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FROM LOCAL TO GLOBAL EQUILIBRIUM STATES: THERMODYNAMIC FORMALISM VIA AN INDUCING SCHEME

RENAUD LEPLAIDEUR

ABSTRACT. We present a method to construct equilibrium states via induction. This method can be used for some non-uniformly hyperbolic dynamical systems and for non-Hölder continuous potentials. It allows to prove the occurrence of phase transition.

1. SETTINGS

1.1. **Goal.** We consider a dynamical system (X, f) , where X is a compact metric space and f is topology mixing and local homeomorphism. For x in X , f_x^{-1} is the inverse branch defined by x . It is a homeomorphism defined on a neighborhood of $f(x)$ onto its image which is a neighborhood of x .

If $\phi : X \rightarrow \mathbb{R}$ is a continuous function or at least a Borel function, the pressure of ϕ is

$$\mathcal{P} := \sup \left\{ h_\mu(f) + \int \phi d\mu \right\},$$

where $h_\mu(f)$ is the Kolmogorov entropy. The supremum is taken over the set of invariant probabilities. A measure which realizes the supremum is called an *equilibrium state for ϕ* . In the following it will be also referred to as a *global equilibrium state*. In the following, *doing the thermodynamic formalism* means study existence, uniqueness and other properties of global equilibrium states.

It is well-known that if (X, f) is uniformly hyperbolic and ϕ is Hölder continuous, then there exists a unique equilibrium state (see *e.g.* [2, 7]). In that case it is also a *Gibbs state* and has full support. The main heuristic explanation for this result is that hyperbolicity and Hölder regularity combine themselves and allow to construct the equilibrium state via the spectral elements of the *transfer operator*.

The existence of equilibrium states for less regular potentials and/or for systems with weaker hyperbolicity properties is a challenging task. In that case, the study of the spectral properties of the transfer operator is usually much harder. It turns out that, for several cases, one strategy is to consider induction.

Induction may also be a solution to deal with another question, related to the notion of *phase transition*. Beyond the question of existence of some equilibrium state, which can follow from abstract properties, one may want to get information on that equilibrium state as *e.g.* if it has or does not have full support. For this kind of question, induction can be a solution; it is actually a way to prove that an equilibrium state gives or does not give positive weight to some special set (see *e.g.* [3]).

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The goal of this note is to present a method based on induction to answer to these two previous questions. This method was actually presented in [9] for the case of uniformly hyperbolic dynamical system and Hölder continuous potential. It was then developed and extended in further works of the author; the note summarizes here all the principal steps using more general settings.

1.2. Markov set and induced system. We assume there exists some Markov set R : this is a proper set $R = \overline{\mathring{R}}$ such that for every n and for every $x \in \mathring{R} \cap f^{-n}(\mathring{R})$, the set $C_{n,f}(x) := f_x^{-n}(R)$ is contained in R and $f^n(C_{n,f}(x)) = R$.

We consider the induced subsystems (R, F) where F is the first return map:

$$F(x) = f^{\tau(x)}(x) \text{ if } \tau(x) := \min\{k > 0, f^k(x) \in R\}.$$

It is not necessarily well-defined everywhere, but it is however defined μ -a.e. for any invariant probability μ satisfying $\mu(R) > 0$ (Poincaré's recurrence theorem). The main point is that the Markov property allows to well define the inverse branches: if x belongs to \mathring{R} and $F(x)$ is well defined and also belongs to \mathring{R} , then we set $C_{1,F}(x) := C_{\tau(x),f}(x)$. It is called the *1-cylinder of x (with respect to F)*. The integer $\tau(x)$ is called the *return-time* of the 1-cylinder. By construction $C_{1,F}(x)$ is a proper set, and the intersection of two different 1-cylinders has empty interior. They also form a *partition* of R ,

- up to points which never return to R by iterations of F ,
- and up to the fact that two 1-cylinders may have non-empty intersection (on their borders).

For x in the interior of a 1-cylinder C , $\tau(x)$ is well-defined and coincides with the return-time of the cylinder; this may not hold for point in ∂C . However we set $F(x) = f^n(x)$ if x belongs to ∂C and the return time for C is n . We point out that F may thus be multi-valued on these points, but these definitions coincide with the inverse branches.

Therefore, the Markov property yields that for each x in R and for each 1-cylinder C , there exists a unique $x' \in C$ such that $F(x') = x$. The set of x' 's is denoted as $Pre(x)$.

The main question we are interested in is :

Question 1. Is there a way to do the Thermodynamic formalism for (R, F) and to recover (or to study the properties of) the equilibrium state for (X, f) and ϕ ?

1.3. Hypothesis on ϕ . Consider some f -invariant probability measure $\widehat{\mu}$. We assume $\widehat{\mu}(R) > 0$. Then, the conditional measure $\mu := \frac{\widehat{\mu}(\cdot \cap R)}{\widehat{\mu}(R)}$ is F -invariant. We remind the notation

$$S_n(\phi) := \phi + \phi \circ T + \dots + \phi \circ T^{n-1}.$$

In particular $S_{\tau(\cdot)}(\phi)(\cdot)$ maps x to $S_{\tau(x)}(\phi)(x)$.

By the Abramov formula (see [13] p. 257-258) we get

$$\begin{aligned}
 h_{\widehat{\mu}}(f) + \int \phi d\widehat{\mu} &\leq \mathcal{P} \text{ with equality iff } \widehat{\mu} = \text{equil. state} \\
 &\Downarrow \\
 h_{\widehat{\mu}}(f) + \int \phi d\widehat{\mu} - \mathcal{P} &\leq 0 \text{ with equality iff } \widehat{\mu} = \text{equil. state} \\
 &\Downarrow \\
 \widehat{\mu}(R) \left(h_{\mu}(F) + \int S_{\tau(\cdot)}(\phi) - \mathcal{P} \cdot \tau(\cdot) d\mu \right) &\leq 0 \text{ with equality iff } \widehat{\mu} = \text{equil. state} \\
 &\Downarrow \\
 h_{\mu}(F) + \int S_{\tau(\cdot)}(\phi) - \mathcal{P} \cdot \tau(\cdot) d\mu &\leq 0 \text{ with equality iff } \widehat{\mu} = \text{equil. state}
 \end{aligned}$$

This simple sequence of inequalities shows that the thermodynamic formalism for (X, f) and ϕ is related to the thermodynamic formalism for (R, F) and $S_{\tau(\cdot)}(\phi)(\cdot)$.

For x in $C_{1,F}(y) = C_{n,f}(y) \subset F$, we set $\Phi(x) := S_n(\phi)(x)$. Our main assumptions on ϕ are that it is possible to study the thermodynamic formalism for Φ . Hypotheses are listed along the way. We first assume:

(H1) Φ is continuous on each 1-cylinder.

(H2) There exists C such that for x and y in the same 1-cylinder, $|\Phi(x) - \Phi(y)| \leq C$.

We set for $Z \in \mathbb{R}$,

$$(1) \quad \mathcal{L}_Z(\psi)(x) := \sum_{y \in \text{Pre}(x)} e^{\Phi(y) - Z \cdot \tau(y)} \psi(y).$$

This is the transfer operator for (R, F) and for the potential $\Phi - Z \cdot \tau(\cdot)$.

Question 2. For which Z can we do thermodynamic formalism ?

Definition 1.1. Any equilibrium state for (R, F) and for $\Phi - Z \cdot \tau(\cdot)$ is called a local equilibrium state (associated to the parameter Z). It will be denoted by μ_Z (if it exists).

Question 3. Among the measures μ_Z , can we recover/find an equilibrium state for (X, f) and ϕ ?

Roughly speaking, Question 3 means that we want to find Z such that the local equilibrium state say μ_Z for $\Phi - Z \cdot \tau(\cdot)$ is the induced measure of a global equilibrium state (for (X, f) and ϕ). It is a reformulation of Question 1.

2. ANSWERS TO QUESTIONS

2.1. Local thermodynamic formalism.

Proposition 2.1. There exists a critical $Z_c \geq -\infty$ such that

–for every $Z < Z_c$ and for every $x \in R$, $\mathcal{L}_Z(\mathbb{1}_R)(x) = +\infty$,

–and for every $Z > Z_c$, for every $\psi : R \rightarrow \mathbb{R}$ continuous and for every $x \in R$, $\mathcal{L}_Z(\psi)(x)$ converges.

Theorem 1 (see [4], Lem. 3.4). If (X, f) is a subshift of finite type and R is a cylinder and ϕ is continuous, then Z_c is the pressure for ϕ of the set of points whose trajectory avoids R .

Proposition 2.2. *If ϕ is continuous, then $Z_c \leq \mathcal{P}$.*

Proof. See Prop. 3.10 in [12] for a proof with a possible discontinuous potential. \square

From now on, we assume that ϕ has some regularity such that for every $Z > Z_c$, \mathcal{L}_Z satisfies the hypothesis of the Ionescu-Tulcea & Marinescu theorem with some Banach spaces $(\mathcal{V}, \|\cdot\|_{\mathcal{V}}) \subset \mathcal{C}^0(R)$:

- (i) if $(\varphi_n)_{n \in \mathbb{N}}$ is a sequence of functions in \mathcal{V} which converges in $\mathcal{C}^0(R)$ to a function φ and if for all $n \in \mathbb{N}$, $\|\varphi_n\|_{\mathcal{V}} \leq C$ for some $C > 0$, then $\varphi \in \mathcal{V}$ and $\|\varphi\|_{\mathcal{V}} \leq C$,
- (ii) \mathcal{L}_Z leaves \mathcal{V} invariant and is bounded for $\|\cdot\|_{\mathcal{V}}$;
- (iii) there exists $M_Z > 0$ such that $\sup_n \{\|\mathcal{L}_Z^n(\varphi)\|_{\infty}, \varphi \in \mathcal{V}, \|\varphi\|_{\infty} \leq 1\} \leq M_Z < \infty$;
- (iv) there exists an integer n_0 and two constants $0 < a < 1$ and $0 \leq b < +\infty$ such that for all $\varphi \in \mathcal{V}$ we have $\|\mathcal{L}_Z^{n_0}(\varphi)\|_{\mathcal{V}} \leq a\|\varphi\|_{\mathcal{V}} + b\|\varphi\|_{\infty}$;
- (v) if \mathcal{X} is a bounded subset of $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ then $\mathcal{L}_Z^{n_0}(\mathcal{X})$ has compact closure in $\mathcal{C}^0(R)$.

Under these hypotheses, \mathcal{L}_Z is quasi-compact on \mathcal{V} : the spectrum is the union of finitely many isolated complex numbers which are eigenvalues with strictly dominating modulus and the essential spectrum contained in an open disk of radius the strictly smaller than the already mentioned isolated eigenvalues. Moreover, the spectral radius λ_Z is a dominating eigenvalue (see [9]). The space \mathcal{V} is ‘‘morally’’ the space of Hölder continuous functions. This is for instance the case in [9] or even in [12] despite the potential not being continuous¹. The space of Hölder functions can also be used for the Hofbauer potential of the Manneville-Pomeau map (see [6, 14] or Subsection 3.3).

Remark 1. We emphasize that in the case that \mathcal{V} is the set of Hölder continuous functions, hypothesis (iii) of Ionescu-Tulcea & Marinescu theorem is a direct consequence of (H2). \blacksquare

Theorem 2. *For every $Z > Z_c$, there exists a unique local equilibrium state for (R, F) and $\Phi - Z\tau(\cdot)$. It is a Gibbs measure (with respect to F) and the pressure is $\log \lambda_Z$.*

The same holds for $Z = Z_c$ if $\mathcal{L}_Z(\mathbb{1}_R)$ converges for $Z = Z_c$ and hypotheses of Ionescu-Tulcea & Marinescu theorem hold too.

Remark 2. Note that in the case that \mathcal{V} is the set of Hölder continuous functions on R , the hypotheses of Ionescu-Tulcea & Marinescu theorem hold if and only if $\mathcal{L}_Z(\mathbb{1}_R)$ converges for $Z = Z_c$. \blacksquare

Assuming Theorem 2 holds, the local equilibrium state μ_Z is then of the form

$$d\mu_Z := H_Z d\nu_Z \text{ with } \mathcal{L}_Z(H_Z) = \lambda_Z H_Z \text{ and } \mathcal{L}_Z^*(\nu_Z) = \lambda_Z \nu_Z.$$

Continuity of H_Z and positivity of \mathcal{L}_Z yield that H_Z is strictly positive. The mixing hypothesis also shows that μ_Z has full support in R .

2.2. Local and global equilibria. The main question we are interested in is to know if among these μ_Z , one could be the restriction of a/the global equilibrium state for (X, f) and ϕ .

It is well known (see [5]) that there exists an f -invariant probability measure $\widehat{\mu}_Z$ such that $\mu_Z = \frac{\widehat{\mu}_Z(\cdot)}{\widehat{\mu}_Z(\cdot \cap R)}$ if and only if

$$(2) \quad \int \tau d\mu_Z < +\infty.$$

¹ Actually in this case the potential is Hölder continuous except on a single point where it is not continuous.

Now we have

Theorem 3. *Inequality (2) holds if and only if for some $x \in R$ $\mathcal{L}_Z(\tau)(x)$ converges. This is in particular the case if $Z > Z_c$.*

In this case the measure $\widehat{\mu}_Z$ has full support (due to mixing) and satisfies

$$(3) \quad h_{\widehat{\mu}_Z}(f) + \int \phi d\widehat{\mu}_Z = Z + \widehat{\mu}_Z(R) \log \lambda_Z.$$

Corollary 2.3. *For every $Z \geq \mathcal{P}$, $\lambda_Z \leq 1$.*

Proof. Equation (3) yields the desired inequality for $Z > \mathcal{P} \geq Z_c$ and continuity² in Z shows it also holds for $Z = Z_c$ \square

One important point is that $Z \mapsto \lambda_Z$ is decreasing and analytic on $]Z_c, +\infty[$ or even on $[Z_c, +\infty[$ if $\mathcal{L}_{Z_c}(\mathbb{1}_R)$ converges. Moreover, for $Z > Z_c$

$$(4) \quad \frac{d \log \lambda_Z}{dZ} = \frac{-1}{\widehat{\mu}_Z(R)}.$$

It follows that $Z \mapsto Z + \widehat{\mu}_Z(R) \log \lambda_Z$ attains its maximum either at the unique point Z where $\lambda_Z = 0$, or at Z_c if $\lambda_Z < 1$ for every $Z > Z_c$.

Then the main theorem is:

Theorem 4. *With the previous assumptions.*

- If $\lim_{Z \rightarrow Z_c} \log \lambda_Z < 1$ or $\lim_{Z \rightarrow Z_c} \log \lambda_Z = 1$ but $\lim_{Z \rightarrow Z_c} \mathcal{L}(\tau) = +\infty$, then, no global equilibrium state $\widehat{\mu}$ for ϕ gives positive weight to R .
- If $\lim_{Z \rightarrow Z_c} \log \lambda_Z > 1$ or $\lim_{Z \rightarrow Z_c} \log \lambda_Z = 1$ but $\lim_{Z \rightarrow Z_c} \mathcal{L}(\tau) < +\infty$, and if there is one global equilibrium state $\widehat{\mu}$ such that $\widehat{\mu}(R) > 0$, then
 - (1) $\widehat{\mu}$ is the unique equilibrium state with full support,
 - (2) The unique Z such that $\lambda_Z = 1$ is $Z = \mathcal{P}$,
 - (3) $\widehat{\mu} = \widehat{\mu}_{\mathcal{P}}$.

Remark 3. If (X, f) is a subshift of finite type and ϕ is continuous, Theorem 1 and existence of the global equilibrium state show that the condition $Z_c < \mathcal{P}$ yields the existence of some global equilibrium state giving positive weight to R . Consequently, there is uniqueness of the global equilibrium state and it is equal to $\widehat{\mu}_{\mathcal{P}}$. \blacksquare

3. APPLICATIONS

3.1. For non-uniformly hyperbolic dynamics. In [12, 11] the method is used for a horseshoe with homoclinic tangency. The potential is $\phi(x) := -\log J^u(x) := -\log \det Df|_{E^u}$. It is non-continuous due to the homoclinic tangency. Authors prove the existence and uniqueness of a global equilibrium state for $\beta \cdot \phi$ and for every $\beta \in \mathbb{R}$.

²following from the theorem of monotone convergence.

3.2. For uniformly hyperbolic dynamics. In [3], the method is used to produce a freezing phase transition at positive temperature with ground state supported on a quasi-crystal. Namely, it is proved that there exists a continuous potential on $\{0, 1\}^{\mathbb{N}}$, say ϕ , and some $\beta_c > 0$ such that the graphs of the pressure function for $\beta \cdot \phi$ is strictly convex for $\beta \in [0, \beta_c]$ and a half-line for $\beta \geq \beta_c$. Moreover, there exists a unique global equilibrium state for $\beta \cdot \phi$ (may be except for $\beta = \beta_c$). It has full support for $\beta < \beta_c$ and is supported on some uniquely ergodic and zero-entropy (and different to a periodic orbit) for $\beta > \beta_c$.

In [8] the method is used to construct a mixing system with a non-flat phase transition. It is also proved that after the transition, the system may have co-existence of several global equilibrium states despite the pressure function remains analytic.

In [10], the method is used to prove convergence at temperature zero of the global equilibrium state in a subshift of finite type and for a locally constant potential. Induction allows to control the different basins of different ergodic ground states, and to estimate how their relative weights vary in function of the inverse of the temperature β .

3.3. Hofbauer potential or Manneville-Pomeau map. To finish with a simple example, we apply the method to the Hofbauer potential in $\{0, 1\}^{\mathbb{N}}$

$$\phi(x) = \begin{cases} -\log(1 + \frac{1}{n}) & \text{if } x = 0^n 1 \dots, \\ -A < 0 & \text{if } x = 1 \dots \end{cases}$$

This case is usually associated to the Manneville-Pomeau map, say *e.g.*

$$f : [0, 1] \circlearrowleft x \mapsto \begin{cases} \frac{x}{1-x} & \text{if } x \in [0, \frac{1}{2}], \\ 2x \bmod 1 & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

One can find in [1] a description of why these two cases are associated, and actually similar.

In that case we induce on the cylinder [1]. Note that only one orbit does not enter into [1], and it is $0^\infty = 000\dots$. Moreover, for any $x \in [1]$, and for every $\beta > 0$

$$\mathcal{L}_{\beta, Z}(\mathbb{1}_{[1]})(x) = e^{-\beta \cdot A} \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \right)^\beta e^{-Z \cdot (n+1)}.$$

For every $\beta \geq 0$, this series converges if $Z > 0$ and diverges for $Z < 0$. Therefore $Z_c = 0$, and we point-out that

$$0 = h_{\delta_{0^\infty}} + \beta \cdot \int \phi d\delta_{0^\infty}.$$

which is the *ad'hoc* reformulation of Theorem 1.

Now, the form of the potential also yields $\lambda_{\beta, Z} = \mathcal{L}_{\beta, Z}(\mathbb{1}_{[1]})(x)$ for any x in [1]. Let us study the critical case $Z = Z_c$:

$$\lambda_{\beta, 0} := e^{-\beta \cdot A} \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \right)^\beta.$$

For $\beta \leq 1$, $\lambda_{\beta, 0} = +\infty$. Furthermore, the function $\beta \mapsto \lambda_{\beta, 0}$ is decreasing on $]1, +\infty[$, goes to $+\infty$ if $\beta \rightarrow 1$ and goes to 0 if $\beta \rightarrow +\infty$. Therefore, there exists a unique β_c such that $\lambda_{\beta_c, 0} = 1$.

For $\beta > \beta_c$, no equilibrium state gives positive weight to [1], which means that δ_{0^∞} is the unique equilibrium state and the pressure is 0.

For $\beta < \beta_c$, the map $Z \mapsto \lambda_{\beta,Z}$ is decreasing, and there is a unique $Z = \mathcal{P}(\beta) > 0$ such that

$$\lambda_{\beta, \mathcal{P}(\beta)} = 1.$$

As $\mathcal{P}(\beta) > 0 = Z_c$, we are in the case of Theorem 3, and the associated measure $\widehat{\mu}_{\mathcal{P}(\beta)}$ satisfies

$$h_{\widehat{\mu}_{\mathcal{P}(\beta)}}(\sigma) + \beta \int \phi d\widehat{\mu}_{\mathcal{P}(\beta)} = \mathcal{P}(\beta) > 0.$$

This last inequality shows that δ_{0^∞} cannot be an equilibrium state, hence, there exists an equilibrium state which gives positive weight to [1], and it is $\widehat{\mu}_{\mathcal{P}(\beta)}$.

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