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SPECIAL BENT AND NEAR-BENT FUNCTIONS

J.WOLFMANN

ABSTRACT. Starting from special near-bent functions in dimension $2t - 1$ we construct bent functions in dimension $2t$ having a specific derivative. We deduce new families of bent functions.

1. INTRODUCTION

\mathbb{F}_2 is the finite field of order 2. A m -boolean function (or boolean function in m dimensions) is a map F from \mathbb{F}_2^m to \mathbb{F}_2 .

Bent functions are the boolean functions whose Fourier coefficients have constant magnitude and were introduced by Rothaus in [8]. It means that a m -boolean function F is bent if all its Fourier coefficients are in $\{-2^{m/2}, 2^{m/2}\}$. Bent functions are of interest for Coding Theory, Cryptology and well-correlated binary sequences. For example, they have the maximum Hamming distance to the set of affine boolean functions. They were the topic of a lot of works (see References) but the complete classification of bent functions and other questions are still open.

By definition, a m -boolean function F is near-bent if all its Fourier coefficients are in $\{-2^{(m+1)/2}, 0, 2^{(m+1)/2}\}$

Since Fourier coefficient are in \mathbb{Z} then bent functions in m dimensions exist only when m is even and near-bent functions in m dimensions exist only when m is odd.

1.1. A two-variable representation.

Assume $m = 2t$. We identify \mathbb{F}_2^{2t} with $\mathbb{F}_{2^{2t}}$ and $\mathbb{F}_{2^{2t}}$ with:

$$\mathbb{F}_{2^{2t-1}} \times \mathbb{F}_2 = \{X = (u, \nu) \mid u \in \mathbb{F}_{2^{2t-1}}, \nu \in \mathbb{F}_2\}.$$

This decomposition can be used to introduce a two-variable representation of $(2t)$ -boolean functions as follows:

Define $\mathbb{F}_{2t}^{(0)} = \{(u, 0) \mid u \in \mathbb{F}_{2^{2t-1}}\}$ and $\mathbb{F}_{2t}^{(1)} = \{(u, 1) \mid u \in \mathbb{F}_{2^{2t-1}}\}$.

The restrictions of a $(2t)$ -boolean function F , respectively to $\mathbb{F}_{2t}^{(0)}$ and to $\mathbb{F}_{2t}^{(1)}$ induces two $(2t - 1)$ -boolean functions f_0 and f_1 defined by $f_0(u) = F(u, 0)$ and $f_1(u) = F(u, 1)$.

The two-variable representation (TVR) of F is then defined by:

$$(1) \quad \phi_F(x, y) = (y + 1)f_0(x) + yf_1(x)$$

Note that $F(u, 0) = f_0(u) = \phi_F(u, 0)$ and $F(u, 1) = f_1(u) = \phi_F(u, 1)$. Hence if $X = (u, \nu)$ then $F(X) = \phi_F(u, \nu)$.

Key words and phrases. Bent Functions, Near-Bent Functions.

It can be proved that if F is a $(2t)$ -bent function then f_0 and f_1 are $(2t - 1)$ -near-bent functions (see 3.1.4). This leads to consider the following inverse problem : construct $(2t)$ -bent functions from $(2t - 1)$ -near-bent functions. This was the purpose of [7] and in a certain sense of [10] where connections with cyclic codes are established.

Using the above representation of $\mathbb{F}_{2^{2t}}$, it could be easily checked that $f_0 + f_1$ is the derivative of F with respect to $(0, 1)$. A possible way to study bent functions is to classify them regarding the degree (as a boolean function) of this derivative. The most simple case to consider is when this degree is one. In the present paper we are concerned with bent functions such that $f_0 + f_1 = tr + \xi$ where tr is the trace of $\mathbb{F}_{2^{2t-1}}$ and $\xi \in \{0, 1\}$. We introduce the new notion of pseudo-duality and as application we present new families of bent functions. By the way we present a generalization of a result on the Gold function (Lemma 16 in 3.5).

2. RESULTS

The proofs of the Theorems of this section will be set up in Section 3 while the other results are proven in the present section.

First recall some usual definitions on a m -boolean function F .

• If $e \in \mathbb{F}_2^m$ then the Derivative of F with respect to e is the m -boolean function $D_e F$ defined by:

$$D_e F(X) = F(X) + F(X + e)$$

Remark: In this paper we consider the following special cases.

If F is a $(2t)$ -boolean function whose two-variable representation is introduced in (1) then a direct calculation shows that:

$$D_{0,1}(F) = f_0 + f_1$$

If f is a $(2t - 1)$ -boolean function:

$$D_1 f(x) = f(x + 1) + f(x)$$

• The Fourier transform (or Walsh transform) \hat{F} of F is the map from \mathbb{F}_2^m into \mathbb{Z} defined by:

$$\hat{F}(v) = \sum_{X \in \mathbb{F}_2^m} (-1)^{F(X) + \langle v, X \rangle}$$

where \langle, \rangle denotes any inner product of \mathbb{F}_2^m over \mathbb{F}_2 .

$\hat{F}(v)$ is called the Fourier coefficient of v .

Remark: \hat{F} is independent of the choice of this inner product.

- F is bent if all its Fourier coefficients are in $\{-2^{m/2}, 2^{m/2}\}$.
 F is near-bent if all its Fourier coefficients are in $\{-2^{(m+1)/2}, 0, 2^{(m+1)/2}\}$

- If $m = 2t$ and if F is a bent function then the dual \tilde{F} of F is the $(2t)$ -boolean function defined by:

$$\hat{F}(v) = (-1)^{\tilde{F}(v)} 2^t$$

where \hat{F} is the Fourier transform of F .

It is well-known, and easy to prove that the dual of a bent function is a bent function

- The algebraic degree, or more simply “the degree” of a m -boolean function is the degree of its m -variable polynomial representation. If it is expressed as $Tr(\pi(X))$ where Tr is the trace function of \mathbb{F}_{2^m} then its degree is the maximum of the binary weight of the monomial exponents of $\pi(X)$. (Recall that the binary weight of an integer is the number of non-zero coefficients of the binary expansion of this integer).

(see References for details on these definitions and results).

2.1. Main theorems.

From now on, we use the definitions and notations of the introduction. Thus \mathbb{F}_2^{2t} is identified with $\mathbb{F}_{2^{2t}}$ and $\mathbb{F}_{2^{2t}}$ with $\mathbb{F}_{2^{2t-1}} \times \mathbb{F}_2$. The two-variable representation (TVR) of a $(2t)$ -boolean function F is defined by $\phi_F(x, y) = (y + 1)f_0(x) + yf_1(x)$ where f_0 and f_1 are specified in Section 1. Obviously, $\phi_{F_1+F_2} = \phi_{F_1} + \phi_{F_2}$.

Definition f_0 and f_1 are called the components of F .

Notation: The trace of $\mathbb{F}_{2^{2t-1}}$ is denoted by tr .

The next theorem set a condition on a near-bent function to be the first component of a bent function.

Theorem 1.

Let f_0 be a $(2t - 1)$ -near-bent function.

If the derivative $D_1 f_0$ is a constant function then the $(2t)$ -boolean function F such that :

$\phi_F(x, y) = (y + 1)f_0(x) + yf_1(x)$ with $f_0 + f_1 = tr$ is a bent function.

We now have a pseudo-reciprocal theorem.

Theorem 2.

Let F be a bent function and let \tilde{F} be its dual bent function with:

$\phi_F(x, y) = (y + 1)f_0(x) + yf_1(x)$ and $\phi_{\tilde{F}}(x, y) = (y + 1)\tilde{f}_0(x) + y\tilde{f}_1(x)$.

If $f_0 + f_1 = tr$ then $D_1 \tilde{f}_0 = 0$ and $D_1 \tilde{f}_1 = 1$

- The converse of Theorem 1 is not true. In other words, it is not true that if $f_0 + f_1 = tr$ then $D_1 f_0$ is a constant function, as it will be seen with Example 2. However, there is a special case given by the next corollary which comes immediately from Theorem 2.

Corollary 3.

Continuing the above notation, let F be bent such that $f_0 + f_1 = tr$. If F is self dual, say $F = \tilde{F}$, then $D_1 f_0 = 0$.

Informations on the components of the dual \tilde{F} of a bent function F such that $f_0 + f_1 = tr$ are now set up in the next result.

Theorem 4.

Let F be a bent function and let \tilde{F} be its dual bent function with:
 $\phi_F(x, y) = (y + 1)f_0(x) + yf_1(x)$ and $\phi_{\tilde{F}}(x, y) = (y + 1)\tilde{f}_0(x) + y\tilde{f}_1(x)$.
 Assume $f_0 + f_1 = tr$.

Let \hat{f}_0 the Fourier transform of f_0 . Define:

$$\mathcal{S} = \{v \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(v) = -2^t\} \text{ and } \mathcal{S}_1 = \{u + 1 \mid u \in \mathcal{S}\}.$$

Let $\mathcal{G} = \{v \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(v) = 0\}$

Let g be the characteristic function of \mathcal{G} .

1) The support of \tilde{f}_0 is $\mathcal{S} \cup \mathcal{S}_1$.

2) $\tilde{f}_1(x) = \tilde{f}_0(x) + g(x)$

The next theorem states properties of a bent function in the case where the hypothesis of Theorem 1 and Theorem 2 are both satisfied.

Theorem 5.

Let H be a bent function and let \tilde{H} be its dual bent function with:
 $\phi_H(x, y) = (y + 1)h_0(x) + yh_1(x)$ and $\phi_{\tilde{H}}(x, y) = (y + 1)\tilde{h}_0(x) + y\tilde{h}_1(x)$.
 Assume $h_0 + h_1 = tr$.

a) If $D_1 h_0 = 0$ then $\tilde{h}_0 + \tilde{h}_1 = tr$.

b) If $D_1 h_0 = 1$ then $\tilde{h}_0 + \tilde{h}_1 = tr + 1$.

2.2. Pseudo-duality.

The previous results lead to the introduction of a new definition (notations as above).

Definition 6. Let G be a $(2t)$ -bent function and let \tilde{G} be its dual bent function with $\phi_{\tilde{G}}(x, y) = (y + 1)\tilde{g}_0(x) + y\tilde{g}_1(x)$.

The Pseudo-duals of G are the two $(2t)$ -boolean function \bar{G}_0 and \bar{G}_1 defined by:

$$\begin{aligned} \phi_{\bar{G}_0}(x, y) &= (y + 1)\tilde{g}_0(x) + y(\tilde{g}_0(x) + tr(x)). \\ \phi_{\bar{G}_1}(x, y) &= (y + 1)\tilde{g}_1(x) + y(\tilde{g}_1(x) + tr(x)). \end{aligned}$$

The meaning of this definition is given by the next theorem:

Theorem 7.

Define two conditions on a $(2t)$ -bent functions \mathcal{G} with:

$$\phi_G(x, y) = (y + 1)g_0(x) + yg_1(x).$$

$$(\mathcal{T}) \quad g_0 + g_1 = tr + \xi \text{ with } \xi \in \{0, 1\}$$

$$(\mathcal{C}) \quad D_1 g_0 = 0$$

If F is a $(2t)$ -bent function meeting condition (\mathcal{T}) then:

- A) The pseudo-duals \bar{F}_0 and \bar{F}_1 are bent functions.
- B) The dual $\tilde{\bar{F}}_0$ of \bar{F}_0 meets (\mathcal{C}) and (\mathcal{T}) with $\xi = 0$.
- C) The dual $\tilde{\bar{F}}_1$ of \bar{F}_1 meets (\mathcal{C}) and (\mathcal{T}) with $\xi = 1$.

2.3. New families of bent functions.

If \mathcal{F} is a family of $(2t)$ -boolean functions, define $\bar{\mathcal{F}}_0 = \{\bar{F}_0 \mid F \in \mathcal{F}\}$ and $\bar{\mathcal{F}}_1 = \{\bar{F}_1 \mid F \in \mathcal{F}\}$.

By applying the previous theorem, if \mathcal{F} is a family of bent functions meeting (\mathcal{T}) , then $\bar{\mathcal{F}}_0$ and $\bar{\mathcal{F}}_1$ are new families of bent functions.

This is the case in the next proposition by using a family introduced in [7] (Theorem 9).

2.3.1. The Kasami-Welch example.

This designation comes from the description of the near-bent function f_0 introduced in this example which is a classical object in the theory of boolean functions and of binary sequence.

Proposition 8.

Let t, s, d be integers such that:

$$\begin{aligned} 2t - 1 & \text{ is not divisible by } 3, \quad 3s \equiv \pm 1 \pmod{(2t - 1)}, \\ s < t, \quad d & = 4^s - 2^s + 1. \end{aligned}$$

Let F be the $(2t - 1)$ -boolean function with:

$$\phi_F(x, y) = (y + 1)tr(x^d) + ytr(x^d + x).$$

The assertions A), B), C) of Theorem 7 hold for F .

Proof.

In Theorem 9 of [7] it is proved that F is a bent function. Since F satisfies (\mathcal{T}) then Theorem 7 applies and gives the expected result. \square

Remark: It is proved in [4] that in the Kasami-Welch case, if g is the function introduced in Theorem 4 then $g(x) = 1 + tr(x^{2^s+1})$.

Example:

$$t = 4, \quad s = 2.$$

$$F : \quad f_0(x) = tr(x^{13}), \quad f_1(x) = f_0(x) + tr(x)$$

$$\tilde{F} : \quad \tilde{f}_0(x) = tr(x^7 + x^{11} + x^{19} + x^{21}), \quad \tilde{f}_1(x) = \tilde{f}_0(x) + tr(x^5 + 1)$$

New bent functions

$$\bar{F}_0 : \quad \bar{f}_0^{(0)}(x) = \tilde{f}_0(x), \quad \bar{f}_1^{(0)}(x) = \tilde{f}_0(x) + tr(x)$$

$$\bar{F}_1 : \quad \bar{f}_0^{(1)}(x) = \tilde{f}_1(x), \quad \bar{f}_1^{(1)}(x) = \tilde{f}_1(x) + tr(x)$$

$$\begin{aligned} \tilde{\bar{F}}_0 : \quad \tilde{\bar{f}}_0^{(0)}(x) & = tr(x + x^3 + x^7 + x^{11} + x^{19} + x^{21}) \\ \tilde{\bar{f}}_1^{(0)}(x) & = \tilde{\bar{f}}_0^{(0)}(x) + tr(x) \end{aligned}$$

$$\begin{aligned}\tilde{F}_1 : \quad \tilde{f}_0^{(1)}(x) &= tr(1 + x^5 + x^7 + x^9 + x^{11} + x^{19} + x^{21}) \\ \tilde{f}_1^{(1)}(x) &= \tilde{f}_0^{(1)}(x) + tr(x + 1)\end{aligned}$$

Remark: In the above example $D_1 f_0$ is not a constant function.

2.3.2. The quadratic case.

In the next proposition we study the case where f_0 quadratic and such that $f_0(x) = tr(\pi(x))$ where all the coefficients of $\pi(x)$ in \mathbb{F}_2

Proposition 9.

Let f_0 be a $(2t - 1)$ -near-bent function such that:

- $f_0(x) = tr(\pi(x))$ with $\pi(x) \in \mathbb{F}_2[x]$.
- The degree of f_0 is 2.

1) Then the $(2t)$ -boolean function F such that:

$\phi_F(x, y) = (y + 1)f_0(x) + yf_1(x)$ with $f_1(x) = f_0(x) + tr(x)$ is a bent function.

2) The dual \tilde{F} of F meets (\mathcal{T}) .

3) The assertions A), B), C) of Theorem 6 hold for F .

Proof.

1) First remark that r and s are in \mathbb{N} then for a convenient exponent l : $tr(x^{2^r} + x^{2^s}) = tr[(x^{2^r} + x^{2^s})^{2^l}] = tr(x^{2^j} + 1)$ for some j .

It follows that we can express f as $f_0(x) = \xi + \sum_J tr(x^{2^j} + 1)$ where J is a subset of \mathbb{N} with $J \neq \{0\}$ and $\xi \in \mathbb{F}_2$.

Now, it is easy to check that $x^{2^j+1} + (x+1)^{2^j+1} = x^{2^j} + x + 1$ and thus $tr(x^{2^j} + 1) + tr((x+1)^{2^j} + 1)$.

Therefore $f_0(x) + f_0(x+1) = \sum_J 1$ and $f_0(x) + f_0(x+1) = 0$ if $|J|$ is even and $f_0(x) + f_0(x+1) = 1$ if $|J|$ is odd. In both cases $D_1 f_0$ is a constant function then Part 1) is a consequence of Theorem 1.

2) Since $f_0 + f_1 = tr$ and $D_1 f_0$ is a constant function then Theorem 5 applies.

3) is a direct consequence of Theorem 7. □

Remarks:

- Observe that for any $(2t)$ -boolean function G whose components are g_0 and g_1 the TVR of G $\phi_G(x, y)$ can be rewritten as:

$\phi_G(x, y) = y(g_0(x) + g_1(x)) + g_0(x)$. If $G = F$ then $\phi_F(x, y) = ytr(x) + f_0(x)$. Since the degree of f_0 is 2 we deduce that the degree of F also is 2. Similarly for \tilde{F} : $\phi_{\tilde{F}}(x, y) = y(\tilde{f}_0(x) + \tilde{f}_1(x)) + \tilde{f}_0(x)$.

We know from [6] (Lemma 2.5) that if the degree of F is 2 then \tilde{F} also has degree 2. From $\phi_{\tilde{F}}(x, y) = y(\tilde{f}_0(x) + \tilde{f}_1(x)) + \tilde{f}_0(x)$ we deduce that degree of \tilde{f}_0 is 2. It follows that for every bent function among $F, \tilde{F}, \bar{F}_0, \bar{F}_1, \tilde{\bar{F}}_0, \tilde{\bar{F}}_1$, the degree is 2 and the components have degree 2.

- Obviously, $\bar{F}_0 = \tilde{\bar{F}}, \tilde{\bar{F}}_0 = F$.

- It is shown in [7] that if $\deg f_0 = 2$ there exists e in $\mathbb{F}_{2^{2t-1}}$ such that $f_0(x) + f_1(x) = \text{tr}(ex)$. Another way to prove 1) is to show $e = 1$ when $\pi(x)$ is binary.

Example:

$t = 4$.

$$F : f_0(x) = \text{tr}(x^3 + x^9), f_1(x) = f_0(x) + \text{tr}(x).$$

$$\tilde{F} : \tilde{f}_0(x) = \text{tr}(x^9 + x), \tilde{f}_1(x) = \tilde{f}_0(x) + \text{tr}(x) = \text{tr}(x^9).$$

$$\bar{F}_0 = \tilde{F}, \tilde{\bar{F}}_0 = F.$$

$$\bar{F}_1 : \bar{f}_0^{(1)}(x) = \tilde{f}_1(x), \bar{f}_1^{(1)}(x) = \tilde{f}_1(x) + \text{tr}(x) = \tilde{f}_0(x).$$

$$\tilde{\bar{F}}_1 : \tilde{\bar{f}}_0^{(1)}(x) = \text{tr}(x + x^3 + x^9), \tilde{\bar{f}}_1^{(1)}(x) = \tilde{\bar{f}}_0^{(1)}(x) + \text{tr}(x + 1).$$

2.4. Comments.

- Note that starting from a near-bent function f whose derivative D_1f is a constant function and applying Theorem 1 and Theorem 7, we are in position to construct six bent functions.

- It is easy to check that, for every boolean function f :

$$D_1(f + 1) = D_1(f), D_1(f + \text{tr}) = D_1(f) + 1,$$

if $f(0) = 1$ then $(f + 1)(0) = 0$.

Now assume that f is a near-bent function. As it will be recall by (R_2) in 3.1.2, $f + 1$, $f + \text{tr}$ and $f + \text{tr} + 1$ also are near-bent functions.

From the above-mentioned results:

$$D_1(f + 1) = 0 \text{ or } D_1(f + \text{tr}) = 0.$$

If $f(0) = 0$ then $(f + 1)(0) = 1$ and $(f + 1 + \text{tr})(0) = 0$.

We deduce that among f , $f + 1$, $f + \text{tr}$, $f + \text{tr} + 1$ there always exists a near-bent function h such that $D_1(h) = 0$ and $h(0) = 0$.

Therefore, in order to apply Theorem 1 it is sufficient to search a near-bent function f such that $D_1(f) = 0$ and $f(0) = 0$.

In this case, consider the following polynomial $p(X) = \sum_{i=0}^{2t-2} f(\alpha^i)X^i$ where α is a primitive root of $\mathbb{F}_{2^{2t-1}}$.

As pointed out in [10], this is the representation of a word of a special cyclic code of length $2t - 1$ over \mathbb{F}_2 which depends on the degree of f .

- The map sending a bent function to its pseudo-dual is not injective. For example the bent function defined by $f_0(x) = \text{tr}(x^7 + x^{13} + x^{19} + x^{21})$ and $f_1(x) = f_0(x) + \text{tr}(x)$ and the bent function such that $f_0(x) = \text{tr}(x^3 + x^{11})$ and $f_1(x) = f_0(x) + \text{tr}(x)$ have different duals but the same pseudo-dual.

- Starting from a bent function F which fulfil (\mathcal{C}) or (\mathcal{T}) it is possible to find other bent functions with the same properties. The bent functions F , the dual \tilde{F} , the pseudo-duals \bar{F}_0 and \bar{F}_1 and the duals of these pseudo-duals could be distinct or not. The examples in Section 4 show different situations.

- In the very interesting paper [7] by Leander and McGuire the authors consider the two-variable representations of boolean functions with other notations. In particular, they introduce a characterization of near-bent functions f such that f and $f + tr$ are the components of a bent function (Theorem 3, $e = 1$). This could be use to obtain an alternative proof of Theorem 1.

3. PROOFS

3.1. Preliminaries.

3.1.1. Notation.

Let F be m -boolean function.

- The weight of F is defined by: $w(F) = \#\{v \in \mathbb{F}_2^m \mid F(v) = 1\}$.
- The TVR of F is defined as in section 1 by:

$$\phi_F(x, y) = (y + 1)f_0(x) + yf_1(x).$$
- The Fourier transform of F is defined as in section 2.
- T_v denotes the linear form of \mathbb{F}_2^m defined by $T_v(X) = \langle v, X \rangle$.
- If $m = 2t - 1$ then tr denotes the trace function of $\mathbb{F}_{2^{2t-1}}$ and the map

$$x \longrightarrow tr(ax)$$
 is denoted by t_a .

3.1.2. Elementary and known results.

We begin by summarizing some of the elementary or classical results on bent and near-bent functions (see References).

$$(R_1) : \hat{F}(v) = 2^m - 2w(F + T_v).$$

(R_2) Let F be a m -boolean function and let L be an affine linear form of \mathbb{F}_2^m .

F is a bent function if and only if $F + L$ is a bent function.

F is a near-bent function if and only if $F + L$ is a near-bent function.

$$(R_3) : w(F) = w(f_0) + w(f_1)$$

(R_4): F is bent if and only if:

$\forall V \in \mathbb{F}_{2^{2t}} D_V F$ is balanced. That is:

$$\#\{U \in \mathbb{F}_{2^{2t}} \mid D_V F(U) = 1\} = \#\{U \in \mathbb{F}_{2^{2t}} \mid D_V F(U) = 0\}.$$

(R_5): Let \tilde{F} be the dual of a $(2t)$ -bent function F . Then:

$$\tilde{F}(v) = 1 \text{ if and only if } \hat{F}(v) = -2^t.$$

(R_6): If $m = 2t - 1$ and f is a near-bent function then the distribution of Fourier coefficients is as follows:

$$\begin{aligned} \hat{f}(v) &= 2^t && \text{number of } v: 2^{2t-3} + (-1)^{f(0)} 2^{t-2} \\ \hat{f}(v) &= 0 && \text{number of } v: 2^{2t-2} \\ \hat{f}(v) &= -2^t && \text{number of } v: 2^{2t-3} - (-1)^{f(0)} 2^{t-2} \end{aligned}$$

Comments: (R_1) , (R_2) and (R_5) derive immediatly from the definitions. (R_3) is obtained with straightforward calculations and (R_4) is classical. The distribution given in (R_6) is a special cases of Proposition 4 in [2].

3.1.3. Representation of $(2t)$ -linear forms.

The purpose of this part is to express linear forms and the inner product \langle, \rangle used in the calculation of the Fourier coefficients in such a way which is consistent with the decomposition of $\mathbb{F}_{2^{2t}}$ as $\mathbb{F}_{2^{2t-1}} \times \mathbb{F}_2$.

Definition 10.

For every (a, η) in $\mathbb{F}_{2^{2t}}$ the map $L_{(a, \eta)}$ from $\mathbb{F}_{2^{2t}}$ into \mathbb{F}_2 is defined by :

$$L_{(a, \eta)}(x, \nu) = tr(ax) + \eta\nu.$$

Remark: $L_{(a, \eta)}(x, \nu)$ is nothing but $tr(ax) + t_r^{(1)}(\eta\nu)$ where $t_r^{(1)}$ is the trace of \mathbb{F}_2 since $t_r^{(1)}(\mu) = \mu$ for every μ in \mathbb{F}_2 .

It can be easily checked that:

- (\star) $L_{(a, \eta)}$ is a linear form of $\mathbb{F}_{2^{2t}}$.
- (\star, \star) The map $(a, \eta) \longrightarrow L_{(a, \eta)}$ from $\mathbb{F}_{2^{2t}}$ to $\{L_{(a, \eta)} \mid (a, \eta) \in \mathbb{F}_{2^{2t}}\}$ is injective.
- (\star, \star, \star) The map $((a, \eta), (x, \nu)) \longrightarrow L_{(a, \eta)}(x, \nu)$ is a non-degenerate symmetric bilinear form of $\mathbb{F}_{2^{2t}}$.

We immediatly deduce from (\star) and (\star, \star) that $\{L_{(a, \eta)} \mid (a, \eta) \in \mathbb{F}_{2^{2t}}\}$ is the set of linear forms of $\mathbb{F}_{2^{2t}}$.

On the other hand, (\star, \star, \star) leads to a choice of the inner product T_v as defined in section 2.

Definition 11.

The inner product \langle, \rangle such that $T_v(X) = \langle v, X \rangle$ is now defined by $T_{(a, \eta)} = L_{(a, \eta)}$. In other words:

$$\langle (a, \eta), (x, \nu) \rangle = tr(ax) + \eta\nu.$$

We immediatly deduce :

Proposition 12.

$$(*) \quad \phi_{T_{(a, \eta)}}(x, y) = (y + 1)tr(ax) + y(tr(ax) + \eta).$$

Let F be a $(2t)$ -boolean function such that:

$$\phi_F(x, y) = (y + 1)f_0(x) + yf_1(x). \text{ Then:}$$

$$(**) \quad \phi_{F+T_{(a, \eta)}}(x, y) = (y + 1)(f_0(x) + tr(ax)) + y(f_1(x) + tr(ax) + \eta).$$

3.1.4. Representation of Bent Functions.

As before, in this subsection we consider a $(2t)$ -boolean F function such that:

$$\phi_F(x, y) = (y + 1)f_0(x) + yf_1(x).$$

Lemma 13.

$$a) \quad \hat{F}(u, 0) = \hat{f}_0(u) + \hat{f}_1(u).$$

$$b) \quad \hat{F}(u, 1) = \hat{f}_0(u) - \hat{f}_1(u).$$

c) If $f_0 + f_1 = tr$ then $\hat{f}_1(u) = \hat{f}_0(u + 1)$

Proof.

From (**) and (R_3):

• If $\eta = 0$: $w(F + T_{(u,0)}) = w(f_0 + t_u) + w(f_1 + t_u)$.

According to (R_1) this means:

$$2^{2t-1} - \frac{1}{2}\hat{F}(u, 0) = 2^{2t-2} - \frac{1}{2}\hat{f}_0(u) + 2^{2t-2} - \frac{1}{2}\hat{f}_1(u)$$

and this leads to a).

• If $\eta = 1$: first remark that $w(f_1 + t_u + 1) = 2^{2t-1} - w(f_1 + t_u)$. Hence $w(F + T_{(u,0)}) = w(f_0 + t_u) + 2^{2t-1} - w(f_1 + t_u)$. By using (R_1) as above, we obtain the result of b).

• $\hat{f}_0(u) = 2^{2t-1} - 2w(f_0 + t_u)$ and $\hat{f}_1(u) = 2^{2t-1} - 2w(f_0 + t_{u+1})$ whence $\hat{f}_1(u) = \hat{f}_0(u + 1)$ \square

The next proposition is a version of a classical result which can be found in several papers on bent functions ([2],[9], etc...). We give now a proof for sake of convenience.

Proposition 14.

F is a bent function if and only if:

(a) f_0 and f_1 are near-bent.

(b) $\forall a \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(a) \mid + \mid \hat{f}_1(a) \mid = 2^t$

Remark: (b) means that one of $\mid \hat{f}_0(a) \mid$ and $\mid \hat{f}_1(a) \mid$ is equal to 2^t and the other one is equal to 0.

Proof.

- Let (a, η) be in $\mathbb{F}_{2^{2t}}$.

• Assume F is bent. From Lemma 13:

$\hat{f}_0(a) = \frac{1}{2}[\hat{F}(a, 0) + \hat{F}(a, 1)]$ and $\hat{f}_1(a) = \frac{1}{2}[\hat{F}(a, 0) - \hat{F}(a, 1)]$. Since F is bent, $\hat{f}_0(a)$ and $\hat{f}_1(a)$ are in $\{-2^t, 2^t\}$. By inspection of all possible case we see that $\hat{f}_0(a)$ and $\hat{f}_1(a)$ are in $\{-2^t, 0, -2^t\}$ for every a , which means that $\hat{f}_0(a)$ and $\hat{f}_1(a)$ are near-bent. Furthermore, we can check that in every case only one of $\hat{f}_0(a)$ and $\hat{f}_1(a)$ is 0.

• Conversely, now assume (a) and (b). By Lemma 13, this immediately implies that for every (a, η) in $\mathbb{F}_{2^{2t}}$ the weight of $\hat{F}(a, \eta) = \epsilon 2^t$ with $\epsilon \in \{-1, +1\}$ and this proves that F is bent. \square

3.2. Proof of Theorem 1.

Proof.

Let f_0 be a $(2t - 1)$ -near bent function. Let F be the $(2t)$ -boolean function whose components are f_0 and f_1 such that $f_1 = f_0 + tr$. Our task is to prove that if $D_1 f_0$ is a constant function, then F is a bent function.

First remark that f_1 also is a near-bent function. This means that \hat{f}_0 and \hat{f}_1 take their values in $\{-2^t, 0, 2^t\}$

Since $f_1 = f_0 + tr$ and $a \in \mathbb{F}_{2^{2t-1}}$, according to Lemma 11, c):

$$\hat{f}_1(a) = \hat{f}_0(a + 1).$$

Because $2t - 1$ is odd, observe that $tr(1) = 1$. Therefore, one element of $\{tr(a), tr(a + 1)\}$ is 0 and the other one is 1.

Step 1: $D_1 f_0 = 0$

Proposition 14 now shows that in this case, one element of $\{\hat{f}_0(a), \hat{f}_1(a)\}$ is 0 and the other one is 2^t or -2^t .

Finally, according to Lemma 13, a) and b):

For every (a, η) in $\mathbb{F}_{2^{2t-1}} : \hat{F}_{a,\eta} = 0 \pm 2^t = \pm 2^t$.

This is the proof that F is bent.

Step 2: $\omega = 1$

Following (R_2) , the $(2t - 1)$ -boolean function f_0^\dagger defined by

$$\begin{aligned} f_0^\dagger(x) &= f_0(x) + tr(x) \text{ is also a near-bent function. From this definition:} \\ D_1 f_0^\dagger(x) &= f_0(x + 1) + f_0(x) + tr(x + 1) + tr(x) = D_1 f_0(x) + tr(x) + \\ &tr(1) + tr(x) = 1 + 1 = 0. \end{aligned}$$

Let us consider the $(2t)$ -boolean function F^\dagger such that

$$\begin{aligned} \phi_{F^\dagger}(x, y) &= (y + 1)f_0^\dagger(x) + y(f_0^\dagger(x) + tr(x)) \\ &= (y + 1)(f_0(x) + tr(x)) + y(f_0(x) + tr(x) + tr(x)) \\ &= (y + 1)(f_0(x) + tr(x)) + yf_0(x). \end{aligned}$$

Since $D_1 f_0^\dagger = 0$, Step 1 shows that F^\dagger is a bent function.

Adding the linear form $T(1, 0)$ to F^\dagger gives another bent function (see (R_2)). This means:

$$(y + 1)(f_0^\dagger(x) + tr(x)) + y(f_0^\dagger(x) + tr(x) + tr(x)) = (y + 1)f_0(x) + y(f_0(x) + tr(x))$$

is the TVR of a bent function. This is the expected result. \square

3.3. Proof of Theorem 2.

First, we need the following proposition:

Proposition 15.

Let f_0 and f_1 be the components of a bent function F .

Let $\omega \in \mathbb{F}_2$.

$$D_1 f_0 = \omega \text{ if and only if } D_1 f_1 = \omega + 1$$

Proof.

$$D_{(0,1)} F(X) = F(X + (0, 1)) + F(X).$$

$$\begin{aligned} \text{The TVR of } D_{(0,1)} F \text{ is } &(y + 1)(f_0(x + 1) + f_0(x)) + y(f_1(x + 1) + f_1(x)) \\ &= (y + 1)D_1 f_0(x) + yD_1 f_1(x). \end{aligned}$$

$$\text{From } (R_3) : w(D_{(0,1)} F) = w(D_1 f_0) + w(D_1 f_1)$$

Furthermore, (R_4) shows that $w(D_{(0,1)} F) = 2^{2t-1}$. Thus:

$$(\div) \quad 2^{2t-1} = w(D_1 f_0) + w(D_1 f_1)$$

On the other hand, if f is any $(2t - 1)$ -boolean function then $w(D_1 f) = 2^{2t-1}$ is equivalent to $D_1 f = 1$ while $w(D_1 f) = 0$ is equivalent to $D_1 f = 0$. Therefore, (\div) proves that if one of the two derivatives $D_1 f_0$ and $D_1 f_1$ is 0, then the other one is 1. \square

Proof. of Theorem 2

$D_1 \tilde{f}_0 = 0$ means $\tilde{f}_0(u) = \tilde{f}_0(u+1)$ for all u in $\mathbb{F}_{2^{2t-1}}$.

Since \tilde{f}_0 is the restriction of \tilde{F} to $\mathbb{F}_{2^t}^{(0)} = \{(u, 0) \mid u \in \mathbb{F}_{2^{2t-1}}\}$, then in order to prove that $D_1 \tilde{f}_0 = 0$ it suffices to show that:

$$\forall u \in \mathbb{F}_{2^{2t-1}} : \tilde{F}(u, 0) = \tilde{F}(u+1, 0)$$

Using Lemma 13, since $f_0 + f_1 = tr$ we have successively:

$$\hat{f}_1(u) = \hat{f}_0(u+1).$$

$$\hat{F}(u, 0) = \hat{f}_0(u) + \hat{f}_1(u) = \hat{f}_0(u) + \hat{f}_0(u+1).$$

$$\hat{F}(u+1, 0) = \hat{f}_0(u+1) + \hat{f}_1(u+1) = \hat{f}_0(u+1) + \hat{f}_0(u+1).$$

$$\hat{F}(u, 0) = \hat{F}(u+1, 0)$$

Following (R_5) we deduce that $\tilde{F}(u, 0) = 1$ if and only if $\tilde{F}(u+1, 0) = 1$ for all $u \in \mathbb{F}_{2^{2t-1}}$ and then $\tilde{F}(u, 0) = \tilde{F}(u+1, 0)$.

$D_1 \tilde{f}_1 = 1$ is a direct consequence of Proposition 14. \square

3.4. Proof of Theorem 4.

Proof. According to Lemma 13, for every a in $\mathbb{F}_{2^{2t-1}}$:

$$\hat{f}_1(a) = \hat{f}_0(a+1) \text{ whence } \hat{F}(a, 0) = \hat{f}_0(a) + \hat{f}_0(a+1).$$

On the other hand, a remark after Proposition 14 says that one of $|\hat{f}_0(a)|$ and $|\hat{f}_1(a)|$ is equal to 2^t and the other one is equal to 0.

It follows that every a in $\mathbb{F}_{2^{2t-1}}$ belongs to one of the following sets:

$$\mathcal{A}_1 = \{a \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(a) = -2^t \text{ and } \hat{f}_0(a+1) = 0\}$$

$$\mathcal{A}_2 = \{a \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(a) = 0 \text{ and } \hat{f}_0(a+1) = -2^t\}$$

$$\mathcal{A}_3 = \{a \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(a) = 2^t \text{ and } \hat{f}_0(a+1) = 0\}$$

$$\mathcal{A}_4 = \{a \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(a) = 0 \text{ and } \hat{f}_0(a+1) = 2^t\}$$

The definition of the dual of F induces that (a, η) is in the support of \tilde{F} if and only if $\hat{F}(a, \eta) = -2^t$. If we remark:

$$\hat{F}(a, 0) = -2^t \text{ if } a \in \mathcal{A}_1 \text{ or } a \in \mathcal{A}_2,$$

$$\hat{F}(a, 0) = 2^t \text{ if } a \in \mathcal{A}_3 \text{ or } a \in \mathcal{A}_4.$$

we deduce that, $(a, 0)$ is in the support of \tilde{F} if and only if $a \in \mathcal{A}_1 \cup \mathcal{A}_2$.

In other words the support of \tilde{f}_0 is $\mathcal{S}_0 = \mathcal{A}_1 \cup \mathcal{A}_2$.

From the descriptions of \mathcal{A}_1 and \mathcal{A}_2 , if $a \in \mathcal{A}_1$ then $a+1 \in \mathcal{A}_2$ and if $b \in \mathcal{A}_2$ then $b = a+1$ with $a = b+1 \in \mathcal{A}_1$. Hence $\mathcal{A}_2 = \{u+1 \mid u \in \mathcal{A}_1\}$.

Finally, by inspection we see that \mathcal{A}_1 is nothing but the set

$$\mathcal{S} = \{v \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(v) = -2^t\} \text{ and this leads to result 1).}$$

By Lemma 13: $\hat{F}(a, 1) = \hat{f}_0(a) - \hat{f}_1(a)$ and since $f_0 + f_1 = tr$ then:

$$\hat{f}_1(u) = \hat{f}_0(u+1). \text{ Hence, } (a, 1) \text{ is in the support of } \tilde{F} \text{ if and only if } a$$

is in \mathcal{A}_1 or in \mathcal{A}_4 . Hence the support of \tilde{f}_1 is $\mathcal{T}_1 = \mathcal{A}_1 \cup \mathcal{A}_4$. It can be easily seen that \mathcal{T}_1 is the symmetric difference of the support of \tilde{f}_0 and of $\mathcal{G} = \{v \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(v) = 0\}$. This immediatly gives result 2). \square

3.5. A fundamental lemma.

We need the following lemma which is an important point for the next proofs and is a generalization of a classical result on the Gold function.

Lemma 16.

Let f be a $(2t - 1)$ -near-bent function.

If $D_1f = 0$ then: $\hat{f}(u) = 0$ if and only if $tr(u) = 1$

If $D_1f = 1$ then: $\hat{f}(u) = 0$ if and only if $tr(u) = 0$

Proof. of the lemma

Assume that $D_1f = \omega$ with $\omega \in \mathbb{F}_2$ which means that $f(x + 1) = f(x) + \omega$. The transform $\tau : x \rightarrow x + 1$ is a permutation of $\mathbb{F}_{2^{2t-1}}$ and then preserves the weight of every $(2t - 1)$ -boolean function. Thus:

$$\#\{x \mid f(x) + tr(ux) = 1\} = \#\{x \mid f(x + 1) + tr(u(x + 1)) = 1\}.$$

$$(E) \quad \#\{x \mid f(x) + tr(ux) = 1\} = \#\{x \mid f(x) + \omega + tr(ux) + tr(u) = 1\}.$$

If $tr(u) + \omega = 1$ the right hand member of (E) is:

$$\#\{x \mid f(x) + tr(ux) = 0\} = 2^{2t-1} - \#\{x \mid f(x) + tr(ux) = 1\}$$

Hence (E) becomes:

$$\#\{x \mid f(x) + tr(ux) = 1\} = 2^{2t-1} - \#\{x \mid f(x) + tr(ux) = 1\}$$

In other words $w(f + t_u) = 2^{2t-1} - w(f + t_u)$ and thus:

If $tr(u) + \omega = 1$: $w(f) + t_u = 2^{2t-2}$ which is equivalent to $\hat{f}(u) = 0$. For $\omega = 0$ or $\omega = 1$ the number of u such that $tr(u) + \omega = 1$ is 2^{2t-2} and (R_6) says that this is also the number of u such that $\hat{f}(u) = 0$. Then, immediately: $\hat{f}(u) = 0$ if and only if $tr(u) + \omega = 1$. Finally if $\omega = 0$ then $\hat{f}(u) = 0$ if and only if $tr(u) = 1$ and if $\omega = 1$ then $\hat{f}(u) = 0$ if and only if $tr(u) = 0$. \square

Comment: In [5], Gold proves that if $f(x) = tr(x^{2^k+1})$ where k prime to $2t - 1$ then $\hat{f}(a) = 0$ if and only if $tr(a) = 0$. Since $D_1f = 1$ as it is mentioned in the proof of Proposition 9, the previous lemma is a generalization of the result of Gold.

3.6. Proof of Theorem 5.

Proof.

Since $h_0 + h_1 = tr$ then Theorem 4 says that $\tilde{h}_0 + \tilde{h}_1$ is the characteristic function of $\mathcal{H} = \{v \in \mathbb{F}_{2^{2t-1}} \mid \hat{h}_0(v) = 0\}$. We know from Lemma 16 that if $D_1h_0 = 0$ then $\hat{h}_0(u) = 0$ is equivalent to $tr(u) = 1$ and if $D_1h_0 = 1$ then $\hat{h}_0(u) = 0$ is equivalent to $tr(u) = 0$. This the same as saying that the characteristic function of \mathcal{H} is tr if $D_1h_0 = 0$ and is $tr + 1$ if $D_1h_0 = 1$ and this is the expected result. \square

3.7. Proof of Theorem 7.

Proof.

Notations: The components of the considered boolean functions are:

f_0, f_1 for F and \tilde{f}_0, \tilde{f}_1 for the dual \tilde{F} of F .

$\bar{f}_0^{(0)}, \bar{f}_1^{(0)}$ for the pseudo-dual \bar{F}_0 and $\bar{f}_0^{(1)}, \bar{f}_1^{(1)}$ for the pseudo-dual \bar{F}_1 .

$\tilde{\tilde{f}}_0^{(0)}, \tilde{\tilde{f}}_1^{(0)}$ for the dual $\tilde{\tilde{F}}_0$ of \bar{F}_0 and $\tilde{\tilde{f}}_0^{(1)}, \tilde{\tilde{f}}_1^{(1)}$ for the dual $\tilde{\tilde{F}}_1$ of \bar{F}_1 .

Proof of A):

Since $f_0 + f_1 = tr$ then $D_1 \tilde{f}_0 = 0$ and $D_1 \tilde{f}_1 = 1$ (Theorem 2).

We deduce from Theorem 1 that \bar{F}_0 and \bar{F}_1 are bent functions.

Proof of B) and C):

From the definitions of the duals:

$$\bar{f}_0^{(0)} = \tilde{f}_0, \bar{f}_0^{(1)} = \tilde{f}_1, \bar{f}_0^{(0)} + \bar{f}_1^{(0)} = tr, \bar{f}_0^{(1)} + \bar{f}_1^{(1)} = tr.$$

Hence, according to Theorem 2: $D_1 \tilde{\tilde{f}}_0^{(0)} = 0$ and $D_1 \tilde{\tilde{f}}_0^{(1)} = 0$ and thus $\tilde{\tilde{F}}_0$ and $\tilde{\tilde{F}}_1$ meet (\mathcal{C}) .

Now applying Theorem 5 to \bar{F}_0 with $h_0 = \bar{f}_0^{(0)}$ and $h_1 = \bar{f}_1^{(0)}$, we get $h_0 + h_1 = tr$. Since $D_1 \bar{f}_0^{(0)} = 0$ then $\tilde{h}_0 + \tilde{h}_1 = \tilde{\tilde{f}}_0^{(0)} + \tilde{\tilde{f}}_1^{(0)} = tr$. That is $\tilde{\tilde{F}}_0$ meets (\mathcal{T}) with $\xi = 0$.

Similarly, for \bar{F}_1 with $h_0 = \bar{f}_0^{(1)}$ and $h_1 = \bar{f}_1^{(1)}$ we have $h_0 + h_1 = tr$. Remark $D_1 \bar{f}_0^{(1)} = D_1 \tilde{f}_1$ whence, again from Theorem 2, $D_1 \bar{f}_0^{(1)} = 1$ and Theorem 5 gives $\tilde{\tilde{f}}_0^{(0)} + \tilde{\tilde{f}}_1^{(0)} = tr + 1$ and thus $\tilde{\tilde{F}}_1$ meets (\mathcal{T}) with $\xi = 1$. \square

4. EXAMPLES

If (\mathcal{C}) or (\mathcal{T}) hold for a bent function F , the bent functions F , the dual \tilde{F} , the pseudo-duals \bar{F}_0, \bar{F}_1 and the duals of these pseudo-duals can be distinct or not. The examples below show different situations. For every following example, condition (\mathcal{T}) is satisfied for the initial bent function F .

$t = 4$ and tr is the trace function of \mathbb{F}_{2^7}

Example 1.

$$F : \begin{aligned} f_0(x) &= tr(x^7 + x^{13}) \quad \text{not } (\mathcal{C}), (\mathcal{T}) \\ f_1(x) &= f_0(x) + tr(x) \end{aligned}$$

$$\tilde{F} : \begin{aligned} \tilde{f}_0(x) &= tr(x^5 + x^7 + x^9 + x^{13} + x^{19} + x^{21}) \\ \tilde{f}_1(x) &= \tilde{f}_0(x) + tr(x + x^5 + x^9) \end{aligned}$$

$$\bar{F}_0 : \begin{aligned} \bar{f}_0^{(0)}(x) &= \tilde{f}_0(x), \bar{f}_1^{(0)}(x) = \tilde{f}_0(x) + tr(x) \end{aligned}$$

$$\bar{F}_1 : \begin{aligned} \bar{f}_0^{(1)}(x) &= \tilde{f}_1(x), \bar{f}_1^{(1)}(x) = \tilde{f}_1(x) + tr(x) \end{aligned}$$

$$\tilde{\tilde{F}}_0 : \begin{aligned} \tilde{\tilde{f}}_0^{(0)}(x) &= tr(x + x^7 + x^9 + x^{13} + x^{19} + x^{21}), \\ \tilde{\tilde{f}}_1^{(0)}(x) &= \tilde{\tilde{f}}_0^{(0)}(x) + tr(x) \end{aligned}$$

$$\begin{aligned}\tilde{\tilde{F}}_1 : \quad & \tilde{\tilde{f}}_0^{(1)}(x) = \text{tr}(x + x^3 + x^7 + x^{13} + x^{19} + x^{21}) \\ & \tilde{\tilde{f}}_1^{(1)}(x) = \tilde{\tilde{f}}_0^{(1)}(x) + \text{tr}(x + 1)\end{aligned}$$

Example 2.

$$\begin{aligned}F : \quad & f_0(x) = \text{tr}(x^{15} + x^{27} + x^{29} + x^{43}), \quad \text{not } (\mathcal{C}), (\mathcal{T}) \\ & f_1(x) = f_0(x) + \text{tr}(x)\end{aligned}$$

$$\begin{aligned}\tilde{F} : \quad & \tilde{f}_0(x) = \text{tr}(x + x^3 + x^5 + x^9) \\ & \tilde{f}_1(x) = \tilde{f}_0(x) + \text{tr}(x^5 + x^7 + x^{11} + x^{19} + x^{21})\end{aligned}$$

$$\bar{F}_0 : \quad \bar{f}_0^{(0)}(x) = \tilde{f}_0(x), \bar{f}_1^{(0)}(x) = \tilde{f}_0(x) + \text{tr}(x)$$

$$\bar{F}_1 : \quad \bar{f}_0^{(1)}(x) = \tilde{f}_1(x), \bar{f}_1^{(1)}(x) = \tilde{f}_1(x) + \text{tr}(x)$$

$$\tilde{\tilde{F}}_0 = \bar{F}_0$$

$$\begin{aligned}\tilde{\tilde{F}}_1 : \quad & \tilde{\tilde{f}}_0^{(1)}(x) = \text{tr}(x + x^3 + x^5 + x^7 + x^9 + x^{11} + x^{19} + x^{21}) \\ & \tilde{\tilde{f}}_1^{(1)}(x) = \tilde{\tilde{f}}_{0,1}^{(1)}(x) + \text{tr}(x + 1)\end{aligned}$$

Example 3.

$$\begin{aligned}F : \quad & f_0(x) = \text{tr}(x + x^3 + x^7 + x^{11} + x^{19} + x^{21}), \quad (\mathcal{C}), (\mathcal{T}) \\ & f_1(x) = f_0(x) + \text{tr}(x)\end{aligned}$$

$$\begin{aligned}\tilde{F} : \quad & \tilde{f}_0(x) = \text{tr}(x^7 + x^{11} + x^{19} + x^{21}) \\ & \tilde{f}_1(x) = \tilde{f}_0(x) + \text{tr}(x)\end{aligned}$$

$$\bar{F}_0 = \tilde{F} \text{ and } \tilde{\tilde{F}}_0 = F \quad \bar{F}_1 = \tilde{F} \text{ and } \tilde{\tilde{F}}_1 = F:$$

Example 4.

$$\begin{aligned}F : \quad & f_0(x) = \text{tr}(x^3 + x^5 + x^7 + x^{11} + x^{19} + x^{21}), \quad (\mathcal{C}), (\mathcal{T}) \\ & f_1(x) = f_0(x) + \text{tr}(x)\end{aligned}$$

$$\tilde{F} = F = \bar{F}_0 = \bar{F}_1 = \tilde{\tilde{F}}_0 = \tilde{\tilde{F}}_1$$

4.1. A special example and an open question.

In [7] the authors recall the definition of non-weakly-normal bent function and they introduce (Fact 13) an example of such a function F in dimension 12 defined by: $\phi_F(x, y) = (y + 1)\text{tr}(x^{241} + x) + y\text{tr}(x^{241})$ where tr is the trace of $\mathbb{F}_{2^{11}}$ over \mathbb{F}_2 .

We see that $f_1 + f_0 = \text{tr}$ and thus Theorem 7 holds.

An interesting open question is: are \tilde{F} , \bar{F}_0 , \bar{F}_1 , $\tilde{\tilde{F}}_0$, $\tilde{\tilde{F}}_1$ also non-weakly normal.

5. CONCLUSION

We have introduced a way to construct bent functions starting from a near-bent functions which fulfil the special condition of Theorem 1. Applying this Theorem and Theorem 7 we obtain six bent functions. An open question now is to describe explicitly the near-bent functions f such that D_1f is a constant function, for example by means of the trace function.

Another question is to express the characteristic function of the set \mathcal{S} which appears in Theorem 4, by using the trace function.

The results of this work could probably be generalized by replacing 1 with e such that $tr(e) = 1$ and $tr(x)$ with $tr(ex)$ in (\mathcal{C}) and (\mathcal{T}) .

6. REFERENCES

- [1] A.Canteault, C.Carlet,P.Charpin,C.Fontaine, On Cryptographic Properties of the Cosets of $R(1,m)$.
IEEE Transactions on Information Theory, vol.47, **4**, (2001),1494-1513.
- [2] A.Canteault,P.Charpin, Decomposing Bent Functions.
IEEE Transactions on Information Theory, vol.49, **8**, (2003),2004-2019.
- [3] J.F.Dillon, Elementary Hadamard Difference Sets.
Ph.D. Thesis, University of Maryland(1974).
- [4] J.F.Dillon, Multiplicative difference sets via additive characters.
Des. Codes Cryptogr., vol.17, **4**, (1999) 225-235.
- [5] R.Gold, Maximal recursive squences with 3-valued recursive cross-correlation functions.
IEEE Transactions on Information Theory, vol.14, (1968),154-156.
- [6] X.D. Hou, Cubic Bent Functions.
Discrete Mathematics,**189**(1998), 149-161.
- [7] G. Leander, G. McGuire, Construction of Bent Functions from Near-Bent Functions.
Journal of Combinatorial Theory, Series A,vol.116,**4**,(2009),960-970.
- [8] O.S.Rothaus, On Bent Functions.
Journal of Combinatorial Theory, series A,**20**(1976), 300-305.
- [9] J.Wolfmann, Bent Functions and Coding Theory.
in *Difference Sets, Sequences and their Correlation properties* (A. Pott, P.V. Kumar, T. Helleseth, D. Jungnickel, Eds), NATO Sciences Series, Series C, vol.542, Kluwer Academic Publishers (1999) 393-418.

[10] J.Wolfmann, Cyclic Code Aspects of Bent Functions.
in *Finite Fields Theory and Applications, Contemporary Mathematics series of the AMS*, vol. 518, Amer.Math Soc. Providence, RI,
(2010), 363-384.

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