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## SPECIAL BENT AND NEAR-BENT FUNCTIONS

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ABSTRACT. Starting from special near-bent functions in dimension  $2t - 1$  we construct bent functions in dimension  $2t$  having a specific derivative. We deduce new families of bent functions.

### 1. INTRODUCTION

Let  $\mathbb{F}_2$  be the finite field of order 2. An  $m$ -boolean function (or boolean function in  $m$  dimensions) is a map  $F$  from  $\mathbb{F}_2^m$  to  $\mathbb{F}_2$ .

Bent functions are the boolean functions whose Fourier coefficients have constant magnitude. They were introduced by Rothaus in [8]. An  $m$ -boolean function  $F$  is bent if all its Fourier coefficients are in  $\{-2^{m/2}, 2^{m/2}\}$ . Bent functions are of interest for Coding Theory, Cryptology and well-correlated binary sequences. For example, they have the maximum possible Hamming distance to the set of affine boolean functions. They were the topic of a lot of works (see [2, 3, 6, 7, 9, 10]) but the complete classification of bent functions and other questions are still open.

By definition, a  $m$ -boolean function  $F$  is near-bent if all its Fourier coefficients are in  $\{-2^{(m+1)/2}, 0, 2^{(m+1)/2}\}$ . Since the Fourier coefficients are in  $\mathbb{Z}$  the bent functions in  $m$  dimensions exist only when  $m$  is even and near-bent functions in  $m$  dimensions exist only when  $m$  is odd.

1.1. A TWO-VARIABLE REPRESENTATION. Assume  $m = 2t$ . We identify  $\mathbb{F}_2^{2t}$  with  $\mathbb{F}_{2^{2t}}$  and  $\mathbb{F}_{2^{2t}}$  with

$$\mathbb{F}_{2^{2t-1}} \times \mathbb{F}_2 = \{X = (u, \nu) \mid u \in \mathbb{F}_{2^{2t-1}}, \nu \in \mathbb{F}_2\}.$$

This decomposition can be used for a two-variable representation of  $(2t)$ -boolean functions as follows

$$\mathbb{F}_{2t}^{(0)} = \{(u, 0) \mid u \in \mathbb{F}_{2^{2t-1}}\} \quad \text{and} \quad \mathbb{F}_{2t}^{(1)} = \{(u, 1) \mid u \in \mathbb{F}_{2^{2t-1}}\}.$$

The restrictions of a  $(2t)$ -boolean function  $F$ , respectively to  $\mathbb{F}_{2t}^{(0)}$  and to  $\mathbb{F}_{2t}^{(1)}$  induce two  $(2t - 1)$ -boolean functions  $f_0$  and  $f_1$  defined by  $f_0(u) = F(u, 0)$  and  $f_1(u) = F(u, 1)$ .

The two-variable representation (TVR) of  $F$  is then defined by

$$(1) \quad \phi_F(x, y) = (y + 1)f_0(x) + yf_1(x).$$

Note that  $F(u, 0) = f_0(u) = \phi_F(u, 0)$  and  $F(u, 1) = f_1(u) = \phi_F(u, 1)$ . Hence if  $X = (u, \nu)$  then  $F(X) = \phi_F(u, \nu)$ .

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It can be proved that if  $F$  is a  $(2t)$ -bent function then  $f_0$  and  $f_1$  are  $(2t - 1)$ -near-bent functions (see 3.1.4). This leads to considering the following inverse problem: construct  $(2t)$ -bent functions from  $(2t - 1)$ -near-bent functions. This was the purpose of [7] and in a certain sense of [10] where connections with cyclic codes were established.

Using the above representation of  $\mathbb{F}_{2^{2t}}$ , it could be easily checked that  $f_0 + f_1$  is the derivative of  $F$  with respect to  $(0, 1)$  (see next section for the definition of the derivative). A possible way to study bent functions is to classify them regarding the degree (as a boolean function) of this derivative. The most simple case to consider is when this degree is one. In the present paper we are concerned with bent functions such that  $f_0 + f_1 = tr + \xi$  where  $tr$  is the trace of  $\mathbb{F}_{2^{2t-1}}$  and  $\xi \in \{0, 1\}$ . We introduce the new notion of pseudo-duality and as application we present new families of bent functions. By the way we present a generalization of a result on the Gold function (Lemma 15 in Section 3.2).

## 2. RESULTS

The proofs of the Theorems of this section will be given in Section 3 while the other results are proven in the present section.

First recall some usual definitions of a  $m$ -boolean function  $F$ .

- If  $e \in \mathbb{F}_2^m$  then the Derivative of  $F$  with respect to  $e$  is the  $m$ -boolean function  $D_e F$  defined by

$$D_e F(X) = F(X) + F(X + e).$$

**Remark.** In this paper we consider the following special cases. If  $F$  is a  $(2t)$ -boolean function whose two-variable representation is given in (1) then direct calculation shows that

$$D_{(0,1)}(F) = f_0 + f_1.$$

If  $f$  is a  $(2t - 1)$ -boolean function then

$$D_1 f(x) = f(x + 1) + f(x).$$

- The Fourier transform (or the Walsh transform)  $\hat{F}$  of  $F$  is the map from  $\mathbb{F}_2^m$  into  $\mathbb{Z}$  defined by

$$\hat{F}(v) = \sum_{X \in \mathbb{F}_2^m} (-1)^{F(X) + \langle v, X \rangle},$$

where  $\langle, \rangle$  denotes any inner product of  $\mathbb{F}_2^m$  over  $\mathbb{F}_2$ .  $\hat{F}(v)$  is called the Fourier coefficient of  $v$ .

**Remark.** The set of  $\hat{F}(v)$  when  $v$  runs through  $\mathbb{F}_2^m$  is independent of the choice of the inner product  $\langle, \rangle$ .

- $F$  is bent if all its Fourier coefficients are in  $\{-2^{m/2}, 2^{m/2}\}$ .  $F$  is near-bent if all its Fourier coefficients are in  $\{-2^{(m+1)/2}, 0, 2^{(m+1)/2}\}$ .

- If  $m = 2t$  and if  $F$  is a bent function then the dual  $\tilde{F}$  of  $F$  is the  $(2t)$ -boolean function defined by

$$\hat{F}(v) = (-1)^{\tilde{F}(v)} 2^t,$$

where  $\hat{F}$  is the Fourier transform of  $F$ . It is well-known, and easy to prove that the dual of a bent function is a bent function (see [3] or [8]).

- The algebraic degree, or more simply “the degree” of an  $m$ -boolean function is the degree of its  $m$ -variable polynomial representation. If it is expressed as  $Tr(\pi(X))$  where  $Tr$  is the trace function of  $\mathbb{F}_{2^m}$  then its degree is the maximum

of the binary weight of the monomial exponents of  $\pi(X)$ . Recall that the binary weight of an integer is the number of non-zero coefficients of the binary expansion of this integer.

See References section for details on the previous definitions and results.

**2.1. MAIN THEOREMS.** From now on, we use the definitions and notations of the introduction. Thus  $\mathbb{F}_2^{2t}$  is identified with  $\mathbb{F}_{2^{2t}}$  and  $\mathbb{F}_{2^{2t}}$  with  $\mathbb{F}_{2^{2t-1}} \times \mathbb{F}_2$ . The two-variable representation (TVR) of a  $(2t)$ -boolean function  $F$  is defined by  $\phi_F(x, y) = (y + 1)f_0(x) + yf_1(x)$  where  $f_0$  and  $f_1$  are defined in Section 1. Obviously,  $\phi_{F_1 + F_2} = \phi_{F_1} + \phi_{F_2}$ .

**Definition.**  $f_0$  and  $f_1$  are called the components of  $F$ .

**Notation.** The trace of  $\mathbb{F}_{2^{2t-1}}$  is denoted by  $tr$  and defined by

$$tr(x) = \sum_{i=0}^{2t-2} x^{2^i}.$$

The next theorem sets a condition on a near-bent function to be the first component of a bent function.

**Theorem 1.** *Let  $f_0$  be a  $(2t - 1)$ -near-bent function. If the derivative  $D_1f_0$  is a constant function then the  $(2t)$ -boolean function  $F$  such that*

$$\phi_F(x, y) = (y + 1)f_0(x) + yf_1(x) \quad \text{with} \quad f_0 + f_1 = tr$$

*is a bent function.*

We now present a pseudo-reciprocal theorem.

**Theorem 2.** *Let  $F$  be a bent function and  $\tilde{F}$  be its dual bent function with*

$$\phi_F(x, y) = (y + 1)f_0(x) + yf_1(x) \quad \text{and} \quad \phi_{\tilde{F}}(x, y) = (y + 1)\tilde{f}_0(x) + y\tilde{f}_1(x).$$

*If  $f_0 + f_1 = tr$ , then  $D_1\tilde{f}_0 = 0$  and  $D_1\tilde{f}_1 = 1$ .*

The converse of Theorem 1 is not true. In other words, it is not true that if  $f_0 + f_1 = tr$  then  $D_1f_0$  is a constant function, as it will be seen in Example 2. However, there is a special case given by the next corollary which follows immediately from Theorem 2.

**Corollary 3.** *With the above notation, let  $F$  be a bent such that  $f_0 + f_1 = tr$ . If  $F$  is self dual, say  $F = \tilde{F}$ , then  $D_1f_0 = 0$ .*

Relations between the components of the dual  $\tilde{F}$  of a bent function  $F$  such that  $f_0 + f_1 = tr$  are now presented in the next result.

**Theorem 4.** *Let  $F$  be a bent function and let  $\tilde{F}$  be its dual bent function with*

$$\phi_F(x, y) = (y + 1)f_0(x) + yf_1(x) \quad \text{and} \quad \phi_{\tilde{F}}(x, y) = (y + 1)\tilde{f}_0(x) + y\tilde{f}_1(x).$$

*Assume  $f_0 + f_1 = tr$ . Let  $\hat{f}_0$  be the Fourier transform of  $f_0$ . Define  $\mathcal{S} = \{v \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(v) = -2^t\}$  and  $\mathcal{S}_1 = \{u + 1 \mid u \in \mathcal{S}\}$ . Let  $\mathcal{G} = \{v \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(v) = 0\}$ . Let  $g$  be the characteristic function of  $\mathcal{G}$ .*

- 1) *The support of  $\tilde{f}_0$  is  $\mathcal{S} \cup \mathcal{S}_1$ ;*
- 2)  *$\tilde{f}_1(x) = \hat{f}_0(x) + g(x)$ .*

The next theorem states properties of a bent function in the case when the hypothesis of Theorems 1 and 2 are both satisfied.

**Theorem 5.** Let  $H$  be a bent function and let  $\tilde{H}$  be its dual bent function with

$$\phi_H(x, y) = (y + 1)h_0(x) + yh_1(x) \quad \text{and} \quad \phi_{\tilde{H}}(x, y) = (y + 1)\tilde{h}_0(x) + y\tilde{h}_1(x).$$

Assume  $h_0 + h_1 = tr$ .

- a) If  $D_1h_0 = 0$  then  $\tilde{h}_0 + \tilde{h}_1 = tr$ ;
- b) If  $D_1h_0 = 1$  then  $\tilde{h}_0 + \tilde{h}_1 = tr + 1$ .

2.2. PSEUDO-DUALITY. The previous results lead to the introduction of a new definition (notations are above).

**Definition 6.** Let  $G$  be a  $(2t)$ -bent function and let  $\tilde{G}$  be its dual bent function with  $\phi_{\tilde{G}}(x, y) = (y + 1)\tilde{g}_0(x) + y\tilde{g}_1(x)$ . The Pseudo-duals of  $G$  are the two  $(2t)$ -boolean function  $\bar{G}_0$  and  $\bar{G}_1$  defined by

$$\begin{aligned} \phi_{\bar{G}_0}(x, y) &= (y + 1)\tilde{g}_0(x) + y(\tilde{g}_0(x) + tr(x)); \\ \phi_{\bar{G}_1}(x, y) &= (y + 1)\tilde{g}_1(x) + y(\tilde{g}_1(x) + tr(x)). \end{aligned}$$

The meaning of this definition is given by the next theorem

**Theorem 7.** Define the following two conditions on a  $(2t)$ -bent functions  $\mathcal{G}$  with  $\phi_{\mathcal{G}}(x, y) = (y + 1)g_0(x) + yg_1(x)$ .

- (T)  $g_0 + g_1 = tr + \xi$  with  $\xi \in \{0, 1\}$ ;
- (C)  $D_1g_0 = 0$ .

If  $F$  is a  $(2t)$ -bent function meeting condition (T) then

- A) The pseudo-duals  $\bar{F}_0$  and  $\bar{F}_1$  are bent functions;
- B) The dual  $\tilde{\bar{F}}_0$  of  $\bar{F}_0$  meets (C) and (T) with  $\xi = 0$ ;
- C) The dual  $\tilde{\bar{F}}_1$  of  $\bar{F}_1$  meets (C) and (T) with  $\xi = 1$ .

2.3. NEW FAMILIES OF BENT FUNCTIONS. Let  $\mathcal{F}$  be a family of  $(2t)$ -boolean functions, define  $\bar{\mathcal{F}}_0 = \{\bar{F}_0 \mid F \in \mathcal{F}\}$  and  $\bar{\mathcal{F}}_1 = \{\bar{F}_1 \mid F \in \mathcal{F}\}$ .

By applying the previous theorem, if  $\mathcal{F}$  is a family of bent functions meeting (T), then  $\bar{\mathcal{F}}_0$  and  $\bar{\mathcal{F}}_1$  are new families of bent functions.

This is the case in the next proposition by using a family introduced in [7, Theorem 9].

2.3.1. *The Kasami-Welch example.* This definition comes from the description of the near-bent function  $f_0$  introduced in this example which is a classical object in the theory of boolean functions.

**Proposition 8.** Let  $t, s, d$  be integers such that

- $2t - 1$  is not divisible by 3,  $3s \equiv \pm 1 \pmod{2t - 1}$ ;
- $s < t$ ,  $d = 4^s - 2^s + 1$ .

Let  $F$  be the  $(2t - 1)$ -boolean function with

$$\phi_F(x, y) = (y + 1)tr(x^d) + ytr(x^d + x).$$

The assertions A), B), C) of Theorem 7 hold for  $F$ .

*Proof.* In [7, Theorem 9] it is proved that  $F$  is a bent function. Since  $F$  satisfies (T) then Theorem 7 applies and gives the expected result.  $\square$

**Remark.** It is proved in [4] that in the Kasami-Welch case, if  $g$  is the function introduced in Theorem 4 then  $g(x) = 1 + tr(x^{2^s+1})$ .

**Example.**  $t = 4, s = 2$ .

$$F : f_0(x) = tr(x^{13}), f_1(x) = f_0(x) + tr(x),$$

$$\tilde{F} : \tilde{f}_0(x) = tr(x^7 + x^{11} + x^{19} + x^{21}), \tilde{f}_1(x) = \tilde{f}_0(x) + tr(x^5 + 1).$$

New bent functions

$$\bar{F}_0 : \bar{f}_0^{(0)}(x) = \tilde{f}_0(x), \bar{f}_1^{(0)}(x) = \tilde{f}_0(x) + tr(x),$$

$$\bar{F}_1 : \bar{f}_0^{(1)}(x) = \tilde{f}_1(x), \bar{f}_1^{(1)}(x) = \tilde{f}_1(x) + tr(x),$$

$$\tilde{\tilde{F}}_0 : \tilde{\tilde{f}}_0^{(0)}(x) = tr(x + x^3 + x^7 + x^{11} + x^{19} + x^{21}),$$

$$\tilde{\tilde{f}}_1^{(0)}(x) = \tilde{\tilde{f}}_0^{(0)}(x) + tr(x),$$

$$\tilde{\tilde{F}}_1 : \tilde{\tilde{f}}_0^{(1)}(x) = tr(1 + x^5 + x^7 + x^9 + x^{11} + x^{19} + x^{21}),$$

$$\tilde{\tilde{f}}_1^{(1)}(x) = \tilde{\tilde{f}}_0^{(1)}(x) + tr(x + 1).$$

**Remark.** In the above example  $D_1 f_0$  is not a constant function.

2.3.2. *The quadratic case.* In the next proposition we study the case where  $f_0$  is quadratic and such that  $f_0(x) = tr(\pi(x))$  where all the coefficients of  $\pi(x)$  are in  $\mathbb{F}_2$ .

**Proposition 9.** *Let  $f_0$  be a  $(2t - 1)$ -near-bent function such that*

- $f_0(x) = tr(\pi(x))$  with  $\pi(x) \in \mathbb{F}_2[x]$ ;
- The degree of  $f_0$  is 2.

1) *Then the  $(2t)$ -boolean function  $F$  such that*

$$\phi_F(x, y) = (y + 1)f_0(x) + yf_1(x) \quad \text{with} \quad f_1(x) = f_0(x) + tr(x)$$

*is a bent function.*

- 2) *The dual  $\tilde{F}$  of  $F$  meets  $(\mathcal{T})$ .*
- 3) *The assertions A), B), C) of Theorem 7 hold for  $F$ .*

*Proof.* 1) First notice that  $r$  and  $s$  are in  $\mathbb{N}$ , then for a convenient exponent  $l$ ,

$$tr(x^{2^r} + x^{2^s}) = tr[(x^{2^r} + x^{2^s})^{2^l}] = tr(x^{2^{j+1}})$$

for some  $j$ . It follows that we can express  $f$  as  $f_0(x) = \xi + \sum_J tr(x^{2^j+1})$  where  $J$  is a subset of  $\mathbb{N}$  with  $J \neq \{0\}$  and  $\xi \in \mathbb{F}_2$ . Now, it is easy to check that  $x^{2^j+1} + (x + 1)^{2^j+1} = x^{2^j} + x + 1$  and thus  $tr(x^{2^j+1}) + tr((x + 1)^{2^j+1}) = 1$ . Therefore  $f_0(x) + f_0(x + 1) = \sum_J 1$  and  $f_0(x) + f_0(x + 1) = 0$  if  $|J|$  is even and  $f_0(x) + f_0(x + 1) = 1$  if  $|J|$  is odd. In both cases  $D_1 f_0$  is a constant function then Part 1) is a consequence of Theorem 1.

2) Since  $f_0 + f_1 = tr$  and  $D_1 f_0$  is a constant function then Theorem 5 applies.

3) This is a direct consequence of Theorem 7. □

**Remark.** Observe that for any  $(2t)$ -boolean function  $G$  whose components are  $g_0$  and  $g_1$  the TVR of  $G$   $\phi_G(x, y)$  can be rewritten as

$$\phi_G(x, y) = y(g_0(x) + g_1(x)) + g_0(x).$$

If  $G = F$  then  $\phi_F(x, y) = y tr(x) + f_0(x)$ . Since the degree of  $f_0$  is 2 we deduce that the degree of  $F$  also is 2. Similarly for  $\tilde{F}$ :  $\phi_{\tilde{F}}(x, y) = y(\tilde{f}_0(x) + \tilde{f}_1(x)) + \tilde{f}_0(x)$ . We know from [6, Lemma 2.5] that if the degree of  $F$  is 2 then  $\tilde{F}$  also has degree 2. From  $\phi_{\tilde{F}}(x, y) = y(\tilde{f}_0(x) + \tilde{f}_1(x)) + \tilde{f}_0(x)$  we deduce that degree of  $\tilde{f}_0$  is 2. It follows that for every bent function among  $F, \tilde{F}, \bar{F}_0, \bar{F}_1, \tilde{\tilde{F}}_0, \tilde{\tilde{F}}_1$ , the degree is 2 and

then the components have degree 2. Obviously,  $\bar{F}_0 = \tilde{F}$ ,  $\tilde{\bar{F}}_0 = F$ . It is shown in [7] that if  $\deg f_0 = 2$  there exists  $e$  in  $\mathbb{F}_{2^{2t-1}}$  such that  $f_0(x) + f_1(x) = \text{tr}(ex)$ . Another way to prove 1) is to show that  $e = 1$  when  $\pi(x)$  is binary.

**Example.**  $t = 4$ .

$$F : f_0(x) = \text{tr}(x^3 + x^9), f_1(x) = f_0(x) + \text{tr}(x).$$

$$\tilde{F} : \tilde{f}_0(x) = \text{tr}(x^9 + x), \tilde{f}_1(x) = \tilde{f}_0(x) + \text{tr}(x) = \text{tr}(x^9).$$

$$\bar{F}_0 = \tilde{F}, \tilde{\bar{F}}_0 = F.$$

$$\bar{F}_1 : \bar{f}_0^{(1)}(x) = \tilde{f}_1(x), \bar{f}_1^{(1)}(x) = \tilde{f}_1(x) + \text{tr}(x) = \tilde{f}_0(x).$$

$$\tilde{\bar{F}}_1 : \tilde{\bar{f}}_0^{(1)}(x) = \text{tr}(x + x^3 + x^9), \tilde{\bar{f}}_1^{(1)}(x) = \tilde{\bar{f}}_0^{(1)}(x) + \text{tr}(x + 1).$$

#### 2.4. COMMENTS.

- Note that starting from a near-bent function  $f$  whose derivative  $D_1f$  is a constant function and applying Theorem 1 and Theorem 7, we are in position to construct six bent functions.

- It is easy to check that, for every boolean function  $f$ ,

$$D_1(f + 1) = D_1(f), \quad D_1(f + \text{tr}) = D_1(f) + 1,$$

if  $f(0) = 1$  then  $(f + 1)(0) = 0$ .

Now assume that  $f$  is a near-bent function. As it will be shown by  $(R_2)$  in Section 3.1.2,  $f + 1$ ,  $f + \text{tr}$  and  $f + \text{tr} + 1$  are also near-bent functions. From the above-mentioned results  $D_1(f + 1) = 0$  or  $D_1(f + \text{tr}) = 0$ .

If  $f(0) = 0$  then  $(f + 1)(0) = 1$  and  $(f + 1 + \text{tr})(0) = 0$ . We deduce that among  $f$ ,  $f + 1$ ,  $f + \text{tr}$ ,  $f + \text{tr} + 1$  there always exists a near-bent function  $h$  such that  $D_1(h) = 0$  and  $h(0) = 0$ . Therefore, in order to apply Theorem 1 it is sufficient to find a near-bent function  $f$  such that  $D_1(f) = 0$  and  $f(0) = 0$ .

In this case, consider the following polynomial  $p(X) = \sum_{i=0}^{2t-2} f(\alpha^i)X^i$  where  $\alpha$  is a primitive root of  $\mathbb{F}_{2^{2t-1}}$ . As pointed out in [10], this is the representation of a word of a special cyclic code of length  $2t - 1$  over  $\mathbb{F}_2$  which depends on the degree of  $f$ .

- The map sending a bent function to its pseudo-dual is not injective. For example the bent function defined by  $f_0(x) = \text{tr}(x^7 + x^{13} + x^{19} + x^{21})$  and  $f_1(x) = f_0(x) + \text{tr}(x)$  and the bent function such that  $f_0(x) = \text{tr}(x^3 + x^{11})$  and  $f_1(x) = f_0(x) + \text{tr}(x)$  have different duals, but the same pseudo-dual.

- Starting from a bent function  $F$  which fulfill  $(\mathcal{C})$  or  $(\mathcal{T})$  it is possible to find other bent functions with the same properties. The bent functions  $F$ , the dual  $\tilde{F}$ , the pseudo-duals  $\bar{F}_0$  and  $\bar{F}_1$  and the duals of these pseudo-duals could be either distinct or not. The examples in Section 4 show different situations.

- In a very interesting paper [7] by Leander and McGuire the authors consider the two-variable representations of boolean functions with other notations. In particular, they introduce a characterization of near-bent functions  $f$  such that  $f$  and  $f + \text{tr}$  are the components of a bent function (Theorem 3,  $e = 1$ ). This could be used to obtain an alternative proof of Theorem 1.

### 3. PROOFS

#### 3.1. PRELIMINARIES.

3.1.1. *Notation.* Let  $F$  be an  $m$ -boolean function.

- The weight of  $F$  is defined by  $w(F) = \#\{v \in \mathbb{F}_2^m \mid F(v) = 1\}$ .
- The TVR of  $F$  is defined as in Section 1 by

$$\phi_F(x, y) = (y + 1)f_0(x) + yf_1(x).$$

- The Fourier transform of  $F$  is defined as in Section 2.
- $T_v$  denotes the linear form of  $\mathbb{F}_2^m$  defined by  $T_v(X) = \langle v, X \rangle$ .
- If  $m = 2t - 1$  then  $tr$  denotes the trace function of  $\mathbb{F}_{2^{2t-1}}$  and the map  $x \rightarrow tr(ax)$  is denoted by  $t_a$ .

3.1.2. *Elementary and known results.* We begin by summarizing some of the elementary or classical results on bent and near-bent functions (see [1, 2, 9]).

( $R_1$ )  $\hat{F}(v) = 2^m - 2w(F + T_v)$ .

( $R_2$ ) Let  $F$  be a  $m$ -boolean function and let  $L$  be an affine linear form of  $\mathbb{F}_2^m$ .  $F$  is a bent function if and only if  $F + L$  is a bent function.  $F$  is a near-bent function if and only if  $F + L$  is a near-bent function.

( $R_3$ )  $w(F) = w(f_0) + w(f_1)$ .

( $R_4$ )  $F$  is bent if and only if  $\forall V \in \mathbb{F}_{2^{2t}} D_V F$  is balanced. That is

$$\#\{U \in \mathbb{F}_{2^{2t}} \mid D_V F(U) = 1\} = \#\{U \in \mathbb{F}_{2^{2t}} \mid D_V F(U) = 0\}.$$

( $R_5$ ) Let  $\tilde{F}$  be the dual of a  $(2t)$ -bent function  $F$ . Then  $\tilde{F}(v) = 1$  if and only if  $\hat{F}(v) = -2^t$ .

( $R_6$ ) If  $m = 2t - 1$  and  $f$  is a near-bent function then the distribution of Fourier coefficients is as follows

$$\begin{aligned} \hat{f}(v) &= 2^t \quad \text{number of } v : 2^{2t-3} + (-1)^{f(0)}2^{t-2}, \\ \hat{f}(v) &= 0 \quad \text{number of } v : 2^{2t-2}, \\ \hat{f}(v) &= -2^t \quad \text{number of } v : 2^{2t-3} - (-1)^{f(0)}2^{t-2}. \end{aligned}$$

**Comment.** ( $R_1$ ), ( $R_2$ ) and ( $R_5$ ) follow immediatly from the definitions. ( $R_3$ ) is obtained with straightforward calculations and ( $R_4$ ) is classical. The distribution given in ( $R_6$ ) is a special cases of Proposition 4 in [2].

3.1.3. *Representation of  $(2t)$ -linear forms.* The purpose of this part is to express linear forms and the inner product  $\langle, \rangle$  used in the calculation of the Fourier coefficients in such a way which is consistent with the decomposition of  $\mathbb{F}_{2^{2t}}$  as  $\mathbb{F}_{2^{2t-1}} \times \mathbb{F}_2$ .

**Definition 10.** For every  $(a, \eta)$  in  $\mathbb{F}_{2^{2t}}$  the map  $L_{(a,\eta)}$  from  $\mathbb{F}_{2^{2t}}$  into  $\mathbb{F}_2$  is defined by

$$L_{(a,\eta)}(x, \nu) = tr(ax) + \eta\nu.$$

**Remark.**  $L_{(a,\eta)}(x, \nu)$  is nothing but  $tr(ax) + t_r^{(1)}(\eta\nu)$  where  $t_r^{(1)}$  is the trace of  $\mathbb{F}_2$  since  $t_r^{(1)}(\mu) = \mu$  for every  $\mu$  in  $\mathbb{F}_2$ .

It can be easily checked that

( $\star$ ):  $L_{(a,\eta)}$  is a linear form of  $\mathbb{F}_{2^{2t}}$ .

( $\star, \star$ ): The map  $(a, \eta) \rightarrow L_{(a,\eta)}$  from  $\mathbb{F}_{2^{2t}}$  to  $\{L_{(a,\eta)} \mid (a, \eta) \in \mathbb{F}_{2^{2t}}\}$  is injective.

( $\star, \star, \star$ ): The map  $((a, \eta), (x, \nu)) \rightarrow L_{(a,\eta)}(x, \nu)$  is a non-degenerate symmetric bilinear form of  $\mathbb{F}_{2^{2t}}$ .



We immediately deduce from  $(\star)$  and  $(\star, \star)$  that  $\{L_{(a,\eta)} \mid (a,\eta) \in \mathbb{F}_{2^{2t}}\}$  is the set of linear forms of  $\mathbb{F}_{2^{2t}}$ . On the other hand,  $(\star, \star, \star)$  leads to a choice of the inner product  $T_v$  as defined in section 2.

**Definition 11.** The inner product  $\langle, \rangle$  such that  $T_v(X) = \langle v, X \rangle$  is now defined by  $T_{(a,\eta)} = L_{(a,\eta)}$ . In other words,

$$\langle (a, \eta), (x, \nu) \rangle = \text{tr}(ax) + \eta\nu.$$

We immediately deduce

**Proposition 12.**  $(\star)$   $\phi_{T_{(a,\eta)}}(x, y) = (y+1)\text{tr}(ax) + y(\text{tr}(ax) + \eta)$ .

Let  $F$  be a  $(2t)$ -boolean function such that  $\phi_F(x, y) = (y+1)f_0(x) + yf_1(x)$ . Then

$(\star\star)$   $\phi_{F+T_{(a,\eta)}}(x, y) = (y+1)(f_0(x) + \text{tr}(ax)) + y(f_1(x) + \text{tr}(ax) + \eta)$ .

3.1.4. *Representation of bent functions.* As before, in this subsection we consider a  $(2t)$ -boolean  $F$  function such that  $\phi_F(x, y) = (y+1)f_0(x) + yf_1(x)$ .

**Lemma 13.**

- a)  $\hat{F}(u, 0) = \hat{f}_0(u) + \hat{f}_1(u)$ .
- b)  $\hat{F}(u, 1) = \hat{f}_0(u) - \hat{f}_1(u)$ .
- c) If  $f_0 + f_1 = \text{tr}$  then  $\hat{f}_1(u) = \hat{f}_0(u+1)$

*Proof.* From  $(\star\star)$  and  $(R_3)$ ,

- If  $\eta = 0$ :  $w(F + T_{(u,0)}) = w(f_0 + t_u) + w(f_1 + t_u)$ . According to  $(R_1)$  this means

$$2^{2t-1} - \frac{1}{2}\hat{F}(u, 0) = 2^{2t-2} - \frac{1}{2}\hat{f}_0(u) + 2^{2t-2} - \frac{1}{2}\hat{f}_1(u),$$

and this leads to a).

- If  $\eta = 1$ : first notice that  $w(f_1 + t_u + 1) = 2^{2t-1} - w(f_1 + t_u)$ . Hence

$$w(F + T_{(u,0)}) = w(f_0 + t_u) + 2^{2t-1} - w(f_1 + t_u).$$

By using  $(R_1)$  as above, we obtain the result of b).

- $\hat{f}_0(u) = 2^{2t-1} - 2w(f_0 + t_u)$  and  $\hat{f}_1(u) = 2^{2t-1} - 2w(f_0 + t_{u+1})$  whence  $\hat{f}_1(u) = \hat{f}_0(u+1)$ .  $\square$

The next proposition is a version of a classical result which can be found in several papers on bent functions ([2, 9]). We give now a proof for sake of convenience.

**Proposition 14.**  $F$  is a bent function if and only if

- (a)  $f_0$  and  $f_1$  are near-bent.
- (b)  $\forall a \in \mathbb{F}_{2^{2t-1}} \mid |f_0(a)| + |f_1(a)| = 2^t$ .

**Remark.** (b) means that one of  $|f_0(a)|$  and  $|f_1(a)|$  is equal to  $2^t$  and the other one is equal to 0.

*Proof.* Let  $(a, \eta)$  be in  $\mathbb{F}_{2^{2t}}$ .

Assume  $F$  is bent. From Lemma 13,  $\hat{f}_0(a) = \frac{1}{2}[\hat{F}(a, 0) + \hat{F}(a, 1)]$  and  $\hat{f}_1(a) = \frac{1}{2}[\hat{F}(a, 0) - \hat{F}(a, 1)]$ . Since  $F$  is bent,  $\hat{f}_0(a)$  and  $\hat{f}_1(a)$  are in  $\{-2^t, 2^t\}$ . By inspection of all possible case we see that  $\hat{f}_0(a)$  and  $\hat{f}_1(a)$  are in  $\{-2^t, 0, -2^t\}$  for every  $a$ , which means that  $\hat{f}_0(a)$  and  $\hat{f}_1(a)$  are near-bent. Furthermore, we can check that in every case only one of  $\hat{f}_0(a)$  and  $\hat{f}_1(a)$  is 0.

Conversely, now assume (a) and (b). By Lemma 13, this immediately implies that for every  $(a, \eta)$  in  $\mathbb{F}_{2^{2t}}$  the weight of  $\hat{F}(a, \eta) = \epsilon 2^t$  with  $\epsilon \in \{-1, +1\}$  and this proves that  $F$  is bent.  $\square$

3.2. A FUNDAMENTAL LEMMA. We need the following lemma which is important for the next proofs and is a generalization of a classical result on the Gold function (see [5]), since if  $f$  is the Gold function then  $D_1 f$  is a constant function.

**Lemma 15.** *Let  $f$  be a  $(2t - 1)$ -near-bent function.*

- If  $D_1 f = 0$ , then  $\hat{f}(u) = 0$  if and only if  $tr(u) = 1$ .
- If  $D_1 f = 1$ , then  $\hat{f}(u) = 0$  if and only if  $tr(u) = 0$ .

*Proof.* Assume that  $D_1 f = \omega$  with  $\omega \in \mathbb{F}_2$  which means that  $f(x + 1) = f(x) + \omega$ . The transform  $\tau : x \rightarrow x + 1$  is a permutation of  $\mathbb{F}_{2^{2t-1}}$  and then preserves the weight of every  $(2t - 1)$ -boolean function. Thus

$$\#\{x \mid f(x) + tr(ux) = 1\} = \#\{x \mid f(x + 1) + tr(u(x + 1)) = 1\},$$

$$(E) \quad \#\{x \mid f(x) + tr(ux) = 1\} = \#\{x \mid f(x) + \omega + tr(ux) + tr(u) = 1\}.$$

If  $tr(u) + \omega = 1$  the right hand member of (E) is

$$\#\{x \mid f(x) + tr(ux) = 0\} = 2^{2t-1} - \#\{x \mid f(x) + tr(ux) = 1\}.$$

Hence (E) becomes

$$\#\{x \mid f(x) + tr(ux) = 1\} = 2^{2t-1} - \#\{x \mid f(x) + tr(ux) = 1\}.$$

In other words  $w(f + t_u) = 2^{2t-1} - w(f + t_u)$  and thus

$$\text{If } tr(u) + \omega = 1: w(f + t_u) = 2^{2t-2} \text{ which is equivalent to } \hat{f}(u) = 0.$$

For  $\omega = 0$  or  $\omega = 1$  the number of  $u$  such that  $tr(u) + \omega = 1$  is  $2^{2t-2}$  and  $(R_6)$  claims that this is also the number of  $u$  such that  $\hat{f}(u) = 0$ . Then, immediately  $\hat{f}(u) = 0$  if and only if  $tr(u) + \omega = 1$ . Finally if  $\omega = 0$  then  $\hat{f}(u) = 0$  if and only if  $tr(u) = 1$  and if  $\omega = 1$  then  $\hat{f}(u) = 0$  if and only if  $tr(u) = 0$ .  $\square$

3.3. PROOF OF THEOREM 1. Let  $f_0$  be a  $(2t - 1)$ -near bent function. Let  $F$  be the  $(2t)$ -boolean function whose components are  $f_0$  and  $f_1$  such that  $f_1 = f_0 + tr$ . Our task is to prove that if  $D_1 f_0$  is a constant function, then  $F$  is a bent function.

First notice that  $f_1$  also is a near-bent function. This means that  $\hat{f}_0$  and  $\hat{f}_1$  take their values in  $\{-2^t, 0, 2^t\}$ . Since  $f_1 = f_0 + tr$  and  $a \in \mathbb{F}_{2^{2t-1}}$ , according to Lemma 13, c):  $\hat{f}_1(a) = \hat{f}_0(a + 1)$ .

Because  $2t - 1$  is odd, observe that  $tr(1) = 1$ . Therefore, one element of  $\{tr(a), tr(a + 1)\}$  is 0 and the other one is 1.

Lemma 15 shows that if  $D_1 f_0$  is a constant function then  $\hat{f}_0(a)$  and  $\hat{f}_0(a + 1)$  are not 0 in the same time. Hence, one element of  $\{\hat{f}_0(a), \hat{f}_1(a)\}$  is 0 and the other one is  $2^t$  or  $-2^t$ .

Finally, according to Proposition 14, this is the proof that  $F$  is bent.

3.4. PROOF OF THEOREM 2. First, we need the following proposition

**Proposition 16.** *Let  $f_0$  and  $f_1$  be the components of a bent function  $F$ . Let  $\omega \in \mathbb{F}_2$ . Then we have*

$$D_1 f_0 = \omega \text{ if and only if } D_1 f_1 = \omega + 1.$$

*Proof.*  $D_{(0,1)}F(X) = F(X + (0, 1)) + F(X)$ . The TVR of  $D_{(0,1)}F$  is

$$(y + 1)(f_0(x + 1) + f_0(x)) + y(f_1(x + 1) + f_1(x)) = (y + 1)D_1f_0(x) + yD_1f_1(x).$$

From  $(R_3)$  :  $w(D_{(0,1)}F) = w(D_1f_0) + w(D_1f_1)$ . Furthermore,  $(R_4)$  shows that  $w(D_{(0,1)}F) = 2^{2t-1}$ . Thus

$$(\div) \quad 2^{2t-1} = w(D_1f_0) + w(D_1f_1).$$

On the other hand, if  $f$  is any  $(2t - 1)$ -boolean function, then  $w(D_1f) = 2^{2t-1}$  is equivalent to  $D_1f = 1$ , while  $w(D_1f) = 0$  is equivalent to  $D_1f = 0$ . Therefore,  $(\div)$  proves that if one of the two derivatives  $D_1f_0$  and  $D_1f_1$  is 0, then the other one is 1.  $\square$

Now we go back to the proof of Theorem 2.  $D_1\tilde{f}_0 = 0$  means  $\tilde{f}_0(u) = \tilde{f}_0(u + 1)$  for all  $u$  in  $\mathbb{F}_{2^{2t-1}}$ . Since  $\tilde{f}_0$  is the restriction of  $\tilde{F}$  to  $\mathbb{F}_{2^t}^{(0)} = \{(u, 0) \mid u \in \mathbb{F}_{2^{2t-1}}\}$ , then in order to prove that  $D_1\tilde{f}_0 = 0$  it suffices to show that

$$\forall u \in \mathbb{F}_{2^{2t-1}} : \tilde{F}(u, 0) = \tilde{F}(u + 1, 0).$$

Using Lemma 13, since  $f_0 + f_1 = tr$ , we have successively

$$\hat{f}_1(u) = \hat{f}_0(u + 1).$$

$$\hat{F}(u, 0) = \hat{f}_0(u) + \hat{f}_1(u) = \hat{f}_0(u) + \hat{f}_0(u + 1).$$

$$\hat{F}(u + 1, 0) = \hat{f}_0(u + 1) + \hat{f}_1(u + 1) = \hat{f}_0(u + 1) + \hat{f}_0(u + 1).$$

$$\hat{F}(u, 0) = \hat{F}(u + 1, 0)$$

Following  $(R_5)$  we deduce that  $\tilde{F}(u, 0) = 1$  if and only if  $\tilde{F}(u + 1, 0) = 1$  for all  $u \in \mathbb{F}_{2^{2t-1}}$  and then  $\tilde{F}(u, 0) = \tilde{F}(u + 1, 0)$ .  $D_1\tilde{f}_1 = 1$  is a direct consequence of Proposition 16.

3.5. PROOF OF THEOREM 4. According to Lemma 13, for every  $a$  in  $\mathbb{F}_{2^{2t-1}}$ ,  $\hat{f}_1(a) = \hat{f}_0(a + 1)$  whence  $\hat{F}(a, 0) = \hat{f}_0(a) + \hat{f}_0(a + 1)$ . On the other hand, a remark after Proposition 14 says that one of  $|\hat{f}_0(a)|$  and  $|\hat{f}_1(a)|$  is equal to  $2^t$  and the other one is equal to 0. It follows that every  $a$  in  $\mathbb{F}_{2^{2t-1}}$  belongs to one of the following sets

$$\mathcal{A}_1 = \{a \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(a) = -2^t \text{ and } \hat{f}_0(a + 1) = 0\},$$

$$\mathcal{A}_2 = \{a \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(a) = 0 \text{ and } \hat{f}_0(a + 1) = -2^t\},$$

$$\mathcal{A}_3 = \{a \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(a) = 2^t \text{ and } \hat{f}_0(a + 1) = 0\},$$

$$\mathcal{A}_4 = \{a \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(a) = 0 \text{ and } \hat{f}_0(a + 1) = 2^t\}.$$

The definition of the dual of  $F$  induces that  $(a, \eta)$  is in the support of  $\tilde{F}$  if and only if  $\hat{F}(a, \eta) = -2^t$ . If we notice

- $\hat{F}(a, 0) = -2^t$  if  $a \in \mathcal{A}_1$  or  $a \in \mathcal{A}_2$ ;
- $\hat{F}(a, 0) = 2^t$  if  $a \in \mathcal{A}_3$  or  $a \in \mathcal{A}_4$ ,

we deduce that,  $(a, 0)$  is in the support of  $\tilde{F}$  if and only if  $a \in \mathcal{A}_1 \cup \mathcal{A}_2$ . In other words the support of  $\tilde{f}_0$  is  $\mathcal{S}_0 = \mathcal{A}_1 \cup \mathcal{A}_2$ .

From the descriptions of  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , if  $a \in \mathcal{A}_1$  then  $a + 1 \in \mathcal{A}_2$  and if  $b \in \mathcal{A}_2$  then  $b = a + 1$  with  $a = b + 1 \in \mathcal{A}_1$ . Hence  $\mathcal{A}_2 = \{u + 1 \mid u \in \mathcal{A}_1\}$ . Finally, by inspection we see that  $\mathcal{A}_1$  is nothing but the set  $\mathcal{S} = \{v \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(v) = -2^t\}$  and this leads to result 1).

Since  $f_0 + f_1 = tr$  then by Lemma 13,  $\hat{f}_1(u) = \hat{f}_0(u + 1)$ . Hence,  $(a, 1)$  is in the support of  $\tilde{F}$  if and only if  $a$  is in  $\mathcal{A}_3$  or in  $\mathcal{A}_3$ . Consequently, the support of  $\tilde{f}_1$  is  $\mathcal{T}_1 = \mathcal{A}_3 \cup \mathcal{A}_4$ . It can be easily seen that  $\mathcal{T}_1$  is the symmetric difference of the support of  $\tilde{f}_0$  and of  $\mathcal{G} = \{v \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(v) = 0\}$ . This immediately gives result 2).

3.6. PROOF OF THEOREM 5. Since  $h_0 + h_1 = tr$  then Theorem 4 claims that  $\tilde{h}_0 + \tilde{h}_1$  is the characteristic function of  $\mathcal{H} = \{v \in \mathbb{F}_{2^{2t-1}} \mid \hat{h}_0(v) = 0\}$ . We know from Lemma 15 that if  $D_1 h_0 = 0$  then  $\hat{h}_0(u) = 0$  is equivalent to  $tr(u) = 1$  and if  $D_1 h_0 = 1$  then  $\hat{h}_0(u) = 0$  is equivalent to  $tr(u) = 0$ . This the same as saying that the characteristic function of  $\mathcal{H}$  is  $tr$  if  $D_1 h_0 = 0$  and is  $tr + 1$  if  $D_1 h_0 = 1$  and this is the expected result.

3.7. PROOF OF THEOREM 7. The components of the considered boolean functions are

- $f_0, f_1$  for  $F$  and  $\tilde{f}_0, \tilde{f}_1$  for the dual  $\tilde{F}$  of  $F$ .
- $\bar{f}_0^{(0)}, \bar{f}_1^{(0)}$  for the pseudo-dual  $\bar{F}_0$  and  $\bar{f}_0^{(1)}, \bar{f}_1^{(1)}$  for the pseudo-dual  $\bar{F}_1$ .
- $\tilde{\tilde{f}}_0^{(0)}, \tilde{\tilde{f}}_1^{(0)}$  for the dual  $\tilde{\tilde{F}}_0$  of  $\bar{F}_0$  and  $\tilde{\tilde{f}}_0^{(1)}, \tilde{\tilde{f}}_1^{(1)}$  for the dual  $\tilde{\tilde{F}}_1$  of  $\bar{F}_1$ .

Proof of A): Since  $f_0 + f_1 = tr$  then  $D_1 \tilde{f}_0 = 0$  and  $D_1 \tilde{f}_1 = 1$  (Theorem 2). We deduce from Theorem 1 that  $\bar{F}_0$  and  $\bar{F}_1$  are bent functions.

Proof of B) and C): From the definitions of the duals,

$$\bar{f}_0^{(0)} = \tilde{f}_0, \bar{f}_0^{(1)} = \tilde{f}_1, \bar{f}_0^{(0)} + \bar{f}_1^{(0)} = tr, \bar{f}_0^{(1)} + \bar{f}_1^{(1)} = tr.$$

Hence, according to Theorem 2,  $D_1 \tilde{\tilde{f}}_0^{(0)} = 0$  and  $D_1 \tilde{\tilde{f}}_0^{(1)} = 0$  and thus  $\tilde{\tilde{F}}_0$  and  $\tilde{\tilde{F}}_1$  meet (C).

Now applying Theorem 5 to  $\bar{F}_0$  with  $h_0 = \bar{f}_0^{(0)}$  and  $h_1 = \bar{f}_1^{(0)}$ , we get  $h_0 + h_1 = tr$ . Since  $D_1 \bar{f}_0^{(0)} = 0$  then  $\tilde{h}_0 + \tilde{h}_1 = \tilde{\tilde{f}}_0^{(0)} + \tilde{\tilde{f}}_1^{(0)} = tr$ . That is  $\tilde{\tilde{F}}_0$  meets (T) with  $\xi = 0$ .

Similarly, for  $\bar{F}_1$  with  $h_0 = \bar{f}_0^{(1)}$  and  $h_1 = \bar{f}_1^{(1)}$  we have  $h_0 + h_1 = tr$ . Remark  $D_1 \bar{f}_0^{(1)} = D_1 \tilde{f}_1$  whence, again from Theorem 2,  $D_1 \bar{f}_0^{(1)} = 1$  and Theorem 5 gives  $\tilde{\tilde{f}}_0^{(0)} + \tilde{\tilde{f}}_1^{(0)} = tr + 1$  and thus  $\tilde{\tilde{F}}_1$  meets (T) with  $\xi = 1$ .

#### 4. EXAMPLES

If (C) or (T) hold for a bent function  $F$ , the bent functions  $F$ , the dual  $\tilde{F}$ , the pseudo-duals  $\bar{F}_0, \bar{F}_1$  and the duals of these pseudo-duals can be distinct or not. The examples below show different situations.

For every following example, condition (T) is satisfied for the initial bent function  $F$ .  $t = 4$  and  $tr$  is the trace function of  $\mathbb{F}_{2^7}$ .

**Example 1.**  $F : f_0(x) = tr(x^7 + x^{13})$  not (C), (T)

$$f_1(x) = f_0(x) + tr(x)$$

$$\tilde{F} : \begin{aligned} \tilde{f}_0(x) &= tr(x^5 + x^7 + x^9 + x^{13} + x^{19} + x^{21}) \\ \tilde{f}_1(x) &= \tilde{f}_0(x) + tr(x + x^5 + x^9) \end{aligned}$$

$$\bar{F}_0 : \bar{f}_0^{(0)}(x) = \tilde{f}_0(x), \bar{f}_1^{(0)}(x) = \tilde{f}_0(x) + tr(x)$$

$$\bar{F}_1 : \bar{f}_0^{(1)}(x) = \tilde{f}_1(x), \bar{f}_1^{(1)}(x) = \tilde{f}_1(x) + tr(x)$$

$$\tilde{\tilde{F}}_0 : \begin{aligned} \tilde{\tilde{f}}_0^{(0)}(x) &= tr(x + x^7 + x^9 + x^{13} + x^{19} + x^{21}), \\ \tilde{\tilde{f}}_1^{(0)}(x) &= \tilde{\tilde{f}}_0^{(0)}(x) + tr(x) \end{aligned}$$

$$\begin{aligned}\tilde{\tilde{F}}_1 : \quad & \tilde{f}_0^{(1)}(x) = \text{tr}(x + x^3 + x^7 + x^{13} + x^{19} + x^{21}) \\ & \tilde{f}_1^{(1)}(x) = \tilde{f}_0^{(1)}(x) + \text{tr}(x + 1)\end{aligned}$$

**Example 2.**  $F : f_0(x) = \text{tr}(x^{15} + x^{27} + x^{29} + x^{43})$ , not  $(\mathcal{C})$ ,  $(\mathcal{T})$   
 $f_1(x) = f_0(x) + \text{tr}(x)$

$$\begin{aligned}\tilde{F} : \quad & \tilde{f}_0(x) = \text{tr}(x + x^3 + x^5 + x^9) \\ & \tilde{f}_1(x) = \tilde{f}_0(x) + \text{tr}(x^5 + x^7 + x^{11} + x^{19} + x^{21})\end{aligned}$$

$$\bar{F}_0 : \quad \bar{f}_0^{(0)}(x) = \tilde{f}_0(x), \bar{f}_1^{(0)}(x) = \tilde{f}_0(x) + \text{tr}(x)$$

$$\bar{F}_1 : \quad \bar{f}_0^{(1)}(x) = \tilde{f}_1(x), \bar{f}_1^{(1)}(x) = \tilde{f}_1(x) + \text{tr}(x)$$

$$\tilde{\tilde{F}}_0 = \bar{F}_0$$

$$\begin{aligned}\tilde{\tilde{F}}_1 : \quad & \tilde{f}_0^{(1)}(x) = \text{tr}(x + x^3 + x^5 + x^7 + x^9 + x^{11} + x^{19} + x^{21}) \\ & \tilde{f}_1^{(1)}(x) = \tilde{f}_0^{(1)}(x) + \text{tr}(x + 1)\end{aligned}$$

**Example 3.**  $F : f_0(x) = \text{tr}(x + x^3 + x^7 + x^{11} + x^{19} + x^{21})$ ,  $(\mathcal{C})$ ,  $(\mathcal{T})$   
 $f_1(x) = f_0(x) + \text{tr}(x)$

$$\begin{aligned}\tilde{F} : \quad & \tilde{f}_0(x) = \text{tr}(x^7 + x^{11} + x^{19} + x^{21}) \\ & \tilde{f}_1(x) = \tilde{f}_0(x) + \text{tr}(x)\end{aligned}$$

$$\bar{F}_0 = \tilde{F} \text{ and } \tilde{\tilde{F}}_0 = F \quad \bar{F}_1 = \tilde{F} \text{ and } \tilde{\tilde{F}}_1 = F:$$

**Example 4.**  $F : f_0(x) = \text{tr}(x^3 + x^5 + x^7 + x^{11} + x^{19} + x^{21})$ ,  $(\mathcal{C})$ ,  $(\mathcal{T})$   
 $f_1(x) = f_0(x) + \text{tr}(x)$

$$\tilde{F} = F = \bar{F}_0 = \bar{F}_1 = \tilde{\tilde{F}}_0 = \tilde{\tilde{F}}_1$$

4.1. A SPECIAL EXAMPLE AND AN OPEN QUESTION. In [7] the authors recall the definition of non-weakly-normal bent function and they introduce (Fact 13) an example of such a function  $F$  in dimension 12 defined by  $\phi_F(x, y) = (y+1)\text{tr}(x^{241} + x) + y\text{tr}(x^{241})$  where  $\text{tr}$  is the trace of  $\mathbb{F}_{2^{11}}$  over  $\mathbb{F}_2$ . We see that  $f_1 + f_0 = \text{tr}(x)$  and thus Theorem 7 holds.

An interesting open question is: Are  $\tilde{F}$ ,  $\bar{F}_0$ ,  $\bar{F}_1$ ,  $\tilde{\tilde{F}}_0$ ,  $\tilde{\tilde{F}}_1$  also non-weakly normal?

## 5. CONCLUSION

We have introduced a way to construct bent functions starting from a near-bent functions which fulfill the special condition of Theorem 1. Applying this Theorem and Theorem 7 we obtain six bent functions. An open question now is to describe explicitly the near-bent functions  $f$  such that  $D_1 f$  is a constant function, for example by means of the trace function.

Another question is to express the characteristic function of the set  $\mathcal{S}$  which appears in Theorem 4, by using the trace function.

The results of this work could probably be generalized by replacing 1 with  $e$  such that  $\text{tr}(e) = 1$  and  $\text{tr}(x)$  with  $\text{tr}(ex)$  in  $(\mathcal{C})$  and  $(\mathcal{T})$ .

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