \mathcal{E} . According to the principle of analytic continuation we may assume without loss of generality that each Ω_n is closed in Ω . From Baire's theorem we deduce that at least one Ω_n has non-empty interior, thus (again by analytic continuation) $\tilde{f}_{\tilde{z}}$ is solution of $Q_n = 0$ for any $\tilde{z} \in \Omega$. Therefore f is solution of

$$Q_n \left(f, \frac{\partial f}{\partial z_1}, \frac{\partial^2 f}{\partial z_1^2}, \dots, \frac{\partial^{k_n} f}{\partial z_1^{k_n}} \right) = 0,$$

and the set $\mathbb{E}(z) \cap \mathbb{C}\{z\}$ is analytically meager in $\mathbb{C}\{z\}$.

1.2.3. Corollary C.

A (germ of a) meromorphic, order one differential equation in \mathbb{C}^2

$$(\diamond) \quad y' = \frac{P(x, y)}{Q(x, y)}$$

induces a (germ of a) foliation at the origin of the complex plane. Roughly speaking, the leaves of such a foliation are the connected Riemann surfaces corresponding to «maximal» solutions. By Cauchy-Lipschitz's theorem if P or Q does not vanish at some point⁵ then the foliation is locally conjugate to a product of two discs. On the contrary at a

⁵Such a point is called regular.

singularity of the foliation, which we locate at $(0,0)$ for convenience, a whole range of complex behaviors can turn up. An obvious fact is that the generic germ of a foliation⁶ is regular, since singular ones correspond to the analytic set of $\mathbb{C}\{x,y\}^2$ defined by

$$\text{Sing} := \{(P, Q) \in \mathbb{C}\{x, y\}^2 : P(0, 0) = Q(0, 0) = 0\}.$$

From now on we solely work in Sing , which is given the analytic structure of $\mathbb{C}\{x, y\}^2$ through the continuous, affine and onto mapping

$$(P, Q) \mapsto (P - P(0, 0), Q - Q(0, 0)).$$

An important question regarding foliations is that of finding solutions of (\diamond) in «closed form», which was originally formulated by J. LIOUVILLE in terms of consecutive quadratures and exponentiation of quadratures of meromorphic functions. In the modern framework of differential Galois theory this notion translates as the request that every germ of a solution near every regular points admit an analytic continuation coinciding with a determination of an abstract solution lying in a finite tower of consecutive extensions of differential fields $\mathbb{K}_0 < \dots < \mathbb{K}_n$ of the following kind:

- ⊗ we start from the field \mathbb{K}_0 of germs of meromorphic functions near $(0, 0)$,
- ⊗ $\mathbb{K}_{n+1} = \mathbb{K}_n \langle f \rangle$ where $f' = a \in \mathbb{K}_n$ (a quadrature),
- ⊗ $\mathbb{K}_{n+1} = \mathbb{K}_n \langle f \rangle$ where $f' = af$ with $a \in \mathbb{K}_n$ (an exponentiation of a quadrature).

For the sake of example let us wander a little away from the path we are currently treading, and consider the case of a linear differential system, where Q is the $n \times n$ identity matrix and $P(x, y) = P(x)y$ is obtained from a $n \times n$ matrix $P(x)$ with entries rational in x , and y is a vector in \mathbb{C}^n . To simplify further imagine that

$$P(x) = \sum_{\ell=1}^k \frac{D_\ell}{x - x_\ell}$$

for some finite collection of constant, diagonal matrices $(D_\ell)_{\ell \leq k}$ and distinct points x_ℓ of \mathbb{C} , so that the system is Fuchsian. Then the solutions are multi-valued mappings

$$x \mapsto y(x) = \prod_{\ell=1}^k (x - x_\ell)^{D_\ell} \times C$$

where $C \in \mathbb{C}^n$ is the vector of «initial conditions». The multi-valuedness of solutions is directly related to $(D_\ell)_\ell$ and is measured by the monodromy group, obtained by performing successive local analytic continuations, starting from some transverse line $\{x = \text{cst}\}$ outside the singular locus $\bigcup_\ell \{x = x_\ell\}$, and returning to it after winding around the singularities (a kind of first-return mapping acting on C). The monodromy group is a representation of the fundamental group of the punctured sphere $\overline{\mathbb{C}} \setminus \{x_\ell : \ell \leq k\}$ into a linear algebraic subgroup in $\text{GL}_n(\mathbb{C})$. E. KOLCHIN (see for instance [PS03]) related the Liouvillian integrability of the system to the solvability of the (connected component of identity of the Zariski-closure of the) monodromy group. It is well known that the generic linear algebraic group is non-solvable.

⁶For the sake of clarity we identify the set of germs of foliations with $\mathbb{C}\{x, y\}^2 = \{(P, Q)\}$, voluntarily forgetting that proportional couples induce the same foliation; in particular the singular locus of a holomorphic foliation in \mathbb{C}^2 is always isolated. This technicality will be dealt with in due time.

Back to the non-linear setting we would like to generalize this non-solvability result. The candidate replacement differential Galois theory has been introduced in a recent past by B. MALGRANGE [Mal01] and subsequently developed by G. CASALE (we refer to [Cas06] for matters regarding our present study). The monodromy group is replaced by the groupoid of holonomy, but its geometric construction is the same up to replacement of the fundamental group of the base space by the fundamental groupoid. Although the tools introduced here are not powerful enough to deal with such a generality we can nonetheless say something in the generic case.

In this paper when we speak of a *reduced* singularity we mean that the linearized differential equation at the singularity, identified with the 2-dimensional square matrix

$$L(P, Q) := \begin{bmatrix} \frac{\partial P}{\partial x}(0, 0) & \frac{\partial P}{\partial y}(0, 0) \\ \frac{\partial Q}{\partial x}(0, 0) & \frac{\partial Q}{\partial y}(0, 0) \end{bmatrix},$$

possesses at least one non-zero eigenvalue. It is obvious again that the generic element of Sing is reduced, since non-reduced foliations can be discriminated by the characteristic polynomial of their linear part, and therefore form the analytic set

$$\text{Sing} \cap \{(P, Q) : \det L(P, Q) = \text{tr} L(P, Q) = 0\}.$$

Liouvillian integrability of foliations with $L(A, B) \neq 0$ is now a well-studied topic [BT99, Cas06] so we dismiss this case and consider only germs of singular foliations belonging to the proper analytic set

$$\text{ZLP} := \{(P, Q) : L(P, Q) = 0\}$$

(ZLP standing for *zero linear part*).

An old result, formalized by A. SEIDENBERG [Sei68], states that every germ $\mathcal{F} \in \text{Sing}$ of a holomorphic foliation with a singularity at $0 \in \mathbb{C}^2$ can be reduced, that is: there exists a complex surface \mathcal{M} and a proper rational morphism $\pi : \mathcal{M} \rightarrow (\mathbb{C}^2, 0)$, obtained as successive blow-ups of singular points, such that

- ⊗ $\mathcal{E} := \pi^{-1}(0)$, called the exceptional divisor, is a finite, connected union of normally-crossing copies of $\mathbb{P}_1(\mathbb{C})$,
- ⊗ the restriction of π to $\mathcal{M} \setminus \mathcal{E}$ is a biholomorphism,
- ⊗ the pulled-back foliation $\pi^* \mathcal{F}$ has only reduced singularities, located on the exceptional divisor.

Either a component \mathcal{D} of the exceptional divisor is transverse to all but finitely many leaves of $\pi^* \mathcal{F}$, in which case we are confronted to a *dicritic component*, or \mathcal{D} is a leaf of $\pi^* \mathcal{F}$. To a non-dicritic component \mathcal{D} and any (small enough) transversal disk Σ , not meeting the (finite) singular locus $\text{Sing}(\mathcal{F})$ of the foliation, we associate the «projective» holonomy group $\text{Hol}(\mathcal{D}, \Sigma)$ of germs of invertible holomorphic first-return maps obtained by following (lifting) cycles of $\mathcal{D} \setminus \text{Sing}(\pi^* \mathcal{F})$, with base-point $\mathcal{D} \cap \Sigma$, in a leaf of $\pi^* \mathcal{F}$. By endowing Σ with an analytic chart, so that $\mathcal{D} \cap \Sigma$ correspond to 0, the group $\text{Hol}(\mathcal{D}, \Sigma)$ is naturally identified with a finitely-generated sub-group of $\text{Diff}(\mathbb{C}, 0)$. As the domain of definition of an element Δ may not equal the whole Σ this is not a sub-group of the biholomorphisms of Σ . For this reason such objects are usually referred to as pseudo-groups in the literature, although we consider them as groups of germs (that is, without considering a geometric realization). If one chooses another transverse $\tilde{\Sigma}$ then $\text{Hol}(\mathcal{D}, \Sigma)$ and $\text{Hol}(\mathcal{D}, \tilde{\Sigma})$ are biholomorphically conjugate.

In the case where \mathcal{F} is reduced after a single blow-up the (conjugacy class of the) holonomy group embodies all the information about Liouvillian integrability. In particular it is solvable if the equation (\diamond) is integrable. Corollary C then follows from combining Corollary A with the facts that such foliations \mathcal{F} are Zariski-full and that we can build an analytic, open mapping

$$\mathcal{F} \mapsto \text{Hol}(\mathcal{D}, \Sigma).$$

A formal proof will be given in the body of the article.

1.3. Strong analyticity.

To the extent of my knowledge the notion presented now has never been thoroughly studied so far, which is a pity since it is fairly common in actual problems. It runs as follows : a continuous map $\Lambda : U \subset \mathbb{C}\{\mathbf{z}\} \rightarrow \mathbb{C}\{\mathbf{w}\}$ is *strongly analytic* if for any finite-dimensional family of functions $(f_{\mathbf{x}})_{\mathbf{x} \in \mathbb{D}^n}$ such that $(\mathbf{x}, \mathbf{z}) \mapsto f_{\mathbf{x}}(\mathbf{z})$ is analytic near $\mathbf{0} \in \mathbb{C}^n \times \mathbb{C}^m$, the corresponding family $(\Lambda(f_{\mathbf{x}}))_{\mathbf{x}}$ is also an analytic function of (\mathbf{x}, \mathbf{w}) near $\mathbf{0}$. This property can be easily checked⁷ and ensures that the composition of two source/range compatible strongly analytic maps is again strongly analytic. For instance the composition and differentiation mappings trivially satisfy this property. The next result enables all the previous theorems to apply in the case of strongly analytic relations:

Theorem D. *Let $\Lambda : U \subset \mathbb{C}\{\mathbf{z}\} \rightarrow \mathbb{C}\{\mathbf{w}\}$ be a strongly analytic map. Then Λ is analytic.*

We will provide a criterion characterizing strongly analytic maps among analytic ones. Notice also that in Theorem C the flow mapping is actually strongly analytic.

The continuity condition on Λ can be notably relaxed and only *ample boundedness* (analogous to local boundedness as in the characterization of continuous linear mappings) is actually required. This is a very natural condition to impose when dealing with analyticity in locally convex space, as the whole theory relies on making sense of the Cauchy's formula in an infinite dimensional space (which is the argument the proof of this theorem also is based upon).

1.4. Applications to dynamics.

Having a «nice» composition and differentiation comes in handy in dynamics, as we already illustrated with the introductory corollaries. We introduce also the *glocal* problem: how can one recognize that a local dynamical system is the local trace of a global one. We establish a tentative program to deal with this problem, which will demand the making of a differential geometry in $\mathbb{C}\{\mathbf{z}\}$.

1.4.1. Infinitely-renormalizable parabolic diffeomorphisms.

The usual framework of unidimensional (discrete) dynamics is the iteration of rational maps $\Delta : \mathbb{C} \rightarrow \mathbb{C}$. However to understand the local structure of, say, the Julia set of Δ one is often led to study its local analytic conjugacy class, which can be quite rich indeed. For instance near a parabolic fixed-point the space of equivalence classes under

⁷When I say that the property is easily checked I mean informally that in actual problems it should not be more difficult to prove strong analyticity than plain analyticity.

local changes of coordinates is huge⁸. It is, roughly speaking, isomorphic to a product of finitely many copies of $\text{Diff}(\mathbb{C}, 0)$ through a one-to-one map

$$\acute{E}V : \Delta \mapsto \acute{E}V(\Delta),$$

known as the «Écalte-Voronin invariants» mapping. Germs of biholomorphisms have same Écalte-Voronin invariants if, and only if, they are locally conjugate by a germ of a biholomorphism.

Understanding how the local dynamics (*i.e.* the local invariants) varies as Δ does is usually a hard but rewarding task, undertaken for example by M. SHISHIKURA [Shi00] to prove that the boundary of the quadratic Mandelbrot set has Hausdorff-dimension 2. This citation is particularly interesting since one ingredient of the proof consists in exploiting the strong analyticity of $\acute{E}V$. The result also relies on the existence of infinitely-renormalizable maps, in the following way. Suppose for the sake of simplicity that Δ is tangent to the identity

$$\Delta(z) = z + \alpha z^2 + \dots, \quad \alpha \neq 0$$

so that $\acute{E}V(\Delta) \in \text{Diff}(\mathbb{C}, 0)^2$. The invariant $\acute{E}V(\Delta)$ is well-defined up to a choice of a linear chart on \mathbb{C} , and therefore we can always arrange that a component $\tilde{\Delta}$ of $\acute{E}V(\Delta)$ itself is tangent to the identity. With corresponding notations, if $\tilde{\alpha} \neq 0$ then $\acute{E}V(\tilde{\Delta}) \in \text{Diff}(\mathbb{C}, 0)^2$ is again well-defined and Δ is renormalizable once. The set of germs of diffeomorphisms which are renormalizable n times is consequently a Zariski-full open set. We particularly derive the

Corollary D. *The generic tangent-to-the-identity germ of a diffeomorphism of the complex line is infinitely-renormalizable.*

I'm grateful to M. YAMPOLSKY for suggesting this application to me.

1.4.2. Application of Corollary A to the topology of foliations.

Non-solvability of finitely-generated subgroups of $\text{Diff}(\mathbb{C}, 0)$ is a key point in studying rigidity properties of holomorphic foliations on compact complex surfaces. In the case of *e.g.* a foliation on $\mathbb{P}_2(\mathbb{C})$ it measures (through the holonomy representation of the line at infinity $\text{Hol}(\Sigma, L_\infty)$) how the leaves are mutually entangled: if the dynamics is sufficiently «mixing» then topological conjugations between such foliations turn out to be holomorphic (*i.e.* homographies). In that respect we should cite a consequence of the Nakai theorem exploited by A. LINS NETO, P. SAD and B. SCÁRDUA in [LSS98]. They show that the set of topologically rigid foliations of given degree in $\mathbb{P}_2(\mathbb{C})$ contains an open and dense set, by proving that holonomy groups $\text{Hol}(\Sigma, L_\infty)$ corresponding to those foliations are non-solvable⁹. The «generic» freeness of $\text{Hol}(\Sigma, L_\infty)$, for a fixed foliation degree, is proved by Y. IL'YASHENKO and A. PYARTLI [IP94]. We point out that the context of both results is of a different nature from ours: for a fixed degree the space of foliations is a finite-dimensional complex projective space.

In the context of germs of singular foliations J.-F. MATTEI, J. REBELO and H. REIS [MRR13] devise a result comparable to Corollary A. It is stronger in the sense that any subgroup

⁸Although it has the same cardinality as \mathbb{R} , its algebraic dimension is infinite.

⁹And therefore, according to Nakai's theorem, have dense orbits in a «big» domain of the transversal Σ .

$G = \langle \Delta_\ell \rangle_{1 \leq \ell \leq n}$ can be perturbed by the action of $\text{Diff}(\mathbb{C}, 0)^n$

$$\varphi = (\varphi_\ell)_{1 \leq \ell \leq n} \mapsto \varphi^* G := \langle \varphi_\ell^* \Delta_\ell \rangle_{1 \leq \ell \leq n},$$

in such a way that for a «generic» choice of φ the corresponding subgroup $\varphi^* G$ is free. They deduce from this result a statement about the corresponding pseudo-group (obtained by realizing the group on a common domain of definition of the generators) that the generic foliation has at most countably many non-simply-connected leaves. Here «generic» refers to G_δ -genericity in $\text{Diff}(\mathbb{C}, 0)$ for the *analytic topology*, introduced by F. TAKENS [Tak84], which we present in Section 2. This topology is Baire but otherwise severely flawed: it does not turn $\text{Diff}(\mathbb{C}, 0)$ into a topological vector space (and does not enjoy a continuous composition) which, ironically enough, forbids any reasonably interesting analytic structure on $\text{Diff}(\mathbb{C}, 0)$. I believe that an effective analytic geometry in $\mathbb{C}\{z\}$ would allow to obtain much the same kind of result, and perhaps more constructively. Indeed the Fréchet calculus allows to identify directions transverse with the tangent space of an analytic set (e.g. the set of generators satisfying a given non-trivial algebraic relation), therefore pointing directions along which the algebraic relation between generators of G will be broken by perturbation. One should now check that these transverse directions can be realized as tangent spaces of curves embedded in the analytic set describing locally the property of being in the same conjugacy class, which is a statement reaching beyond the limits of the present article. However it is related to what comes now.

1.4.3. The glocal problem for diffeomorphisms.

It is not clear how to distinguish which local objects are actually global objects having been processed through a local change of coordinates. The *glocal problem* in the context of germs of diffeomorphisms refers to the following question:

« Is any element of $\text{Diff}(\mathbb{C}, 0)$ locally conjugate to a rational one ? If not, how do we recognize that some of them are ? »

In his thesis A. EPSTEIN underlines that $\acute{E}V(\Delta)$, for rational Δ , behave (dynamically) very much like a rational map itself (a «finite-type map»), except for the fact that it is transcendental and must admit a frontier for analytic continuation, so the answer to the former question is «no». Yet no answer to the latter one is known. Somehow glocality must be readable in the map $\acute{E}V$, but this task is a difficult one hindered by the fact that obtaining general properties on the invariants map is hard work.

Another approach consists in acknowledging that being glocal is a sub-analytic property. Indeed the set of biholomorphisms locally conjugate to a polynomial of degree d is the projection on the second factor of the analytic set Ω defined by the zero-locus of

$$\begin{aligned} \text{Diff}(\mathbb{C}, 0) \times \text{Diff}(\mathbb{C}, 0) &\longrightarrow \mathbb{C}\{z\} \\ (\varphi, \Delta) &\longmapsto (\text{Id} - J_d)(\varphi^{\circ-1} \circ \Delta \circ \varphi), \end{aligned}$$

where J_d is the d^{th} -jet of a germ and Id stands for the identity mapping from $\mathbb{C}\{z\}$ to $\mathbb{C}\{z\}$. Therefore it should be possible to gain knowledge from the study of the tangent space of Ω . This can be done using the Fréchet calculus detailed in this paper, although we need a more powerful tool to derive existential (or explicit) results by geometrical arguments. An article is in preparation regarding the glocal problem in $\text{Diff}(\mathbb{C}, 0)$.

1.4.4. The glocal problem for foliations.

A similar problem can be stated in the realm of germs of singular holomorphic foliations in \mathbb{C}^2 :

« Is any element of Sing locally conjugate to a polynomial one ? »

A non-constructive, negative answer is given to the question in [GT10]: the generic local conjugacy class of germs of saddle-node¹⁰ singularities do not contain polynomial representative. The methods used there generalize flawlessly to the case of resonant-saddle singularities¹¹, which are 2-dimensional counterparts to parabolic diffeomorphisms through the holonomy correspondence. Using the enhanced theorems that we prove here, we can be slightly more precise:

Corollary E. *The generic germ of a 2-dimensional resonant-saddle foliation is not glocal.*

We point out that, unlike the discrete case, no explicit example of non glocal foliation is yet known, as no characterization (even partial ones) of glocal foliations exists. I hope that from the developments to come for discrete dynamics will emerge a general framework in which explicit examples and/or criteria can be devised using perturbations along non-tangential directions provided by the Fréchet calculus.

1.5. Structure of the paper and table of contents.

- ⊗ In Section 2 we review «usual» topologies on $\mathbb{C}\{z\}$ and introduce the factorial and «useful» normally convex topologies we will use. We compare their relative thinness. We also explicit the sequential completion of these spaces.
- ⊗ In Section 3 we give a short survey of the standard definitions and general properties of analytic maps between locally convex spaces. We also introduce strong analyticity and Theorem D is proved.
- ⊗ In Section 4 we introduce analytic sets and prove the Baire analyticity and related properties of $\mathbb{C}\{z\}$, including Theorem A and Theorem B.
- ⊗ In Section 5 we introduce and give examples of the Fréchet calculus in $\mathbb{C}\{z\}$.
- ⊗ In Section 6 we present more applications of the main results to complex analysis, particularly by equipping the field of germs of meromorphic functions with an analytical structure modelled on $\mathbb{C}\{z\}$. In doing so we study coprimality in the ring $\mathbb{C}\{z\}$ and prove that coprime families $(f_1, \dots, f_k) \in \mathbb{C}\{z\}^k$ form an open, Zariski-dense set.
- ⊗ In Section 7 we give a complete proof to Theorem C and Corollary C.

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¹⁰The linear part at $(0,0)$ of (\diamond) has exactly one non-zero eigenvalue.

¹¹The linear part at $(0,0)$ of (\diamond) has two non-zero eigenvalues whose ratio lies in $\mathbb{Q}_{\leq 0}$, not formally linearizable.

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1.6. Notations and conventions.

- ⊗ The usual sets of numbers $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are used with the convention that \mathbb{N} is the set of non-negative integers.
- ⊗ Let $m \in \mathbb{N}$; we use bold-typed letters to indicate m -dimensional vectors $\mathbf{z} = (z_1, \dots, z_m)$ and multi-indexes $\mathbf{j} = (j_1, \dots, j_m)$. We define as usual
 - ⊙ $|\mathbf{j}| := \sum_{\ell=1}^m j_\ell$,
 - ⊙ $\mathbf{j}! := \prod_{\ell=1}^m j_\ell!$,
 - ⊙ $\mathbf{z}^{\mathbf{j}} := \prod_{\ell=1}^m z_\ell^{j_\ell}$.

- ⊛ $\mathbb{N}^{(\mathbb{N})}$ denotes the set of all finitely-supported sequences of non-negative integers, that can be identified with $\coprod_{m \geq 0} \mathbb{N}^m$.
- ⊛ We also define the **insertion symbol**

$$\mathbf{z} \triangleleft_j a := (z_1, \dots, z_{j-1}, a, z_{j+1}, \dots, z_m).$$

- ⊛ We use the notation « \oplus » to concatenate vectors

$$(z_1, \dots, z_k) \oplus (z_{k+1}, \dots, z_m) := (z_1, \dots, z_m),$$

and by a convenient abuse of notations we set $\mathbf{z} \oplus a := \mathbf{z} \oplus (a) = (z_1, \dots, z_m, a)$. In the same way we define

$$\begin{aligned} \mathbf{z}^{\oplus 0} &:= () \\ \mathbf{z}^{\oplus k+1} &:= \mathbf{z}^{\oplus k} \oplus \mathbf{z}. \end{aligned}$$

- ⊛ The symbol « \bullet » stands for the argument of a mapping, for instance $\exp \bullet$ stands for the function $z \mapsto \exp z$. We use it when the context renders the notation unambiguous.
- ⊛ If X and Y are topological spaces with the same underlying set, we write $X \geq Y$ if the identity mapping $\text{Id} : X \rightarrow Y$ is continuous. We write $X > Y$ if in addition the spaces are not homeomorphic.
- ⊛ A complex locally convex space whose topology is induced by a family $(\|\bullet\|_a)_{a \in A}$ of norms will be called a **normally convex space**.
- ⊛ $\mathbb{C}[[\mathbf{z}]]$ is the complex algebra of formal power series, $\mathbb{C}\{\mathbf{z}\}$ the sub-algebra consisting of those that converge (nontrivial domain of convergence). We distinguish between the formal power series $\sum_{n \geq 0} f_n z^n \in \mathbb{C}[[z]]$ with its sum, when it exists, $z \mapsto \sum_{n=0}^{\infty} f_n z^n$ understood as a holomorphic function on a suitable domain.
- ⊛ $\mathbb{C}[\mathbf{z}]_{\leq d}$ is the complex vector space of all polynomial of degree at most d , while $\mathbb{C}[\mathbf{z}]_{=d}$ is the Zariski-full open set consisting of those with degree exactly d .
- ⊛ For the sake of keeping notations as simple as possible we often identify the symbol « \mathbf{z} » both with an element of \mathbb{C}^m and with the identity mapping of the ambient space. This ambiguity will resolve itself according to the context. It may not be orthodoxically sound but it will prove quite convenient in some places.
- ⊛ In the context of the previous item the notation Id will be kept for the linear operator of an underlying, implicit vector space such as $\mathbb{C}[\mathbf{z}]$ or $\mathbb{C}\{\mathbf{z}\}$.
- ⊛ $\text{Diff}(\mathbb{C}^m, 0)$ is the group of germs of biholomorphisms fixing the origin of \mathbb{C}^m .
- ⊛ Throughout the article the notation \mathbb{D} stands for the open unit disc of \mathbb{C} .
- ⊛ A **domain** of a topological space is a non-empty, connected open set.

2. ABOUT USEFUL TOPOLOGIES ON $\mathbb{C}\{\mathbf{z}\}$

For the sake of simplicity we only present the case $m = 1$, from which the general case is easily derived. We say that the space $\mathbb{C}\{z\}$ of germ of a holomorphic function at $0 \in \mathbb{C}$ is a **compositing differential algebra** when it is endowed with a structure of topological algebra for which the differentiation $\frac{\partial}{\partial z}$ and the right- (*resp.* left-) composition $g^* : f \mapsto f \circ g$ with a given germ g vanishing at 0 (*resp.* $f_* : g \mapsto f \circ g$ with a given germ f), are continuous operations. We review here some topologies usually put on the space of convergent power series. We prove that some of them fail to be «useful» in the sense that they do not induce a structure of compositing differential algebra. For the sake of

example if $\mathbb{C}\{z\}$ is given a normed topology then it cannot be a compositing differential algebra. The sequence $z \mapsto \exp(kz)$, for $k \in \mathbb{N}$, indeed shows that differentiation is not continuous, while $z \mapsto z^k$ provides a sequence for which the right-composition with e.g. $z \mapsto 2z$ does not satisfy the axiom of continuity.

Notice that the «most» natural topology on $\mathbb{C}\{z\}$, the inductive topology, is useful. Yet it is not so easy to handle as compared to the sequential topologies we introduce (in particular because the inductive topology is not metrizable). We end this section by proving that some of the former, for instance the factorial ones, are useful.

2.1. The inductive topology.

The usual definition of the space of germs of holomorphic functions at $0 \in \mathbb{C}$ introduces $\mathbb{C}\{z\}$ as an inductive space

$$\mathbb{C}\{z\} := \varinjlim (\mathcal{B}_r)_{r>0},$$

where \mathcal{B}_r denotes the Banach space of bounded holomorphic functions on the disk $r\mathbb{D}$ equipped with the sup norm

$$\|f\|_{r\mathbb{D}} := \sup_{|z|<r} |f(z)|,$$

transition maps $\mathcal{B}_r \rightarrow \mathcal{B}_{r'}$ for $r' \leq r$ being defined by the restriction morphisms

$$\iota_{r \rightarrow r'} : f \in \mathcal{B}_r \mapsto f|_{r'\mathbb{D}} \in \mathcal{B}_{r'}.$$

An element $f \in \mathbb{C}\{z\}$ is therefore understood as an equivalence class of all couples $(r, f_r) \in \coprod_{r>0} \{r\} \times \mathcal{B}_r$ such that

$$(r, f_r) \rightsquigarrow (r', f_{r'}) \iff (\exists 0 < \rho \leq \min(r, r')) \ f_r|_{\rho\mathbb{D}} = f_{r'}|_{\rho\mathbb{D}}.$$

We denote by

$$\underline{(r, f_r)}$$

the equivalence class of (r, f_r) . The «convergence radius» function

$$\begin{aligned} \mathcal{R} : \mathbb{C}\{z\} &\longrightarrow]0, \infty] \\ f &\longmapsto \sup \left\{ r > 0 : \exists (r, f_r) \in \mathcal{B}_r, \underline{(r, f_r)} = f \right\} \end{aligned}$$

is well-defined.

Definition 2.1. The topological space obtained as the direct limit $\varinjlim \mathcal{B}_r$ equipped with the inductive topology will be denoted by

$$\varinjlim \mathbb{C}\{z\} := \varinjlim (\mathcal{B}_r)_{r>0}.$$

Notice that this direct limit is the same as that obtained by considering the countable family $(\mathcal{B}_{1/k})_{k \in \mathbb{N}_{>0}}$.

Remark 2.2. A mapping $\Lambda : \varinjlim \mathbb{C}\{z\} \rightarrow X$, where X is a topological space, is continuous if, and only if, for all $r > 0$ the map $\Lambda \circ \underline{\bullet} : \{r\} \times \mathcal{B}_r \rightarrow X$ is.

By definition $U \subset \varinjlim \mathbb{C}\{z\}$ is a neighborhood of 0 if, and only if

$$r^*U := \left\{ f_r : \underline{(r, f_r)} \in U \right\}, \ r > 0$$

is a direct system of neighborhoods of $0 \in \mathcal{B}_r$, that is

$$(2.1) \quad \begin{aligned} &(\forall r > 0) (\exists \delta(r) > 0) \quad \|f_r\|_{r\mathbb{D}} < \delta(r) \implies f_r \in r^*U \\ &(\forall 0 < r' \leq r) \quad r^*U = \iota_{r \rightarrow r'}^{-1}(r'^*U). \end{aligned}$$

Therefore the inductive topology is generated by the family $U(\delta)$ of neighborhoods of 0 (and by their image under the translations) indexed by a positive function $\delta : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$

$$U(\delta) := \bigcup_{r>0} \underline{B_{r\mathbb{D}}(0, \delta(r))},$$

where $B_{r\mathbb{D}}(0, \delta(r))$ is the ball of center 0 and radius $\delta(r)$ in \mathcal{B}_r . The set $U(\delta)$, being an absorbing and balanced neighborhood of 0, is the preimage of $[0, 1[$ by the Minkowski functional Ω_δ of $U(\delta)$:

$$\begin{aligned} \Omega_\delta(f) &:= \inf\{\lambda > 0 : f \in \lambda U(\delta)\} \\ &= \inf_{0 < r < \mathcal{R}(f)} \frac{\|f_r\|_{r\mathbb{D}}}{\delta(r)}. \end{aligned}$$

Definition 2.3. Without changing the topology we can make the following supplementary assumptions on δ .

- ⊗ δ is continuous and increasing, because $r > 0 \mapsto \|f_r\|_{r\mathbb{D}}$ is.
- ⊗ δ is a bounded, C^∞ function on $\mathbb{R}_{>0}$, flat at 0 such that $\ln \circ \delta \circ \exp$ is strictly concave on $\mathbb{R}_{<0}$. We use here the fact that $\Omega_\bullet : \delta \mapsto \Omega_\delta$ is decreasing.

If these conditions are fulfilled we say that δ is a **proper comparison function**.

Remark 2.4. The nice fact about proper comparison functions δ is that for every non-constant $f = \sum_{n \geq 0} f_n z^n \in \mathbb{C}\{z\}$ there exists a unique $0 < \rho < \mathcal{R}(f)$ depending on f and δ such that

$$(2.2) \quad \Omega_\delta(f) = \frac{\|f\|_{\rho\mathbb{D}}}{\delta(\rho)}.$$

According to Hadamard's Three Circles theorem $\tilde{f} := \ln \circ \|f_\bullet\|_{\bullet\mathbb{D}} \circ \exp$ is convex, it particularly admits left- and right-derivatives, respectively \tilde{f}'_- and \tilde{f}'_+ , which are increasing functions. Their limit at $-\infty$ is $\nu(f)$, the standard valuation of the germ f at 0, while they tend to $\deg f$ in the other direction (we conventionally set $\deg f := \infty$ when f is not a polynomial). On the other hand $\tilde{\delta} := \ln \circ \delta \circ \exp$ has a derivative which decreases strictly from ∞ to 0. Therefore there exists a unique ρ where the minimum of $\tilde{f} - \tilde{\delta}$ is reached, namely

$$\begin{aligned} \rho &= \sup\{\exp u : \tilde{f}'_-(u) < \tilde{\delta}'(u)\} \\ &= \inf\{\exp u : \tilde{f}'_+(u) > \tilde{\delta}'(u)\} \end{aligned}$$

which is stuck between r_f^+ and r_f^- , defined as the unique solution of

$$\tilde{f}'_\pm(\ln \rho) = \tilde{\delta}'(\ln r_f^\pm).$$

When \tilde{f} is differentiable at ρ all these numbers coincide.

It is actually possible to show that the inductive topology on $\overrightarrow{\mathbb{C}\{z\}}$ is that of a (non-metrizable) locally convex space, which is ultrabornological and nuclear, satisfying the following property.

Proposition 2.5. *The space $(\mathbb{C}\{z\}, \cdot, +, \times, \frac{\partial}{\partial z})$ is a compositing differential algebra, which is complete but not Baire.*

2.2. The analytic topology.

Another topology on $\mathbb{C}\{z\}$ has been introduced by F. TAKENS [Tak84]. It is worth noticing that although $\mathbb{C}\{z\}$ becomes a Baire space it is not a topological vector space, and for that reason will not be of great interest to us in the sequel; we only mention it for the sake of being as complete as possible. This topology is spanned by the following collection of neighborhoods of 0 (and their images by the translations of $\mathbb{C}\{z\}$):

$$U_{\rho, \delta} := \left\{ f \in \mathbb{C}\{z\} : \exists (\rho, f_\rho) \in f \text{ and } \|f_\rho\|_{\rho\mathbb{D}} < \delta \right\}, \quad \rho, \delta > 0.$$

The resulting topological space is written $\mathbb{C}\{z\}^\omega$. This space is not a topological vector space since no $U_{\rho, \delta}$ is absorbing¹². Let us conclude this paragraph by mentioning the

Proposition 2.6. *$\mathbb{C}\{z\}^\omega$ is a Baire space. The multiplication, differentiation or the right-composition with a dilatation are not continuous operations.*

2.3. The projective topology.

Because we work in the holomorphic world the space $\mathbb{C}\{z\}$ can also be analyzed through the Taylor linear one-to-one (but not onto) mapping

$$\begin{aligned} \mathbb{C}\{z\} &\xrightarrow{\tau} \mathbb{C}[[z]] \\ \underline{(r, f_r)} &\mapsto \sum_{n \geq 0} \frac{f_r^{(n)}(0)}{n!} z^n. \end{aligned}$$

The space $\mathbb{C}\{z\}$ is therefore isomorphic to the sub-algebra of the formal power series

$$\mathbb{C}[[z]] := \left\{ \sum_{n \geq 0} f_n z^n : (f_n)_n \in \mathbb{C}^{\mathbb{N}} \right\},$$

which differs from $\mathbb{C}^{\mathbb{N}}$ by the choice of the Cauchy product instead of the Hadamard (term-wise) product, characterized by the condition

$$\liminf |f_n|^{-1/n} > 0.$$

Hadamard's formula stipulates that this value is nothing but the radius of convergence $\mathcal{R}(\sum_{n \geq 0} f_n z^n)$ of the corresponding germ. The latter is given by the evaluation map

$$\begin{aligned} \mathbb{C}\{z\} &\xrightarrow{\varepsilon} \mathbb{C}\{z\} \\ f = \sum_{n \geq 0} f_n z^n &\mapsto \left(\frac{\mathcal{R}(f)}{1 + \mathcal{R}(f)}, z \mapsto \sum_{n=0}^{\infty} f_n z^n \right). \end{aligned}$$

There is no special meaning in taking $\frac{\mathcal{R}(f)}{1 + \mathcal{R}(f)}$, any finite number less than $\mathcal{R}(f)$ would work (e.g. $\min\{1, \frac{1}{2}\mathcal{R}(f)\}$).

¹²If the radius of convergence of f is strictly less than ρ then f does not belong to any $\lambda U_{\rho, \delta}$ whatever $\lambda \in \mathbb{C}$ may be.

We can therefore equip $\mathbb{C}\{z\}$ with the projective topology inherited from $\mathbb{C}[[z]]$ and defined by the N^{th} -jet projectors:

$$\begin{aligned} J_N : \mathbb{C}\{z\} &\longrightarrow \mathbb{C}[z]_{\leq N} \\ \sum_{n \geq 0} f_n z^n &\longmapsto \sum_{n \leq N} f_n z^n \end{aligned}$$

where the topology on each $\mathbb{C}[z]_{\leq N}$ is the standard normed one. It will also be convenient to introduce the Taylor-coefficient map of degree N as:

$$\begin{aligned} T_N : \mathbb{C}\{z\} &\longrightarrow \mathbb{C} \\ \sum_{n \geq 0} f_n z^n &\longmapsto f_N \end{aligned}$$

so that $J_N(\bullet) = \sum_{n=0}^N T_n(\bullet) z^n$. The induced topology is that of the product topology on $\prod_{N \geq 0} \mathbb{C}$; this coarseness renders the projective topology useless for our purposes in this article¹³.

Definition 2.7. The topological space obtained as the restriction to $\mathbb{C}\{z\}$ of the inverse limit $\lim_{\leftarrow} (\mathbb{C}[z]_{\leq N})_{N \in \mathbb{N}}$, equipped with the projective topology, will be denoted by

$$\mathbb{C}\{z\} := \lim_{\leftarrow} (\mathbb{C}[z]_{\leq N})_{N \in \mathbb{N}} \cap \mathbb{C}\{z\}.$$

Proposition 2.8. $(\mathbb{C}\{z\}, +, \times)$ is a non-Baire, non-complete topological algebra. Neither is it a compositing differential algebra.

Proof. This space clearly is not complete. Besides the decomposition $\mathbb{C}\{z\} = \bigcup_{N \in \mathbb{N}} F_N$, where

$$(2.3) \quad F_N := \left\{ \sum_{n \geq 0} f_n z^n : |f_n| \leq N^n \right\} = \bigcap_{n \in \mathbb{N}} |T_n|^{-1}([0, N^n])$$

is a closed set with empty interior, shows the space cannot be Baire. \square

2.4. Sequential topologies.

We define a norm on $\mathbb{C}\{z\}$ by making use of the Taylor coefficients at 0 of a germ at 0. Being given a sequence $a = (a_n)_{n \in \mathbb{N}}$ of positive numbers we can formally define

$$\left\| \sum_{n \geq 0} f_n z^n \right\|_a := \sum_{n=0}^{\infty} a_n |f_n|.$$

It is a genuine norm on $\mathbb{C}\{z\}$ if, and only if, a is asymptotically sufficiently flat, *i.e.* that it belongs to

$$\mathcal{A} := \left\{ a \in \mathbb{R}_{>0}^{\mathbb{Z}_{\geq 0}} : \lim_{n \rightarrow \infty} a_n^{1/n} = 0 \right\}.$$

Definition 2.9.

- (1) For any $a \in \mathcal{A}$ the above norm will be called the **a -norm** on $\mathbb{C}\{z\}$.

¹³We also mention that $\mathbb{C}\{z\}$ could be equipped with the restriction of the normed topology offered by the Krull distance on $\mathbb{C}[[z]]$, but this topology is rougher yet and even less interesting.

(2) The entire function

$$\mathfrak{r}_a : x \in \mathbb{C} \mapsto \sum_{n=0}^{\infty} a_n x^n$$

is called the **comparison function** of a . The **amplitude** of a is the function

$$\mathfrak{a}_a : r > 0 \mapsto \max_{n \in \mathbb{N}} \{a_n r^n\} < \infty.$$

- (3) For every non-empty subset $A \subset \mathcal{A}$ we define the A -**topology** of $\mathbb{C}\{z\}$ as the normally convex topology associated to the family of norms $(\|\bullet\|_a)_{a \in A}$. The topological vector space $(\mathbb{C}\{z\}, (\|\bullet\|_a)_{a \in A})$ will be written $\mathbb{C}\{z\}_A$. Such a topology is also called a **sequential topology**.
- (4) Two collections $A, A' \subset \mathcal{A}$ will be deemed **equivalent** if they induce equivalent topologies, that is the identity mapping is a homeomorphism $\mathbb{C}\{z\}_A \rightarrow \mathbb{C}\{z\}_{A'}$. For all intents and purposes we then say that both sequential topologies are the same but we write $A \simeq A'$ for precision.

Remark. With a similar construction it is possible to derive norms induced by hermitian inner products on $\mathbb{C}\{z\} \times \mathbb{C}\{z\}$ as defined by

$$\left\langle \sum_{n \geq 0} f_n z^n, \sum_{n \geq 0} g_n z^n \right\rangle_a := \sum_{n=0}^{\infty} a_n f_n \overline{g_n}.$$

Since we have not felt the need to use the associated extra structure this viewpoint provides, we will not particularly develop it (although the results presented here should continue to hold).

Every A -topology is spanned by the family of finite intersections of open a -balls of some radius $\varepsilon_a > 0$ and center 0

$$B_a(0, \varepsilon_a) := \{f \in \mathbb{C}\{z\} : \|f\|_a < \varepsilon_a\}$$

for $a \in A$. We recall that a linear mapping $L : \mathbb{C}\{z\}_A \rightarrow \mathbb{C}\{w\}_B$ is continuous if, and only if, for all $b \in B$ there exists a finite set $F \subset A$ and some $C_b \geq 0$ such that

$$(\forall f \in \mathbb{C}\{z\}) \quad \|L(f)\|_b \leq C_b \max_{a \in F} \|f\|_a.$$

We denote by $\mathcal{L}(\mathbb{C}\{z\}_A \rightarrow \mathbb{C}\{w\}_B)$ the space of all linear, continuous mappings and endow it with the natural locally convex topology induced by that of the source and range spaces:

$$(2.4) \quad (\forall L \in \mathcal{L}(\mathbb{C}\{z\}_A \rightarrow \mathbb{C}\{z\}_B)) (\forall a \in A, b \in B) \quad \|L\|_{a \rightarrow b} := \sup_{f \neq 0} \frac{\|L(f)\|_b}{\|f\|_a}.$$

We finish this section by mentioning the following easy lemma, based on the observation that if $f(z) = \sum_{n \geq \nu} f_n z^n \in \mathbb{C}\{z\}$ with $f_\nu \neq 0$ then for every $a \in \mathcal{A}$ and $g \in B_a(f, \frac{a_\nu}{|f_\nu|})$ either the derivative $\frac{d^\nu g}{dz^\nu}(0)$ does not vanish or some derivative of lesser order does not.

Lemma 2.10. *For any A -topology on $\mathbb{C}\{z\}$ the valuation map*

$$\begin{aligned} \mathbb{C}\{z\}_A \setminus \{0\} &\longrightarrow \mathbb{N} \\ \sum_{n \geq 0} f_n z^n \neq 0 &\longmapsto \inf \{n : f_n \neq 0\} \end{aligned}$$

is lower semi-continuous.

2.4.1. Naive polydiscs.

Definition 2.11. Let $r := (r_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers and take $f \in \mathbb{C}\{z\}$. The **naive-polydisc** of center f and poly-radius r is the set

$$D(f, r) := \{g \in \mathbb{C}\{z\} : (\forall n \in \mathbb{N}) |T_n(f - g)| < r_n\}.$$

Lemma 2.12. A naive-polydisc $D(f, r)$ contains an open a -ball of center f if, and only if,

$$\liminf_{n \rightarrow \infty} a_n r_n > 0.$$

This particularly means that $\frac{1}{r} \in \mathcal{A}$ and $\lim r = \infty$. Beside the above limit represents the maximum radius of a -balls that can be included in $D(f, r)$.

Proof. We can suppose that $f = 0$ and write $r = (r_n)_n$. Assume first that there exists $\varepsilon > 0$ such that $B_a(0, \varepsilon) \subset D(0, r)$. In particular for a given $n \in \mathbb{N}$ the polynomial ηz^n belongs to $D(0, r)$ for every $\eta \in \mathbb{C}$ such that $|\eta| a_n < \varepsilon$, which implies $\varepsilon \leq r_n a_n$. Conversely if $\liminf a_n r_n > 0$ then there exists $\varepsilon > 0$ such that $a_n r_n \geq \varepsilon$ for every $n \in \mathbb{N}$. If we choose $f \in B_a(0, \varepsilon)$ then for all integer n we particularly have the estimate $a_n |f_n| \leq \varepsilon \leq a_n r_n$ so that $B_a(0, \varepsilon) \subset D(0, r)$. \square

2.4.2. Topological completion.

Any sequential topology induces a uniform structure on $\mathbb{C}\{z\}$, allowing to contemplate the notion of topological completeness¹⁴. We give without proof the following statements.

Proposition 2.13. Consider some A -topology on $\mathbb{C}\{z\}$.

- (1) The Cauchy (sequential) completion of $\mathbb{C}\{z\}_A$ is canonically isomorphic, as \mathbb{C} -algebras, to the locally convex subspace of $\mathbb{C}[[z]]$ defined by

$$\widehat{\mathbb{C}\{z\}}_A := \bigcap_{a \in A} \left\{ \sum_{n \geq 0} f_n z^n : \sum_{n=0}^{\infty} a_n |f_n| < \infty \right\}$$

endowed with the family of norms $(\|\bullet\|_a)_{a \in A}$.

- (2) Take $f = \sum_n f_n z^n \in \widehat{\mathbb{C}\{z\}}_A$ and the associated sequence of jets $J_N(f) = \sum_{n=0}^N f_n z^n$ as $N \in \mathbb{N}$. Then for every $a \in A$

$$\lim_{N \rightarrow \infty} \|f - J_N(f)\|_a = 0.$$

In particular the subspace of polynomials is dense in $\widehat{\mathbb{C}\{z\}}_A$.

- (3) We have

$$\mathbb{C}\{z\} = \widehat{\mathbb{C}\{z\}}_A,$$

which means the space $\mathbb{C}\{z\}_A$ is sequentially complete. Besides no other (non-equivalent) A -topology can be.

¹⁴In general one only needs sequential completeness, and the corresponding completed space will be named Cauchy's completion as opposed to Hausdorff's topological completion of the uniform space $\mathbb{C}\{z\}_A$.

Remark 2.14. When A is at most countable the space $\widehat{\mathbb{C}\{z\}}_A$ is a special case of a Köthe sequential space [Köt69], some of which have been extensively studied (e.g. rapidly decreasing sequences) in particular regarding the property of tameness to be used in Nash-Moser local inversion theorem [Ham82]. Unfortunately it is known that the Köthe spaces presented here do not fulfill Nash-Moser's theorem hypothesis although, as we see further down, some of them are nuclear.

2.4.3. Radius of convergence and Baire property.

We relate now the Baire property on $\mathbb{C}\{z\}_A$ to the absence of a positive lower bound for $\mathcal{R}(\bullet)$ on any domain. This relationship was suggested by R. SCHÄPFKE.

Proposition 2.15. *Fix some A -topology on $\mathbb{C}\{z\}$.*

- (1) \mathcal{R} is upper semi-continuous on $\widehat{\mathbb{C}\{z\}}_A$ and for all non-empty open set $U \subset \mathbb{C}\{z\}_A$ we have $\mathcal{R}(U) =]0, \infty]$. In particular \mathcal{R} can never be positively lower-bounded on U .
- (2) The space $\mathbb{C}\{z\}_A$ can never be Baire (in particular not Fréchet when A is countable).

Proof.

- (1) Each map $H_N : \sum_{n \geq 0} f_n z^n \mapsto |f_N|^{1/N}$ is continuous and ranges in $[0, \infty[$. As a consequence each member of the sequence of functions indexed by $n \in \mathbb{N}$ and defined by

$$R_n := \sup_{N \geq n} H_N$$

is lower semi-continuous. Therefore $\frac{1}{\mathcal{R}(\bullet)}$ is the limit of an increasing sequence of lower semi-continuous functions and consequently is itself lower semi-continuous. According to Theorem 2.13 (2) the value ∞ belongs to $\mathcal{R}(U)$, so let us now take an arbitrary $r > 0$. For all $\lambda \in \mathbb{C}_{\neq 0}$ the radius of convergence of the power series $f_\lambda := \lambda \sum_{n \geq 0} r^{-n} z^n$ is precisely r . By taking λ small enough and by picking $f \in U \cap \mathbb{C}[z]$ the germ $f + f_\lambda$ belongs to U and its radius of convergence is precisely r .

- (2) $\mathbb{C}\{z\}$ is covered by the countable family of closed sets F_N defined in (2.3). Every one of these has empty interior because of (1).

□

2.5. Comparing the topologies.

Proposition 2.16. *Let $A \subset \mathcal{A}$ be given. We have the following (strict) ordering of topologies*

$$\underleftarrow{\mathbb{C}\{z\}} < \mathbb{C}\{z\}_A < \underrightarrow{\mathbb{C}\{z\}}.$$

The main part of this proposition is just a rephrasing of the following two fairly obvious lemmas.

Lemma 2.17. *The Taylor map $\mathcal{T} : \underrightarrow{\mathbb{C}\{z\}} \rightarrow \mathbb{C}\{z\}_A$ is continuous. More precisely the standard Cauchy's estimate gives, for all $r > 0$, all $f_r \in \mathcal{B}_r$ and all $a \in A$:*

$$\|\mathcal{T}(f_r)\|_a \leq \left\| z \mapsto \frac{r}{r-z} \right\|_a \|f_r\|_{r\mathbb{D}}.$$

Lemma 2.18. *The N^{th} -jet projector $J_N : \mathbb{C}\{z\}_A \rightarrow \mathbb{C}[z]_{\leq N}$ is continuous. More precisely for all $a \in \mathcal{A}$ and $f \in \mathbb{C}\{z\}$*

$$\|J_N(f)\|_a \leq \|f\|_a.$$

The fact that $\mathbb{C}\{z\}_A > \mathbb{C}\{z\}$ is clear enough. To see that $\mathbb{C}\{z\}_A < \varinjlim \mathbb{C}\{z\}$ we are going to consider the Minkowski functional Ω_δ of the proper comparison function

$$\delta : r > 0 \mapsto \exp\left(-\frac{1}{r}\right).$$

Let $a \in \mathcal{A}$ be given and fix $d \in \mathbb{N}_{>0}$. Define

$$\begin{aligned} \rho_d &:= a_d^{1/d} \\ \alpha_d &:= \rho_d^d (d\rho_d - 1) \end{aligned}$$

and

$$f_d : z \mapsto \alpha_d + z^d.$$

By construction ρ satisfies (2.2) for $f := f_d$, and

$$\Omega_\delta(f_d) = \frac{f_d(\rho_d)}{\delta(\rho_d)} = (\alpha_d + a_d) \exp\left(a_d^{-1/d}\right).$$

Since $a \in \mathcal{A}$ the ratio

$$\frac{\Omega_\delta(f_d)}{\|f_d\|_a} = \frac{\alpha_d + a_d}{\alpha_d a_0 + a_d} \exp\left(a_d^{-1/d}\right)$$

goes to infinity as d does, which proves that the evaluation mapping $\mathcal{E} : \mathbb{C}\{z\}_A \rightarrow \varinjlim \mathbb{C}\{z\}$ cannot be continuous.

2.6. Useful topologies.

Definition 2.19. Let $A \subset \mathcal{A}$ be non-empty.

- (1) An A -topology is **finite** if it is equivalent to some A' -topology, A' being a finite set.
- (2) An A -topology is **useful** when $(\mathbb{C}\{z\}_A, \cdot, +, \times, \frac{\partial}{\partial z})$ is a compositing differential topological algebra.
- (3) The **factorial topology** is the metrizable sequential topology spanned by the family

$$\begin{aligned} \text{AF} &:= \left\{ a\left(\frac{1}{k}\right) : k \in \mathbb{N} \right\}, \\ a(\alpha) &:= (n!^{-\alpha})_{n \in \mathbb{N}}, \alpha \in \mathbb{R}_{>0}. \end{aligned}$$

Remark 2.20. In case one wants to study the factorial topology on $\mathbb{C}\{z\}$ with $\mathbf{z} = (z_1, \dots, z_m)$ then one defines for $\alpha > 0$

$$\left\| \sum_{\mathbf{n} \in \mathbb{N}^m} f_{\mathbf{n}} z^{\mathbf{n}} \right\|_{a(\alpha)} := \sum_{\mathbf{n}=0}^{\infty} |f_{\mathbf{n}}| (\mathbf{n}!)^{-\alpha}.$$

2.6.1. General properties of useful topologies.

Proposition 2.21. *The \mathcal{A} -topology is useful, while no finite \mathcal{A} -topology can be.*

Proof. The fact that no finite \mathcal{A} -topology may be useful has been hinted at in the introduction of this section, the argument being the same as in the normed case. The remaining of the proposition is mainly a consequence of the next trivial lemma:

Lemma 2.22. *For every $a \in \mathcal{A}$ there exists $b, c \in \mathcal{A}$ such that for all $m \in \mathbb{N}$ and all $\mathbf{j} \in \mathbb{N}^m$ we have $c_n \leq 1$ and*

$$a_{|\mathbf{j}|} \leq c_m \prod_{\ell=1}^m b_{j_\ell}.$$

Back to our proposition, let us first address the continuity of the multiplication. For $a \in \mathcal{A}$:

$$\|f \times g\|_a = \sum_{n=0}^{\infty} a_n \left| \sum_{p+q=n} f_p g_q \right|.$$

According to the lemma there exist $b, c \in \mathcal{A}$ such that $a_n \leq b_p b_q$ for every $p+q=n$, which means that

$$\|f \times g\|_a \leq \|f\|_b \|g\|_b.$$

Now consider the action of the derivation:

$$(2.5) \quad \frac{\partial f}{\partial z} = \sum_{n \geq 0} (n+1) f_{n+1} z^n$$

so that for every $a, b \in \mathcal{A}$

$$\left\| \frac{\partial f}{\partial z} \right\|_a = \sum_{n=0}^{\infty} a_n (n+1) |f_{n+1}| \leq \sum_{n=0}^{\infty} b_{n+1} |f_{n+1}| \times \frac{(n+1) a_n}{b_{n+1}}.$$

We can always find b such that the sequence $\left(\frac{(n+1) a_n}{b_{n+1}} \right)_{n \in \mathbb{N}}$ is bounded by some¹⁵ $C > 0$ so that

$$\left\| \frac{\partial f}{\partial z} \right\|_a \leq C \|f\|_b.$$

We end the proof by using the composition formula, being given $g \in \mathbb{C}\{z\}$,

$$(2.6) \quad f \circ g = \sum_n \left(\sum_{m \leq n} f_m \sum_{\mathbf{j} \in \mathbb{N}^m, |\mathbf{j}|=n} \prod_{\ell=1}^m g_{j_\ell} \right) z^n.$$

For $a \in \mathcal{A}$ we have

$$\|f \circ g\|_a \leq \sum_{n=0}^{\infty} a_n \sum_{m \leq n} |f_m| \sum_{\mathbf{j} \in \mathbb{N}^m, |\mathbf{j}|=n} \prod_{\ell=1}^m |g_{j_\ell}|.$$

¹⁵Take for instance $b_0 := 1$ and $b_{n+1} := \sqrt{a_n}$.

Invoking once more the previous lemma we conclude the existence of $b, c \in \mathcal{A}$ such that

$$\begin{aligned} \|f \circ g\|_a &\leq \sum_{m=0}^{\infty} c_m |f_m| \sum_{n=0}^{\infty} \sum_{j \in \mathbb{N}^m, |j|=n} \prod_{\ell=1}^m b_{j_\ell} |g_{j_\ell}| \\ &\leq \sum_{m=0}^{\infty} \sqrt{c_m} |f_m| \sqrt{c_m} \|g\|_b^m \leq \|f\|_{\sqrt{c}} \mathfrak{a}_{\sqrt{c}}(\|g\|) \end{aligned}$$

where the amplitude function \mathfrak{a}_\bullet is introduced in Definition 2.9. \square

We show now a nice feature of useful topologies:

Proposition 2.23. *Assume that the sequential topology $\mathbb{C}\{z\}_A$ is useful. Then $\mathbb{C}\{z\}_A$ is a nuclear space. In particular if A is countable then $\widehat{\mathbb{C}\{z\}_A}$ is a Montel space (i.e. every closed, bounded¹⁶ set of $\widehat{\mathbb{C}\{z\}_A}$ is compact).*

Proof. To show that $\mathbb{C}\{z\}_A$ is nuclear we must show that for every $a \in A$ the natural embedding $\iota_a : \mathbb{C}\{z\}_A \hookrightarrow \widehat{\mathbb{C}\{z\}_a}$ is nuclear. By definition this boils down to proving the existence of a sequence $(g_n)_{n \in \mathbb{N}}$ of $\widehat{\mathbb{C}\{z\}_a}$ and a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of continuous linear forms $\mathbb{C}\{z\}_A \rightarrow \mathbb{C}$ such that

- ⊗ there exists $b \in A$ with $\sum_{n=0}^{\infty} \|\varphi_n\|_b \|g_n\|_a < \infty$
- ⊗ $\iota_a(\bullet) = \sum_{n=0}^{\infty} \varphi_n(\bullet) g_n$ for the above normal convergence.

The natural choice is $g_n := z^n$ and $\varphi_n := T_n$, so that $\|g_n\|_a = a_n$ and $\|T_n\|_b \leq \frac{1}{b_n}$. Since A is useful there exists $b \in A$ and $C > 0$ such that for all $f \in \mathbb{C}\{z\}$

$$\|f \circ (2z)\|_a \leq C \|f\|_b.$$

Take now $f(z) := z^n$ so that for all $n \in \mathbb{N}$

$$2^n a_n \leq C b_n,$$

and $\sum_{n=0}^{\infty} \frac{a_n}{b_n} < \infty$ as expected. It is besides well known [Trè06, Section 50] that nuclear Fréchet spaces are Montel spaces. \square

2.6.2. Factorial topology.

We study now the factorial topology, which we will use in most applications because of its nice combinatorial and analytical properties. The choice of the exponents $1/k$ is rather arbitrary and one could chose any strictly decreasing to zero sequence of positive numbers in what follows. The topological completion of the space $\widehat{\mathbb{C}\{z\}_{\text{AF}}}$ corresponds to the Köthe sequence space of finite order $\Lambda_0(n!^{-1/k})$ which has been well studied, and the choice of exponent we make is therefore «standard»¹⁷.

Proposition 2.24. *The factorial topology is useful. Moreover the next three additional quantitative properties hold.*

- (1) For every $f, g \in \mathbb{C}\{z\}$ and all $\alpha > 0$:

$$\|f \times g\|_{a(\alpha)} \leq \|f\|_{a(\alpha)} \|g\|_{a(\alpha)}.$$

¹⁶We recall that a subset Ω of a locally convex space is bounded if every neighborhood of 0 can be rescaled to contain Ω .

¹⁷Another reason why this choice may be deemed natural is that $\widehat{\mathbb{C}\{z\}_{\text{AF}}}$ is the space of «sub-Gevrey» formal power series, that is those power series $\sum_{n \geq 0} f_n z^n$ such that $\sum_{n \geq 0} (n!)^{-\alpha} f_n z^n$ converges for all $\alpha > 0$.

(2) For all $\alpha > \beta > 0$ and every $f, g \in \mathbb{C}\{z\}$ with $g = 0$:

$$\|f \circ g\|_{a(\alpha)} \leq \|f\|_{a(\beta)} \mathfrak{a}_{a(\alpha-\beta)}(\|g\|_{a(\alpha)})$$

where the amplitude \mathfrak{a}_\bullet is defined in Definition 2.9. It is equal to $\sup_{n \in \mathbb{N}} n!^{\beta-\alpha} \|g\|_{a(\alpha)}^n$, whose maximum is reached for some rank

$$n_0 \in \{-1, 0, 1\} + \left\lceil \|g\|_{a(\alpha)}^{1/\alpha-\beta} \right\rceil.$$

This constant is optimal.

(3) For all $\alpha > \beta > 0$ there exists a sequence of positive real numbers $D_{k,\alpha,\beta}$ such that for every $f \in \mathbb{C}\{z\}$:

$$\left\| \frac{\partial^k f}{\partial z^k} \right\|_{a(\alpha)} \leq D_{k,\alpha,\beta} \|f - J_k(f)\|_{a(\beta)}.$$

The optimal constant is given by

$$D_{k,\alpha,\beta} = \sup_{n \in \mathbb{N}} \frac{(n+k)!^{\beta+1}}{n!^{\alpha+1}}$$

whose maximum is reached for some rank

$$n_0 \in \{-1, 0, 1\} + \left\lceil r_{k,\alpha,\beta} \right\rceil,$$

where $r_{k,\alpha,\beta}$ is the positive solution of the equation $(x+k)^{\beta+1} = x^{\alpha+1}$. We also have

$$r_{k,\alpha,\beta} \sim_{k \rightarrow \infty} k^{\beta+1/\alpha+1} \text{ and } r_{k,\alpha,\beta} > k^{\beta+1/\alpha+1}.$$

Besides there exists $\chi_{k,\alpha,\beta} \in]r_{k,\alpha,\beta}, r_{k,\alpha,\beta} + 1[$ such that for all k , writing χ instead of $\chi_{k,\alpha,\beta}$,

$$D_{k,\alpha,\beta} \leq \frac{e^{\beta+1}}{(2\pi)^{\alpha+1/2}} \chi^{k(\alpha+1)} \exp((\alpha-\beta)\chi - (\beta+1)k).$$

One has

$$k^{\beta+1/\alpha+1} < \chi_{k,\alpha,\beta} < e^{\sqrt{\beta+1/\alpha-\beta}} k^{\beta+1/\alpha+1} + 1.$$

Proof. The proof first relies on the following trivial combinatorial estimate, which is a counterpart to Lemma 2.22:

Lemma 2.25. Let $m \in \mathbb{N}_{>0}$ and $\mathbf{j} \in \mathbb{N}^m$. Then

$$|\mathbf{j}|! \geq \mathbf{j}!$$

and if moreover $j_\ell > 0$ for all ℓ we have

$$|\mathbf{j}|! \geq m! \mathbf{j}!.$$

The addition and scalar multiplication are of course continuous. The three estimates (1), (2) and (3) guarantee the usefulness of the factorial topology. Take two elements f and g in $\mathbb{C}\{z\}$ and write them down respectively as $\sum_n f_n z^n$ and $\sum_n g_n z^n$.

(1) According to the lemma we have $(p+q)! \geq p!q!$ so that

$$\begin{aligned} \|f \times g\|_{a(\alpha)} &= \sum_{n=0}^{\infty} n!^{-\alpha} \left| \sum_{p+q=n} f_p g_q \right| \\ &\leq \sum_{n=0}^{\infty} \sum_{p+q=n} p!^{-\alpha} q!^{-\alpha} |f_p g_q| = \|f\|_{a(\alpha)} \|g\|_{a(\alpha)}. \end{aligned}$$

(2) Because of the lemma we can show in the same spirit as Proposition 2.21 that

$$\|f \circ g\|_{a(\alpha)} \leq \|f\|_{a(\beta)} \sup_{m \in \mathbb{N}} m!^{\beta-\alpha} \|g\|_{a(\alpha)}^m.$$

We can try to estimate $\mathfrak{a}_{a(\beta-\alpha)}(\|g\|_{a(\alpha)})$ using the Gamma and Digamma functions, through the study of the auxiliary function

$$\begin{aligned} \phi : \mathbb{R}_{>1} &\longrightarrow \mathbb{R} \\ x &\longmapsto \Gamma(x)^{-\delta} r^x \end{aligned}$$

where $\delta, r > 0$ are given. This function admits a unique maximum, located at the zero of its logarithmic derivative (expressed in terms of the Digamma function $\psi = \frac{\Gamma'}{\Gamma}$)

$$\frac{\phi'}{\phi}(x) = -\delta\psi(x) + \ln r.$$

It is well known that for $x > 1$

$$(2.7) \quad \ln(x-1) \leq \psi(x) \leq \ln x$$

so that the equality $\psi(x) = \frac{1}{\delta} \ln r$ happens only when

$$x \in [r^{1/\delta}, r^{1/\delta} + 1].$$

Since this interval has length 1 the result follows by setting $r := \|g\|_{a(\alpha)}$ and $\delta := \alpha - \beta$.

(3) We apply formula (2.5) repeatedly so that

$$\begin{aligned} \left\| \frac{\partial^k f}{\partial z^k} \right\|_{a(\alpha)} &= \sum_{n=0}^{\infty} a(\alpha)_n \frac{(n+k)!}{n!} |f_{n+k}| \\ &\leq \sum_{n=0}^{\infty} (n+k)!^{-\beta} |f_{n+k}| \times \frac{(n+k)!^{\beta+1}}{n!^{\alpha+1}} \\ &\leq D_{k,\alpha,\beta} \|f - J_k(f)\|_{a(\beta)}. \end{aligned}$$

The constant is optimal: take for f the monomial z^d where d is any integer such that $\frac{(d+k)!^{\beta+1}}{d!^{\alpha+1}}$ equals $D_{k,\alpha,\beta}$. In order to derive the final estimate we need to determine d ; for this we study the auxiliary function

$$\begin{aligned} \varphi : \mathbb{R}_{>1} &\longrightarrow \mathbb{R} \\ x &\longmapsto \frac{\Gamma(x+k)^{\beta+1}}{\Gamma(x)^{\alpha+1}} \end{aligned}$$

much in the same way as we did just before. This function admits a unique maximum, which is the positive zero of its logarithmic derivative

$$\begin{aligned}\frac{\varphi'}{\varphi}(x) &= (\beta + 1)\psi(x+k) - (\alpha + 1)\psi(x) \\ &= (\beta - \alpha)\psi(x) + (\beta + 1)\sum_{j=0}^{k-1} \frac{1}{x+j}.\end{aligned}$$

We therefore seek the positive real χ such that

$$(2.8) \quad \delta\psi(\chi) = \sum_{j=0}^{k-1} \frac{1}{\chi+j}$$

where

$$\begin{aligned}\delta &:= \frac{\alpha - \beta}{\beta + 1} > 0 \\ \delta + 1 &= \frac{\alpha + 1}{\beta + 1}.\end{aligned}$$

It is completely elementary that for $x > 1$ we have

$$\ln \frac{x+k}{x} < \sum_{j=0}^{k-1} \frac{1}{x+j} < \ln \frac{x+k-1}{x-1}.$$

This estimate, coupled with (2.7), implies particularly that the equality (2.8) is possible only if $\ln \frac{x+k}{x} < \delta \ln x$ and $\ln \frac{x+k-1}{x-1} < \delta \ln(x-1)$. As a consequence x_* lies in the interval $]r, r+1[$ where r is the positive solution of

$$(x+k)^{\beta+1} = x^{\alpha+1}$$

or, equivalently,

$$x^{\delta+1} - x = k.$$

Since this interval has length 1 one has

$$d \in \{-1, 0, 1\} + [r].$$

Plugging $x := k^{1/\delta+1}$ in the previous expression shows that

$$k^{1/\delta+1} < r$$

(in particular the sequence r is unbounded). The relation $\frac{k}{r} = r^\delta - 1$ proves $r = o(k)$. Both relations hold also for χ instead of r . Moreover let $\sigma > 0$ be given such that

$$\sigma \geq e^{1/\sqrt{\delta}} > 1.$$

Then

$$\begin{aligned}
\delta &\geq \frac{1}{(\ln \sigma)^2} > \frac{1}{\sigma \ln \sigma} \geq \frac{\ln(1 + 1/\sigma)}{\ln \sigma} \\
\delta + 1 &> \frac{\ln(\sigma + 1)}{\ln \sigma} \\
1 &> \frac{\sigma}{\sigma^{\delta+1} - 1} \\
k \geq 1 &> \left(\frac{\sigma}{\sigma^{\delta+1} - 1} \right)^{\delta+1/\delta} \\
k(\sigma^{\delta+1} - 1) &> \sigma k^{1/\delta+1} \\
(\sigma k^{1/\delta+1})^{\delta+1} - \sigma k^{1/\delta+1} &> k
\end{aligned}$$

so that

$$r < \sigma k^{1/\delta+1}.$$

We use finally the classical relation for $x > 0$

$$(2.9) \quad \sqrt{2\pi} x^{x-1/2} e^{-x} \leq \Gamma(x) \leq e^{1-x} x^{x-1/2}$$

and the fact that $\chi^{\alpha+1} > (\chi + k)^{\beta+1}$ to derive

$$\varphi(\chi) \leq \frac{e^{\beta+1}}{(2\pi)^{\alpha+1/2}} \chi^{k(\alpha+1)} \exp((\alpha - \beta)\chi - (\beta + 1)k).$$

□

3. ANALYTICITY

In this section we consider two Hausdorff, locally convex spaces $(E, (\|\bullet\|_{a \in A}))$ and $(F, (\|\bullet\|_{b \in B}))$, i.e. topological linear spaces whose topologies are induced by a collection of separating semi-norms. We begin with giving general definitions and properties, summarizing some important results of the references [Bar85] and [Maz84]. We then present specifics for the spaces $\mathbb{C}\{z\}_A$ and $\mathbb{C}\{z\}$, as well as introducing the notion of strong analyticity. Notice that because of Proposition 2.16 and Definition 3.2 below, as soon as a map $\Lambda : \mathbb{C}\{z\} \rightarrow \mathbb{C}\{x\}$ is proved analytic for some sequential A -topology, which is comparatively an easier task to perform, it will automatically be analytic for the inductive topology.

A notion we will need is that of ample boundedness, which mimics the criterion for the continuity of linear mappings.

Definition 3.1. A mapping $\Lambda : U \rightarrow F$ from an open set U of E is **amply bounded** if for every $f \in U$ and all $b \in B$ there exists a neighborhood $W \subset E$ of 0 such that

$$\sup_{h \in W} \|\Lambda(f + h)\|_b < \infty.$$

Continuous mappings are amply bounded.

3.1. Polynomials, power series and analyticity.

Definition 3.2.

- (1) A **polynomial** on E of degree at most d with values in F is a finite sum

$$P(f) = \sum_{p=0}^d P_p(f^{\oplus p})$$

of continuous, symmetric p -linear mappings $P_p \in \mathcal{L}_p(E \rightarrow F)$. The least d for which the above expansion holds for $P \neq 0$ is its degree $\deg P$. As usual we conventionally set $\deg 0 := -\infty$, and recall that

$$f^{\oplus p} = (f, \dots, f) \in E^p.$$

- (2) A **formal power series** Φ from E to F is a series built from a sequence of continuous, symmetric p -linear mappings $(P_p)_{p \in \mathbb{N}} \in \prod_{p \in \mathbb{N}} \mathcal{L}_p(E \rightarrow F)$:

$$\Phi(f) := \sum_{p \geq 0} P_p(f^{\oplus p}).$$

The space of all such objects is a complex algebra with the standard sum and Cauchy product operations on series.

- (3) A mapping Λ defined on some neighborhood U of $f \in E$, with values in F , is said to be **analytic at f** if for all $b \in B$ there exists a neighborhood W of $0 \in E$ such that

$$\lim_{N \rightarrow \infty} \left\| \Lambda(f+h) - \sum_{p=0}^N P_p(h^{\oplus p}) \right\|_b = 0$$

uniformly in $h \in W$.

- (4) We say that Λ is **analytic** on U if it is analytic at any point of U . If Λ is analytic on the whole E we say that Λ is **entire**.
- (5) The p -linear mapping $p!P_p$ is called the p -th **Taylor coefficient** of Λ at f and is written as

$$\frac{\partial^p \Lambda}{\partial f^p} \Big|_f := p!P_p.$$

The formal power series $\sum_{p \geq 0} \frac{1}{p!} \frac{\partial^p \Lambda}{\partial f^p} \Big|_f$ is called the **Taylor series** of Λ at f .

Theorem 3.3. *Let $U \subset E$ be an open set. In the following hatted spaces denotes their Hausdorff topological completion (as uniform spaces).*

- (1) ([Bar85, p177]) *If Λ is analytic at some point $f \in U$ then its Taylor coefficients are unique. In that case $\Lambda(f) = \frac{\partial^0 \Lambda}{\partial f^0} \Big|_f$.*
- (2) ([Bar85, p195]) *Analytic mappings $U \rightarrow F$ are continuous.*
- (3) ([Bar85, p196]) *An amply bounded mapping $U \rightarrow F$ is analytic if, and only if, for every $f \in U$ there exists a sequence $(P_p)_{p \in \mathbb{N}}$ of p -linear mappings, not necessarily continuous but satisfying nonetheless the rest of condition (3) in the above definition. Particularly in that case each P_p is a posteriori continuous.*
- (4) ([Bar85, p187]) *Polynomials with values in F are entire. If Φ is a polynomial then $\frac{\partial^p \Phi}{\partial f^p} \Big|_f = 0$ for all $p > \deg P$ and $f \in E$.*

- (5) ([Bar85, p192]) Let $\Lambda : U \rightarrow F$ be analytic and $\Xi : F \rightarrow G$ be a continuous linear mapping with values in a normally convex space G . Then $\Xi \circ \Lambda$ is analytic and

$$\frac{\partial^p \Xi \circ \Lambda}{\partial f^p} \Big|_f = \Xi \circ \frac{\partial^p \Lambda}{\partial f^p} \Big|_f.$$

- (6) ([Maz84, p63]) More generally the composition of source/range compatible analytic maps remains analytic.
- (7) ([Maz84, p68]) Any analytic map $\Lambda : U \rightarrow F$ extends in a unique fashion to an analytic map $\hat{\Lambda} : \hat{U} \rightarrow \hat{F}$ from an open set of \hat{E} . Notice that \hat{U} might not be a topological completion of U .

Notice that from (1) is deduced the identity theorem, using a standard connectedness argument:

Corollary 3.4. (Identity Theorem [Bar85, p241]) Let $U \subset E$ be a domain and Λ analytic on U such that Λ vanishes on some open subset of U . Then Λ is the zero map.

3.2. Analytical spaces.

As usual with analyticity, the definition is given for maps defined on open sets. Yet in practice we would like to speak of analyticity of maps defined on sets with empty interior (e.g. proper analytic sets). Although there exists a notion of analytic variety [Maz84] we do not want to venture into those territories. As we will be in general able to parameterize the sets of interest to us, we introduce instead the notion of induced analytic structure:

Definition 3.5. Let X be a topological space, $\Psi : U \subset E \rightarrow X$ be continuous and onto from an open set U of E .

- (1) We say that a map $\Lambda : X \rightarrow F$ is **analytic with respect to the analytic structure induced by Ψ** if $\Psi^* \Lambda = \Lambda \circ \Psi : U \rightarrow F$ is analytic.
- (2) Let $V \subset F$ be an open set. We say that a map $\Lambda : V \rightarrow X$ is **analytic with respect to the analytic structure induced by Ψ** if there exists an analytic map $\tilde{\Lambda} : V \rightarrow U$ such that $\tilde{\Lambda}^* \Psi = \Lambda$.
- (3) We say in this case that X is an **analytic space modeled on E** , equipped with the analytic structure induced by Ψ .
- (4) These definitions allow to speak of analytic maps between analytic spaces X and Y , each one equipped with an induced analytic structure, respectively $\Psi_X : U_X \rightarrow X$ and $\Psi_Y : U_Y \rightarrow Y$. A map $\Lambda : X \rightarrow Y$ is **analytic** if there exists an analytic map $\tilde{\Lambda} : U_X \rightarrow U_Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\Lambda} & Y \\ \Psi_X \uparrow & & \uparrow \Psi_Y \\ U_X & \xrightarrow{\tilde{\Lambda}} & U_Y \end{array}$$

Basic examples include vector subspaces $F \subset E$ with a continuous projector $E \rightarrow F$ and, more generally, quotient spaces with respect to equivalence relations \sim whose canonical map $E \rightarrow E/\sim$ is continuous. Also the space of diffeomorphisms $\text{Diff}(\mathbb{C}, 0)$ is

the range of the continuous map (see Proposition 3.20)

$$\begin{aligned}\Psi : \mathbb{C}\{z\} &\longrightarrow \text{Diff}(\mathbb{C}, 0) \subset \mathbb{C}\{z\} \\ f &\longmapsto (z \mapsto z \exp f(z))\end{aligned}$$

and is therefore an analytic space for the analytic structure induced by Ψ . According to Proposition 3.20 the group law

$$(\Delta_1, \Delta_2) \in \text{Diff}(\mathbb{C}, 0)^2 \longmapsto \Delta_1 \circ \Delta_2 \in \text{Diff}(\mathbb{C}, 0)$$

is continuous since it corresponds *via* Ψ to the entire mapping

$$(f_1, f_2) \longmapsto f_2 + f_1 \circ (z \exp f_2) .$$

Corresponding statements for the m -dimensional case follow.

3.3. Cauchy integrals and estimates.

In the finite-dimensional setting a key element of the theory of analytic functions is the Cauchy's formula and related estimates. The infinite-dimensional setting is no exception to that rule and those estimates likewise play a major role in all the theory. If Λ is analytic on some open set U then, for fixed $f \in U$ and h small enough we can pick $\eta > 0$ in such a way that the mapping

$$x \in \eta \mathbb{D} \longmapsto \Lambda(f + xh) \in F$$

is also analytic (for the standard normed topology on \mathbb{C}). In particular it is continuous and the integral $\oint_{|x|=\eta} \Lambda(f + xh) \frac{dx}{x^{p+1}}$ makes perfectly sense in some topological completion of F . Yet the very nature of the formula below implies *a posteriori* that this integral belongs to F .

Proposition 3.6. [Bar85, p210-212] *Let $\Lambda : U \rightarrow F$ be analytic and pick $f \in U$. For every $h \in E$, $p \in \mathbb{N}$ and $\eta > 0$ such that $f + \eta \overline{\mathbb{D}}h \subset U$ we have*

$$\frac{\partial^p \Lambda}{\partial f^p} |_f (h^{\oplus p}) = \frac{1}{2i\pi} \oint_{\eta \mathbb{S}^1} \Lambda(f + xh) \frac{dx}{x^{p+1}} .$$

Besides if $a \in A$ and $b \in B$ then for all $h \in E$ and all $\eta > 0$ so that $B_a(f, \eta) \subset U$ we have¹⁸

$$\left\| \frac{1}{p!} \frac{\partial^p \Lambda}{\partial f^p} |_f (h^{\oplus p}) \right\|_b \leq \frac{\|h\|_a^p}{\eta^p} \sup_{\|u\|_a=\eta} \|\Lambda(f + u)\|_b .$$

Notice that a variation on this presentation of Cauchy's formula can be used to define completely the values of the p -symmetric linear mapping (see [Bar85, p229])

$$(3.1) \quad \frac{\partial^p \Lambda}{\partial f^p} |_f (h_1, \dots, h_p) = \frac{1}{2i\pi} \oint_{\eta/p \mathbb{S}^1} \dots \oint_{\eta/p \mathbb{S}^1} \Lambda \left(f + \sum_{j=1}^p x_j h_j \right) \prod_{j=1}^p \frac{dx_j}{x_j^2} ,$$

so that

$$(3.2) \quad \left\| \frac{1}{p!} \frac{\partial^p \Lambda}{\partial f^p} |_f (h_1, \dots, h_p) \right\|_b \leq \frac{p^p}{p!} \times \frac{\prod_{j=1}^p \|h_j\|_a}{\eta^p} \sup_{\|u\|_a \leq \eta} \|\Lambda(f + u)\|_b .$$

¹⁸We point out that the right-hand side can be infinite if the choice of a, b is not done so that $\|\Lambda\|_b$ is bounded on $B_a(f, \eta)$.

This estimate is optimal, and can still be useful for studying convergence of power series since $\frac{p^p}{p!}$ is sub-geometric.

3.4. Fréchet- and Gâteaux-holomorphy.

Definition 3.7. Let $\Lambda : U \rightarrow F$ be a mapping defined on a non-empty open set $U \subset E$. In the following \hat{F} represents the topological Hausdorff completion of F (as a uniform space).

- (1) We say that it is **G-holomorphic** (meaning Gâteaux-holomorphic) if for all finite dimensional linear subspace $S \subset E$ the restriction $\Lambda|_{S \cap U}$ is analytic. This is equivalent ([Maz84, p51]) to requiring that for all $f \in U$ and $h \in E$ the map $x \mapsto \Lambda(f + xh)$ be analytic at $0 \in \mathbb{C}$.
- (2) We say that it is **F-holomorphic** (meaning Fréchet-holomorphic) if it is differentiable at every point $f \in U$ in the following sense : there exists a continuous linear map $\partial_f \Lambda \in \mathcal{L}(E \rightarrow \hat{F})$ such that for all $b \in B$ we can find $a \in A$ and some function $\varepsilon : U - f \rightarrow \mathbb{R}_{\geq 0}$ with
 - ⊗ for all $h \in U - f$ we have

$$\|\Lambda(f + h) - \Lambda(f) - \partial_f \Lambda(h)\|_b \leq \varepsilon(h) \|h\|_a,$$

$$\otimes \lim_{h \rightarrow 0} \varepsilon(h) = 0.$$

Remark 3.8. If Λ is analytic then it is F-holomorphic and

$$\frac{\partial^1 \Lambda}{\partial f^1} |_f = \partial_f \Lambda.$$

If Λ is F-holomorphic then it is continuous.

Theorem 3.9. [Bar85, p246][Maz84, p62] Consider a mapping $\Lambda : U \rightarrow F$ with U an open set of E . The following propositions are equivalent

- (1) Λ is analytic,
- (2) Λ is amply bounded and G-holomorphic,
- (3) Λ is F-holomorphic.

Remark 3.10. Non-continuous linear maps provide examples of G-holomorphic functions which are not analytic. It is likewise possible to build two G-holomorphic mappings whose composition is not G-holomorphic anymore ([Maz84, p63]). In view of the results presented earlier in this section these maps obviously fail to be amply bounded.

3.5. Multivariate analyticity.

Many good properties of finite dimensional complex analysis persist. In particular we dispose of Hartogs theorems regarding separate analyticity for analytic mappings defined on a Cartesian product of finitely many locally convex spaces equipped with the product topology. This is indeed a direct consequence of the characterization of Theorem 3.9.

Theorem 3.11. (Hartogs lemma) Let $\Lambda : W_1 \times \cdots \times W_m \rightarrow F$ be a map from a product of open sets in locally convex spaces. Then Λ is analytic if, and only if, Λ is separately analytic, meaning that for all choices of $p \in \{1, \dots, m\}$ and of $\mathbf{f} \in \prod_j W_j$ the mapping $f \mapsto \Lambda(\mathbf{f} \triangleleft_p f)$ is analytic on W_p .

3.6. The case of ultrabornological spaces.

The following universal characterization will be useful in practice:

Theorem 3.12. (Universal property of the analytic inductive limit [Bar85, p254]) *Let $E = \lim_{\rightarrow} (E_k)_{k \in \mathbb{N}}$ be ultrabornological, each E_k being endowed with its Banach topology; denote by $\iota_r : E_r \hookrightarrow E$ the canonical (continuous) embedding. Then a map $\Lambda : U \subset E \rightarrow F$, where F is any locally convex space, is analytic if, and only if, every $\Lambda \circ \iota_r$ is analytic for $r > 0$.*

3.7. Specifics for spaces of germs.

From now on $(E, (\|\bullet\|_a)_{a \in A})$ (resp. $(F, (\|\bullet\|_b)_{b \in B})$) stands for the vector space $\mathbb{C}\{z\}$ (resp. $\mathbb{C}\{w\}$) endowed with a Hausdorff, locally convex topology such as a sequential topology or the inductive topology. Before diving into more details, let us first state once and for all that the analytic structure put on the finite-dimensional vector subspaces $\mathbb{C}[z]_{\leq N} \simeq \mathbb{C}^{N+1}$ is that induced, in the sense of Definition 3.5, by the (continuous) jets projectors $J_N : \mathbb{C}\{z\} \rightarrow \mathbb{C}[z]_{\leq N}$. It coincides with the standard analytic structure, and the space of analytic functions with respect to J_N is the same as the usual holomorphic functions on open sets of \mathbb{C}^{N+1} .

3.7.1. Uniform convergence of the Taylor series.

Proposition 3.13. *Let Λ be analytic on a neighborhood of some $f \in E$, with Taylor coefficients $\Lambda_p \in \mathcal{L}_p(E \rightarrow F)$. For any $b \in B$ there exists $a \in A$ such that the formal power series $\sum_p \Lambda_p(h^{\oplus p})$ converges $\|\bullet\|_b$ -normally towards $\Lambda(f+h)$, uniformly in $\|h\|_a$ small enough. Besides for such values of h*

$$\begin{aligned} \Lambda(f+h) &= \lim_{m \rightarrow \infty} \sum_{j \in \mathbb{N}^m} \frac{|j|!}{j!} \Lambda_{|j|}(\oplus_n (z^n)^{\oplus j_n}) \prod_n h_n^{j_n} \\ &= \sum_{j \in \mathbb{N}^{(\mathbb{N})}} \frac{|j|!}{j!} \Lambda_{|j|}(\oplus_n (z^n)^{\oplus j_n}) \prod_n h_n^{j_n}. \end{aligned}$$

Proof. Take a sequence $(f_n)_{n \in \mathbb{N}} \subset \mathbb{C}\{z\}$. Using the multinomial formula we derive for every $m \in \mathbb{N}$

$$\Lambda_p \left(\left(\sum_{n=0}^m f_n \right)^{\oplus p} \right) = \sum_{j \in \mathbb{N}^m, |j|=p} \frac{p!}{j!} \Lambda_p(\oplus_{n=0}^m (f_n)^{\oplus j_n}).$$

Since Λ is analytic for all $b \in B$ there exists $\eta > 0$ small enough and $a \in A$ such that $\sup_{\|u\|_a=\eta} \|\Lambda(f+u)\|_b$ is finite ; let us write K this value. We have for for all $m \in \mathbb{N}$ and all $h = \sum_n h_n z^n \in \mathbb{C}\{z\}$, invoking Cauchy's extended estimate (3.2) and Stirling estimate (2.9):

$$\begin{aligned} \left\| \Lambda_p \left(\left(\sum_{n=0}^m h_n z^n \right)^{\oplus p} \right) \right\|_b &= \left\| \sum_{j \in \mathbb{N}^m, |j|=p} \frac{p!}{j!} \Lambda_p(\oplus_{n=0}^m (h_n z^n)^{\oplus j_n}) \right\|_b \\ &\leq K \left(\frac{e}{\eta} \|h\|_a \right)^p. \end{aligned}$$

The left-hand side thereby admits a limit as $m \rightarrow \infty$ and because Λ_p is continuous we obtain

$$\begin{aligned}\Lambda_p(h^{\oplus p}) &= \lim_{m \rightarrow \infty} \sum_{j \in \mathbb{N}^m, |j|=p} \frac{p!}{j!} \Lambda_p(\oplus_n (h_n z^n)^{\oplus j_n}) \\ &= \sum_{j \in \mathbb{N}^{\mathbb{N}}, |j|=p} \frac{p!}{j!} \Lambda_p(\oplus_{n=0}^{\infty} (h_n z^n)^{\oplus j_n})\end{aligned}$$

with convergence of the right-hand side in $\mathbb{C}\{z\}$, and if $\|h\|_a < \frac{\eta}{e}$ we have

$$\sum_p \left\| \Lambda_p(h^{\oplus p}) \right\|_b \leq K \frac{\eta}{\eta - e \|h\|_a}.$$

□

3.7.2. Quasi-strong analyticity.

We introduce now a notion of holomorphy between spaces of germs that does not require any *a priori* topology on $\mathbb{C}\{z\}$. This notion is traditionally referred to as «the» notion of holomorphy in $\mathbb{C}\{z\}$ in the context of dynamical systems.

Definition 3.14. Let U be a non-empty subset of $\mathbb{C}\{z\}$.

- (1) A **germ of a holomorphic** map $\lambda : (\mathbb{C}^m, 0) \rightarrow U$ is a mapping defined on some neighborhood of $0 \in \mathbb{C}^m$ and such that the map

$$\lambda^* : (x, z) \mapsto \lambda(x)(z)$$

belongs to $\mathbb{C}\{x, z\}$. The set of all such germs will be denoted by $\mathcal{O}((\mathbb{C}^m, 0) \rightarrow U)$.

- (2) A map $\Lambda : U \rightarrow \mathbb{C}\{w\}$ is said to be **quasi-strongly analytic** on U if for any $m \in \mathbb{N}_{>0}$ and any $\lambda \in \mathcal{O}((\mathbb{C}^m, 0) \rightarrow U)$ the composition $\Lambda \circ \lambda$ belongs to $\mathcal{O}((\mathbb{C}^m, 0) \rightarrow \mathbb{C}\{w\})$.
- (3) We extend in the obvious way these definitions to analytic maps between analytic spaces as in Definition 3.5.

Remark.

- (1) The notion (2) is called «strong analyticity» in [GT10]. Here we reserve this name to the yet stronger notion we introduce below.
- (2) Notice that by definition the composition of source/range-compatible quasi-strongly analytic mappings is again quasi-strongly analytic.
- (3) Each affine map $x \mapsto f + xh$ is quasi-strongly analytic for fixed $f \in U$ and $h \in \mathbb{C}\{z\}$.
- (4) It is *not* sufficient to consider the case $m = 1$, as opposed to what happens for G-holomorphic mappings.

We begin with a characterization of quasi-strongly analytic maps in terms of power-wise holomorphy:

Proposition 3.15. Let a mapping $\Lambda : U \rightarrow F$ be given, where $U \subset \mathbb{C}\{z\}$, and write $\Lambda(f) = \sum_{n \geq 0} T_n(\Lambda(f)) w^n$. The following assertions are equivalent:

- (1) Λ is quasi-strongly analytic,
- (2) for all $\lambda \in \mathcal{O}((\mathbb{C}^m, 0) \rightarrow U)$ the following two conditions are fulfilled
 - ⊗ $T_n(\Lambda \circ \lambda) \in \mathcal{O}((\mathbb{C}^m, 0) \rightarrow \mathbb{C})$ is analytic on a disc whose size does not depend on n ,

⊗

$$\liminf_{\mathbf{x} \rightarrow 0} \mathcal{R}(\Lambda(\lambda(\mathbf{x}))) > 0.$$

In that sense quasi-strongly analytic maps are precisely those who are uniformly power-wise quasi-strongly analytic and who respect lower-boundedness of the radius of convergence.

Proof. Let $\lambda \in \mathcal{O}((\mathbb{C}^m, 0) \rightarrow U)$ with $\lambda(0) = f \in U$ be given. If (1) holds then $\Lambda^* : (\mathbf{x}, w) \mapsto \Lambda(\lambda(\mathbf{x}))(w)$ is analytic, and

$$T_n(\Lambda(\lambda(\mathbf{x}))) = \frac{1}{n!} \frac{\partial^n \Lambda^*}{\partial w^n}(\mathbf{x}, 0)$$

so that $T_n \circ \Lambda : U \rightarrow \mathbb{C}$ is quasi-strongly analytic on $\varepsilon \mathbb{D}^m$ for some $\varepsilon > 0$ independent on n . Besides $\inf_{\|\mathbf{x}\| < \varepsilon} \mathcal{R}(\Lambda(\lambda(\mathbf{x}))) > 0$ by definition. Assume now that (2) holds and take $\eta > 0$ strictly less than $\liminf_{\mathbf{x} \rightarrow 0} \mathcal{R}(\Lambda(\lambda(\mathbf{x})))$, together with ε for which $T_n \circ \Lambda \circ \lambda$ is holomorphic on $\varepsilon \mathbb{D}^m$. Then there exists $\varepsilon \geq \gamma > 0$ such that

$$\inf_{\|\mathbf{x}\| = \gamma} \liminf_{n \rightarrow +\infty} |T_n \circ \Lambda(\lambda(\mathbf{x}))|^{-1/n} > \eta > 0.$$

In particular $\sum_{n \geq 0} T_n(\Lambda(\lambda(\mathbf{x})))w^n$ is absolutely convergent on $\gamma \mathbb{D}^m \times \frac{\eta}{2} \mathbb{D}$. \square

We can establish a link between this notion of holomorphy and that of G-holomorphy, which is only natural since the concepts look alike.

Proposition 3.16. *Let $\Lambda : U \rightarrow F$ be quasi-strongly analytic and assume that the ordering between topologies $\mathbb{C}\{z\} \geq F$ holds. Then Λ is G-holomorphic.*

Proof. We exploit the fact that it possible to use the usual Cauchy integral formula of $(x, w) \mapsto \Lambda(f + xh)(w)$ to derive the Taylor coefficients of $x \mapsto \Lambda(f + xh)$.

Lemma 3.17. *Assume $\Lambda : U \rightarrow F$ is quasi-strongly analytic. Let $f \in U$ and $h \in E$; we can choose $\eta > 0$ such that $f + 2\eta \mathbb{D}h \subset U$. Then for all $p \in \mathbb{N}$*

$$\Lambda_p(h^{\oplus p}) := w \mapsto \oint_{\eta \mathbb{S}^1} \Lambda(f + xh)(w) \frac{dx}{x^{p+1}}$$

defines a p -linear symmetric mapping, whose values do not depend on the choice of η provided it is kept sufficiently small.

The p -linear mappings Λ_p need not be continuous and Theorem 3.3 (3) will be of help here. Because of the extended Cauchy estimates in F (Proposition 3.6) we have for all $b \in B$

$$\left\| \Lambda_p(h^{\oplus p}) \right\|_b \leq \eta^{-p} \sup_{|x|=\eta} \|\Lambda(f + xh)\|_b.$$

Since $F \leq \mathbb{C}\{z\}$ for every $b \in B$ and every

$$0 < \rho < \inf_{|x|=\eta} \mathcal{R}(\Lambda(f + xh))$$

there exists a constant $C = C_{b,\rho} > 0$ such that

$$\|\Lambda(f + xh)\|_b \leq C \|\Lambda(f + xh)\|_{\rho \mathbb{D}}.$$

For example if F is given by a sequential B -topology then the Cauchy estimates in \mathbb{C} (Lemma 2.17) states that $C := \left\| w \mapsto \frac{\rho}{\rho-w} \right\|_b$. As a consequence

$$\left\| \Lambda_p(h^{\oplus p}) \right\|_b \leq C \eta^{-p} \|(x, w) \mapsto \Lambda(f + xh)(w)\|_{\eta \mathbb{D} \times \rho \mathbb{D}}$$

defines the general term of a normally convergent power series

$$x \mapsto \sum_{p=0}^{\infty} \Lambda_p(h^{\oplus p}) x^p,$$

uniformly in $|x| \leq \frac{\eta}{2}$. By construction this mapping is analytic at 0 and its sum is $\Lambda(f + xh)$. This is the required property of G -holomorphy. \square

3.7.3. Strong analyticity.

It is tempting to declare that the Taylor coefficient Λ_p computed in Lemma 3.17 coincides with the value of the integral

$$\oint_{\eta \mathbb{S}^1} \Lambda(f + xh) \frac{dx}{x^{p+1}}.$$

Yet nothing guarantees that this integral exists and belongs to $\mathbb{C}\{w\}$. If we require that Λ be amply bounded and quasi-strongly analytic then it does and equals $\Lambda_p(h^{\oplus p})$. For instance its value can be obtained as the limit of the sequence of Riemann sums

$$I_N := \frac{2i\pi\eta}{N+1} \sum_{j=0}^N \frac{\Lambda(f + x_j h)}{x_j^p}$$

where

$$x_j := \eta \exp\left(j \frac{2i\pi}{N+1}\right).$$

There exists $\varepsilon > 0$ such that $(x, w) \in \frac{3}{2}\eta \mathbb{D} \times \varepsilon \mathbb{D} \mapsto \Lambda(f + xh)(w)$ is holomorphic and bounded, in which case

$$I_N(w) = \frac{2i\pi\eta}{N+1} \sum_{j=0}^N \frac{\Lambda(f + x_j h)(w)}{x_j^p}$$

converges as $N \rightarrow \infty$ toward $\oint_{\eta \mathbb{S}^1} \Lambda(f + xh)(w) \frac{dx}{x^{p+1}}$ on the one hand, while it converges toward $\oint_{\eta \mathbb{S}^1} \Lambda(f + xh) \frac{dx}{x^{p+1}}(w)$ on the other hand. This motivates the

Definition 3.18. Assume $F \leq \overrightarrow{\mathbb{C}\{z\}}$. An amply bounded map $\Lambda : U \subset E \rightarrow F$ which is also quasi-strongly analytic will be called **strongly analytic**.

Remark.

- ⊗ Notice again that the composition of source/range compatible strongly analytic mappings remains strongly analytic.
- ⊗ We recall that ample boundedness derives from continuity, and the latter property will in fact be automatically guaranteed in virtue of Theorem 3.9 and as a consequence of Theorem C, which we prove now.

Proposition 3.19. *Let a map $\Lambda : U \subset E \rightarrow F$ be given, where U is an open set of E and $F \leq \mathbb{C}\{z\}$. The following assertions are equivalent:*

- (1) Λ is strongly analytic,
- (2) Λ is analytic and for every $\lambda \in \mathcal{O}((\mathbb{C}^m, 0) \rightarrow U)$ we have simultaneously

$$\liminf_{\mathbf{x} \rightarrow 0} \mathcal{R}(\Lambda \circ \lambda(\mathbf{x})) > 0$$

$$\limsup_{\mathbf{x} \rightarrow 0, r \rightarrow 0} \|\Lambda \circ \lambda(\mathbf{x})\|_{r\mathbb{D}} < \infty.$$

Proof. (1) \Rightarrow (2) is a direct consequence of the definition of holomorphy and of the application of Theorem 3.9 to Proposition 3.16. Let us prove (2) \Rightarrow (1). Since an analytic map is amply bounded we only need to prove the holomorphy of Λ . Let $\lambda \in \mathcal{O}((\mathbb{C}^m, 0) \rightarrow U)$ be given; without loss of generality we can assume that $\lambda(0) = 0 \in U$. Since $(u, \mathbf{x}) \mapsto u\lambda(\mathbf{x})$ belongs to $\mathcal{O}((\mathbb{C}^{m+1}, 0) \rightarrow U)$ we can consider a positive number $0 < r' < \liminf_{(u, \mathbf{x}) \rightarrow 0} \mathcal{R}(\Lambda(u\lambda(\mathbf{x})))$. Repeating the construction of Lemma 3.17 and of the beginning of this paragraph we deduce that there exists $\rho, \rho' > 0$ such that for all integer p

$$(\forall |w| < r', \forall \|\mathbf{x}\| < \rho') \quad \Lambda_p(\lambda(\mathbf{x})^{\oplus p})(w) = \frac{1}{2i\pi} \oint_{\rho\mathbb{S}^1} \Lambda(u\lambda(\mathbf{x}))(w) \frac{du}{u^{p+1}},$$

where Λ_p is the Taylor coefficient of Λ . This formula implies particularly that $\Lambda_p(\lambda(\mathbf{x})^{\oplus p}) \in \mathcal{B}_{r'}$ and

$$\left| \Lambda_p(\lambda(\mathbf{x})^{\oplus p})(w) \right| \leq \rho^{-p} \sup_{|u|=\rho} |\Lambda(u\lambda(\mathbf{x}))(w)|.$$

By choosing $\rho > \eta > 0$, and using that $\limsup_{\mathbf{x} \rightarrow 0, r \rightarrow 0} \|\Lambda(\lambda(\mathbf{x}))\|_{r\mathbb{D}} < \infty$ we deduce that, at the expense of decreasing r' and ρ' , there exists $K > 0$ such that

$$\left| \Lambda_p((\eta\lambda(\mathbf{x}))^{\oplus p})(w) \right| \leq K \left(\frac{\eta}{\rho} \right)^p.$$

Hence the functional series $(\mathbf{x}, w) \mapsto \sum_{p \geq 0} \Lambda_p(\lambda(\mathbf{x})^{\oplus p})(w)$ is uniformly convergent and the sum is thereby analytic on $\rho'\mathbb{D} \times r'\mathbb{D}$. \square

Let us conclude this part by proving the

Proposition 3.20. *Assume that $\mathbb{C}\{z\}$ is endowed with a useful topology and pick $f \in \mathbb{C}\{z\}$ with radius of convergence $\mathcal{R}(f)$. The right-composition mapping*

$$f_* : \mathbb{C}\{z\} \longrightarrow \mathbb{C}\{z\}$$

$$g \longmapsto f \circ g$$

is strongly analytic on the domain $U := J_0^{-1}(\mathcal{R}(f)\mathbb{D}) = \{g : |g(0)| < \mathcal{R}(f)\}$.

We refer to Proposition 5.2 for the computation of the coefficients of the power series representing f_* .

Proof. By definition of a useful topology f_* is continuous, so we only need to show that it is quasi-strongly analytic. The latter property is trivial because being given $\lambda \in \mathcal{O}((\mathbb{C}^{m'}, 0) \rightarrow U)$ with $\lambda(0) = g \in U$, the map $f_* \circ \lambda$ also belongs to $\mathcal{O}((\mathbb{C}^{m'}, 0) \rightarrow \mathbb{C}\{z\})$ as $(f_* \circ \lambda)^*(\mathbf{x}, z) = f(\lambda^*(\mathbf{x}, z))$ defines a convergent power series in (\mathbf{x}, z) . \square

4. ANALYTIC SETS

This section is devoted to introducing and studying properties of analytic sets, as defined by the vanishing locus of analytic maps $E \rightarrow F$ where $(E, (\|\bullet\|_a)_{a \in A})$ and $(F, (\|\bullet\|_b)_{b \in B})$ are Hausdorff, locally convex spaces.

Definition 4.1.

- (1) A closed subset $\Omega \subset E$ is an **analytic set** if for all $f \in \Omega$ there exists a domain $U \ni f$ and some collection $(\Lambda_i)_{i \in I}$ of analytic mappings $\Lambda_i : U \rightarrow F$ such that $\Omega \cap U = \bigcap_{i \in I} \Lambda_i^{-1}(0)$.
- (2) An analytical set is called **proper** at some point $f \in \Omega$ if for all collection $((\Lambda_i)_{i \in I}, U)$, with $U \ni f$ and $\Omega \cap U = \bigcap_{i \in I} \Lambda_i^{-1}(0)$, we have $\Omega \cap U \neq U$. We say that Ω is proper if it is proper at one of its points.
- (3) A subset $M \subset E$ is said to be **analytically meager** if M is included in a countable union of proper analytic sets.
- (4) Let X be an analytic space whose analytic structure is induced by a continuous and onto map $\varphi : U \subset E \rightarrow X$ as in Definition 3.5. A closed subset $\Omega \subset X$ is an analytic set if $\varphi^{-1}(\Omega)$ is an analytic set of U . We likewise extend to this setting the notion of analytically meager subsets of X , and say that X is an **analytical Baire space** if every analytically meager subset of X has empty interior.

Remark 4.2.

- (1) There is no *a priori* hypothesis on the cardinality of the set I in the definition of analytic sets.
- (2) It is obvious from the definition that finite unions and unspecified intersections of analytic sets still are analytic sets.
- (3) We will show shortly that a proper analytic set is actually proper at each one of its points (Proposition 4.8).
- (4) In (4) above we expressly do *not* define analytic sets of X as the vanishing loci of analytic maps. This condition is indeed far too restrictive, as will be illustrated in Section 6 when speaking about the analytic space of meromorphic functions (Remark 6.3).

Because taking the N^{th} -jet of a germ is an analytic operation, each subspace of polynomials with given upper-bound N on their degree

$$\mathbb{C}[z]_{\leq N} = \ker \left(\sum_n a_n z^n \mapsto \sum_{n > N} a_n z^n \right) = \ker (\text{Id} - J_N)$$

is a proper analytic set. The countable union $\mathbb{C}[z]$ of all these subspaces is consequently analytically meager (and dense) in $\mathbb{C}\{z\}$, and therefore $\mathbb{C}[z]$ is not an analytical Baire space.

A nice feature of analytic sets is that they are negligible, in the sense that the removable singularity theorem holds:

Theorem 4.3. (Removable Singularity Theorem [Maz84]) *Let $U \subset E$ be an open set, $\Omega \subset U$ a proper analytic set and Λ be analytic on $U \setminus \Omega$. We assume that for every $a \in A$ and every $f \in \Omega$ there exists a neighborhood $V \subset U$ of f such that $\|\Lambda|_{V \setminus \Omega}\|_a$ is bounded. Then Λ admits a unique analytic extension to U .*

The aim of this section is mainly to show the next result:

Theorem 4.4. *Assume there exists a Banach space \mathcal{V} and a continuous, linear mapping $\iota : \mathcal{V} \rightarrow E$ such that $\iota(\mathcal{V})$ is dense in E . Then any analytic space X modeled on E is an analytical Baire space.*

We deduce Theorem A from this statement and the next trivial lemma:

Lemma 4.5. *Let \mathcal{V} be the vector subspace of $\mathbb{C}\{z\}$ defined by*

$$\mathcal{V} := \left\{ \sum_n f_n z^n : |f_n| \text{ is bounded} \right\}$$

equipped with the norm $\|\bullet\|_\infty$:

$$\|f\|_\infty := \sup_n |f_n|.$$

Then $(\mathcal{V}, \|\bullet\|_\infty)$ is a Banach space and the inclusion $(\mathcal{V}, \|\bullet\|_\infty) \hookrightarrow \mathbb{C}\{z\}$ is continuous. More precisely for each $f \in \mathcal{V}$ and each proper comparison function δ (Definition 2.3)

$$\Omega_\delta(f) \leq \|f\|_\infty \Omega_\delta\left(\frac{1}{1-z}\right).$$

We also intend to prove the Theorem B:

Theorem 4.6. *Let X be an analytic space modeled on $\mathbb{C}\{z\}$ and consider an analytic map $\Lambda : U \rightarrow X$ on an open set $U \subset E$. The image by Λ of the trace on U of any linear subspace of E with at most countable dimension is analytically meager.*

Remark 4.7.

- (1) It is worth noticing that the analyticity condition on Λ can be loosened a little. Indeed we are only considering the countable union of range of analytic maps restricted to finite-dimensional open sets, so that only G-holomorphy (or quasi-strong analyticity) is actually required. In this form the spine of the proof is borrowed from [GT10], but some flesh is added to cover the more general context.
- (2) Be careful that «analytically meager» does not imply «empty interior» in general, according to Theorem 4.4.

4.1. Proper analytic sets.

Proposition 4.8. *Let Ω be a connected, non-empty analytic set. The following assertions are equivalent :*

- (1) Ω is proper,
- (2) Ω is proper at any one of its points,
- (3) the interior of Ω is empty,
- (4) for all $f \in \Omega$ there exists a domain $U \ni f$ and a non-empty collection $(\Lambda_i)_{i \in I}$ such that $U \cap \Omega = \bigcap_{i \in I} \Lambda_i^{-1}(0)$ while $\Lambda_i \neq 0$,
- (5) there exists a domain U meeting Ω and a non-empty collection $(\Lambda_i)_{i \in I}$ such that $U \cap \Omega = \bigcap_{i \in I} \Lambda_i^{-1}(0)$ while $\Lambda_i \neq 0$.

A consequence of this result is that proper analytical meager subsets of $\mathbb{C}\{z\}$ are genuine meager subsets in the sense of Baire.

Proof. The implications (3) \Rightarrow (2) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1) are trivial. Because of the identity theorem (Corollary 3.4) we also have (4) \Rightarrow (3). Let us show (1) \Rightarrow (2) by a connectedness argument; for this consider the set

$$C := \{f \in \Omega : \Omega \text{ is not proper at } f\},$$

which is an open subset of Ω . Let us consider a sequence $(f_n)_{n \in \mathbb{N}} \subset C$ converging in Ω to some f_∞ and prove that $f_\infty \in C$. Pick a domain $U \ni f_\infty$ such that $U \cap \Omega = \bigcap_{i \in I} \Lambda_i^{-1}(0)$ for some collection of maps $(\Lambda_i)_{i \in I}$ analytic on U , acknowledging that for n big enough we have $f_n \in U$. Therefore Λ_i must vanish on some small ball around f_n and the identity theorem applies once more proving that Λ_i vanishes on the whole U , which is $f_\infty \in C$. Because by hypothesis Ω is connected and $\Omega \setminus C \neq \emptyset$ we showed $C = \emptyset$, which is (2), as required. \square

4.2. Analytical Baire property : proof of Theorem 4.4. By definition of the analytic sets of X it is sufficient to show the property for E . Assume then that there exists a countable collection of analytic sets $(C_n)_{n \in \mathbb{N}}$ such that $\bigcup_n C_n$ has non-empty interior W , and show that at least one C_n is not proper. Pick some point f in W ; the affine subspace

$$\mathcal{V}_f := f + \iota(\mathcal{V})$$

is Baire. Since W is open the intersection $W \cap \mathcal{V}_f$ is a non-void open subset in \mathcal{V}_f (i.e. for the topology induced by $\|\bullet\|_{\mathcal{V}}$), and $C_n \cap \mathcal{V}_f$ is closed in \mathcal{V}_f . Therefore at least one $C_n \cap \mathcal{V}_f$ has a non-empty interior W_n . Let $\Lambda : U \rightarrow F$ be an analytic map on a neighborhood U of some point $\tilde{f} \in W_n$ such that $C_n \cap U \subset \Lambda^{-1}(0)$. Because $\Lambda|_{\mathcal{V}_f}$ is analytic for the analytic structure of \mathcal{V}_f inherited from its Banach topology, necessarily Λ vanishes on \mathcal{V}_f . By denseness of \mathcal{V}_f in E we deduce that $\Lambda = 0$ and C_n cannot be proper, according to the characterization given in Proposition 4.8.

4.3. Range of a finite-dimensional analytic map : proof of Theorem B.

Proposition 4.9. *Take an analytic map $\lambda : V \rightarrow \mathbb{C}\{w\}$ defined on some neighborhood V of $0 \in \mathbb{C}^m$ such that $\text{rank}(\partial_0 \lambda) = m$. There exists an open neighborhood $W \subset V$ of the origin such that the following properties hold:*

- (1) $\lambda|_W$ is one-to-one,
- (2) $\lambda(W)$ is the trace of a proper analytic set on some domain $U \subset \mathbb{C}\{w\}$ containing $\lambda(0)$, defined by the vanishing of a non-zero analytic mapping $\Lambda : U \rightarrow \mathbb{C}\{w\}$.

Proof. We can assume, without loss of generality, that $\lambda(0) = 0$.

- (1) We adapt here the proof done in [GT10]. If there exists $N \in \mathbb{N}$ such that the map $J_N \circ \lambda$ is one-to-one, then so is λ . We claim that a family $f_1, \dots, f_k \in \mathbb{C}\{w\}$ is free (over \mathbb{C}) if, and only if, there exists $N \in \mathbb{N}$ such that their N -jets are free (as elements of the vector space $\mathbb{C}[w]_{\leq N}$). Suppose that for any $N \in \mathbb{N}$ there exists a non-trivial relation

$$L_N := (\lambda_{1,N}, \dots, \lambda_{k,N}) \neq 0$$

for the family $C_N := (J_N(f_1), \dots, J_N(f_k))$, that is

$$J_N \left(\sum_{j=1}^k \lambda_{j,N} f_j \right) = 0.$$

Up to rescaling of L_N one can suppose that it belongs to the unit sphere of \mathbb{C}^k and thereby one can consider some accumulation point $L_\infty := (\lambda_{1,\infty}, \dots, \lambda_{k,\infty})$. Because $J_{p+1}(f) = 0$ implies $J_p(f) = 0$, by taking the limit $N \rightarrow \infty$ while fixing an arbitrary p we obtain that L_∞ is a non-trivial relation for C_p (by continuity of J_N), and thus is a non-trivial relation for (f_1, \dots, f_k) , proving our claim. If λ is of maximal rank at $\mathbf{0}$, i.e. the rank of $\partial_0 \lambda$ is m , there accordingly exists $N \in \mathbb{N}$ such that the mapping $J_N \circ \lambda$ is of maximal rank. Therefore the mapping $J_N \circ \lambda$ is locally one-to-one around $\mathbf{0}$, say on some domain $V' \subset V$, and so is λ .

- (2) Let $N \in \mathbb{N}$ and V' be given as above, so that $J_N \circ \lambda|_{V'}$ is one-to-one. Let $(b_0, \dots, b_{m-1}) \subset \mathbb{C}[z]_{\leq N}$ be a basis of $\text{im}(\partial_0(J_N \circ \lambda))$, which we complete with a basis $(b_m, \dots, b_N) \subset \mathbb{C}[z]_{\leq N}$ of the cokernel of $\partial_0(J_N \circ \lambda)$, so that $\mathbb{C}[z]_{\leq N} = \mathbb{C}\langle b_0, \dots, b_N \rangle$. Write $J_N \circ \lambda(\mathbf{x}) = \sum_{j=0}^N \lambda_j(\mathbf{x}) b_j$ with $\lambda_j : V' \rightarrow \mathbb{C}$ holomorphic. According to the local inversion theorem there exists a domain $W \subset V'$ containing $\mathbf{0}$ such that $\tilde{\lambda} : \mathbf{x} \in W \mapsto \sum_{j < m} \lambda_j(\mathbf{x}) b_j$ is a biholomorphism onto its image. Let us write Φ the inverse map, holomorphic from the open set $\tilde{\lambda}(W)$ of $\mathbb{C}\langle b_0, \dots, b_{m-1} \rangle$ into W and introduce the natural (continuous) projector

$$\begin{aligned} P : \mathbb{C}\{w\} &\longrightarrow \mathbb{C}\langle b_0, \dots, b_{m-1} \rangle \\ \sum_{n \leq N} f_n b_n + \sum_{n > N} f_n \text{Id}^n &\longmapsto \sum_{n < m} f_n b_n, \end{aligned}$$

so that $P \circ \lambda = \tilde{\lambda}$. Consider the (analytic) mapping

$$\begin{aligned} \Lambda : J_N^{-1}(\tilde{\lambda}(W)) \subset \mathbb{C}\{w\} &\longrightarrow \mathbb{C}\{w\} \\ f &\longmapsto f - \lambda \circ \Phi \circ P(f). \end{aligned}$$

By construction $\Lambda(f) = 0$ if, and only if, $f \in \lambda(W)$. In particular $\lambda(W)$ is closed in $U := J_N^{-1}(\tilde{\lambda}(W))$.

□

Corollary 4.10. *Let $\lambda : V \subset \mathbb{C}^m \rightarrow \mathbb{C}\{w\}$ be analytic. Then $\lambda(V)$ is analytically meager.*

Since a countable union of analytically meager subsets remains analytically meager Theorem B derives from this result.

Proof. It is done by induction on m , the result being trivially true for $m = 0$. The reasoning is essentially the same as in [GT10], and consists in splitting V into at most countably many sets on which the restriction of λ has relative maximal rank. Outside a proper analytic subset Σ of V the rank of the map λ is constant; let us call $\mu(\lambda)$ this generic rank. If $\mu(\lambda) < m$ then we can find at most countably many analytic discs $D_k \subset V$ of dimension $\mu(\lambda)$ such that $\bigcup_k \lambda(D_k) = \lambda(V)$. Thanks to our induction hypothesis this case has already been dealt with and $\lambda(V)$ is analytically meager. We therefore assume that $\mu(\lambda) = m$. The set Σ admits a decomposition $\Sigma = \bigcup_k C_k$ into at most countably many analytic discs of dimension p_k with $0 \leq p_k < m$ (see e.g. [Cos82]). We apply the induction hypothesis to each $\lambda|_{C_k}$ as before to obtain that $\lambda(\Sigma)$ is analytically meager. We can apply Proposition 4.9 at each point of the open set $V \setminus \Sigma$, which is therefore covered by countably many domains $(W_k)_k$ such that each $\lambda(W_k)$ is included in a proper analytic set. This settles the induction. □

4.4. Remarks on tangent spaces.

As in the finite dimensional case we can build (at least) two candidate tangent spaces.

Definition 4.11. Let Ω be a non-empty analytic set and pick $f \in \Omega$, together with a collection of locally defining maps $(\Lambda_i)_i$.

- (1) We define the **algebraic tangent space** of Ω at f

$$\mathcal{T}_f^A \Omega := \bigcap_i \ker \frac{\partial \Lambda_i}{\partial f} |_f.$$

- (2) We define the **geometric tangent space** of Ω at f

$$\mathcal{T}_f^G \Omega := \{ \lambda'(0) : \lambda \text{ holomorphic } (\mathbb{C}, 0) \rightarrow (\Omega, f) \}.$$

Obviously $\mathcal{T}_f^G \Omega < \mathcal{T}_f^A \Omega$ and equality is equivalent, for finite-dimensional spaces and irreducible Ω , to the fact that f is a regular point of Ω . Therefore we propose the following definition:

Definition 4.12. Assume Ω is irreducible. We say that f is a **regular point** of Ω if $\mathcal{T}_f^G \Omega = \mathcal{T}_f^A \Omega$.

We conjecture that regular points of (at least strongly) analytic sets enjoy «nice» geometric features, maybe reaching as far as the existence of a local analytic parameterization.

5. FRÉCHET CALCULUS

In this section we fix a choice of a useful A -topology on $\mathbb{C}\{z\}$. For $a \in \mathbb{C}\{z\}$ the notation $a \times \bullet$ stands for the endomorphism $h \mapsto a \times h$ of $\mathbb{C}\{z\}$. More generally in the formulas below the symbol « \bullet » will stand for the argument of a continuous linear mapping.

5.1. Actually computing derivatives.

Computing actual derivatives of «simple» operations can be performed easily. In fact one can compute them in many less simple cases using the following formula, which is only a consequence of the continuity of the derivative:

Lemma 5.1. Let $\Lambda : U \rightarrow \mathbb{C}\{w\}$ be analytic and $f \in U$. Then

$$\begin{aligned} \frac{\partial \Lambda}{\partial f} |_f \left(\sum_{n \geq 0} h_n z^n \right) &= \sum_{n \geq 0} h_n \frac{\partial \Lambda}{\partial f} |_f (z^n) \\ &= \sum_{n \geq 0} h_n \frac{\partial \Lambda(f + x \times z^n)}{\partial x} (0). \end{aligned}$$

We shall present an example: differentiating the strongly analytic map (Proposition 3.20) $g \mapsto f \circ g$ with respect to g (i.e. for fixed f). This allows to compute $\frac{\partial f \circ g}{\partial g} |_g$ for any $g \in U$:

$$\frac{\partial f \circ g}{\partial g} |_{(f,g)} (z^n) = \frac{\partial f \circ (g + x z^n)}{\partial x} (0) = z^n \times f' \circ g$$

so that

Proposition 5.2. *Let $f \in \mathbb{C}\{z\}$ be given. The right-composition mapping $g \mapsto f \circ g$ is analytic on the domain $J_0^{-1}(\mathcal{R}(f)\mathbb{D})$ and for all g within this domain we have*

$$(5.1) \quad \frac{\partial f \circ g}{\partial g} \Big|_{(f,g)} = (f' \circ g) \times \bullet.$$

More generally we have, for all $p \in \mathbb{N}$,

$$\frac{\partial^p f \circ g}{\partial g^p} \Big|_g (\bullet^{\oplus p}) = (f^{(p)} \circ g) \times \bullet^p.$$

5.2. The chain rule.

A corollary of Theorem 3.11, and of the usual rules of calculus, is that if $\Lambda : W_1 \times W_2 \subset \mathbb{C}\{z\} \times \mathbb{C}\{z\} \rightarrow \mathbb{C}\{w\}$ is (strongly) analytic, and if $\Lambda_j : U \rightarrow W_j$ are (strongly) analytic maps, $j \in \{1, 2\}$, then $\Lambda(\Lambda_1, \Lambda_2)$ is also (strongly) analytic and, for all $f \in U$,

$$\frac{\partial \Lambda(\Lambda_1, \Lambda_2)}{\partial f} \Big|_f = \frac{\partial \Lambda}{\partial f_1} \Big|_{(\Lambda_1(f), \Lambda_2(f))} \left(\frac{\partial \Lambda_1}{\partial f} \Big|_f \right) + \frac{\partial \Lambda}{\partial f_2} \Big|_{(\Lambda_1(f), \Lambda_2(f))} \left(\frac{\partial \Lambda_2}{\partial f} \Big|_f \right).$$

Let us give applications of this calculation and of Proposition 5.2.

Proposition 5.3.

(1) *The diffeomorphing map*

$$\mathcal{D} : g \in \mathbb{C}\{z\} \mapsto z \times \exp \circ g \in \text{Diff}(\mathbb{C}, 0),$$

is strongly analytic and for all $g \in \mathbb{C}\{z\}$

$$\frac{\partial \mathcal{D}}{\partial g} \Big|_g = \mathcal{D}(g) \times \bullet.$$

More generally, with convergence in $\mathbb{C}\{z\}$,

$$\mathcal{D}(g) = z \times \sum_{p=0}^{\infty} \frac{g^p}{p!}.$$

(2) *The inversion map*

$$\begin{aligned} \iota : \mathbb{C}\{z\} &\longrightarrow \text{Diff}(\mathbb{C}, 0) \\ g &\longmapsto \mathcal{D}(g)^{\circ-1} \end{aligned}$$

is strongly analytic and for all $g \in \mathbb{C}\{z\}$ we have

$$\frac{\partial \iota}{\partial g} \Big|_g = - \left(\frac{z}{1 + z \times g'} \times \bullet \right) \circ \iota(g).$$

(3) *There exists a unique strongly analytic map*

$$\mathcal{H} : \mathbb{C}\{z\} \longrightarrow \mathbb{C}\{z\}$$

such that $\mathcal{H}(0) = 0$ and

$$\iota = \mathcal{D} \circ \mathcal{H}.$$

If Log denotes the principal determination of the logarithm then

$$\mathcal{H}(g) = \text{Log} \circ \frac{\iota(g)}{z}$$

for all $g \in \mathbb{C}\{z\}$. We have also

$$\frac{\partial \iota}{\partial g}|_g = \iota(g) \times \frac{\partial \mathcal{H}}{\partial g}|_g,$$

that is

$$\frac{\partial \mathcal{H}}{\partial g}|_g = - \left(\frac{\bullet}{1 + z \times g'} \right) \circ \iota(g).$$

Every other holomorphic map $\tilde{\mathcal{H}}$ such that $\iota = \mathcal{D} \circ \tilde{\mathcal{H}}$ is obtained from \mathcal{H} by adding an element of $2i\pi\mathbb{Z}$.

Proof. We use the rules established in the previous sections. For the sake of clarity we use the notation \mathcal{C}_f to denote the right composing mapping $g \mapsto f \circ g$.

(1) We have

$$\frac{\partial \mathcal{D}}{\partial g}|_g = z \times \frac{\partial \exp g}{\partial g}|_g = \mathcal{D}(g) \times \bullet.$$

(2) Using the previous lemma to differentiate with respect to g the relation

$$\mathcal{C}_{\iota(g)}(\mathcal{D}(g)) = z$$

we derive the formula:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{C}_f(\mathcal{D}(g))}{\partial f}|_{\iota(g)} \left(\frac{\partial \iota}{\partial g}|_g \right) + \frac{\partial \mathcal{C}_{\iota(g)}(h)}{\partial h}|_{\mathcal{D}(g)} \left(\frac{\partial \mathcal{D}}{\partial g}|_g \right) \\ &= \mathcal{C}_{\frac{\partial \iota}{\partial g}|_g}(\mathcal{D}(g)) + \mathcal{C}_{\iota(g)'}(\mathcal{D}(g)) \times \mathcal{D}(g) \times \bullet. \end{aligned}$$

The same relation differentiated with respect to z yields the usual formula

$$\mathcal{D}(g)' \times \mathcal{C}_{\iota(g)'}(\mathcal{D}(g)) = 1,$$

completing the proof since $\mathcal{D}(g)' = (1 + \text{Id} \times g') \times \exp \circ g$.

(3) The equality regarding the derivatives, whenever they exist, comes from differentiating $\iota = \mathcal{D} \circ \mathcal{H}$. Let us now prove the existence of \mathcal{H} . Consider

$$L : h \in \mathbb{C}\{z\} \mapsto \sum_{p>0} \frac{(-1)^{p+1}}{p} h^p,$$

which is a convergent series for the topology of $\mathbb{C}\{z\}$. Obviously $\mathcal{C}_{\exp} \circ L(h) = 1 + h$ for all $h \in \widehat{\mathbb{C}\{h\}}$. Therefore setting

$$\mathcal{H}(g) := L\left(\frac{\iota(g)}{z}\right) - 1$$

does the trick. □

Corollary 5.4. *The universal covering of $\text{Diff}(\mathbb{C}, 0)$ can be represented by the analytic covering*

$$\begin{aligned} \mathcal{D} : \mathbb{C}\{z\} &\longrightarrow \text{Diff}(\mathbb{C}, 0) \\ g &\longmapsto z \exp \circ g. \end{aligned}$$

The fiber is canonically isomorphic to $2i\pi\mathbb{Z}$: the Galois group of the covering is generated by the shift $g \mapsto g + 2i\pi$.

6. APPLICATION TO COMPLEX ANALYSIS

For a family of k germs $F := (f_\ell)_{1 \leq \ell \leq k}$ we let $\mathcal{I}(F)$ be the ideal of $\mathbb{C}\{\mathbf{z}\}$ spanned by the family. Define the **Milnor number** of F as

$$\mu(F) := \dim_{\mathbb{C}} \mathbb{C}\{\mathbf{z}\} / \mathcal{I}(F) \in \mathbb{N} \cup \{\infty\}.$$

It is a well-known consequence of the *Nullstellensatz* for complex analytic functions that the following statements are equivalent:

- ⊗ $\mu(F) < \infty$,
- ⊗ $\mathcal{I}(F)$ contains a power of \mathfrak{Z} , the maximal ideal of $\mathbb{C}\{\mathbf{z}\}$,
- ⊗ no element of \mathfrak{Z} is a common factor to all the f_ℓ 's.

As it turns out the set of collections $F \in \mathbb{C}\{\mathbf{z}\}^m$ with infinite Milnor number has the structure of an analytic set.

Theorem 6.1. *Let m and k be integers greater than 1. We understand μ here as the mapping*

$$\begin{aligned} \mu : \mathbb{C}\{\mathbf{z}\}^k &\longrightarrow \mathbb{N} \cup \{\infty\} \\ F &\longmapsto \mu(F). \end{aligned}$$

Endow each factor $\mathbb{C}\{\mathbf{z}\}$ with a sequential topology. Then the set $\mathcal{I} := \mu^{-1}(\infty)$ is a proper analytic set of \mathfrak{Z}^k .

We actually show that $\mu^{-1}(\infty)$ is «algebraic» in the sense that for each $N \in \mathbb{N}$ there exists a polynomial map $\Lambda_N : \mathbb{C}[\mathbf{z}]_{\leq N}^k \rightarrow \mathbb{C}^{d(N)}$ such that

$$\mu^{-1}(\infty) = \bigcap_{N \in \mathbb{N}} (\Lambda_N \circ J_N)^{-1}(0),$$

where J_N is the Cartesian product of the N^{th} -jet operator of each copy of $\mathbb{C}\{\mathbf{z}\}$. From this particular decomposition stems the fact that the theorem derives from the same result in the factorial ring $\mathbb{C}[[\mathbf{z}]]$.

We show, in the rest of this section, that determining if $k \geq 2$ formal power series have a non-trivial common factor, *i.e.* belonging to the maximal ideal (still written) \mathfrak{Z} of $\mathbb{C}[[\mathbf{z}]]$, is an «algebraic» condition. Observe the set $\mathcal{I} \subset \mathfrak{Z}^k$ formed by non-coprime families is never empty since $(f, \dots, f) \in \mathcal{I}$ whenever $f \in \mathfrak{Z}$. The case $k = 2$ encompasses all technical difficulties so it is completed first, in Section 6.2. We finally reduce the general case $k \geq 2$ to the latter study, in Section 6.4. In Section 6.3 we present an effective computable process which stops in finite time if, and only if, the given family $(f_\ell)_{\ell \leq k}$ is not coprime, using a growing sequence of Macaulay-like matrices.

Before performing all these tasks we shall use the above structure theorem to equip the space of germs of meromorphic functions with an analytic structure.

6.1. The analytic space of meromorphic germs.

Let $\mathbb{C}(\{z\})$ stand for the space of germs of meromorphic functions (for short, a meromorphic germ) at the origin of \mathbb{C}^m . This space is by definition the fractions field of the ring $\mathbb{C}\{z\}$, that is the set of equivalence classes of couples $(P, Q) \in \mathbb{C}\{z\} \times (\mathbb{C}\{z\} \setminus \{0\})$ given by

$$(6.1) \quad (P_1, Q_1) \leftrightarrow (P_2, Q_2) \iff P_1 Q_2 = P_2 Q_1.$$

Of course we write $\frac{P}{Q}$ the equivalence class of (P, Q) , and name Quot the canonical projection

$$\begin{aligned} \text{Quot} : \mathbb{C}\{z\} \times (\mathbb{C}\{z\} \setminus \{0\}) &\longrightarrow \mathbb{C}(\{z\}) \\ (P, Q) &\longmapsto \frac{P}{Q}. \end{aligned}$$

Definition 6.2.

- (1) We say that (P, Q) is a **proper representative** of $\frac{P}{Q}$ if $\mu(P, Q) < \infty$. Two proper representatives differ by the multiplication with a multiplicatively invertible holomorphic germ.
- (2) We say that a meromorphic germ is **purely meromorphic** if it does not admit a representative of the form $(P, 1)$ or $(1, Q)$. This is equivalent to requiring that any proper representative belongs to \mathbb{Z}^2 . The set of all purely meromorphic germs is written

$$\mathbb{C}(\{z\})_0 := \left\{ \frac{P}{Q} : P(0) = Q(0) = 0, \mu(P, Q) < \infty \right\}.$$

Since $\mathbb{C}\{z\}$ is factorial any meromorphic germ admits a proper representative. Therefore we can equip the space $\mathbb{C}(\{z\})$ with the analytic structure (as in Definition 3.5) induced by the quotient map Quot restricted to the complement of $\mu^{-1}(\infty)$, which is a nonempty open set by Theorem 6.1.

Remark 6.3.

- (1) Working with proper representative is quite natural since a lot of constructions involving meromorphic germs (e.g. analytic maps Λ from some open set $U \subset \mathbb{C}(\{z\})$) fail to be possible for non-proper representatives of meromorphic germs. Besides if $\text{Quot}^* \Lambda$ is amply bounded near $\mu^{-1}(\infty)$ then Theorem 4.3 applies and allows to extend analytically $\text{Quot}^* \Lambda$ to the whole $\text{Quot}^{-1}(U)$.
- (2) Let us say a few words about the definition of analytic sets of $\mathbb{C}(\{z\})$ as in Definition 4.1 (4). If analytic sets of $\mathbb{C}(\{z\})$ were defined solely as the vanishing loci of analytic maps then the set of germs $\frac{P}{Q}$ where, for instance, $J_N(P) = J_N(Q) = 0$ would not form an analytic set, since the natural map $j_N : (P, Q) \mapsto (J_N(P), J_N(Q))$ is neither left- nor right-invariant by Quot. Yet the vanishing locus of j_N is Quot-invariant, in the sense that $J_N(P) = J_N(Q) = 0$ implies $J_N(uP) = J_N(uQ) = 0$ for any $u \in \mathbb{C}\{z\}$.
- (3) The map

$$\begin{aligned} M : \mathbb{C}\{z\} \times (\mathbb{C}\{z\} \setminus \mathbb{C}) \setminus \mu^{-1}(\infty) &\longrightarrow \mathbb{C}\{z\} \times (\mathbb{C}\{z\} \setminus \mathbb{C}) \\ (P, Q) &\longmapsto (P - J_0(P), Q - J_0(Q)) \end{aligned}$$

is analytic and onto, so we can equip $\mathbb{C}(\{z\})_0$ with the analytic structure induced by

$$M_0 := \text{Quot} \circ M .$$

6.2. Coprime pairs of $\mathbb{C}[[z]]^2$.

6.2.1. Reduction of the proof of Theorem 6.1 when $k = 2$.

The result is «only» an exercise in linear algebra. Assume there exists $(h_1, h_2) \in \mathbb{C}[[z]]^k$ such that

$$(6.2) \quad f_1 h_1 = f_2 h_2$$

with $\nu(h_1) < \nu(f_2)$ and $\nu(h_2) < \nu(f_1)$, where ν is the valuation associated to the gradation by homogeneous degree on $\mathbb{C}[[z]]$. We say in that case, for short, that the collection (f_1, f_2) is **composite**. It follows that \mathbb{I} is the set of all composite collections; notice in particular that if $\nu(h_\ell) < \nu(f_{3-\ell})$ for some $\ell \in \{1, 2\}$ then the relation equally holds for the other one. We set

$$(6.3) \quad \nu := \min(\nu(f_1), \nu(f_2)) > 0,$$

and write $f_\ell =: \sum_{\mathbf{n} \in \mathbb{N}^m} f_{\ell, \mathbf{n}} \mathbf{z}^{\mathbf{n}}$ with $f_{\ell, \mathbf{n}} = 0$ for $|\mathbf{n}| < \nu$. We set likewise $h_\ell =: \sum_{\mathbf{n} \in \mathbb{N}^m} h_{\ell, \mathbf{n}} \mathbf{z}^{\mathbf{n}}$ then express the relationship (6.2) coefficients-wise:

$$(6.4) \quad (\forall \mathbf{n} \in \mathbb{N}^m) \quad \sum_{\mathbf{p}+\mathbf{q}=\mathbf{n}} (f_{1, \mathbf{p}} \times h_{1, \mathbf{q}} - f_{2, \mathbf{p}} \times h_{2, \mathbf{q}}) = 0.$$

We say that (f_1, f_2) is **composite at rank** $d \in \mathbb{N}$ if, and only if, one can find a tuple $(h_{\ell, \mathbf{q}})_{|\mathbf{q}| \leq d, \ell \leq k}$ such that

- ⊗ the previous relation holds for every $|\mathbf{n}| \leq d + 1$,
- ⊗ at least one $h_{\ell, \mathbf{q}}$ is non-zero for $|\mathbf{q}| < \nu$.

If a collection $(f_\ell)_\ell$ is composite then it is composite at every rank $d \in \mathbb{N}$. If a collection fails to be composite at some rank then it also does at every other bigger rank.

We work with the family of linear spaces $E_d := \mathbb{C}[z]_{\leq d}^k$, indexed by $d \in \mathbb{N}$. Define the linear mapping

$$\begin{aligned} \varphi_d : E_d &\longrightarrow \mathbb{C}[z]_{\leq d+1} \cap \mathfrak{Z} \\ (h_1, h_2) &\longmapsto J_{d+1}(f_1 h_1 - f_2 h_2) . \end{aligned}$$

Proposition 6.4. *The following properties are equivalent:*

- (1) *The family $(f_\ell)_\ell$ is composite.*
- (2) *The family $(f_\ell)_\ell$ is composite at every rank $d \geq 0$.*
- (3) *For all $d \geq 0$ we have*

$$\text{rank}(\varphi_d) \leq \binom{m+d}{d}.$$

We deduce that composite collections form a proper analytic subset of $\mathbb{C}[[z]]^2$, expressing for all d the vanishing of every minor of size greater than $\binom{m+d}{d}$ of a given matrix of φ_d .

6.2.2. Proof of (2) \Rightarrow (1).

Only the case $f_1 f_2 \neq 0$ is non-trivial, for which $0 < \nu < \infty$ (see (6.3)). We prove that if (f_1, f_2) is composite at every rank $d \in \mathbb{N}$ then there exists two formal power series $h_1, h_2 \in \mathbb{C}[[\mathbf{z}]]$ such that $f_1 h_1 = f_2 h_2$ and $\nu(h_\ell) < \nu(f_{3-\ell})$. Consider the restriction $j_d : E_{d+1} \twoheadrightarrow E_d$ of J_d , each one yielding a canonical section $E_d \hookrightarrow E_d \oplus \ker j_d = E_{d+1}$. For that identification we have

$$(6.5) \quad \varphi_{d+1}|_{E_d} = \varphi_d$$

and $\varphi_{d+1}(\ker j_d) \subset \ker j_{d+1}$. Write $\kappa_d : \ker \varphi_{d+1} \rightarrow \ker \varphi_d$ the restriction of j_d . We choose a direct system of one-to-one maps $(\kappa_d : K_{d+1} \hookrightarrow K_d)_{d \in \mathbb{N}}$ of complement subspaces of $\ker \varphi_d \cap J_d(\ker J_{\nu-1})$ in $\ker \varphi_d$ in the following fashion:

- ⊗ $K_d := \ker \varphi_d$ if $d \leq \nu - 1$
- ⊗ define $K := \kappa_d^{-1}(K_d)$ and, observing that $K \cap \ker \kappa_d = \ker j_d \cap \ker \kappa_d$, define K_{d+1} as some complement in K of $K \cap \ker \kappa_d$.

Fact 6.5. *A collection $(f_\ell)_\ell$ is composite at some rank d if, and only if,*

$$\dim K_d \neq 0.$$

Now if (f_1, f_2) is composite at every rank $d \in \mathbb{N}$ then the inverse limit K_∞ is embeddable as a linear subspace of finite positive dimension in $\mathbb{C}[[\mathbf{z}]]^2$, solving our problem.

6.2.3. Proof of (1) \Rightarrow (3).

If $f_1 = f_2 = 0$ then (f_1, f_2) is composite and $\varphi_d = 0$. Assume now that, say, $f_1 \neq 0$ and $\nu = \nu(f_1)$. We introduce a notation regarding the number of multi-indexes whose length satisfy some property \mathcal{P}

$$\mathfrak{C}_{\mathcal{P}} := \# \{ \mathbf{n} \in \mathbb{N}^m : \mathcal{P}(|\mathbf{n}|) \}.$$

For instance the number of multi-indexes of length less than or equal to d is

$$\mathfrak{C}_{\leq d} = \binom{m+d}{d},$$

which incidentally is the sought bound on $\text{rank}(\varphi_d)$ for composite collections. We observe that E_d has dimension $2\mathfrak{C}_{\leq d}$, hence the

Fact 6.6. *Condition (3) holds if, and only if, for all $d \in \mathbb{N}$*

$$\dim \ker \varphi_d \geq \mathfrak{C}_{\leq d}.$$

Because $E_d \cap \ker J_{d+1-\nu} \subset \ker \kappa_d$ contributes for a subspace of dimension $2\mathfrak{C}_{d+2-\nu \leq \bullet \leq d}$ we are only interested in studying the dimension of a complement A_d of $E_d \cap \ker J_{d+1-\nu}$ in $\ker \varphi_d$. Notice also that

$$\begin{aligned} (\forall d < 2(\nu - 1)) \quad 2\mathfrak{C}_{d+2-\nu \leq \bullet \leq d} &= \mathfrak{C}_{\leq d} + (\mathfrak{C}_{\leq d} - 2\mathfrak{C}_{\leq d+1-\nu}) \\ &\geq \mathfrak{C}_{\leq d} + \mathfrak{C}_{\leq 2(\nu-1)} - 2\mathfrak{C}_{\leq \nu-1} \\ &\geq \mathfrak{C}_{\leq d} \end{aligned}$$

so that (3) holds for all such d . In the sequel we thus assume $d \geq 2(\nu - 1)$. As before we choose:

- ⊗ two direct systems $(\kappa_d : K_{d+1} \hookrightarrow K_d)_{d \in \mathbb{N}}$ and $(\kappa_d : A_{d+1} \rightarrow A_d)_{d \in \mathbb{N}}$,

⊗ for each d , a subspace $C_d < \ker \varphi_d \cap \ker J_{\nu-1}$ such that

$$(6.6) \quad A_d = \kappa_d(A_{d+1}) \oplus C_d.$$

Lemma 6.7. *Assume that $K_\infty \neq 0$ and $d \geq 2(\nu-1)$. Then*

$$\dim A_d \geq \mathfrak{C}_{\leq d-2(\nu-1)}.$$

Now we have

$$\begin{aligned} \ker \varphi_d &\geq 2\mathfrak{C}_{d+2-\nu \leq \bullet \leq d} + \mathfrak{C}_{\leq d-2(\nu-1)} = \mathfrak{C}_{\leq d} + \sum_{j=d+2-\nu}^d (\mathfrak{C}_{=j} - \mathfrak{C}_{=j-\nu+1}) \\ &\geq \mathfrak{C}_{\leq d} \end{aligned}$$

as expected.

Proof. We begin with the trivial remark that $J_{d+1-\nu}(\mathfrak{Z}K_\infty) \cap \ker J_{\nu-1} < A_d \cap \ker J_{\nu-1}$ then bound from below the dimension of the leftmost space. The «worst-case scenario» corresponds to the support of $J_{\nu-1}(K_\infty)$ consisting in a single point \mathbf{n}_0 located on the homogeneous segment $\{\mathbf{n} : |\mathbf{n}| = \nu-1\}$. Yet in that case $\mathfrak{Z}K_\infty$ contains an embedding of the ideal $\mathbf{z}^{\mathbf{n}_0}\mathfrak{Z}$ so that

$$\begin{aligned} \dim J_{d+1-\nu}(\mathfrak{Z}K_\infty) \cap \ker J_{\nu-1} &\geq \dim J_{d+1-\nu}(\mathbf{z}^{\mathbf{n}_0}\mathfrak{Z}) = \sum_{p=1}^{d-2(\nu-1)} \mathfrak{C}_{=j} \\ &= \mathfrak{C}_{\leq d-2(\nu-1)} - 1. \end{aligned}$$

Finally

$$\begin{aligned} \dim A_d &\geq (\mathfrak{C}_{\leq d-2(\nu-1)} - 1) + \dim K_d \\ &\geq \mathfrak{C}_{\leq d-2(\nu-1)}. \end{aligned}$$

□

6.2.4. Proof of (3)⇒(2).

This is the most involved part of the proof. We start from the decomposition (6.6). Because the restriction $\hat{\kappa}_d := \kappa_d|_{A_d}$ is still a projector we have $A_{d+1} = \ker \hat{\kappa}_d \oplus \text{Im } \hat{\kappa}_d$. Whence:

Fact 6.8. *For any collection (f_1, f_2) and any $d \in \mathbb{N}$*

$$A_d \oplus \ker \hat{\kappa}_d = A_{d+1} \oplus C_d.$$

In particular

$$\dim A_d + \dim \ker \hat{\kappa}_d = \dim A_{d+1} + \dim C_d.$$

As before we would like to use the image $Z_p := J_{d+p+1-\nu}(\mathfrak{Z}C_d)$ to bound from below $\dim C_{d+p}$, but Z_p need not be included in C_{d+p} :

- ⊗ an element $h \in Z_p$ might not lie in $\ker J_{d+1-\nu}$, and $J_{d+2-\nu}(\varphi_{d+p}(h))$ has no reason to vanish,

- ⊛ contrary to what happened in the previous paragraph's lemma, we do not dispose of a global object like K_∞ and there is no reason why the question

$$0 = J_{d+p+2-\nu}(g\varphi_d(h)) \stackrel{?}{=} \varphi_{d+p}(J_{d+p+1-\nu}(gh))$$

should admit a positive answer when $g \in \mathfrak{Z}$ and $h \in C_d$.

Yet both conditions can be resolved if on the one hand

$$p \geq \hat{d} := d + 1 - \nu$$

and on the other hand we replace \mathfrak{Z} by its trace $\mathfrak{Z}_{=p}$ on the space of homogeneous polynomials of degree d .

Lemma 6.9. *If $\dim C_d > 0$ then $\mathfrak{Z}_{=p}C_d$ contains at least a vector space of dimension $\mathfrak{C}_{=p}$, since each map*

$$\begin{aligned} \tau_{\mathbf{n}} : C_d &\longrightarrow C_{d+p} \\ h &\longmapsto \mathbf{z}^{\mathbf{n}}h \end{aligned}$$

is a monomorphism whenever $|\mathbf{n}| = p$.

Proof. Take any $h \in C_d \setminus 0$ and show that $h_{\mathbf{n}} := \mathbf{z}^{\mathbf{n}}h \in C_{d+p} \setminus 0$. Suppose now, by contradiction, that $h_{\mathbf{n}} \notin C_{d+p}$: there exists $\hat{h}_{\mathbf{n}} \in A_{d+p+1}$ such that $\kappa_{d+p}(\hat{h}_{\mathbf{n}}) = h_{\mathbf{n}}$. Because $h_{\mathbf{n}}$ has support inside $\{\mathbf{m} : m_j \geq n_j\}$ we may require without loss of generality that $\hat{h}_{\mathbf{n}}$ also has. Then $\hat{h}_{\mathbf{n}} = \mathbf{z}^{\mathbf{n}}\hat{h}$ for some $\hat{h} \in A_{d+1}$, as indeed $\mathbf{z}^{\mathbf{n}}\varphi_{d+1}(\hat{h}) = \varphi_{d+\alpha+1}(\hat{h}_{\mathbf{n}}) = 0$. We reach the contradiction $h = \kappa_d(\hat{h}) \in C_d \setminus 0$. \square

Assume now that (2) does not hold: there exists $d \geq 2(\nu - 1)$ such that $\dim K_{d+1} < \dim K_d$, so that $\dim C_d > 0$. Then for all $p \geq \hat{d}$

$$\dim A_{d+p} \leq \dim A_{\hat{d}} + \sum_{j=\hat{d}}^p (\dim \ker \hat{\kappa}_{d+j-1} - \mathfrak{C}_{=j}).$$

Lemma 6.10. *For every $d \geq 2(\nu - 1)$ the following estimate holds:*

$$\dim \ker \hat{\kappa}_d \leq \mathfrak{C}_{=d+2-\nu}.$$

Proof. This is clear from the matrix representation of φ_d given in Section 6.3. \square

Now we can write for $p \geq \hat{d}$

$$\dim A_d \leq \mathfrak{C}_{\leq \hat{d}+p} - \mathfrak{C}_{\leq p} + c,$$

where c is some term constant with respect to p . Whence

$$\begin{aligned} \dim \ker \varphi_{d+p} &\leq 2\mathfrak{C}_{\hat{d}+p+1 \leq \bullet \leq d+p} + \mathfrak{C}_{\leq \hat{d}+p} - \mathfrak{C}_{\leq p} + c \\ &\leq \mathfrak{C}_{\leq d+p} + (\mathfrak{C}_{\hat{d}+p+1 \leq \bullet \leq d+p} - \mathfrak{C}_{\leq p} + c). \end{aligned}$$

Because $\mathfrak{C}_{\leq p} = \frac{p^m}{p!} + O(p^{m-1})$ and $\mathfrak{C}_{\hat{d}+p+1 \leq \bullet \leq d+p} = O(p^{m-1})$ we deduce that the parenthesized, rightmost term is eventually negative.

The set \mathbb{N}^m of multi-indexes of given dimension m comes with the lexicographic order \ll , that is $(0, 2) \ll (1, 0) \ll (1, 1)$. Define the «lexicomogeneous» order \leq on \mathbb{N}^m by

$$\mathbf{a} \preceq \mathbf{b} \iff \begin{cases} |\mathbf{a}| < |\mathbf{b}| \\ \text{or} & |\mathbf{a}| = |\mathbf{b}| \text{ and } \mathbf{a} \lll \mathbf{b} \end{cases}$$

so that e.g. $(2, 0) \succeq (1, 1) \succeq (0, 2) \succeq (1, 0) \succeq (0, 1) \succeq (0, 0)$. For $d \in \mathbb{N}$ we form the matrix of multi-indexes

$$M_d := [\mathbf{n}_{\mathbf{p}, \mathbf{q}}]_{0 \leq |\mathbf{p}| \leq d, 0 \leq |\mathbf{q}| \leq d+1}$$

defined by the relation

$$\mathbf{n}_{\mathbf{p},\mathbf{q}} := \begin{cases} \mathbf{q} - \mathbf{p} & \text{if } \mathbf{q} - \mathbf{p} \in \mathbb{N}^m \\ 0 & \text{otherwise} \end{cases}$$

where \mathbf{p}, \mathbf{q} are ordered counter-lexicogeneously with $\mathbf{0} \leq \mathbf{p} \leq d \oplus \mathbf{0}$ and $\mathbf{0} < \mathbf{q} \leq (d+1) \oplus \mathbf{0}$. See Table 1 below for an example. This matrix is upper $(d+1) \times (d+1)$ block-triangular, the blocks $[\mathbf{n}_{\mathbf{p}, \mathbf{q}}]$ corresponding to constant lengths $|\mathbf{p}|$ and $|\mathbf{q}|$. Its size is $\mathbb{C}_{\leq d+1} \times \mathbb{C}_{\leq d}$. Each M_d is naturally nested within M_{d+1} as the right-bottom-most submatrix formed by the last d blocks both vertically and horizontally, accounting for the relation (6.5).

$(1, 0, 0)$ $(0, 1, 0)$ $(1, 0, 0)$ $(0, 0, 1)$ $(1, 0, 0)$ $(0, 1, 0)$ $(1, 0, 0)$ $(0, 0, 1)$ $(0, 1, 0)$ $(1, 0, 0)$ $(0, 0, 1)$ $(1, 0, 0)$ $(0, 1, 0)$ $(1, 0, 0)$ $(0, 0, 1)$ $(0, 1, 0)$ $(0, 0, 1)$ $(0, 1, 0)$ $(0, 0, 1)$ $(0, 1, 0)$ $(0, 0, 1)$	$(2, 0, 0)$ $(1, 1, 0)$ $(2, 0, 0)$ $(1, 0, 1)$ $(2, 0, 0)$ $(0, 2, 0)$ $(1, 1, 0)$ $(0, 1, 1)$ $(1, 0, 1)$ $(1, 1, 0)$ $(0, 0, 2)$ $(1, 0, 1)$ $(0, 2, 0)$ $(0, 1, 1)$ $(0, 2, 0)$ $(0, 0, 2)$ $(0, 1, 1)$ $(0, 0, 2)$	$(3, 0, 0)$ $(2, 1, 0)$ $(2, 0, 1)$ $(1, 2, 0)$ $(1, 1, 1)$ $(1, 0, 2)$ $(0, 3, 0)$ $(0, 2, 1)$ $(0, 1, 2)$ $(0, 0, 3)$
	$(1, 0, 0)$ $(0, 1, 0)$ $(1, 0, 0)$ $(0, 0, 1)$ $(1, 0, 0)$ $(0, 1, 0)$ $(0, 0, 1)$ $(0, 1, 0)$ $(0, 0, 1)$	$(2, 0, 0)$ $(1, 1, 0)$ $(1, 0, 1)$ $(0, 2, 0)$ $(0, 1, 1)$ $(0, 0, 2)$
		$(1, 0, 0)$ $(0, 1, 0)$ $(0, 0, 1)$

TABLE 1. The matrix M_2 when $m = 3$. Null entries are not shown. Homogeneous blocks are drawn in solid lines.

We build a matrix $M_d(f_1, f_2)$ from M_d , that of the mapping φ_d . It has twice as many rows and is obtained from the latter by replacing each column $(\mathbf{n}_{\mathbf{p}, \mathbf{q}})_{0 < |\mathbf{q}| \leq d+1}$ with the pair of columns whose respective entries are $(-1)^{\ell+1} f_{\ell, \mathbf{n}_{\mathbf{p}, \mathbf{q}}}$ for $\ell \in \{1, 2\}$.

0	-1	0	0	0	0	-1	0	0	0	1	0
0	-1	0	-1	0	0	-2	3	-1	0	2	0
0	0	0	-1	0	-1	-1	3	-2	3	1	0
0	0	0	0	0	-1	0	0	-1	3	0	0
0	0	0	0	0	0	0	-1	0	0	-1	0
0	0	0	0	0	0	0	-1	0	-1	-2	3
0	0	0	0	0	0	0	0	0	-1	-1	3
0	0	0	0	0	0	0	0	0	0	0	-1
0	0	0	0	0	0	0	0	0	0	0	-1

TABLE 2. The matrix $M_2((x-1)(x+y)^2, (1-3y)(x+y))$ when $m = 2$. Its rank is $6 = \rho(2, 3, 2)$ and $\nu = 1$.

Remark 6.11. When $m = 2$ the first non-trivial case occurs for $d = 2(\nu - 1)$. The condition $\dim \ker \varphi_d \geq \binom{m+d}{d}$ is equivalent to the vanishing of the determinant of the matrix occupying the homogeneous bloc $(2\nu - 1) \times \nu - 1$. This matrix is nothing but the Sylvester matrix (up to columns/rows permutation) of the homogeneous part of degree ν of (f_1, f_2) . Indeed if (f_1, f_2) is composite then also is its part of lowest homogeneous degree, which can be written $\frac{1}{x^\nu} (f_1(1, t), f_2(1, t))$ with $t = \frac{y}{x}$. This situation persists for other values of m as the Macaulay matrix of the homogeneous part of (f_1, f_2) is embedded in $M_{2(\nu-1)}(f_1, f_2)$. I am aware that the topic discussed here is well-known in the setting of commutative algebra on the \mathbb{C} -module of polynomials, but I have not been able to locate a similar construction in the framework of formal power series.

6.4. Coprime families of $\mathbb{C}[[\mathbf{z}]]^k$.

The general case is not more difficult. Proposition 6.4 holds actually for any $k \geq 2$ as we explain now.

Fact 6.12. *The collection $f = (f_\ell)_{1 \leq \ell \leq k}$ is composite if, and only if, there exists $(h_1, \dots, h_k) \in \mathbb{C}[[\mathbf{z}]]^k$ such that $f_\ell h_\ell$ does not depend on ℓ and for every $1 \leq \ell \leq k$*

$$\nu(h_\ell) < \varepsilon_\ell(f) := \sum_{j \neq \ell} \nu(f_j) - \nu \left(\frac{\prod_{j=1}^k f_j}{\gcd(f)^{k-1} \text{lcm}(f)} \right).$$

Notice that when $k = 2$ the right-hand side equals $\nu(f_{3-\ell})$ as before.

The result follows from the study of the linear maps

$$\begin{aligned} \hat{\varphi}_d : E_d &\longrightarrow \mathbb{C}[[\mathbf{z}]]_{\leq d+1}^{\times \binom{k}{2}} \\ (h_1, \dots, h_k) &\longmapsto \left(I_{d+1} (f_p h_p - f_q h_q) \right)_{1 \leq p < q \leq k} \end{aligned}$$

and the property of being composite is expressed in terms of the dimension of $\hat{K}_d := \ker \hat{\phi}_d / \ker J_\varepsilon$ where the extra parameter

$$\varepsilon := \min_{\ell} \varepsilon_{\ell}(f)$$

may in general not be equal to $\nu = \min_{\ell} \nu(f_{\ell})$. This in turn is equivalent to $\dim \ker \hat{\phi}_d \geq (k-1) \binom{m+d}{d}$, as can be recovered from repeating the arguments developed in the case $k=2$.

7. APPLICATION TO DIFFERENTIAL EQUATIONS

In this section every space of germs is given the factorial topology (Definition 2.19). In particular $\mathbb{C}\{\mathbf{z}\}$ becomes a metric space.

7.1. Analyticity of the flow of a vector field: proof of Theorem C.

Theorem 7.1. *Fix $m \in \mathbb{N}$ and consider the space VF of germs at $\mathbf{0} \in \mathbb{C}^m$ of a holomorphic vector field, identified with $\mathbb{C}\{\mathbf{z}\}^m$. For $X \in \text{VF}$ we name Φ_X the flow of X , that is the unique germ of a holomorphic mapping near $(\mathbf{0}, 0)$*

$$\begin{aligned} \Phi_X : \mathbb{C}^m \times \mathbb{C} &\longrightarrow \mathbb{C}^m \\ (\mathbf{p}, t) &\longmapsto \Phi_X^t(\mathbf{p}) \end{aligned}$$

that is solution of the differential system

$$\begin{cases} \dot{\mathbf{z}}(\mathbf{p}, t) &= X(\mathbf{z}(\mathbf{p}, t)) \\ \mathbf{z}(\mathbf{p}, 0) &= \mathbf{p}. \end{cases}$$

Then the «flow mapping»

$$\begin{aligned} \text{VF} &\longrightarrow \mathbb{C}\{\mathbf{z}, t\}^m \\ X &\longmapsto \Phi_X, \end{aligned}$$

where the target space is also given the factorial topology (see Remark 2.20), is strongly analytic.

I believe this theorem should hold for every useful topology. Yet the proof of this result is simplified by the fact that we know the norms of differentiation operators in the factorial topology.

Proof. Let $\chi \in \mathcal{O}((\mathbb{C}^q, 0) \rightarrow \text{VF})$ as in Definition 3.18; for the sake of keeping notations unobtrusive we write $\chi_{\mathbf{x}}$ instead of $\chi(\mathbf{x})$. From the differential system $\dot{\mathbf{z}} = \chi_{\mathbf{x}}(\mathbf{z})$ we build a new differential system incorporating the extra parameter $\mathbf{x} \in (\mathbb{C}^q, 0)$ in the new variable $\mathbf{w} = \mathbf{z} \oplus \mathbf{x}$

$$\begin{cases} \dot{\mathbf{w}}(\mathbf{p}, \mathbf{x}, t) &= \chi_{\mathbf{x}}(\mathbf{z}) \\ \mathbf{w}(\mathbf{p}, \mathbf{x}, 0) &= (\mathbf{p}, \mathbf{x}). \end{cases}$$

By assumption on χ the vector field $(\mathbf{z}, \mathbf{x}) \mapsto \chi_{\mathbf{x}}(\mathbf{z})$ is holomorphic on a neighborhood of $\mathbf{0} \oplus \mathbf{0}$. Therefore the Cauchy-Lipschitz theorem applies to this system and yields a flow which is a germ of a holomorphic mapping with respect to $(\mathbf{p}, \mathbf{x}, t)$. In particular $\chi^* \Phi_{\bullet} : \mathbf{x} \mapsto \Phi_{\chi_{\mathbf{x}}}$ belongs to $\mathcal{O}((\mathbb{C}^q, 0) \rightarrow \mathbb{C}\{\mathbf{z}, t\})$ and the flow mapping is quasi-strongly analytic in the sense of Definition 3.18.

To complete the proof that the flow is strongly analytic on VF we need to establish that $\chi^* \Phi_\bullet$ is amply bounded for the factorial topology. In fact we show that Φ_\bullet is amply bounded on the whole VF, so that Theorem 4.3 will automatically apply. This can be achieved using Lie's formula for the flow:

$$\Phi_X^t(\mathbf{p}) = \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} X \cdot^k \text{Id} \right)(\mathbf{p})$$

with uniform convergence, as a power series in (\mathbf{p}, t) , on a neighborhood of $\mathbf{0} \times 0$. The iterated Lie derivatives are given for germs of functions by $X \cdot^0 f = f$ and $X \cdot^{k+1} := X \cdot (X \cdot^k f)$, and then extended to vectors of functions by acting component-wise. The factorial topology on $\mathbb{C}\{z, t\}$ is induced by $\mathbf{a}(\bullet) = a(\bullet) \oplus (k!^{-\bullet})_{k \in \mathbb{N}}$ while $a(\bullet)$ induces the factorial topology on VF. We thus have for $\alpha > 0$

$$\|\Phi_X\|_{\mathbf{a}(\alpha)} = \sum_{k=0}^{\infty} \frac{1}{(k!)^{\alpha+1}} \|X \cdot^k \text{Id}\|_{a(\alpha)}$$

and want to show that this quantity is uniformly bounded when X lies in some $a(\beta)$ -ball for $\alpha > \beta > 0$. In all the following the letter C denotes a positive number depending only on α and β , whose exact value does not matter much and may vary from place to place.

Let us show the claim when $m = 1$, the general case being not very more difficult but hampered with cumbersome notations. We write

$$X(z) := \xi(z) \frac{\partial}{\partial z}.$$

It is easy to check that $X \cdot^k z$ is of the form, for $k > 0$,

$$\begin{aligned} X \cdot^k z &= \xi \frac{\partial}{\partial z} \left(\xi \frac{\partial}{\partial z} (\dots \xi) \right) \\ &= \xi \sum_{\mathbf{j} \in J_k} \prod_{\ell} \left(\frac{\partial^{j_\ell} \xi}{\partial z^{j_\ell}} \right) \end{aligned}$$

with J_k a finite set of multi-indexes $\mathbf{j} \in \mathbb{N}^{k-1}$ of length $|\mathbf{j}| = k - 1$. Proposition 2.24 (3) taken into account we can write

$$\left\| \left(\frac{\partial^{j_\ell} \xi}{\partial z^{j_\ell}} \right) \right\|_{a(\alpha)} \leq C \exp \left(\sigma(\alpha - \beta) j_\ell^{\beta+1/\alpha+1} - (\beta+1) j_\ell \right) \sigma^{(\alpha+1)j_\ell} j_\ell^{(\beta+1)j_\ell} \|\xi\|_{a(\beta)}$$

for some positive σ . We use Lemma 2.25, more specifically the relation $|\mathbf{j}|! \geq \mathbf{j}!$, and relation (2.9) to derive (with the convention $0^0 = 1$)

$$\|\Phi_X\|_{a(\alpha)} \leq \sum_{k=0}^{\infty} (C \|\xi\|_{a(\beta)})^k \sum_{|\mathbf{j}|=k-1} \left(\prod_{\ell} j_\ell^{j_\ell} \right)^{\beta-\alpha}.$$

Define

$$E_{n,k} := \sum_{\mathbf{j} \in \mathbb{N}^k, |\mathbf{j}|=n} \prod_{\ell} j_\ell^{(\beta-\alpha)j_\ell},$$

the n^{th} Taylor coefficient of φ^k , where φ is the entire function

$$\varphi : z \in \mathbb{C} \mapsto \sum_{n \geq 0} z^n n^{(\beta-\alpha)n}.$$

For any $\rho > 0$ Cauchy's formula yields

$$\begin{aligned} \rho^n E_{n,k} &\leq \varphi(\rho)^k \\ E_{k-1,k-1} &\leq \left(\frac{\varphi(\rho)}{\rho} \right)^{k-1}. \end{aligned}$$

Consequently

$$\|\Phi_X\|_{a(\alpha)} \leq \sum_{k=0}^{\infty} \left(C \|\xi\|_{a(\beta)} \right)^k,$$

with convergence of the right-hand side if $\|\xi\|_{a(\beta)}$ is small enough.

We have just proved that the flow is amply bounded near the null vector field. Let us show now that this property implies that of the flow being amply bounded near any point of VF. Since for any $\lambda \in \mathbb{C}$

$$\Phi_{\lambda X}^{\bullet} = \Phi_X^{\lambda \bullet}$$

we can rescale any given vector field X so that λX belongs to a convenient neighborhood of the null vector field. Since the composition $\lambda^* : \Phi_X^{\bullet} \mapsto \Phi_X^{\lambda \bullet}$ is continuous (Proposition 2.24) we derive the sought property of ample boundedness near X . \square

7.2. Solvability of first-order, ordinary differential equations: proof of Corollary C.

In this section we consider a germ, defined near $(0,0) \in \mathbb{C}^2$, of an ordinary differential equation

$$(\diamond) \quad y' = f(x, y)$$

where $f \in \mathbb{C}(\{x, y\})$ is a germ of a meromorphic function. We refer to Remark 6.3 for the description of the analytic structure we put on $\mathbb{C}(\{x, y\})$.

The geometric object underlying this equation is the foliation it defines: local solutions of (\diamond) define the local leaves of the foliation. If (P, Q) is a proper representative of f the singular locus of the germ of a foliation is $P^{-1}(0) \cap Q^{-1}(0)$ and consists of at most the origin. As was discussed in the introduction we are only concerned with singular germs of foliations whose first jet vanishes at $(0,0)$, that is

- ⊗ f is purely meromorphic (i.e. $P(0,0) = Q(0,0) = 0$), see Definition 6.2,
- ⊗ the matrix

$$L(P, Q) := \begin{bmatrix} \frac{\partial P}{\partial x}(0,0) & \frac{\partial P}{\partial y}(0,0) \\ \frac{\partial Q}{\partial x}(0,0) & \frac{\partial Q}{\partial y}(0,0) \end{bmatrix}$$

is zero. In particular it is non-reduced in the sense that $L(P, Q)$ is nilpotent.

Notice that these conditions do not depend on the choice of a proper representative of f . In particular the set ZLP (for Zero Linear Part) of those foliations can be identified with the analytic subset $\mathbb{C}(\{x, y\})_1$ of $\mathbb{C}(\{x, y\})_0$ defined by the vanishing locus of

$$\frac{P}{Q} \in \mathbb{C}(\{x, y\}) \mapsto (J_1(P), J_1(Q)) \in \mathbb{C}[x, y]^2,$$

which is the range of the analytic mapping

$$M_1 : (P, Q) \in \mathbb{C}\{x, y\} \times (\mathbb{C}\{x, y\} \setminus \{0\}) \setminus \mu^{-1}(\infty) \mapsto \frac{P - J_1(P)}{Q - J_1(Q)} \in \mathbb{C}(\{x, y\})_1,$$

and we equip from now ZLP with the analytic structure induced by this map (Definition 3.5).

The usual process when one is faced with such a singularity is to study a foliated complex surface obtained by pulling-back the germ of a foliation by the standard blow-up of the origin. This foliation has «simpler» singularities, and the repetition of the process at any successive singular point eventually stops at the stage where every singular point is reduced [Sei68]. We are interested in those foliations for which the reduction procedure stops after the first step. Let us describe this process in more details. The complex manifold \mathcal{M} obtained by blowing-up the origin of \mathbb{C}^2 is defined by the gluing of

$$\begin{aligned} \mathcal{M}_x &:= \{(x, u) \in \mathbb{C}^2\} \\ \mathcal{M}_y &:= \{(v, y) \in \mathbb{C}^2\} \end{aligned}$$

through the map

$$\begin{aligned} \mathcal{M}_x \setminus \{u = 0\} &\longrightarrow \mathcal{M}_y \setminus \{v = 0\} \\ (x, u) &\longmapsto \left(\frac{1}{u}, xu\right). \end{aligned}$$

The blow-up morphism π is therefore given in these charts by

$$\begin{aligned} \pi_x : \mathcal{M}_x &\longrightarrow \mathbb{C}^2 \\ (x, u) &\longmapsto (x, xu) \end{aligned}$$

and

$$\begin{aligned} \pi_y : \mathcal{M}_y &\longrightarrow \mathbb{C}^2 \\ (v, y) &\longmapsto (vy, y). \end{aligned}$$

The exceptional divisor \mathcal{E} of \mathcal{M} is the projective line $\pi^{-1}(0)$ of self-intersection -1 , and $\mathcal{M} \setminus \mathcal{E}$ is biholomorphic to $\mathbb{C}^2 \setminus \{0\}$. The pulled-back foliation \mathcal{F} is well-defined on a neighborhood of \mathcal{E} in \mathcal{M} , which we still call \mathcal{M} for the sake of simplicity. \mathcal{F} is said dicritic if \mathcal{E} is not a leaf, otherwise \mathcal{F} only admits a finite number of singularities on \mathcal{E} . A finite set of algebraic conditions in the second jet of P and Q can be included to ensure that \mathcal{F} is reduced and non-dicritic:

Proposition 7.2. *The set RND (for Reduced Non-Dicritic) of non-dicritic foliations reduced after one blow-up is Zariski-full and open in ZLP. Besides the following conditions describe a Zariski-full open set $\text{RND}^* \subset \text{RND}$:*

- ⊗ \mathcal{F} admits exactly three distinct singular points on \mathcal{E} and $\xi_* := (0, 0) \in \mathcal{M}_x$ is not among them,
- ⊗ the linear part of the foliation at every singular point admits two non-zero eigenvalues.

We postpone the proof of this property till Subsection 7.2.1 below. Since \mathcal{F} is transverse, outside \mathcal{E} and maybe three regular analytic curves¹⁹, to the fibers of the natural

¹⁹They correspond to the local separatrices near singular points on \mathcal{E} .

projection $\sigma : (x, u) \in \mathcal{M}_x \mapsto u$, we can build the holonomy group $\text{Hol}(f)$ of \mathcal{F} based at ξ_* , which is canonically identified with a subgroup of $\text{Diff}(\mathbb{C}, 0)$ spanned by two generators \mathfrak{h}_1 and \mathfrak{h}_2 . Simply identifying the group $\langle \mathfrak{h}_1, \mathfrak{h}_2 \rangle$ with the pair $(\mathfrak{h}_1, \mathfrak{h}_2) \in \text{Diff}(\mathbb{C}, 0)^2$ will not do, since one can follow a loop in RND^* in order to exchange \mathfrak{h}_1 and \mathfrak{h}_2 . This difficulty is overcome by considering the quotient $\text{Diff}(\mathbb{C}, 0)^2 / \mathfrak{S}_2$ of the pairs (a, b) modulo the action of the 2-symmetric group

$$(a, b) \mapsto (b, a).$$

This quotient space is endowed with the induced analytic structure.

Theorem 7.3. *The map*

$$\begin{aligned} \text{Hol}(\bullet) : \text{RND}^* &\longrightarrow \text{Diff}(\mathbb{C}, 0)^2 / \mathfrak{S}_2 \\ f &\longmapsto \{\mathfrak{h}_1, \mathfrak{h}_2\} \end{aligned}$$

is strongly analytic. Its image contains the open set $\{\mathfrak{h}'_1(0) \notin \exp(i\mathbb{R}) \text{ or } \mathfrak{h}'_2(0) \notin \exp(i\mathbb{R})\}$.

Remark 7.4. The map $\text{Hol}(\bullet)$ is not onto according to a result by Y. IL'YASHENKO [IL'97], which states that every group $\langle \mathfrak{h}_1, \mathfrak{h}_2 \rangle$ generated by non-linearizable, small-divisors-impaired diffeomorphisms (such that $\mathfrak{h}_1 \circ \mathfrak{h}_2$ is equally not linearizable) cannot be realized as the projective holonomy of any germ of a foliation at the origin of \mathbb{C}^2 . The range of $\text{Hol}(\bullet)$ however contains a non-empty open set, so we say for short that it is **quasi-onto**.

The proof of the theorem splits into three parts:

- ⊗ first we recall how the «projective» holonomy group of \mathcal{F} is built in Subsection 7.2.2,
- ⊗ we then study in Subsection 7.2.3 the local strong analyticity of the holonomy mapping with respect to $\frac{P}{Q}$ by restricting ourselves to a small neighborhood $V \subset \text{RND}^*$ of some $\frac{P^0}{Q^0}$,
- ⊗ we perform next in Subsection 7.2.4 its analytic continuation to obtain a globally strongly analytic, quasi-onto map on RND^* .

This theorem is the key to Theorem D: for RND foliations the differential equation (\diamond) is solvable by quadratures (in the sense of Liouville) only if its image by $\text{Hol}(\bullet)$ is solvable. It is indeed classical that a «solution» lying in a Liouvillian extension of the differential field $\mathbb{C}(\{x, y\})$ must have a solvable monodromy, property which translates into a solvable projective holonomy. Using Corollary A we then deduce that the former condition defines a proper analytic set of RND (since the image of $\text{Hol}(\bullet)$ contains a non-empty open set) whose complement is a Zariski-full open set of ZLP and consists only of non-solvable equations.

7.2.1. Proof of Proposition 7.2.

Let $f \in \text{ZLP}$ be given and fix a proper representative (P, Q) of f ; for the sake of clarity we write $P(x, y) := \sum_{p+q>1} P_{p,q} x^p y^q$ and $Q(x, y) := \sum_{p+q>1} Q_{p,q} x^p y^q$. To compute the pull-back \mathcal{F} of the foliation defined by (\diamond) we consider the vector field

$$X_{\mathcal{F}}(x, y) := Q(x, y) \frac{\partial}{\partial x} + P(x, y) \frac{\partial}{\partial y}$$

whose integral curve $t \in (\mathbb{C}, 0) \mapsto \Phi_X^t(x, y)$ coincides with the local leaf of \mathcal{F} passing through (x, y) . Its pull-back by *e.g.* π_x is given by

$$\pi_x^* X_{\mathcal{F}} = xQ(x, xu) \frac{\partial}{\partial x} + (P(x, xu) - uQ(x, xu)) \frac{\partial}{\partial u}$$

whose components belong to $\mathbb{C}\{x, u\}$. The affine trace $\{x = 0\}$ of \mathcal{E} on \mathcal{M}_x is a line of singularities for $\pi_x^* X_{\mathcal{F}}$, since both of its components belong to the ideal $\mathcal{I}(x^2) < \mathbb{C}\{x, u\}$ spanned by x^2 . We indeed have

$$\begin{aligned} xQ(x, xu) &= x^3(Q_{0,2}u^2 + Q_{1,1}u + Q_{2,0}) + O(x^4) \\ uQ(x, xu) - P(x, xu) &= x^2(Q_{0,2}u^3 + (Q_{1,1} - P_{0,2})u^2 + (Q_{2,0} - P_{1,1})u - P_{2,0}) + O(x^3). \end{aligned}$$

If \mathcal{F} is dicritic then

$$uQ(x, xu) - P(x, xu) \in \mathcal{I}(x^3).$$

If we require that

$$(7.1) \quad Q_{0,2} = 0$$

does not hold then we ensure that the foliation is non-dicritic. Notice that \mathcal{F} is given in the chart \mathcal{M}_x by the holomorphic vector field with isolated singularities

$$(7.2) \quad X := \frac{1}{x^2} \pi^* X_{\mathcal{F}}.$$

This vector field admits three singularities (counted with multiplicity) on \mathcal{E} whose affine coordinates are given by the roots of the polynomial

$$(7.3) \quad \phi(u) := -Q_{0,2}u^3 + (P_{0,2} - Q_{1,1})u^2 + (P_{1,1} - Q_{2,0})u + P_{2,0}.$$

The discriminant of ϕ is a polynomial in the variables $(P_n, Q_n)_{|n|=2}$: requiring that it vanishes describes the analytic set defined by

$$(7.4) \quad \begin{aligned} &Q_{0,2}(27P_{0,2}Q_{0,2} - 4(Q_{2,0} - P_{1,1})^3 - 18(Q_{1,1} - P_{0,2})(Q_{2,0} - P_{1,1})P_{2,0}) \\ &+ (Q_{1,1} - P_{0,2})^2((Q_{2,0} - P_{1,1})^2 + 4(Q_{1,1} - P_{0,2})P_{2,0}) = 0. \end{aligned}$$

Under the condition that this quantity does not vanish we particularly derive that the linearized part of \mathcal{F} at any singular point of $\mathcal{E} \cap \mathcal{M}_x$, the matrix

$$\begin{bmatrix} Q_{0,2}u^2 + Q_{1,1}u + Q_{2,0} & \star \\ 0 & \phi'(u) \end{bmatrix},$$

cannot be nilpotent. The condition $Q_{0,2} \neq 0$ also ensures that \mathcal{F} has no singularity at $(0, \infty) = \mathcal{E} \setminus \mathcal{M}_x$, and the foliation is reduced. As announced above, if we assume that

$$(7.5) \quad P_{2,0} = 0$$

is not fulfilled then 0 is never a root of ϕ .

The union of the analytic sets defined by each one of the three conditions 7.1, 7.4 and 7.5 forms a proper analytic subset Ω of $\{(P, Q) : J_1(P) = J_1(Q) = 0\}$. Since any other proper representative of $\frac{P}{Q}$ defines the same foliation on a (maybe smaller) neighborhood of \mathcal{E} , in particular the roots (and their multiplicity) of the polynomial ϕ remain unchanged. Therefore Ω corresponds to a proper analytic set of ZLP, whose complement RND^* is Zariski-full.

7.2.2. Building the holonomy.

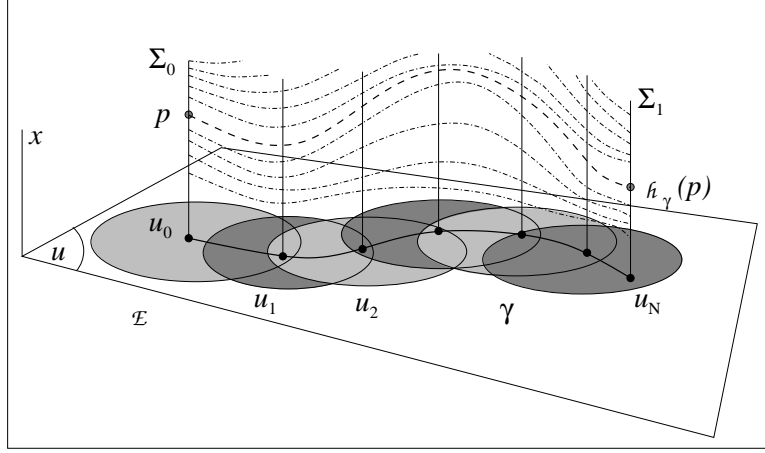


FIGURE 7.1. Computing the holonomy by composing local flows.

Choose a vector field $X = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial u}$ holomorphic on a neighborhood of \mathcal{E} and a path $\gamma : [0, 1] \rightarrow \mathcal{E} \setminus \text{Sing}(X)$ of base-point $\xi_* = (0, u_0)$, such that X is tangent to \mathcal{E} . We require that X be transverse to the lines $\{u = \text{cst}\}$ outside its singular locus and define

$$\hat{X} := \frac{a}{b} \frac{\partial}{\partial x} + \frac{\partial}{\partial u},$$

whose restriction to \mathcal{E} is holomorphic outside $\text{Sing}(X)$. It defines the same foliation as X . The image of γ does not meet any singularity of X therefore at each point $\gamma(s)$ there exists a polydisc $\Sigma_s \times D_s$ on which the mapping

$$(7.6) \quad h_s : (x, u) \mapsto \Phi_{\hat{X}}^u(x, u(s)) = (\eta_s(x, u), u + u(s))$$

is holomorphic. From the open cover $(\Sigma_s \times D_s)_{s \in [0, 1]}$ of the image of γ we can extract a finite sub-cover corresponding to discs centered at points $\gamma(s_\ell)$ for $0 \leq \ell \leq N$ with

- ⊗ $s_0 = 0$ and $s_N = 1$,
- ⊗ $s_{\ell+1} > s_\ell$,
- ⊗ $\gamma(s_{\ell+1}) \in \{0\} \times D_{s_\ell}$.

We say that the collection of times $(s_\ell)_{0 \leq \ell \leq N}$ is **adapted** to (X, γ) , and write

$$\gamma(s_\ell) = (0, u_\ell).$$

Since each h_s is holomorphic there exists $\varepsilon > 0$ such that the mapping

$$\begin{aligned} \mathfrak{h}_\gamma : \Sigma_0 \times \{u_0\} &\longrightarrow \Sigma_1 \times \{u_N\} \\ p = (x, u_0) &\longmapsto \mathfrak{h}_\gamma(p) := h_{s_{N-1}} \left(h_{s_{N-2}} \left(\cdots h_0(x, u_1 - u_0) \cdots \right), u_N - u_{N-1} \right) \end{aligned}$$

is a well-defined mapping. We call this germ of a function the **holonomy of X** (or of the underlying foliation) **along γ with respect to the projection $(x, u) \mapsto u$** . Although \mathfrak{h}_γ should depend on the particular choice of an adapted collection this is not so, as is asserted in the well-known folk theorem:

Theorem 7.5. (Holonomy theorem) *Let a vector field X be holomorphic on a neighborhood of \mathcal{E} , transverse to the fibers of the projection $(x, u) \mapsto u$ outside the singular locus. For every path γ of $\mathcal{E} \setminus \text{Sing}(X)$ the following properties hold:*

- (1) *the holonomy \mathfrak{h}_γ of X along γ depends only on the homotopy class of γ in $\mathcal{E} \setminus \text{Sing}(X)$,*
- (2) *\mathfrak{h}_γ is a germ of a biholomorphism of $\Sigma := \Sigma_0 \times \{u_0\}$ and $\mathfrak{h}_\gamma(0, u_0) = (0, u_N)$,*
- (3) *if ξ_* is a given base-point in $\mathcal{E} \setminus \text{Sing}(X)$ the mapping*

$$\mathfrak{h}_\bullet : \pi_1(\mathcal{E} \setminus \text{Sing}(X), \xi_*) \longrightarrow \text{Diff}(\Sigma, \xi_*) \simeq \text{Diff}(\mathbb{C}, 0)$$

$$\gamma \longmapsto \mathfrak{h}_\gamma$$

*is a group morphism. Its image is the **holonomy group** of X (or better, of the induced foliation).*

7.2.3. Local study.

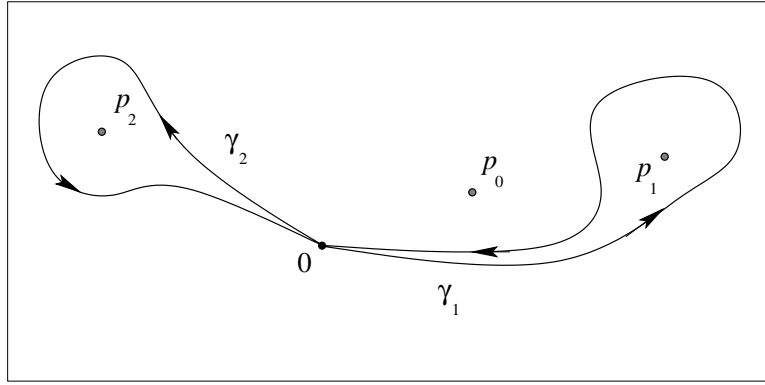


FIGURE 7.2. Generators of the fundamental group of $\mathbb{P}_1(\mathbb{C}) \setminus \{p_0, p_1, p_2\}$.

We continue to work in the affine coordinates (x, u) through the chart π_x and we fix the base-point ζ_* whose coordinates in \mathcal{M}_x is $(0, 0)$. For $\frac{P}{Q} \in \text{RND}^*$ we fix a proper representative (P, Q) and consider the holomorphic vector field X defined in (7.2) by

$$X(x, u) := \frac{Q(x, xu)}{x} \frac{\partial}{\partial x} + \frac{P(x, xu) - uQ(x, xu)}{x^2} \frac{\partial}{\partial u}.$$

Its integral curves define the foliation \mathcal{F} . The (isolated) singular set $\text{Sing}(X)$ of X coincides with that of \mathcal{F} , and because $X(0, u) = \phi(u) \frac{\partial}{\partial u}$ the singularities located on \mathcal{E} are given by the three simple roots of the polynomial ϕ defined by (7.3). It is obvious that the map

$$(P, Q) \in \text{Quot}^{-1}(\text{RND}^*) \longmapsto X \in \text{VF}$$

is strongly analytic, and from now on we only work with vector fields X instead of foliations \mathcal{F} .

In the sequel the superscript «0» refers to objects computed from a fixed proper representative (P^0, Q^0) of a meromorphic germ belonging to RND^* . We also fix a particular set of generators γ_1, γ_2 of the fundamental group $\pi_1(\mathcal{E} \setminus \text{Sing}(X^0), \zeta_*)$, where

$\text{Sing}(X^0) = \{p_0^0, p_1^0, p_2^0\}$ is the singular locus of X^0 . For instance one can choose two simple loops, each turning once around the singular point p_j^0 . Because RND^* does not cross the vanishing locus of the discriminant (7.4) there exists a biholomorphic map in the variables $(P_n)_{|n|=2}$ and $(Q_n)_{|n|=2}$, well-defined in a neighborhood of $(P_n^0, Q_n^0)_{|n|=2}$, whose range is the collection of roots of ϕ . It is in particular possible to find a neighborhood U' of $(P_n^0, Q_n^0)_{|n|=2} \in \mathbb{C}^6$ so that the roots of ϕ does not cross the images of γ_1 and γ_2 for values of the second jet of (P, Q) in U' . Let $V' := J_2^{-1}(U')$ be the corresponding neighborhood of (P^0, Q^0) .

Take $(P, Q) \in V'$; by construction γ_1 and γ_2 generates $\pi_1(\mathcal{E} \setminus \text{Sing}(\mathcal{F}), \zeta_*)$. Therefore the holonomy group of \mathcal{F} is spanned by $\mathfrak{h}_1 := \mathfrak{h}_{\gamma_1}$ and $\mathfrak{h}_2 := \mathfrak{h}_{\gamma_2}$. In the neighborhood V' of (P^0, Q^0) it is possible to keep track of the pair $(\mathfrak{h}_1, \mathfrak{h}_2)$, with ordering, since the paths γ_1 and γ_2 do not vary. The mapping

$$(P, Q) \in V' \mapsto (\mathfrak{h}_1, \mathfrak{h}_2) \in \text{Diff}(\mathbb{C}, 0)^2$$

is thereby well-defined. Let us show it is strongly analytic on a maybe smaller $V \subset V'$. The way the holonomy is constructed guarantees that it is Quot-invariant, meaning that the map $\frac{P}{Q} \in \text{Quot}(V') \mapsto (\mathfrak{h}_1, \mathfrak{h}_2)$ also is well-defined and strongly analytic.

Lemma 7.6. *Let γ be a loop of $\mathcal{E} \setminus \text{Sing}(X^0)$ with base-point ξ_* and consider a sequence $(s_\ell)_{0 \leq \ell \leq N}$ adapted to (X^0, γ) . There exists a neighborhood $V \subset V'$ of (P^0, Q^0) such that the sequence $(s_\ell)_{0 \leq \ell \leq N}$ continues to be adapted to (X, γ) .*

Before we prove this lemma we describe briefly how to derive Theorem 7.3 from it. Consider a neighborhood $V \subset V'$ of (P^0, Q^0) such that γ_1 and γ_2 both admit an adapted covering uniform with respect to $(P, Q) \in V$. Each holonomy generator \mathfrak{h}_1 and \mathfrak{h}_2 is obtained by composing N local flows, and because N does not depend on (P, Q) this composition is strongly analytic with respect to (P, Q) whenever each flow is. But the former statement precisely is the content of Theorem (7.1), which ends our proof.

Proof. Since this property is local, and because there is only finitely many times $(s_k)_{0 \leq k < N}$, we prove it only for the first point $u_0 = 0$. We can request without loss of generality that $\eta > |u_1|$ be chosen so small that

$$\sup_{|u|=\eta} |\phi(u) - \phi(0)| < \frac{1}{4} |\phi(0)|,$$

with a suitable, fixed choice of a thinner adapted covering of the image of γ . Since ϕ depends only on the second jet of (P, Q) we can find a neighborhood $V \subset V'$ such that the above estimate holds for fixed η and all $(P, Q) \in V$.

For $(P, Q) \in V$ let us introduce

$$\begin{aligned} \hat{X}(x, u) &:= \xi(x, xu) x \frac{\partial}{\partial x} + \frac{\partial}{\partial u} \\ \xi(x, u) &:= \frac{Q(x, xu)}{P(x, xu) - uQ(x, xu)} = \frac{Q(x, xu)}{\phi(u) + x(f(x, xu) + ug(x, xu))}. \end{aligned}$$

Because of the form of the arguments in the functions making ξ up, we can always find $\varepsilon > 0$ sufficiently small so that ξ is holomorphic on $\varepsilon \mathbb{D} \times \eta \mathbb{D}$, in particular because there

exists $\varepsilon > 0$ such that for all $|x| < \varepsilon$ and all $|u| \leq \eta$ we have

$$\left| \phi(u) - \phi(0) + x(f(x, xu) + u g(x, xu)) \right| < \frac{1}{2} |\phi(0)|.$$

Let us define

$$C := \|\xi\|_{\varepsilon \mathbb{D} \times \eta^0 \mathbb{D}}.$$

We need to prove that the parameterization of the integral curves of \hat{X} by the flow, starting from points of $\{u = 0, |x| < \varepsilon\}$, is holomorphic on a domain which contains the disc $\eta \mathbb{D}$ for a maybe smaller $\varepsilon > 0$. Let $t \mapsto z(t, x, u)$ be the x -component of the flow $\Phi_{\hat{X}}^t(x, 0)$, i.e. the solution of

$$\begin{cases} \dot{z}(t) &= z(t) \xi(z(t), tz(t)) \\ z(0) &= x \end{cases}.$$

Setting $t := e^{i\theta} \tau$ with $\tau, \theta \in \mathbb{R}_{\geq 0}$ and differentiating $|z|^2 = z\bar{z}$ with respect to τ brings the equation

$$\frac{d|z(t)|^2}{d\tau} = 2|z(t)|^2 \Re(e^{i\theta} \xi(z(t), tz(t))),$$

therefore

$$|z(t)| \leq |x| \exp(C|t|).$$

In particular the integral curve $t \mapsto \Phi_{\hat{X}}^t(x, 0)$ does not escape the polydisc $\varepsilon \mathbb{D} \times \eta \mathbb{D}$ provided $|x| < \varepsilon \exp(-\eta C)$ and $|t| < \eta$, which means that the local holonomy h_0 defined in (7.6) is holomorphic on $\Sigma_0 \times D_0$ where $\Sigma_0 := \varepsilon \exp(-\eta C) \mathbb{D}$ and $D_0 := \eta \mathbb{D}$. Since η depends only on V and not on a particular choice of $(P, Q) \in V$ the result is proved. \square

7.2.4. $\text{Hol}(\bullet)$ is globally analytic and quasi-onto.

The local strong-analyticity we just established implies that $\text{Hol}(\bullet)$ is a well-defined strongly analytic map on the universal covering $\mathcal{C} : \widehat{\text{RND}}^* \rightarrow \text{RND}^*$. We perform the analytic continuation of $\text{Hol}(\bullet)$ by deforming continuously the loops γ_1 and γ_2 so that no singular point ever crosses the image of any loop. The fiber of \mathcal{C} thereby corresponds to foliations with same singular points on the exceptional divisor but with generators of the holonomy group that may not appear in the same order. Yet the group generated is the same, so the mapping $\text{Hol}(\bullet)$ is well-defined as a map from RND^* to the orbifold quotient $\text{Diff}(\mathbb{C}, 0)^2 / \mathbb{S}_2$.

To show the holonomy map is quasi-onto we use a result by A. LINS-NETO:

Theorem 7.7. [Lin87] *Any finitely-generated subgroup $\langle \mathfrak{h}_1, \dots, \mathfrak{h}_n \rangle$ of $\text{Diff}(\mathbb{C}, 0)$ such that $\bigcirc_{k=1}^n \mathfrak{h}_k = \text{Id}$ and at least one generator is analytically linearizable, can be obtained as the projective holonomy group computed along the exceptional divisor of a foliation reduced after one blow-up.*

The construction, based on Grauert's theorem, also ensures that the foliation realizing a given subgroup $\langle \mathfrak{h}_1, \mathfrak{h}_2 \rangle$ belongs to RND^* . Besides it is well known that if $\frac{1}{2i\pi} \log \mathfrak{h}_1'(0)$ or $\frac{1}{2i\pi} \log \mathfrak{h}_2'(0)$ is not real then \mathfrak{h}_1 or \mathfrak{h}_2 is hyperbolic and therefore linearizable.

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