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# A universal set of qubit quantum channels

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We investigate the set of quantum channels acting on a single qubit. We provide an alternative, compact generalization of the Fujiwara-Algoet conditions for complete positivity to non-unital qubit channels, which we then use to characterize the possible geometric forms of the pure output of the channel. We provide universal sets of quantum channels for all unital qubit channels as well as for all extremal (not necessarily unital) qubit channels, in the sense that all qubit channels in these sets can be obtained by concatenation of channels in the corresponding universal set. We also show that our universal sets are essentially minimal.

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## I. INTRODUCTION

The science of quantum information has attracted tremendous interest over the last twenty years after it was realized that certain computational tasks might be solvable more efficiently on a quantum computer than on a classical computer [1]. For example, a quantum algorithm exists that can factorize a large integer in a time polynomial in the number of digits, whereas no such classical algorithm is known [2]. Other examples of quantum speed-up include algorithms for searching an unsorted database [3], the hidden subgroup problem [4], the approximation of Jones polynomials [5], or the solution of linear systems of equations [6] with a recent application to data fitting [7].

Traditionally, quantum information science was mostly concerned with unitary time evolution, which is appropriate for well isolated quantum systems. An important step in the development of quantum information theory was the realization that all unitary operations in the exponentially large Hilbert space of many qubits can be broken down into elementary unitary operations that act on only one or two qubits at the same time [8]. In fact, the combination of a single fixed entangling unitary gate that can be applied on any two qubits and the continuous set of all unitary operations on all single qubits form a “universal gate set”, which is at the heart of the circuit paradigm of quantum computing [9–13].

However, in reality no quantum system is perfectly isolated from its environment. At the very least, the need of state-preparation and the read-out of the results require interaction with the

external world. Interactions with the environment lead typically to decoherence and destroy the quantum effects such as interference [14–16] and entanglement which are vital for the quantum computational speed-up [17]. But couplings to the environment can also have beneficial effects. In particular, it is possible to create entanglement through purely dissipative processes [18–20]. Dissipative processes may be used to effectively confine the dynamics to a part of Hilbert space where decoherence is strongly reduced (“decoherence-free subspaces”, see [21–25]) and at the same time enable entangling quantum gates in a simple fashion [26, 27]. Indeed, non-unitary propagation of quantum states opens up a much larger field of operations, and it is desirable to achieve a thorough understanding of the set of these “quantum channels”, defined quite generally as linear, completely positive trace preserving maps of a density matrix [1, 28]. Quantum channels have also played an important role in terms of error models and for the development of quantum error correction [29–31].

Given the importance of universal unitary gate sets for unitary quantum computation, one might expect that simple universal sets of quantum channels might become equally important for exploring the full power of the most general quantum operations allowed by nature. However, the set of quantum channels is much larger and has more complicated geometry than the set of unitary operations [32]. Important early work, long before the rise of quantum information theory, has provided us with powerful tools that allow us to assess the crucial complete positivity of a channel, such as the Kraus decomposition [33], the Choi matrix [34], or the Lindblad form of Markovian equations of motion [35–37]. But the full understanding of completely positive maps, especially with respect to composition, remains a formidable mathematical problem.

Here we will restrict ourselves largely to single qubit channels. Ruskai and co-workers [38, 39], and Wolf and Cirac [40] have made important contributions which we will heavily use. Fujiwara and Algoet provided simple inequalities that characterize the set of all unital qubit channels, i.e. qubit channels that map the identity matrix on itself [41]. We provide a compact generalization of these conditions to the general non-unital case where the Bloch sphere is mapped to an ellipsoid contained in the Bloch sphere. These new conditions have more natural geometric interpretation than (equivalent) conditions presented in [39] and allow us to classify qubit channels in terms of their pure output (PO). We show in particular that a PO in the form of a circle of non-zero radius on the Bloch sphere is forbidden by the requirement of complete positivity. For unital qubit channels we derive a universal set of qubit channels in the sense that all unital qubit channels can be obtained by concatenation of channels from the universal set. Furthermore, we provide a set of universal channels for extremal (but not necessarily unital) qubit channels. Since all qubit channels

can be obtained by simple convex combination (i.e. random classical sampling) of extremal ones, this essentially solves the problem of a universal channel set for a single qubit.

Very recently, a different approach to universal families of qubit quantum channels has been pursued in [42]. The authors use universal unitary gates to approximate the unitary appearing in the Stinespring dilation of a qubit channel (realizing the channel as a unitary evolution on a larger space). Our approach is more intrinsic, since we do not refer to any dilations of channels. Our universal set contains only single qubit operations, whereas the CNOT gate, acting on 2 qubits, is used in [42]. However, the mathematical objects used in [42] are similar to ours: the superoperator matrix  $T$  (3) and the description of extreme channels, see Proposition V.2.

We begin with the introductory Section II. and III.A, where we recall well-known facts about qubit channels and their geometry. In Sections III.B and III.C we present new results about the minimal set of universal channels for all unital qubit channels. In Section IV.A we derive an alternative necessary and sufficient inequality that characterizes all non-unital qubit channels. In Section IV.B we classify all qubit channels by the number of PO that they have <sup>1</sup>. In Section V. we present new results on universal sets of channels for all extremal qubit channels.

## II. PARAMETRIZATION OF QUBIT CHANNELS

Let  $\mathcal{M}_d(\mathbb{C})$  be the set of complex  $d \times d$  matrices, and  $\mathcal{D}_d \subset \mathcal{M}_d(\mathbb{C})$  the set of density matrices, that is, positive Hermitian  $d \times d$  matrices with trace one. We denote by  $\mathcal{P}_d \subset \mathcal{D}_d$  the set of pure states (density matrices of rank one), and by  $\mathcal{U}_d$  the set of all  $d \times d$  unitary matrices. A channel  $\Phi : \mathcal{M}_d(\mathbb{C}) \rightarrow \mathcal{M}_d(\mathbb{C})$  is a *completely positive* and *trace preserving* linear map; in particular, it maps density matrices to density matrices. We denote by  $\mathcal{C}_d$  the set of channels acting on a  $d$ -dimensional quantum system.

An important class of channels is the class of unitary channels. These are linear mappings  $\Phi_U : \mathcal{M}_d(\mathbb{C}) \rightarrow \mathcal{M}_d(\mathbb{C})$  which map  $\rho \in \mathcal{M}_d(\mathbb{C})$  by a unitary conjugation to  $\Phi_U(\rho) = U\rho U^\dagger$ , where  $U \in \mathcal{U}_d$ . Slightly abusing notation, we shall write  $\mathcal{U}_d$  for the set of unitary channels.

It is known that for a  $d$ -dimensional Hilbert space (of, say,  $k$  qubits,  $d = 2^k$ ), an arbitrary unitary operation can be obtained by concatenation of the controlled-NOT (CNOT)-gate and the continuous set of all single qubit unitaries [8, 10–13]. Our aim is to find a minimal set such that any channel can be realized by iterated application of channels from that minimal set. More specifically,

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<sup>1</sup> After finishing this work we have learned that the main result in Section IV.B essentially follows from the result on the number of POs of quasi-extreme channels presented in [39] (we would like to thank Mary-Beth Ruskai for pointing that out). Our presentation of this result only involves elementary geometric considerations.

we have the following definition.

**Definition II.1.** A set of quantum channels  $\mathcal{F}_d \subset \mathcal{C}_d$  is said to be universal for a set of channels  $\tilde{\mathcal{C}}_d \subset \mathcal{C}_d$  if for all channels  $\Phi \in \tilde{\mathcal{C}}_d$  there exist channels  $\Phi_1, \dots, \Phi_n \in \mathcal{F}_d$  such that

$$\Phi = \Phi_n \circ \Phi_{n-1} \circ \dots \circ \Phi_2 \circ \Phi_1. \quad (1)$$

The two subsets  $\tilde{\mathcal{C}}_d \subset \mathcal{C}_d$  that we will investigate are unital qubit channels and extremal qubit channels, to be defined below. The CNOT gate together with single qubit unitaries form a universal set of *unitary* quantum channels in the case  $\tilde{\mathcal{C}} = \mathcal{U}_d$ .

For any set of channels  $\mathcal{F}$ , we denote by  $\langle \mathcal{F} \rangle_n$  the set of channels generated by  $n$  concatenations of elements from  $\mathcal{F}$  (as in (1)), and by  $\langle \mathcal{F} \rangle$  the set of all channels generated by  $\mathcal{F}$ , with no restriction on the number of concatenations. Our aim is to find a set  $\mathcal{F}$ , as small as possible, such that  $\langle \mathcal{F} \rangle = \tilde{\mathcal{C}}_d$ .

### A. Qubit channels

From now on we fix  $d = 2$  and we consider qubit channels  $\Phi \in \mathcal{C}_2$ . Any state  $\rho \in \mathcal{M}_2(\mathbb{C})$  can be expanded in the basis of Pauli matrices  $\sigma_i$  as  $\rho = \frac{1}{2} \sum_{i=0}^3 r_i \sigma_i$ , where  $r_i \in \mathbb{R}$  and  $\sigma_0 = I_2$  is the identity matrix in  $\mathcal{M}_2(\mathbb{C})$ . Normalization of the trace  $\text{tr} \rho = 1$  implies  $r_0 = 1$ . The components  $r_i$ ,  $i = 1, 2, 3$ , form the Bloch vector  $\mathbf{r} = (r_1, r_2, r_3)$ . Pure states have  $\|\mathbf{r}\| = 1$  and form the Bloch sphere, whereas the set of all other states inside the Bloch sphere correspond to mixed states ( $\text{tr} \rho^2 < 1$ ).

Any linear map  $\Phi \in \mathcal{C}_2$  acting on  $\rho = \frac{1}{2} I_2 + \frac{1}{2} \mathbf{r} \cdot \boldsymbol{\sigma}$ , with  $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \equiv (\sigma_x, \sigma_y, \sigma_z)$ , can be represented by a real  $4 \times 4$  matrix  $T$  that maps the components of  $r_i$  to new ones,

$$(1, r'_1, r'_2, r'_3)^t = T_\Phi (1, r_1, r_2, r_3)^t. \quad (2)$$

The most general linear completely positive map of  $\rho$  is then given by [32, 38, 39]

$$T_\Phi = \begin{pmatrix} 1 & 0_{1 \times 3} \\ \mathbf{t}_\Phi & M_\Phi \end{pmatrix} \quad (3)$$

with  $0_{1 \times 3} = (0, 0, 0)$ ,  $M_\Phi$  is a real  $3 \times 3$  matrix, and  $\mathbf{t}_\Phi \in \mathbb{R}$  a vector. It induces an affine map  $\mathbf{r}' = M_\Phi \mathbf{r} + \mathbf{t}_\Phi$  on the Bloch vector. Composition of two channels,  $\Phi = \Phi_2 \circ \Phi_1$ , implies

$$M_\Phi = M_{\Phi_2} M_{\Phi_1}, \quad (4)$$

$$\mathbf{t}_\Phi = M_{\Phi_2} \mathbf{t}_{\Phi_1} + \mathbf{t}_{\Phi_2}. \quad (5)$$

In the following we drop the index  $\Phi$  when it is clear what channel  $\mathbf{t}_\Phi$  and  $M_\Phi$  refer to. Qubit channels can thus be seen as maps acting on Bloch vectors thanks to the isomorphism

$$SU(2)/\mathbb{Z}_2 \cong SO(3). \quad (6)$$

In particular, the channel  $\Phi_U$  corresponding to unitary conjugation with  $U = \exp(i\phi \mathbf{n} \cdot \boldsymbol{\sigma}) = \cos(\phi)I_2 + i \sin(\phi)\mathbf{n} \cdot \boldsymbol{\sigma}$  is equivalent to a rotation  $R_U \in SO(3)$  of the Bloch vector about axis  $\mathbf{n}$  by an angle  $2\phi$ .

Complete positivity of a qubit channel can be characterized by the positivity of its Choi matrix  $C_\Phi$ . The *Choi matrix* of a channel  $\Phi$  is defined by

$$C_\Phi = [\Phi \otimes I_2](|\text{Bell}\rangle\langle\text{Bell}|), \quad (7)$$

where  $|\text{Bell}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$  is one of the Bell states for two qubits. Alternatively, the Choi matrix can be defined by a reshuffling of the indices of the propagator in the computational basis [32]. A channel is completely positive, if and only if the Choi matrix is non-negative [34]. The eigenvectors of the Choi matrix yield, after reshaping them to a matrix and multiplication with the square root of the corresponding eigenvalue, the Kraus operators  $A_i$  of the channel, defined through  $\Phi : \rho \mapsto \sum_{i=1}^r A_i \rho A_i^\dagger$ . The minimal number  $r$  of Kraus operators is equal to the number of non-zero eigenvalues of the Choi matrix and is called the Kraus rank [32].

## B. Signed singular values for qubit channels

For any matrix  $M_\Phi$ , there exist two orthogonal matrices  $M_1, M_2$  such that the singular value decomposition (SVD) of  $M_\Phi$  reads  $M_\Phi = M_1 D M_2$ , with  $D$  a diagonal matrix with non-negative entries. Any orthogonal matrix  $M$  is such that either  $M$  or  $-M$  is in  $SO(3)$  (in the latter case  $M$  corresponds to an improper rotation, i.e. a concatenation of a proper rotation with a central inversion). Let  $U_i \in SU(2)$  be a unitary matrix corresponding to  $M_i$  via the isomorphism (6) if  $M_i \in SO(3)$ , or corresponding to  $-M_i$  otherwise. Then  $M_\Phi = R_{U_1} \Lambda R_{U_2}$  with  $\Lambda = D$  if both  $M_1$  and  $M_2$  are in  $SO(3)$  or  $\Lambda = -D$  if exactly one of the  $M_i$  is in  $SO(3)$ . The channel  $\Phi$  can thus be decomposed into  $\Phi = \Phi_{U_1} \circ \Phi_\Lambda \circ \Phi_{U_2}$ , where  $\Phi_{U_i}$  are unitary conjugations and  $\Phi_\Lambda$  a channel whose matrix  $M = \Lambda$  is diagonal. We call the diagonal values of  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$  the ‘‘signed singular values’’ of  $\Phi$  [43]. There is of course arbitrariness in the order in which the signed singular values are labeled. Changing the order of the  $\lambda_i$  just amounts to changing the order in which the eigenvectors of  $M_\Phi$  appear in matrices  $R_{U_i}$ . More precisely, for a permutation of three elements,

say  $\sigma = (123)$ , we consider the permutation channel  $\Phi_\sigma$  defined by

$$M_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (8)$$

and  $\mathbf{t}_\sigma = \mathbf{0}$ . This channel allows to permute the eigenvalues of a matrix  $M$ . Namely, if  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ , we get  $M_\sigma \Lambda M_\sigma^\dagger = \text{diag}(\lambda_3, \lambda_1, \lambda_2)$ . Moreover, one can simultaneously change the signs of (exactly) two singular values of  $D$  by concatenating with a unitary channel. For example, concatenation with  $e^{i\pi\sigma_z/2} = \text{diag}(-1, -1, 1)$  changes the signs of  $\lambda_1$  and  $\lambda_2$ .

Summarizing, up to unitary rotations, any qubit channel can be written as

$$T_\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{pmatrix}, \quad (9)$$

where  $\boldsymbol{\lambda}_\Phi = (\lambda_1, \lambda_2, \lambda_3)$  is the vector of (signed) singular values of the matrix  $M$  from (3) and  $\mathbf{t}_\Phi = (t_1, t_2, t_3)$  are the coordinates (in the Pauli basis) of  $\Phi(I_2/2)$ . In the next section we consider the simpler case of unital qubit channels, for which  $\mathbf{t}_\Phi = \mathbf{0}$ . We shall turn to non-unital channels in Section IV.

### III. UNIVERSAL SET OF UNITAL QUBIT CHANNELS

#### A. Geometry of unital channels

Unital qubit channels  $\Phi$  are defined as channels which leave the fully mixed state  $\rho_0 = I_2/2$  invariant. In the representation (3), a channel  $\Phi$  is unital if and only if  $\mathbf{t}_\Phi = \mathbf{0}$ . Using  $\Phi(\sigma_i) = \lambda_i \sigma_i$  for  $i = 0, 1, 2, 3$  with  $\lambda_0 = 1$ , we obtain the Choi matrix (7) in the computational basis for unital qubit channels,

$$C_\Phi = \frac{1}{4} \begin{pmatrix} 1 + \lambda_3 & 0 & 0 & \lambda_1 + \lambda_2 \\ 0 & 1 - \lambda_3 & \lambda_1 - \lambda_2 & 0 \\ 0 & \lambda_1 - \lambda_2 & 1 - \lambda_3 & 0 \\ \lambda_1 + \lambda_2 & 0 & 0 & 1 + \lambda_3 \end{pmatrix}. \quad (10)$$



Using the obvious block-structure of  $C_\Phi$ , its eigenvalues  $q_0, q_1, q_2, q_3$  are easily computed as

$$q_0 = (1 + \lambda_1 + \lambda_2 + \lambda_3)/4 \quad (11)$$

$$q_1 = (1 + \lambda_1 - \lambda_2 - \lambda_3)/4 \quad (12)$$

$$q_2 = (1 - \lambda_1 + \lambda_2 - \lambda_3)/4 \quad (13)$$

$$q_3 = (1 - \lambda_1 - \lambda_2 + \lambda_3)/4. \quad (14)$$

According to Choi's theorem [34], the linear map  $\Phi$  is completely positive iff its Choi matrix  $C_\Phi$  is positive, i.e.  $q_i \geq 0$ ,  $i = 0, 1, 2, 3$ . These four inequalities are exactly equivalent to the celebrated Fujiwara-Algoet conditions (FAC) for the complete positivity of a unital qubit channel [41],

$$\begin{cases} 1 + \lambda_3 \geq |\lambda_1 + \lambda_2| \\ 1 - \lambda_3 \geq |\lambda_1 - \lambda_2|. \end{cases} \quad (15)$$

The FAC (15) provide four inequalities; equality in any one of them is equivalent to  $q_i = 0$  for some  $i$ . Note that similar conditions were obtained in [41] for a particular subclass of non-unital channels, but we shall address this question in Section IV A.

To each channel of the form (9) one can associate a point in  $\mathbb{R}^3$  specified by its coordinates  $(\lambda_1, \lambda_2, \lambda_3)$ . Let  $V_1 \equiv (1, 1, 1)$ ,  $V_2 \equiv (1, -1, -1)$ ,  $V_3 \equiv (-1, 1, -1)$ , and  $V_4 \equiv (-1, -1, 1)$  be four points in  $\mathbb{R}^3$ . Point  $V_1$  corresponds to the identity channel; points  $V_i$ ,  $2 \leq i \leq 4$ , correspond respectively to deterministic bit flip, bit-phase flip, and phase flip [32]. The vertices  $V_1, V_2, V_3, V_4$  define a regular tetrahedron  $\mathcal{T}$ . Rewriting relations (11)-(14) as

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = q_0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + q_1 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + q_2 \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + q_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \quad (16)$$

$$= q_0 V_1 + q_1 V_2 + q_2 V_3 + q_3 V_4, \quad (17)$$

one can interpret the FAC (15) geometrically by saying that the signed singular values  $\lambda_i$  of a quantum channel  $\Phi$  must lie inside the tetrahedron  $\mathcal{T}$ . Equality in one of the four inequalities (15) defines a face of  $\mathcal{T}$ .

Because of the arbitrariness in the ordering of the  $\lambda_i$  mentioned in the previous section, distinct points of  $\mathcal{T}$  can be exchanged with one another via permutation channels. Moreover, rotation by an angle  $\pi$  about any of the three coordinate axes  $x$ ,  $y$  or  $z$  flips the signs of the two coordinates corresponding to the directions perpendicular to the rotation axis. Starting from the identity channel represented by  $V_1$ , concatenations with unitary channels allow to reach the channels  $\Phi_i$

represented by points  $V_i$ . Therefore, the four vertices of  $\mathcal{T}$  are equivalent up to unitary transformations. Note that an "inversion channel" with  $\boldsymbol{\lambda} = -V_1$  cannot exist, since the corresponding Choi matrix is not positive. However, there are channels with all entries negative, e.g. the one with  $\boldsymbol{\lambda} = (-1, -1, -1)/3$  [40].

There is a connection between the Kraus rank of a channel  $\Phi$ , defined in II A, and the dimension of the boundary on which its representing point lies:

**Proposition III.1.** *For all unital qubit channels  $\Phi$ , the Kraus rank of  $\Phi$  is one plus the dimension of the face of the tetrahedron  $\mathcal{T}$  to which the point  $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$  belongs. Namely, rank-one channels (unitary conjugations) correspond to the vertices of  $\mathcal{T}$ , rank-2 channels correspond to interior of edges, rank-3 channels correspond to interior of faces and full-rank channels to the interior of  $\mathcal{T}$ .*

*Proof.* Since positivity of the Choi matrix is equivalent to complete positivity of the qubit channel, a single vanishing  $q_i$  defines a face of the tetrahedron. Two vanishing  $q_i$ s give the intersection of the corresponding two faces, i.e. an edge, and three vanishing  $q_i$ s a vertex. If no  $q_i$  is zero, we have a generic point inside the tetrahedron. Since at the same time the number of non-vanishing  $q_i$ s is the Kraus rank of the channel (the rank of its Choi matrix) the result holds.  $\square$

In the following subsections, we investigate the decomposition of qubit channels with given Kraus rank.

## B. Edges of tetrahedron

As mentioned above, all edges of  $\mathcal{T}$  are equivalent up to permutation of the vertices. Therefore we only need to consider one of the edges, e.g. the edge  $\overline{V_1V_4}$ . Points belonging to this edge correspond to channels  $\Phi_{\text{PF}}(t)$  given by  $M = \text{diag}(1 - 2t, 1 - 2t, 1)$ ,  $t \in [0, 1]$ . These are phase flip channels, where the probability for a phase flip (conjugation with  $\sigma_z$ ) is equal to  $t$ . This follows from the fact that the vertex  $V_1$  corresponds to the identity channel, whereas the vertex  $V_4$  corresponds to a unitary conjugation by the  $\sigma_z$  Pauli matrix, giving

$$\Phi_{\text{PF}}(t) : \rho \mapsto (1 - t)\rho + t\sigma_z\rho\sigma_z. \quad (18)$$

The entire edge represents the set  $\mathcal{F}_{\text{PF}} = \{\Phi_{\text{PF}}(t), t \in [0, 1]\}$ . More generally, for  $0 < T < 1/2$ , we define the restricted set

$$\mathcal{F}_{\text{PF}}(T) = \{\Phi_{\text{PF}}(t), t \in [0, T]\} \quad (19)$$

Note that the set  $\{\Phi_{\text{PF}}(1-t), t \in [0, T]\}$  can be obtained from the set  $\mathcal{F}_{\text{PF}}(T)$  by unitary conjugation with  $U = \exp(i\frac{\pi}{2}\sigma_z)$ . Since unitary channels are included in our minimal set  $\mathcal{F}$ , it suffices to generate channels corresponding to the half-axis  $T = 1/2$ . In the case of phase flip channels, we have the following technical result.

**Lemma III.2.** *For any fixed  $0 < T < 1/2$  and a given maximum number  $n$  of concatenations, one has*

$$\mathcal{F}_{\text{PF}}(T) \subset \langle \mathcal{F}_{\text{PF}}(\varepsilon) \rangle_n. \quad (20)$$

with  $\varepsilon = \frac{1}{2}(1 - (1 - 2T)^{1/n})$ .

*Proof.* Concatenation of  $n$  phase flips,  $\Phi_{\text{PF}}(\varepsilon)^n$ , leads to the  $M$  matrix  $M = \text{diag}((1 - 2\varepsilon)^n, (1 - 2\varepsilon)^n, 1)$ . For any fixed  $T$  with  $0 < T < 1/2$ , the first two entries are equal to  $1 - 2T$  when  $\varepsilon = \frac{1}{2}(1 - (1 - 2T)^{1/n})$ . Since this is an increasing function of  $T$ , for any  $t \in [0, T]$  there exists a  $\varepsilon' < \varepsilon$  such that  $\Phi_{\text{PF}}(t) = \Phi_{\text{PF}}(\varepsilon)^n$ . □

If one does not place any limit on the number of concatenations, the situation is much simpler: for any  $\varepsilon > 0$ , we have

$$\{\Phi_{\text{PF}}(t), t \in [0, 1/2)\} = \langle \mathcal{F}_{\text{PF}}(\varepsilon) \rangle. \quad (21)$$

Together with unitary channels, this set generates all phase flips along the edge  $\overline{V_1 V_4}$ , apart from the  $1/2$ -phase flip channel. In fact,  $\Phi_{\text{PF}}(1/2)$  must necessarily be included in the universal channel set, because of the following result.

**Proposition III.3.** *For any decomposition  $\Phi_{\text{PF}}(1/2) = \Phi_2 \circ \Phi_1$ , at least one of  $\Phi_1$  or  $\Phi_2$  is unitarily equivalent to  $\Phi_{\text{PF}}(1/2)$ .*

*Proof.* Let  $\Phi_{\text{PF}}(1/2) = \Phi_2 \circ \Phi_1$  be such a decomposition, and let  $M_{1,2}$  be the  $3 \times 3$  matrices defining the channels  $\Phi_{1,2}$ . One has

$$M_{\text{PF}(1/2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = M_2 M_1, \quad (22)$$

and thus  $1 = \|M_{\text{PF}(1/2)}\| \leq \|M_1\| \|M_2\|$ , where  $\|\cdot\|$  denotes the usual operator norm  $\|M\| = \sup_x |Mx|/|x|$ . Since both norms of  $M_{1,2}$  are smaller than or equal to 1, it must be that  $\|M_1\| =$

$\|M_2\| = 1$ . Without loss of generality, we can assume that  $\lambda_3(M_{1,2}) = 1$ . Using the Fujiwara-Algoet conditions, this implies that the signed singular values of  $M_1$  and  $M_2$  are of the form  $(a, a, 1)$  and  $(b, b, 1)$  respectively, for some  $a, b$ . Taking the determinant in Eq. (22), one gets  $ab = 0$ , so that one of  $M_1$  or  $M_2$  must be equal to the initial phase flip channel  $\Phi_{\text{PF}}(1/2)$ , up to unitary conjugation.  $\square$

As mentioned, all the edges are obtained by switching the signs of eigenvalues, permuting the eigenvalues, or by combination of the two procedures. This way, we only need to include  $\mathcal{F}_{\text{PF}}(\varepsilon) \cup \{\Phi_{\text{PF}}(1/2)\}$  in our universal set of quantum channels. We gather the results in this subsection in the following proposition.

**Proposition III.4.** *The unital qubit channels situated on the edges of the tetrahedron  $\mathcal{T}$  can be obtained by the concatenation of channels from the following edge-universal set ( $\varepsilon$  is an arbitrarily small positive constant):*

$$\mathcal{G}_{\text{edge}}^\varepsilon = \mathcal{F}_{\text{PF}}(\varepsilon) \cup \{\Phi_{\text{PF}}(1/2)\} \cup \mathcal{U}_2. \quad (23)$$

### C. A universal set of unital qubit channels

In the seminal paper [40], the divisibility of quantum channels was investigated. A quantum channel  $\Phi \in \mathcal{C}_d$  was called *indivisible* if every possible decomposition of the form  $\Phi = \Phi_2 \circ \Phi_1$  is such that one of the  $\Phi_i$  is a unitary conjugation. We recall one of the main results from [40].

**Proposition III.5** ([40], Theorem 23). *A non-unitary qubit quantum channel  $\Phi \in \mathcal{C}_2$  is indivisible if and only if it has Kraus rank 3.*

Therefore, *any* universal set of qubit channels must include the indivisible channels represented by the faces of the tetrahedron. We shall denote by  $\mathcal{I}_2$  the set of all indivisible qubit channels

$$\mathcal{I}_2 = \{\Phi \in \mathcal{C}_2 : \Phi \text{ has Choi rank } 3\}. \quad (24)$$

According to Proposition III.5, all channels on the edges of the tetrahedron are divisible. However, divisibility does not guarantee reduction to a more basic set of channels. For instance, the phase flip channel  $\Phi_{\text{PF}}(1/2)$  is divisible in the sense of proposition III.5, as  $\Phi_{\text{PF}}(1/2) = \Phi_{\text{PF}}(1/2) \circ \Phi_{\text{PF}}(1/2)$ .

We now state the main result of this section, an  $\varepsilon$ -small universal set of unital qubit channels.

**Theorem III.6.** *For any  $\varepsilon > 0$ , the set*

$$\mathcal{G}^\varepsilon = \mathcal{I}_2 \cup \mathcal{G}_{edge}^\varepsilon \quad (25)$$

$$= \mathcal{I}_2 \cup \mathcal{F}_{\text{PF}}(\varepsilon) \cup \{\Phi_{\text{PF}}(1/2)\} \cup \mathcal{U}_2 \quad (26)$$

is a universal set of unital qubit channels. It is minimal in the sense that all elements of  $\mathcal{I}_2$  and  $\{\Phi_{\text{PF}}(1/2)\}$  are needed, as well as a channel  $\Phi_{\text{PF}}(\varepsilon')$  where  $\varepsilon' \leq \varepsilon$ .

*Proof.* According to Proposition III.4, the set  $\mathcal{F}_{\text{PF}}(\varepsilon) \cup \{\Phi_{\text{PF}}(1/2)\}$  together with unitaries, generates all channels on the edges of the tetrahedron, i.e. all Kraus rank 1 and 2 channels. The set  $\mathcal{I}_2$  contains all Kraus rank 3 channels. It remains to show that the union of these two sets generates all the Kraus rank 4 channels (interior of the tetrahedron). To this end, consider the four edges  $e_1, \dots, e_4$  of the tetrahedron with  $e_1 = \overline{V_1V_2}$ ,  $e_2 = \overline{V_1V_3}$ ,  $e_3 = \overline{V_4V_2}$ , and  $e_4 = \overline{V_4V_3}$ , connecting vertices  $(V_1, V_4)$  to vertices  $(V_2, V_3)$ . The plane representing the set of channels with fixed  $\lambda_3 = z$ ,  $z \in [-1, 1]$ , intersects these four edges in four points  $A_1, \dots, A_4$ , respectively, with  $A_1 = (1, z, z)$ ,  $A_2 = (z, 1, z)$ ,  $A_3 = (-z, -1, z)$ , and  $A_4 = (-1, -z, z)$ . These four points form a rectangle, see Fig. 1.

Consider the set of channels  $\mathcal{R}_1 = \{\Phi(s, z), s \in (0, 1)\} \subset \mathcal{I}_2$ , where  $\Phi(s, z)$  is the channel associated with the matrix  $T$  of the form (3) with  $\mathbf{t} = \mathbf{0}$  and  $M = M(s, z) = \text{diag}(1 + s(z - 1), z + s(1 - z), z)$  defined by the edge  $\overline{A_1A_2}$  of the rectangle with given  $z$ . The concatenation  $\Phi(s, z) \circ \Phi_{\text{PF}}(t)$  with  $\Phi(s, z) \in \mathcal{R}_1$  and  $\Phi_{\text{PF}}(t) \in \mathcal{F}_{V_1V_4}$  (top edge) has the  $M$  matrix  $M = \text{diag}((1 - 2t)(1 + s(z - 1)), (1 - 2t)(z + s(1 - z)), z)$ . At fixed  $s$  the two points corresponding to  $t = 0$  and  $t = 1$  are on the edge  $\overline{A_1A_2}$  and  $\overline{A_3A_4}$ , respectively, and are diametrical with respect to the centre of the rectangle at  $(0, 0, z)$ . Because as  $t$  varies it linearly interpolates between these two points, it fills a line connecting the two points. Therefore, when varying  $s \in [0, 1]$  and  $t \in [0, 1]$ , the channel  $\Phi(s, z, t)$  completely fills a bow-tie shape (brown/dark-colored region in Fig. 1), corresponding to half of the rectangle  $A_1A_2A_3A_4$ . The other complementary half of the rectangle (green-colored region in Fig. 1) is obtained in a similar manner by concatenating channels from the edge  $\overline{A_2A_4}$  described by  $M(s, z) = \text{diag}(z + s(-1 - z), 1 + s(-z - 1), z)$  with channels  $\Phi_{\text{PF}}(t)$  from  $\mathcal{F}_{V_1V_4}$ . Varying  $s, z, t$  over their allowed ranges fills the entire tetrahedron. Since  $\Phi(s, z) \in \mathcal{I}_2$  for  $z \in (-1, 1)$  and  $s \in (0, 1)$ , and since, according to Proposition III.4, all the channels from the top and bottom edges of the tetrahedron can be obtained from  $\mathcal{F}_{\text{PF}}(\varepsilon)$  and  $\Phi_{\text{PF}}(1/2)$ , this completes the proof of the universality part of the theorem.

Regarding the minimality of the set  $\mathcal{G}^\varepsilon$ , note that any universal set needs to contain  $\mathcal{I}_2$  (indivisible channels) and  $\Phi_{\text{PF}}(1/2)$  (because of Proposition III.3). All that remains to be shown is

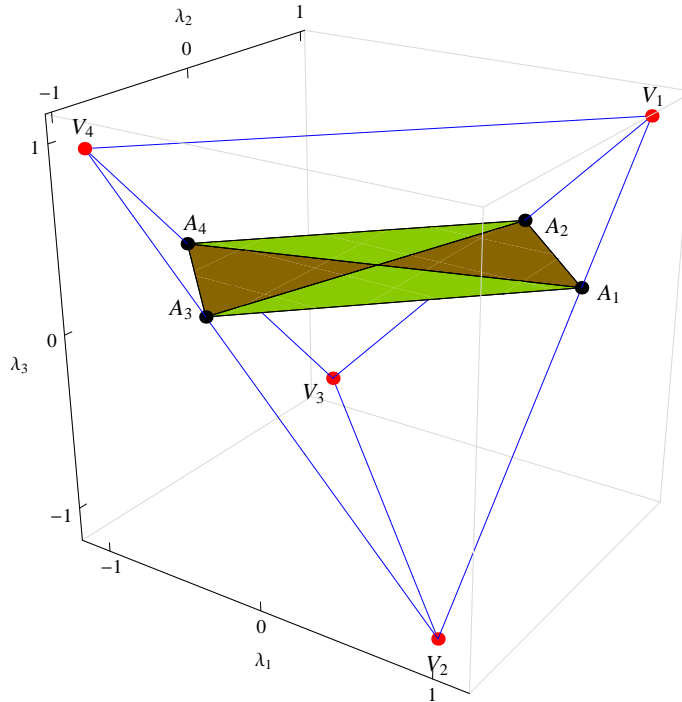


FIG. 1. The Fujiwara-Algoet tetrahedron  $\mathcal{T}$  of admissible signed singular values of a quantum channel with a section  $A_1 A_2 A_3 A_4$ , corresponding to  $\lambda_3 = 1/2$ , and the bow-tie regions obtained by concatenating channels from the edges. For details see the proof of Theorem III.6.

therefore that any universal set also needs to contain phase flip channels of arbitrarily small parameters. To this end, consider a non-trivial decomposition of a phase flip channel  $\Phi_{\text{PF}}(\varepsilon) = \Phi_2 \circ \Phi_1$ . As in Proposition III.3, since the matrix  $M$  associated to the phase flip channel  $\Phi_{\text{PF}}(\varepsilon)$  has operator norm 1, both  $\Phi_1$  and  $\Phi_2$  need to be, up to unitary conjugations, phase flip channels also, of respective parameters  $\delta_1$ ,  $\delta_2$ . Taking the determinant in the equation  $M = M_2 M_1$ , we get  $(1 - 2\varepsilon)^2 = (1 - 2\delta_1)^2 (1 - 2\delta_2)^2$ , so that at least one of  $\delta_{1,2}$  has to be smaller than  $\varepsilon$ . Therefore, any universal set needs to contain  $\Phi_{\text{PF}}(\delta)$  with  $\delta < \varepsilon$ , finishing the proof for the optimality of the set  $\mathcal{G}^\varepsilon$ .  $\square$

#### IV. GEOMETRY OF NON-UNITAL QUBIT CHANNELS

We now consider the more general case of non-unital channels. We first provide two relatively simple forms of generalized Fujiwara-Algoet conditions which allow one to determine the combinations of  $\mathbf{t}$  and  $M$  that represent completely positive maps. We use these conditions to subsequently classify qubit channels by their pure output (PO). Such classification is useful because concatena-

tion of channels results in a channel whose number of pure state outputs can be at most equal to the minimal number of pure state outputs among the used channels.

### A. Condition for complete positivity of non-unital channels

The Choi matrix (7) for a general qubit channel with a  $T$  matrix of the form (3) is given by [32]

$$C_\Phi = \begin{pmatrix} \frac{1}{2}(1 + \lambda_3 + t_3) & 0 & \frac{1}{2}(t_1 + it_2) & \frac{\lambda_1 + \lambda_2}{2} \\ 0 & \frac{1}{2}(1 - \lambda_3 + t_3) & \frac{\lambda_1 - \lambda_2}{2} & \frac{1}{2}(t_1 + it_2) \\ \frac{1}{2}(t_1 - it_2) & \frac{\lambda_1 - \lambda_2}{2} & \frac{1}{2}(1 - \lambda_3 - t_3) & 0 \\ \frac{\lambda_1 + \lambda_2}{2} & \frac{1}{2}(t_1 - it_2) & 0 & \frac{1}{2}(1 + \lambda_3 - t_3) \end{pmatrix}. \quad (27)$$

By a simple change of basis  $RC_\Phi R^\dagger$ , with

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad (28)$$

the Choi matrix can be rewritten as

$$C_\Phi = \frac{1}{2} \begin{pmatrix} 4q_0 & t_1 & t_2 & t_3 \\ t_1 & 4q_1 & it_3 & -it_2 \\ t_2 & -it_3 & 4q_2 & it_1 \\ t_3 & it_2 & -it_1 & 4q_3 \end{pmatrix}, \quad (29)$$

where  $q_i$  are the linear combinations of  $\lambda_i$  introduced in (11)-(14). Note that a necessary condition for  $C_\Phi \geq 0$  is  $q_i \geq 0$  for all  $i = 0, \dots, 3$ , so that  $\lambda_i$  still satisfy the original FAC conditions (15). The following result generalizes the FAC to the case of non-unital channels, beyond the simple case where only one of the  $t_i$  is non-zero. Equivalent necessary and sufficient conditions were obtained in [39, Corollary 2], in the form of three inequalities; we claim that our re-formulation has a more natural geometric interpretation.

**Theorem IV.1** (Generalized Fujiwara-Algoet conditions). *Let  $\Phi : \mathcal{M}_2(\mathbb{C}) \rightarrow \mathcal{M}_2(\mathbb{C})$  be a non-unital linear map whose matrix in the Pauli basis is given by (9). Let  $t = \|\mathbf{t}\|$  and  $\mathbf{u} = \mathbf{t}/t$  the corresponding unit vector. Then the map  $\Phi$  is a quantum channel if and only if*

$$q_i \geq 0, \quad i = 0, 1, 2, 3 \quad \text{and} \\ t^2 \leq r - \sqrt{r^2 - q}, \quad (30)$$

where the  $q_i$  are defined in (11)-(14) and

$$r = 1 - \sum_i \lambda_i^2 + 2 \sum_i \lambda_i^2 u_i^2, \quad (31)$$

$$q = 256 \prod_{i=0}^3 q_i. \quad (32)$$

*Proof.* Since trace preservation of the map  $\Phi$  follows from (9), the only property that needs to be checked is complete positivity. By Choi's theorem,  $\Phi$  is completely positive if and only if the Choi matrix  $C_\Phi$  is positive. The characteristic polynomial  $p(x) = \det(C_\Phi - xI_4)$  of  $C_\Phi$  reads

$$p(x) = x^4 - 2x^3 + \frac{a}{2}x^2 - \frac{b}{2}x + \det C_\Phi, \quad (33)$$

where

$$a = 3 - \sum_i \lambda_i^2 - t^2 \quad (34)$$

$$b = 1 - \sum_i \lambda_i^2 - t^2 + 2\lambda_1\lambda_2\lambda_3 \quad (35)$$

$$\det C_\Phi = \frac{1}{16}(t^4 - 2rt^2 + q). \quad (36)$$

Since  $C_\Phi$  is Hermitian its roots are real. By Descartes' rule of signs, all roots  $x_i$  are positive iff the coefficients of the powers of  $x$  change sign from one coefficient to the next, that is, iff  $\det C_\Phi \geq 0$ ,  $a \geq 0$ , and  $b \geq 0$ . Since  $q_i$  are diagonal elements of  $C_\Phi$  in (29), a necessary condition for positivity of  $C_\Phi$  is that  $q_i$  be all positive, that is,  $\lambda_i$  lie within the tetrahedron  $\mathcal{T}$  of admissible values of the unital case. As  $q_i \geq 0$  ( $i = 0, \dots, 3$ ) implies  $|\lambda_i| \leq 1$  ( $i = 1, \dots, 3$ ) and thus  $|\prod_{i=1}^3 \lambda_i| \leq 1$ , one always has  $a \geq b$ , thus condition  $a \geq 0$  is a consequence of  $b \geq 0$ . We are therefore left with just two generalized Fujiwara-Algoet (GFA) conditions,

$$\det C_\Phi \geq 0 \Leftrightarrow t^4 - 2rt^2 + q \geq 0, \quad \text{and} \quad (37)$$

$$b \geq 0 \Leftrightarrow t^2 \leq 1 - \sum_i \lambda_i^2 + 2\lambda_1\lambda_2\lambda_3. \quad (38)$$

Note that these conditions only depend on the square of the  $t_i$ , whereas the signs of the  $\lambda_i$  matter. Condition (37) is a polynomial of degree 2 in  $t^2$ , whose discriminant  $r^2 - q$  is always positive. Indeed, let  $\lambda_3$  be the signed singular value with the smallest absolute value; then  $r$ , as a function of  $\mathbf{u}$ , reaches its minimal value which is  $r_{\min} = 1 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2$ . Therefore,

$$r^2 - q \geq r_{\min}^2 - q = 4(\lambda_1\lambda_2 - \lambda_3)^2 \geq 0. \quad (39)$$



Because of positive discriminant the polynomial in (37) has two real roots  $r \pm \sqrt{r^2 - q}$  and, due to positive coefficient in front of  $t^4$ , the condition (37) is thus satisfied iff

$$t^2 \leq r - \sqrt{r^2 - q} \text{ or } t^2 \geq r + \sqrt{r^2 - q}. \quad (40)$$

We shall now show that the condition  $b \geq 0$  selects the left root as the relevant one.

It turns out that, fixing  $\lambda_j$ , for any value of the  $u_i$  two further inequalities hold,

$$r - \sqrt{r^2 - q} \leq 1 - \sum_i \lambda_i^2 + 2\lambda_1\lambda_2\lambda_3 \leq r + \sqrt{r^2 - q}. \quad (41)$$

To show these, one first notices that the quantities  $r - \sqrt{r^2 - q}$  and  $r + \sqrt{r^2 - q}$  are, respectively, decreasing and increasing functions of  $r$  ( $r - \sqrt{r^2 - q}$  is decreasing because, taking a derivative, we get the condition  $\sqrt{r^2 - q} \leq r$ , which is always satisfied within the tetrahedron). Therefore,  $r - \sqrt{r^2 - q}$  is always smaller than or equal to  $r_{\min} - \sqrt{r_{\min}^2 - q}$ , while  $r + \sqrt{r^2 - q}$  is always larger than or equal to  $r_{\min} + \sqrt{r_{\min}^2 - q}$ , where  $r_{\min} = 1 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2$  is the smallest possible value of  $r$  when  $u_i$  varies (we again denote by  $\lambda_3$  the one with the smallest absolute value). Inequalities (41) will therefore follow if we show that

$$r_{\min} - \sqrt{r_{\min}^2 - q} \leq 1 - \sum_i \lambda_i^2 + 2\lambda_1\lambda_2\lambda_3 \leq r_{\min} + \sqrt{r_{\min}^2 - q}. \quad (42)$$

Showing (42) is equivalent to showing  $f_- \geq 0$  and  $f_+ \leq 0$ , where we defined  $f_{\pm} = 1 - \sum_i \lambda_i^2 + 2\lambda_1\lambda_2\lambda_3 - (r_{\min} \pm \sqrt{r_{\min}^2 - q})$ . Plugging explicit expressions for  $q = (1 + \lambda_1 + \lambda_2 + \lambda_3)(1 + \lambda_1 - \lambda_2 - \lambda_3)(1 - \lambda_1 + \lambda_2 - \lambda_3)(1 - \lambda_1 - \lambda_2 + \lambda_3)$  and  $r_{\min} = 1 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2$  into  $f_{\pm}$ , and simplifying, results in

$$\begin{aligned} f_- &= 2|\lambda_1\lambda_2 - \lambda_3| + 2\lambda_3(\lambda_1\lambda_2 - \lambda_3), \\ f_+ &= -2|\lambda_1\lambda_2 - \lambda_3| + 2\lambda_3(\lambda_1\lambda_2 - \lambda_3). \end{aligned} \quad (43)$$

As  $|\lambda_3| \leq 1$ , indeed  $f_- \geq 0$  and  $f_+ \leq 0$ . Inequalities (42) are therefore true, and so are (41). The logic of the two directions of the proof of Theorem IV.1 can now be seen summarized as follows:

1.) If  $\Phi$  is a quantum channel, then  $C_{\Phi} \geq 0$  and thus inequalities (37) and (38) hold. In addition, since  $q_i$  are the diagonal matrix elements of  $C_{\Phi}$  in an appropriate basis (eq.(29)), we also have  $q_i \geq 0$  and thus  $q \geq 0$  which implies inequalities (41). Since by (38)  $t^2 \leq 1 - \sum_i \lambda_i^2 + 2\lambda_1\lambda_2\lambda_3$ , we have from (41) that  $t^2 \leq 1 - \sum_i \lambda_i^2 + 2\lambda_1\lambda_2\lambda_3 \leq r + \sqrt{r^2 - q}$ . Therefore, the second inequality in (40) is satisfied only when  $t^2 = r + \sqrt{r^2 - q}$ . This equality implies equality in the second inequality of (41), which in turn implies that  $r = r_{\min}$ . Therefore,  $f_+ = 0$  and thus  $|\lambda_i| = 1$  which corresponds to a unitary, and thus unital, channel, which is excluded in the statement of the Theorem. The

only remaining possibility is that the left inequality in (40) be satisfied. In any case we have that  $C_\Phi \geq 0$  which implies  $t^2 \leq r - \sqrt{r^2 - q}$  and  $q_i \geq 0$ .

2.) If  $t^2 \leq r - \sqrt{r^2 - q}$  we have from (40) that  $\det C_\Phi \geq 0$ . Furthermore, since by assumption  $q_i \geq 0$ , inequalities (41) are valid and we have thus  $t^2 \leq r - \sqrt{r^2 - q} \leq 1 - \sum_i \lambda_i^2 + 2\lambda_1\lambda_2\lambda_3$ . The latter chain of inequalities implies by (38) that  $b \geq 0$ , which together with  $\det C_\Phi \geq 0$  gives  $C_\Phi \geq 0$  and thus the complete positivity of channel  $\Phi$ .  $\square$

Let us make now some remarks on the conditions appearing in the theorem above. First, note that we exclude unital channels ( $t = 0$ ), since in that case the vector  $\mathbf{u}$  is not defined; one can use the usual Fujiwara-Algoet conditions (15) in that case. Also, note that for any fixed set of  $\lambda_j$ , the condition  $\det C_\Phi \geq 0$  is necessary, but not sufficient. The set of translation vectors  $\mathbf{t}$  satisfying  $\det C_\Phi \geq 0$  is composed of a bounded part corresponding to  $t^2 \leq r - \sqrt{r^2 - q}$ , and an unbounded part with  $t^2 \geq r + \sqrt{r^2 - q}$ . The second condition  $b \geq 0$  then selects  $t^2 \leq r - \sqrt{r^2 - q}$  as the one resulting in a completely positive map.

Conditions equivalent to (30) were already found in [39, Corollary 2], in the form of three inequalities. Inequality (30) gives the maximal modulus square of the translation vector  $\mathbf{t}$  of the ellipsoid compatible with positivity of  $C_\Phi$ . This has a very natural geometric interpretation: it gives the maximum displacement of a given ellipsoid in a given direction such that the corresponding linear map is completely positive. In particular, if one of the  $q_i$  is zero (i.e., the corresponding channel is represented by a point on a face of the tetrahedron), then  $q = 0$  and the condition (30) implies that one must have  $\mathbf{t} = 0$ . The implication holds in the opposite direction, namely, if the GFA conditions (30) only allow  $\mathbf{t} = 0$ , then  $q = 0$ . Thus, one of the  $q_i$  vanishes, implying that  $\boldsymbol{\lambda}$  is a point on the surface of the tetrahedron. As soon as  $\boldsymbol{\lambda}$  is within the tetrahedron the right-hand side of the GFA condition (30) is nonzero and also non-unital channels with such  $\lambda$ s exist.

Note also that in the particular case discussed in [41], where  $t_1 = t_2 = 0$  (so that, in our notation,  $u_1 = u_2 = 0$  and  $u_3 = 1$ ), the quantity  $\sqrt{r^2 - q}$  appearing in the second equation in (30) simplifies to

$$\sqrt{r^2 - q} = 2|\lambda_3 - \lambda_1\lambda_2|, \quad (44)$$

in such a way that (30) is equivalent to the (generalized) FAC

$$t_3^2 \leq (1 \pm \lambda_3)^2 - (\lambda_1 \pm \lambda_2)^2. \quad (45)$$

## B. Classifying channels by their pure output

The goal of this subsection is to classify all qubit channels by the number of pure outputs that they can have. Our main result, Theorem IV.9, prohibits qubit channels (apart from unitary conjugations) that would have more than 2 pure outputs. We prove the theorem by elementary geometric means, however, note that it follows also from the results presented in [39]. Namely, it has been shown [39] that extreme or quasi-extreme channels (i.e., those from the interior of tetrahedron edges) that are not unitary conjugations can have at most 2 pure outputs. Because every channel can be written as a convex combination of extremal channels, as soon as one of the channels in the convex sum is not a unitary conjugation, at most 2 pure outputs are possible. If we have a convex combination of unitary conjugations only, then we know that a convex combination of two unitary conjugations is a quasi-extremal channel, again having at most 2 pure outputs, leading to the same conclusion.

**Definition IV.2.** *The pure output (PO) of a quantum channel  $\Phi$  is the set of pure states in the image of  $\Phi$ :*

$$\text{PO}(\Phi) = \Phi(\mathcal{D}_d) \cap \mathcal{P}_d. \quad (46)$$

Recall that  $\mathcal{D}_d$  is the set of density matrices and  $\mathcal{P}_d \subset \mathcal{D}_d$  the set of pure states.

The pure output of a unital channel inherits the central symmetry of the output ellipsoid in the following precise sense:

**Lemma IV.3.** *The pure output of a unital channel  $\Phi$  is centrally symmetric, i.e.*

$$\rho(\mathbf{r}) = \frac{1}{2}(I_2 + \mathbf{r} \cdot \boldsymbol{\sigma}) \in \text{PO}(\Phi) \Leftrightarrow \rho(-\mathbf{r}) = \frac{1}{2}(I_2 - \mathbf{r} \cdot \boldsymbol{\sigma}) \in \text{PO}(\Phi). \quad (47)$$

*Proof.* Since both  $\Phi(\mathcal{D}_d)$  and  $\mathcal{P}_d$  are centrally symmetric, so is their intersection.  $\square$

In the following we show that quantum channels can be classified according to their pure output. An arbitrary single qubit channel maps the input states – a Bloch ball – to output states within a shifted ellipsoid [38, 39]. Therefore, we start by proving the following elementary Euclidean geometry results that will help us understand possible intersections between a sphere and an ellipsoid. In what follows, we shall abuse language and say that a set is *contained inside* a circle (resp. a sphere) if it is a subset of the corresponding disc (resp. ball).

**Lemma IV.4.** *Let  $C$  be a circle in  $\mathbb{R}^2$  and consider an ellipse  $E$  contained inside the circle. If the intersection  $C \cap E$  contains three distinct points  $M, N, O$ , then  $E = C$ .*

*Proof.* This immediately follows from the fact that an ellipse (and, more generally, a conic section) is uniquely determined by the condition that it passes through three non-collinear points and is tangent to two given lines passing through two of these points (for a proof, see [44], p. 114). Since the hypothesis implies that both  $E$  and  $C$  have this property, one must have  $E = C$  by uniqueness.  $\square$

**Lemma IV.5.** *Let  $S$  be the unit sphere in  $\mathbb{R}^3$  and consider an ellipsoid  $E$  contained inside the sphere. If the intersection  $S \cap E$  contains three distinct points  $M, N, O$ , then it contains the circle determined by those three points.*

*Proof.* Consider the plane determined by the points  $M, N, O$ . Its intersection with the sphere  $S$  defines a circle  $C$  and its intersection with the ellipsoid  $E$  defines an ellipse  $F$ . Obviously,  $M, N, O \in C \cap F$  and  $F$  is inside in  $C$ . Using lemma IV.4, we have  $C = F$ , which is the conclusion.  $\square$

**Proposition IV.6.** *Let  $S$  be the unit sphere in  $\mathbb{R}^3$  and consider an ellipsoid  $E$  contained inside  $S$ . Then, the intersection  $S \cap E$  is one of the following:*

1. *the empty set;*
2. *a point;*
3. *two points;*
4. *a circle;*
5. *the whole sphere  $S$ .*

*Proof.* Using the previous lemma, we only need to consider the case when the intersection contains 4 non-coplanar points  $M, N, O, P$  and to show that the ellipsoid coincides with the sphere. By the results already proved, the intersection contains in fact the whole circle  $C$  determined by  $M, N, O$ . Let  $Q$  be any point on the sphere; we will show that  $Q$  belongs to the intersection  $S \cap E$ . To this end, consider the plane determined by the points  $P, Q$  and  $F$ , where  $F$  is the center of the circle  $C$ . This plane cuts the sphere in a circle  $C'$ , which intersects  $C$  in two points  $X$  and  $Y$ . Since  $X, Y$  and  $P$  belong to the intersection  $S \cap E$ , so does the circle  $C'$  (by Lemma IV.5), and thus, in particular, the point  $Q$ , finishing the proof.  $\square$

The case of an intersection in the form of a circle can be further restricted. First, we state the following lemma, which is Theorem LXXIII in the Supplement of [45].

**Lemma IV.7.** *Let  $C$  be a circle and  $E$  an ellipse contained inside the circle and touching the circle from the inside in exactly two points  $P, Q$ . Then the large ellipse axis is parallel to  $PQ$ , while the line of the small axis bisects the segment  $PQ$ .*

A circle intersection is now of the following type.

**Lemma IV.8.** *The only way for an ellipsoid  $E$  with half axes  $0 < a, b, c \leq 1$  to touch the sphere  $S$  from the inside in the form of a circle of non-zero radius, is to have an ellipsoid that is rotationally symmetric about one of its three axes and displaced in the direction of this axis.*

*Proof.* Let us call  $P$  the plane containing the touching circle and  $C$  the circle centre. We orient the  $z$ -axis of  $\mathbb{R}^3$  so that it is perpendicular to  $P$  and passes through  $C$ , while the origin  $O$  of  $\mathbb{R}^3$  is the center of the sphere  $S$ . First, we shall show that one of the ellipsoid's axes has to be the  $z$ -axis. Then we will show that the ellipsoid has to be rotationally symmetric about the  $z$ -axis.

Let us consider an arbitrary plane  $R$  containing the  $z$ -axis. Such a plane cuts the sphere  $S$  in a circle, while it cuts the ellipsoid  $E$  in an ellipse. It also contains two points  $M$  and  $N$  from the intersecting circle  $C$ . Therefore, in plane  $R$  one has an ellipse that touches a circle from inside in exactly two points,  $M$  and  $N$ . Note that the  $z$ -axis bisects the segment  $MN$  perpendicular to it. From Lemma IV.7, it follows that the small ellipse axis is the  $z$ -axis. We shall consider now two such particular planes  $R$ .

First, choose  $R_1$  to be the plane containing the  $z$  axis and the center  $L$  of the ellipsoid. It is a known fact that the intersection of the ellipsoid  $E$  with the plane passing through its center  $L$  is an ellipse with center  $L$ . Since the small axis of the ellipse is  $Oz$ , we infer  $L \in Oz$ , so the center of the ellipsoid lies on the  $z$  axis.

Let us now choose a second plane  $R_2$ , containing the  $z$  axis (and thus the center  $L$  of  $E$ ) and the smallest axis of the ellipsoid  $E$ . The intersection  $E \cap R_2$  contains the points  $X, Y$  which are the antipodal points ( $L$  is the middle of  $XY$ ) closest to each other of  $E$ . They are also the points closest to each other and symmetric with respect to  $L$  of the ellipse  $E \cap R_2$ . It follows that  $X, Y \in Oz$  so that one of the axes of the ellipsoid is the  $z$  axis.

Points on the surface of the ellipsoid  $E$  therefore satisfy

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z - z_0)^2}{c^2} = 1, \quad (48)$$

while the touching circle can be parametrized as  $(r \cos \phi, r \sin \phi, z_1)$  with some nonzero  $r$  and fixed  $z_1$ . For any  $\phi$  these points should lie on the surface of the ellipsoid, therefore

$$r^2 + (a^2/b^2 - 1)r^2 \sin^2 \phi = a^2 - \frac{a^2}{c^2}(z_1 - z_0)^2 \quad (49)$$

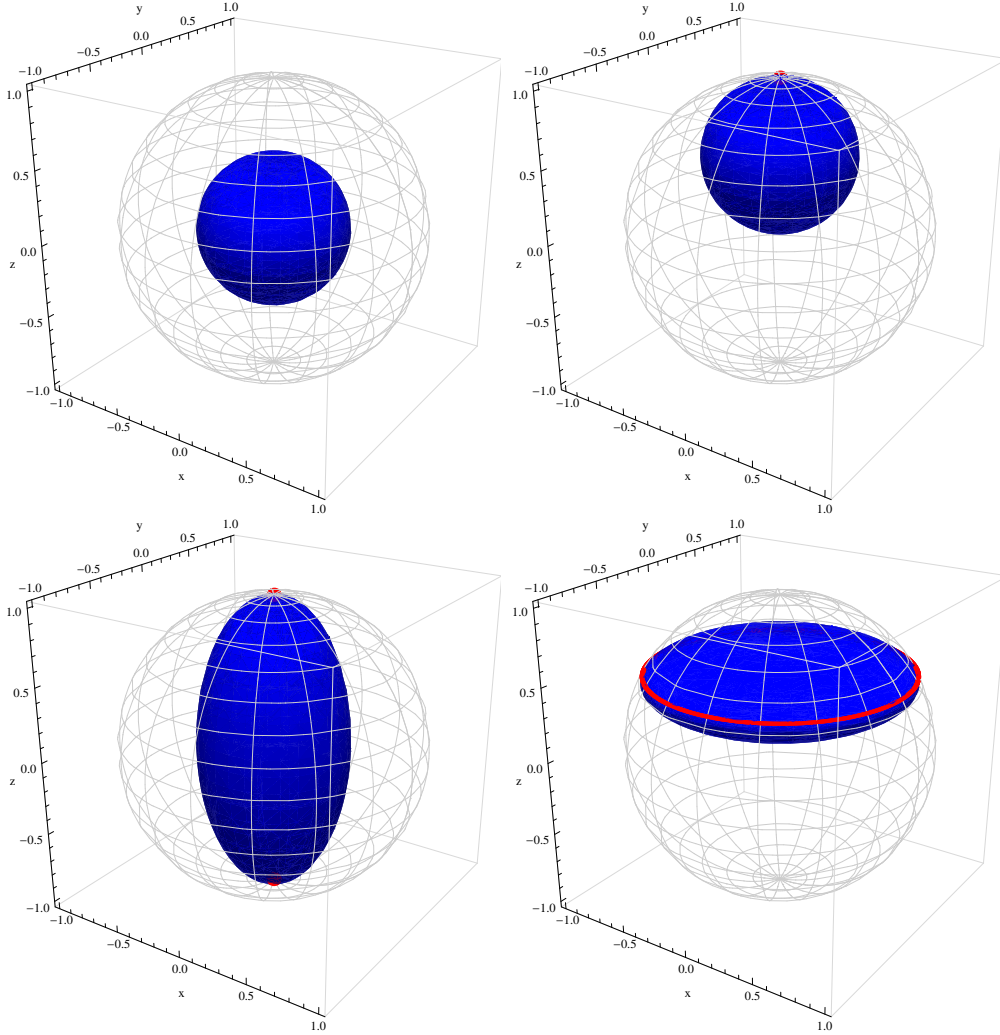


FIG. 2. Different ellipsoids inside the Bloch sphere with, respectively, empty, 1 point, 2 points and circular pure outputs that could be possible from purely geometrical considerations. The circular case does not correspond to the output of a completely positive quantum channel.

should hold. The RHS is independent of  $\phi$  and so should be the LHS. Therefore, we conclude that  $a = b$ . The ellipsoid must be rotationally symmetric, and its displacement can be only along the symmetry axis  $Oz$ .  $\square$

One might think that qubit channels can realize the five different types of pure output suggested by Lemma IV.6. However, we will now show that this is not the case. Rather, the following result holds (see Figure 2):

**Theorem IV.9.** *Let  $\Phi \in \mathcal{C}_2$  be a qubit channel. One of the following holds:*

1.  $\text{PO}(\Phi) = \emptyset$ , the channel has no pure output, all output states are mixed;

2.  $\text{PO}(\Phi) = \{\xi\}$ ,  $\xi \in \mathcal{P}_2$ , the channel has a unique pure output  $\xi$ ;
3.  $\text{PO}(\Phi) = \{\xi, \zeta\}$ ,  $\xi, \zeta \in \mathcal{P}_2$ , the channel has exactly two pure outputs  $\xi, \zeta$ ;
4.  $\text{PO}(\Phi) = \mathcal{P}_2$ , all pure states are outputs of  $\Phi$ . In this case,  $\Phi$  is a unitary conjugation  $\Phi(X) = UXU^\dagger$ , for some unitary matrix  $U$ .

*Proof.* The only allowed forms of pure outputs have the geometric forms given in Proposition IV.6. Examples of pure outputs different from a circle are easily found [32]: The fully mixing channel  $\rho \rightarrow I_2/2$  maps the entire Bloch sphere to its center so that the pure output is the empty set. A decaying channel leads to an ellipsoid that touches the sphere in the South pole and nowhere else. A phase flip channel  $\Phi_{PF}$  (see eq.(18)) shrinks the Bloch sphere in  $x$  and  $y$ -directions, but leaves the  $z$ -direction untouched, such that the resulting ellipsoid touches the Bloch sphere in the North and South poles and nowhere else. Finally, unitary conjugation corresponds to a rotation of the Bloch sphere, and thus has as pure output all pure states  $\mathcal{P}_2$ . It remains to be shown that a pure output in the form of a circle on the Bloch sphere does not correspond to a completely positive qubit quantum channel. According to Lemma IV.8, the demonstration can be reduced to quantum channels with  $t_1 = t_2 = 0$  and  $|\lambda_1| = |\lambda_2| = a$ ,  $|\lambda_3| = c$ . Note that  $a, b, c$  are geometrical quantities (half-axes of the ellipsoid) and are always positive, whereas the signed singular values  $\lambda_i$  can have either sign. In order for the ellipsoid to touch the sphere in a circle (and thus at  $x^2 + y^2 > 0$ ), one needs  $c < a$ . Due to rotational symmetry we need only consider the plane  $y = 0$  in order to obtain a relation between  $t_3, a, c$  required for the ellipsoid to touch the sphere from inside. The  $z$  coordinates of the ellipsoid and sphere in the upper half space read, respectively,

$$z_E = c\sqrt{1 - \left(\frac{x-t_1}{a}\right)^2 - \left(\frac{y-t_2}{b}\right)^2} + t_3 \quad (50)$$

$$z_S = \sqrt{1 - x^2 - y^2}. \quad (51)$$

If  $E$  touches  $S$  in a point  $x, y, z$ , we must have  $z_E = z_S = z$  in that point, and the tangential planes to  $S$  and  $E$  in that point must be identical. Two non-trivial solutions of the touching condition  $dz_E/dx = dz_S/dx$  are then found as  $x = \pm\sqrt{\frac{c^2 - a^4}{c^2 - a^2}}$ . In order that the solutions be real, we need  $c < a^2$ . Reinserting this into the second touching condition  $z_E = z_S$  leads to the shift

$$t_3 = \frac{\sqrt{(1-a^2)(a^2-c^2)}}{a}. \quad (52)$$

The GFA condition (30) in the present case reads  $t^2 \leq (c \pm 1)^2 - 4a^2$ , where  $\pm$  comes from the two possible signs of  $\lambda_3$ . Since here  $t^2 = t_3^2$  and

$$(c \pm 1)^2 - 4a^2 - t_3^2 = \frac{c^2 \pm 2ca^2 - 3a^4}{a^2} \leq 0 \quad (53)$$

for any  $c < a^2$ , the GFA condition can never be satisfied, and thus the solutions are not qubit channels. Therefore there is no quantum channel with a pure output in the form of a circle on the Bloch sphere.  $\square$

Note that when comparing Proposition IV.6 with Theorem IV.9, one sees that the case of the circle is not physical, i.e. there is no completely positive map which has a circle as the pure output set. This is a generalization of the “no pancake” theorem, which states that there is no qubit quantum channel that maps the Bloch sphere to a disk touching the sphere (see [46]).

Channels of type 3 in Theorem IV.9 can be either unital or non-unital. For distinguishing the two the following proposition is useful.

**Proposition IV.10.** *If the two output states  $\xi, \zeta$  in type 3 states from Theorem IV.9 are orthogonal (i.e. antipodal on the Bloch sphere), then the channel is unital, otherwise it is non-unital.*

*Proof.* The two most distant points on an ellipsoid are on its largest axis, so the center of the ellipsoid is the middle of the segment  $\xi\zeta$ , i.e. the center of the Bloch sphere, if  $\xi, \zeta$  are antipodal. It follows that the channel must be unital,  $t = 0$ .

The other direction follows from Lemma IV.3: the pure output is non symmetric, so the channel cannot be unital.  $\square$

Lemma IV.3 can be used to show that all channels with one PO, i.e. of type 2 in Theorem IV.9, are non-unital. Channels with zero PO can be either unital or non-unital.

## V. UNIVERSAL SET FOR EXTREMAL QUBIT CHANNELS

We investigate in this section the important role *extremal* quantum qubit channels have to play with respect to divisibility and universal families.

For general dimensions, necessary and sufficient conditions for a quantum channel to be extreme have been found by [47], using ideas from [34].

**Theorem V.1** ([47]). *A quantum channel  $\Phi$  having Kraus operators  $\{A_i\}_{i=1}^k$  is an extremal point of the convex set of quantum channels iff the set of matrices  $\{A_i^\dagger A_j\}_{i,j=1}^k$  is linearly independent.*

This result was used in Ref. [39] to provide a more geometric picture in the qubit case, which we recall below.



**Proposition V.2** ([39]). *A map  $\Phi : \mathcal{M}_2(\mathbb{C}) \rightarrow \mathcal{M}_2(\mathbb{C})$  as in (9) is an extremal quantum channel if, up to some permutation of indices,*

$$\lambda_3 = \lambda_1 \lambda_2, \quad (54)$$

$$t_3^2 = (1 - \lambda_1^2)(1 - \lambda_2^2), \quad t_1 = t_2 = 0. \quad (55)$$

*This is equivalent to the existence of angles  $u \in [0, 2\pi)$  and  $v \in [0, \pi)$  such that*

$$\begin{aligned} \lambda_1 &= \cos u, \\ \lambda_2 &= \cos v, \\ \lambda_3 &= \cos u \cos v, \\ t_3 &= \sin u \sin v. \end{aligned} \quad (56)$$

*Without sacrificing generality one can assume  $|\lambda_1| \geq |\lambda_2|$  in the above parametrization. Channels with  $u = 0$  and  $v \neq 0$  are the phase flip channels considered in Section III B; they are not extremal channels and must be excluded from the set of parameters above.*

For generic angles such a channel maps the Bloch sphere to a shifted ellipsoid that touches the sphere in two points (the two points might degenerate to one point for some special values of the angles). As shown in Ref. [39], two pure input states at points  $(\pm \cos \theta, 0, \sin \theta)$  are mapped to two pure output states at points  $(\pm \cos \omega, 0, \sin \omega)$ . Angles  $\theta$  and  $\omega$  can be interpreted as latitudes of the pure inputs and outputs. They are related to  $u$  and  $v$  by  $\sin \theta = \tan u / \tan v$  and  $\sin \omega = \sin u / \sin v$ , or, inversely, channel matrix elements are  $\cos u = \cos \omega / \cos \theta$ ,  $\cos v = \tan \theta / \tan \omega$  and  $t_3 = (\cos^2 \theta - \cos^2 \omega) / (\cos^2 \theta \sin \omega)$ . Sometimes it will be more useful to use a parametrization with angles  $\theta$  and  $\omega$  instead of  $u, v$ .

Regarding the signs of generalized singular values  $\lambda_j$  we can see that either none or two can be negative, while  $t_3$  can be either positive or negative. Because we are interested in channels obtained by concatenation, and because we always allow for any number of unitary conjugations, we can always remove any negative signs in  $\lambda_j$  or  $t_3$  by unitary channels, as the following lemma shows.

**Lemma V.3.** *Any extremal qubit channel  $\Phi$ , written in the canonical form of Eq.(56), can be transformed by unitary conjugations into an extremal channel  $\Phi'$  of the same form, but with all  $\lambda_j$  and  $t_3$  non-negative.*

*Proof.* Let  $\Phi_U$  denote a channel corresponding to a unitary conjugation. Unitary conjugations can change the sign of arbitrary two  $\lambda_j$ . First, using the composition rule in Eqs. (4),(5), we observe

that by composing  $\Phi' = \Phi_U \circ \Phi$  we can change the sign of  $t_3$  and the sign of  $\lambda_3$  and either  $\lambda_1$  or  $\lambda_2$ . On the other hand, with  $\Phi' = \Phi \circ \Phi_U$  we can change the sign of any two  $\lambda_j$  while leaving the sign of  $t_3$  intact. Combining concatenations with a unitary before and after the channel  $\Phi$ , we can change any allowed combination of signs. Let us discuss all possible cases: (i) if  $t_3 \geq 0$  and any two  $\lambda_j$  are negative we simply change the sign of these two  $\lambda_j$  by  $\Phi' = \Phi \circ \Phi_U$ ; (ii) if  $t_3 < 0$  and  $\lambda_3 < 0$  as well as one of  $\lambda_{1,2}$  is negative we can change all signs at once with  $\Phi' = \Phi_U \circ \Phi$ ; (iii) if  $t_3 < 0$  and  $\lambda_{1,2} < 0$ , or  $t_3 < 0$  and all  $\lambda_j$  are non-negative we can, using  $\Phi' = \Phi \circ \Phi_U$ , bring the channel to the form discussed under (ii).  $\square$

We therefore see that with unitary conjugations we can always bring an extremal qubit channel to the form given by Eq. (56) with all  $\lambda_j$  and  $t_3$  being non-negative, that is to the set with angles  $u \in (0, \pi/2]$  and  $v \in (0, \pi/2]$  (we must also have  $u \leq v$  due to  $|\lambda_1| \geq |\lambda_2|$ ), plus an additional point  $u = v = 0$ . From now on we shall limit our discussion to that subset of extremal qubit channels. They can be further classified according to the number of different pure output points one gets for different values of  $u$  and  $v$ . In addition, it will be useful to classify channels also depending on whether they represent an injective map.

**Definition V.4.** *A channel  $\Phi$  is called degenerate iff the determinant of  $T_\Phi$  (3) is zero; otherwise it is called non-degenerate. Equivalently, a channel is non-degenerate iff all  $\lambda_j$  are non-zero, i.e., iff the volume of the set of output states is nonzero.*

Depending on the degeneracy and the number of output pure states, extremal qubit channels (56) can be classified as follows.

**Lemma V.5** ([39]). *The set of extremal qubit channels  $\mathcal{X}$  can be classified according to the number of pure outputs as  $\mathcal{X} = \mathcal{U}_2 \cup \mathcal{X}_{1\text{PO}}^{\text{deg}} \cup \mathcal{X}_{1\text{PO}}^{\text{nd}} \cup \mathcal{X}_{2\text{PO}}^{\text{deg}} \cup \mathcal{X}_{2\text{PO}}^{\text{nd}}$  where the pure outputs of the subclasses are as follows:*

1. *Unitary conjugations  $\mathcal{U}_2$  for which  $\text{PO}(\Phi) = \mathcal{P}_2$ . They correspond to  $u = v = 0$ , or  $\omega = \theta \neq \pi/2$ .*
2.  *$\text{PO}(\Phi) = \{\xi\}$ : the set given by  $u = v$  that can be conveniently parametrized as  $\mathbf{t}_\Phi = (0, 0, 1 - \lambda^2)$  and  $\boldsymbol{\lambda}_\Phi = (\lambda, \lambda, \lambda^2)$ . Note that the mapping from  $u, v$  to  $\theta, \omega$  is not injective in this case and all such channels correspond to a single point  $\theta = \omega = \pi/2$  in the  $\theta - \omega$  plane (Fig. 3).*
  - (a) *Degenerate channels  $\mathcal{X}_{1\text{PO}}^{\text{deg}}$ :  $\lambda = 0$ , i.e.,  $u = v = \pi/2$ . The set of output states is a single point on the Bloch sphere.*

(b) Non-degenerate channels  $\mathcal{X}_{1\text{PO}}^{\text{nd}}$ : the set given by  $\lambda \in (0, 1)$ , i.e.,  $u = v$  and  $u \in (0, \pi/2)$ .

3.  $\text{PO}(\Phi) = \{\xi, \zeta\}$ , with  $\xi \neq \zeta$ :

(a) Degenerate channels  $\mathcal{X}_{2\text{PO}}^{\text{deg}}$ : the set given by  $v = \pi/2$  and  $u \in (0, \pi/2)$ , i.e.,  $\theta = 0$  and  $\omega \in (0, \pi/2)$ . The set of output states is a degenerate ellipsoid – a line segment.

(b) Non-degenerate channels  $\mathcal{X}_{2\text{PO}}^{\text{nd}}$ : the set given by  $0 < \theta < \omega < \pi/2$ , or, in angles  $u, v$ , the set  $0 < u < v < \pi/2$  ( $u = 0$  with  $v > 0$  is excluded). This is the interior of the shaded triangle in Fig. 3.

We shall investigate these classes of extremal channels and show that any universal set of qubit quantum channels needs to contain some of these maps. We analyze each individual case in the next two subsections. Before that, let us state few general statements that will be useful in proving decompositions.

**Lemma V.6.** *Let  $\Phi : \mathcal{M}_d(\mathbb{C}) \rightarrow \mathcal{M}_d(\mathbb{C})$  be a quantum channel with the property that there exists a full rank input state  $\rho$  such that  $\Phi(\rho) = |\psi\rangle\langle\psi|$ , a rank-one projector. Then, the channel  $\Phi$  is constant, i.e. for all input states  $\sigma$ ,  $\Phi(\sigma) = |\psi\rangle\langle\psi|$ .*

*Proof.* Let  $\sigma$  be any input state. Since  $\rho$  is full rank, there exists a positive constant  $\varepsilon$  such that  $\varepsilon\sigma \leq \rho$ . Being a quantum channel, the map  $\Phi$  preserves positivity, hence  $\varepsilon\Phi(\sigma) \leq \Phi(\rho) = |\psi\rangle\langle\psi|$ . This means that  $\Phi(\sigma)$  has support only in the  $|\psi\rangle\langle\psi|$  subspace, i.e.  $\Phi(\sigma) = c|\psi\rangle\langle\psi|$  with some  $c > 0$ . Using trace preservation, we conclude that  $c = 1$  and thus  $\Phi(\sigma) = |\psi\rangle\langle\psi|$ .  $\square$

Note that the full-rank hypothesis in the above lemma is necessary; taking a direct sum of two constant channels shows that a mixed input is not enough to guarantee that the channel is constant. However, for qubits, the notions of mixed state and full-rank state are equivalent. Among qubit channels such constant channels are exactly channels of the type 2a in Lemma V.5.

**Lemma V.7.** *In a decomposition of a non-degenerate channel there can be only non-degenerate channels.*

*Proof.* Writing  $\Phi = \Phi_n \circ \dots \circ \Phi_1$  and taking the determinant on both sides of  $T_\Phi = T_{\Phi_n} \dots T_{\Phi_1}$ , we see that the determinant of the right side can be non-zero (i.e.,  $\Phi$  is non-degenerate) only if all channels  $\Phi_j$  are non-degenerate.  $\square$

**Proposition V.8.** *In a decomposition of a non-degenerate extremal channel  $\Phi$  there can be only non-degenerate extremal channels.*

*Proof.* Consider a decomposition  $\Phi = \Phi_1 \circ \Phi_2$ . Due to Lemma V.7 we know that both  $\Phi_1$  and  $\Phi_2$  must be non-degenerate. Suppose  $\Phi_2$  is not extremal, that is, we can write it as a nontrivial convex sum  $\Phi_2 = c_1\Psi_1 + c_2\Psi_2$ , with  $\Psi_1 \neq \Psi_2$ . Using this sum  $\Phi$  can be written as  $\Phi = c_1\Phi_1 \circ \Psi_1 + c_2\Phi_1 \circ \Psi_2$ . Because  $\Phi$  is supposed to be extremal  $\Phi_1 \circ \Psi_1$  must be equal to  $\Phi_1 \circ \Psi_2$ , otherwise  $\Phi$  would be a nontrivial convex combination. But because  $\Phi_1$  is non-degenerate, i.e., an injection, and  $\Psi_1 \neq \Psi_2$ , there is at least one point whose image under  $\Psi_1$  is different from its image under  $\Psi_2$ , and therefore  $\Phi_1 \circ \Psi_2$  can not be equal to  $\Phi_1 \circ \Psi_1$ .  $\Phi_2$  must therefore be extremal. For the case when  $\Phi_1$  would be a convex combination, the argument is analogous. Therefore, neither  $\Phi_1$  nor  $\Phi_2$  can have a nontrivial convex combination.  $\square$

Finally, let us make a remark regarding the relation between extremal and indivisible channels. Somewhat unintuitively, all extremal channels are divisible. This follows from the characterization of extremal [39] and indivisible [40, Theorem 23] channels. Indeed, indivisible channels are unital and this implies  $t_3 = 0$  in Proposition V.2. This, in turn, implies  $u = 0$ , which is an excluded parameter. Most notably, the indivisible channel  $\rho \mapsto (\rho^t + (\text{tr}\rho)I_2)/3$  from [40] is not extremal, as it is unital with  $\boldsymbol{\lambda} = (1/3, -1/3, 1/3)$ , which corresponds to the center of a face of the tetrahedron.

### A. Extremal qubit channels with one pure output

#### 1. Degenerate channels

Consider a *generalized depolarizing* (or *constant*) channel  $\mathcal{Q}_\rho \in \mathcal{C}_d$  defined as

$$\mathcal{Q}_{\rho_0}(\rho) = (\text{tr}\rho)\rho_0, \quad (57)$$

where  $\rho_0 \in \mathcal{D}_d$  is a fixed density operator. The usual depolarizing channel is a particular case obtained by considering for  $\rho_0$  the maximally mixed state  $I_d/d$ . An important feature of generalized depolarizing channels is that their image (as quantum channels) is trivial:  $\mathcal{Q}_{\rho_0}(\mathcal{D}_d) = \{\rho_0\}$ , that is, all states are mapped onto a single point.

We now look at generalized depolarizing qubit channels ( $d = 2$ ). If  $\mathbf{r}$  is the Bloch vector of the state  $\rho_0 \in \mathcal{D}_2$ , then  $M_{\mathcal{Q}_{\rho_0}} = \text{diag}(0, 0, 0)$  and  $\mathbf{t}_{\mathcal{Q}_{\rho_0}} = \mathbf{r}$ . Of special interest to us are channels with pure  $\rho_0$ , that is, extremal channels of the form 2a in Lemma V.5.

**Proposition V.9.** *For a pure state  $|\psi\rangle\langle\psi| \in \mathcal{P}_2$ , consider a decomposition*

$$\mathcal{Q}_{|\psi\rangle\langle\psi|} = \Phi_2 \circ \Phi_1. \quad (58)$$

*Then, at least one of  $\Phi_{1,2}$  is a constant channel  $\mathcal{Q}_{|\phi\rangle\langle\phi|}$ , for some pure state  $|\phi\rangle\langle\phi| \in \mathcal{P}_2$ .*

*Proof.* Let  $\rho$  be an arbitrary mixed (and thus full-rank) input state for the channel  $\mathcal{Q}_{|\psi\rangle\langle\psi|}$ ,

$$\Phi_2(\Phi_1(\rho)) = |\psi\rangle\langle\psi|, \quad (59)$$

and consider the intermediary state  $\sigma = \Phi_1(\rho)$ . If  $\sigma$  is a pure state, then the channel  $\Phi_1$  satisfies the hypothesis of Lemma V.6, and it is therefore constant. Otherwise,  $\sigma$  is a mixed state, but then  $\Phi_2(\sigma) = |\psi\rangle\langle\psi|$ , and, by the same Lemma V.6,  $\Phi_2$  is constant.  $\square$

**Corollary V.10.** *Any set of universal qubit channels contains at least one generalized depolarizing channel  $\mathcal{Q}_\rho$  for some pure state  $\rho \in \mathcal{P}_2$ . As all other generalized depolarizing channels can be obtained from  $\mathcal{Q}_\rho$  by concatenation with some unitary conjugation, it is also sufficient to have a single generalized depolarizing channel in the universal set for the creation of all generalized depolarizing qubit channels with pure output.*

## 2. Non-degenerate channels

Let  $\Phi_{\text{IPO}}(\lambda)$  be an extremal IPO channel (type 2b in Lemma V.5) with a parametrization

$$\mathbf{t}_\Phi = (0, 0, 1 - \lambda^2) \text{ and } \boldsymbol{\lambda}_\Phi = (\lambda, \lambda, \lambda^2). \quad (60)$$

We define a set of length  $\varepsilon$  by  $\mathcal{X}_{\text{IPO}}^{\text{nd}}(\varepsilon)$ ,

$$\mathcal{X}_{\text{IPO}}^{\text{nd}}(\varepsilon) = \{\Phi_{\text{IPO}}(\lambda), \lambda \in (1 - \varepsilon, 1)\}, \quad (61)$$

where  $\varepsilon$  is any positive number less than 1.

**Lemma V.11.** *The set of all IPO non-degenerate extremal channels  $\mathcal{X}_{\text{IPO}}^{\text{nd}}$  can be obtained by concatenation from the IPO non-degenerate extremal universal set  $\mathcal{X}_{\text{IPO}}^{\text{nd}}(\varepsilon)$ .*

*Proof.* It is straightforward to check that the following concatenation rule holds for IPO non-degenerate extremal channels,  $\Phi_{\text{IPO}}(\lambda\mu) = \Phi_{\text{IPO}}(\lambda) \circ \Phi_{\text{IPO}}(\mu)$ ,  $\lambda, \mu \in (0, 1)$ . Therefore, by a completely analogous argument as in the case of phase-flip channels, Lemma III.2, we can see that concatenating at most  $n$  channels from  $\mathcal{X}_{\text{IPO}}^{\text{nd}}(\varepsilon)$ , where  $\varepsilon = 1 - T^{1/n}$ , we can get  $\Phi_{\text{IPO}}(T)$ , with any  $T \in (0, 1)$ .  $\square$

The following proposition shows that any universal set must contain at least some element of the set  $\mathcal{X}_{1\text{PO}}^{\text{nd}}(\varepsilon)$ .

**Proposition V.12.** *Let  $\Phi_{1\text{PO}}^{\text{nd}}$  be a non-degenerate extremal channel with 1 PO and  $\Phi_{1\text{PO}}^{\text{nd}} = \Phi_2 \circ \Phi_1$  an arbitrary decomposition. Then, up to unitary conjugations, both  $\Phi_{1,2}$  must be 1PO non-degenerate extremal channels.*

*Proof.* According to Proposition V.8  $\Phi_{1,2}$  can be either unitaries, 1PO or 2PO non-degenerate extremal channels. A 1PO non-degenerate extremal channel in the parametrization (60) maps a pure input state with  $\theta = \pi/2$ , i.e. the North Pole  $\eta$  of the Bloch sphere to itself,  $\Phi_{1\text{PO}}^{\text{nd}}(\eta) = \eta$ . If  $\Phi_1(\eta)$  is a mixed state, then  $\Phi_2$  maps a mixed state to a pure output, so, by Lemma V.6, it is a constant, hence degenerate channel. This would contradict the non-degeneracy of  $\Phi_{1\text{PO}}^{\text{nd}}$ . Therefore, we must have that  $\Phi_1$  maps  $\eta$  to some pure state  $\zeta$ ,  $\Phi_1(\eta) = \zeta$ , and then  $\Phi_2(\zeta) = \eta$ . This shows that  $\Phi_1$  and  $\Phi_2$  have at least one pure output. In order to exclude possible 2PO channels from the decomposition we shall use a local argument about the curvature of the boundary of output sets at these pure output points.

The output set of  $\Phi_{1\text{PO}}^{\text{nd}}$  as parametrized by (60) is an ellipsoid touching the Bloch sphere at the north pole. Any plane containing the north pole and the origin intersects this output ellipsoid in an ellipse with the major axis  $a = \lambda$  and the minor axis  $b = \lambda^2$ . The radius of curvature of an ellipse at its vertices that are closest to the ellipse centre is  $a^2/b$  and is therefore  $R = 1$  in our case. We also observe that for any extremal channel, i.e., an ellipsoid touching a sphere from inside, the radius of curvature at the touching point in any plane containing a pure output state is upper-bounded by the radius of curvature of the Bloch sphere (which is 1). For a plane containing two pure output points and the origin, where the output set is an ellipse with a major axis  $a = \cos u$  and a minor axis  $b = \cos u \cos v$  (56), one can explicitly calculate that the radius of curvature at the touching point is  $R = (\cos v / \cos u)^2$ . In particular, for unitary conjugations it is of course 1, whereas for a 2PO non-degenerate extremal channel it is always less than 1. As the output set of the concatenation must be in the output set of  $\Phi_2$ , the curvatures of the ellipsoid at a PO can never decrease under concatenation, or, equivalently, the radius of curvature can never increase. Because the curvature of the final  $\Phi_{1\text{PO}}^{\text{nd}}$  must be 1, we conclude that  $\Phi_{1,2}$  can never be non-degenerate 2PO extremal channels.  $\square$

Combining Corollary V.10, Lemma V.11 and Proposition V.12, we obtain an  $\varepsilon$ -small universal set for 1PO extremal channels.

**Corollary V.13.** *For any  $\varepsilon > 0$ , the following set is universal for 1PO extremal channels:*

$$\mathcal{X}_{1\text{PO}}(\varepsilon) = \{\Phi_{1\text{PO}}(\lambda), \lambda \in (1 - \varepsilon, 1) \cup \{0\}\}, \quad (62)$$

where  $\Phi_{1\text{PO}}(\lambda)$  is a channel with  $\mathbf{t}_\Phi = (0, 0, 1 - \lambda^2)$  and  $\boldsymbol{\lambda}_\Phi = (\lambda, \lambda, \lambda^2)$ .

### B. Extremal qubit channels with two pure outputs

Let  $\Phi_{1,2}$  be two extremal 2PO channels of form (56) with parameters  $(\omega_i, \theta_i)$  such that  $\omega_1 = \theta_2$ . Then,  $\Phi = \Phi_2 \circ \Phi_1$  is of the form (56) with parameters  $(\omega_2, \theta_1)$ , that is  $\Phi(\omega_2, \theta_1) = \Phi_2(\omega_2, \omega_1) \circ \Phi_1(\omega_1, \theta_1)$ . Such channels, with parameters  $(\omega, \theta)$  will be denoted by  $\Phi_{2\text{PO}}(\omega, \theta)$ .

#### 1. Degenerate channels

2PO Degenerate extremal channels map two orthogonal pure input states ( $\theta = 0$ ) to two pure output states with  $0 < \omega < \pi/2$  (type 3a in Lemma V.5). The set of output states is a line segment touching the Bloch sphere in two points with an angle  $\omega$ .

**Lemma V.14.** *For any  $\varepsilon > 0$ , the set of all 2PO degenerate extremal channels  $\mathcal{X}_{2\text{PO}}^{\text{deg}}$  can be obtained by concatenation of channels from the set*

$$\mathcal{X}_{2\text{PO}}^{\text{deg}}(\varepsilon) = \{\Phi_{2\text{PO}}(\omega, 0), \omega \in (0, \varepsilon)\}, \quad (63)$$

and channels from the set  $\mathcal{X}_{2\text{PO}}^{\text{nd}}$ .

*Proof.* The statement immediately follows from a general composition rule for two genuine 2PO extremal channels (those with non-equal 2 POs) saying that  $\Phi_{2\text{PO}}(\omega, 0) = \Phi_{2\text{PO}}(\omega, x) \circ \Phi_{2\text{PO}}(x, 0)$ , with any  $x \in (0, \omega)$ , and the fact that  $\mathcal{X}_{2\text{PO}}^{\text{nd}}$  contains all  $\Phi_{2\text{PO}}(\omega, x)$  with  $0 < x < \omega < \pi/2$ .  $\square$

**Lemma V.15.** *If a qubit channel  $\Phi$  maps two orthogonal pure states to two distinct non-orthogonal pure states, it must be a 2PO degenerate extremal channel, i.e., of the type 3a in Lemma V.5.*

*Proof.* Let us denote two orthogonal pure input states by  $\xi_a$  and  $\xi_b$ , and their pure output states by  $\xi'_a$  and  $\xi'_b$ ,  $\Phi(\xi_a) = \xi'_a$ ,  $\Phi(\xi_b) = \xi'_b$ .  $\Phi$  is an affine map and so it maps a line segment  $\overline{\xi_a \xi_b}$  to a line segment  $\overline{\xi'_a \xi'_b}$ . The midpoint of  $\overline{\xi_a \xi_b}$ , which is a centre of the Bloch sphere, is mapped to a midpoint of  $\overline{\xi'_a \xi'_b}$ , which is the centre of the output ellipsoid. Let us denote by  $\mathcal{P}$  a plane containing the ellipsoid centre and the points  $\xi'_a$  and  $\xi'_b$ . In the plane  $\mathcal{P}$  the output ellipsoid is an ellipse touching a circle from inside in points  $\xi'_a$  and  $\xi'_b$ . Due to Lemma IV.7 we know that the large ellipse axis

is parallel to  $\overline{\xi'_a \xi'_b}$  and, because it must also pass through the midpoint of  $\overline{\xi'_a \xi'_b}$ , we conclude that the large axis must be equal to  $\overline{\xi'_a \xi'_b}$ . If the ellipse is supposed to only touch the circle and not intersect it, its small axis must be zero. The plane  $\mathcal{P}$  therefore intersects the ellipsoid in a line segment of nonzero length. Orienting the coordinate system so that the  $x$  axis is parallel to  $\overline{\xi'_a \xi'_b}$ , while the  $z$  axis is in the plane  $\mathcal{P}$  and perpendicular to  $\overline{\xi'_a \xi'_b}$ , we have  $\lambda_3 = 0$  as well as  $t_2 = t_1 = 0$ . Because also  $t_3^2 + \lambda_1^2 = 1$ , we see that the coefficient  $b$ , Eq. (35) in the GFA Theorem IV.1, is equal to  $b = 1 - \lambda_1^2 - \lambda_2^2 - t_3^2 = -\lambda_2^2$ , which is non-negative only if  $\lambda_2 = 0$ . The channel  $\Phi$  is therefore of the form 3a in Lemma V.5.  $\square$

Using that Lemma we can now show that 2PO degenerate extremal channels are also necessary.

**Proposition V.16.** *Let  $\Phi = \Phi_2 \circ \Phi_1$  be an arbitrary decomposition of a 2PO degenerate extremal channel  $\Phi$ . Then exactly one of the channels  $\Phi_{1,2}$  must be a 2PO degenerate extremal channel.*

*Proof.* A 2PO degenerate extremal channel  $\Phi$  maps two pure orthogonal states  $\xi_a$  and  $\xi_b$  to two pure non-orthogonal states  $\xi'_a$  and  $\xi'_b$ ,  $\Phi(\xi_a) = \xi'_a$ ,  $\Phi(\xi_b) = \xi'_b$ . Due to Lemma V.6, and because the states  $\xi'_a$  and  $\xi'_b$  are distinct, we know that neither of  $\Phi_{1,2}$  can be a 1PO degenerate extremal channel, i.e., a channel that would map a mixed state to a pure state. Therefore, the image of pure states  $\xi_a$  and  $\xi_b$  under  $\Phi_1$  must be two pure states, say  $\zeta_a$  and  $\zeta_b$ ,  $\Phi_1(\xi_a) = \zeta_a$ ,  $\Phi_1(\xi_b) = \zeta_b$ . If  $\zeta_a$  and  $\zeta_b$  are non-orthogonal, then  $\Phi_1$  maps two orthogonal states  $\xi_a$  and  $\xi_b$  to two non-orthogonal pure states and, according to Lemma V.15, must be a 2PO degenerate extremal channel, whereas  $\Phi_2$  cannot be a 2PO degenerate extremal channel as it maps two non-orthogonal states to two non-orthogonal states. If on the other hand  $\zeta_a$  and  $\zeta_b$  are orthogonal, then  $\Phi_2$  must in turn map these two orthogonal pure states to two non-orthogonal pure states and must therefore be a 2PO degenerate extremal channel, whereas  $\Phi_1$  maps two orthogonal pure states to two orthogonal pure states, and is therefore a non-extremal unital channel.  $\square$

## 2. Non-degenerate channels

According to Lemma V.5, a channel  $\Phi$  belonging to class 3b takes the form (56) with parameters  $\theta$  and  $\omega$  such that  $0 < \theta < \omega < \pi/2$  (shaded triangle in Fig. 3). From Proposition V.8, up to unitary conjugation, such a channel can only be decomposed into channels of the same form. We obtain in this way a universal set for 2PO non-degenerate channels.



**Proposition V.17.** *For any  $\varepsilon > 0$ , the set  $\{\Phi_{2\text{PO}}(\theta, \omega); 0 < \omega - \theta < \varepsilon\}$  is a universal set for 2PO-non-degenerate channels. Moreover, given any non-trivial decomposition of a 2PO-non-degenerate channel  $\Phi = \Phi_2 \circ \Phi_1$ , both channels  $\Phi_{1,2}$  must be extremal 2PO-non-degenerate.*

*Proof.* Any concatenation  $\Phi = \Phi_n \circ \dots \circ \Phi_2 \circ \Phi_1$  of maps with parameters  $(\theta_i, \omega_i)$  such that  $\omega_i = \theta_{i+1}$  is of the form (56) with parameters  $(\theta_1, \omega_n)$ . Concatenating  $n$  channels with parameters lying in the strip  $\{(\theta, \omega); 0 < \omega - \theta < \varepsilon\}$  (blue area in Fig. 3) allows to reach any final angle  $\omega_n \in [\theta, \theta + n\varepsilon)$  from an initial angle  $\theta$ . Therefore any channel with parameters  $(\theta, \omega)$  can be decomposed into a sequence of channels with  $n = \lfloor (\omega - \theta)/\varepsilon \rfloor + 1$ .

The second statement follows from Proposition V.8 and the fact that both  $\Phi_2$  and  $\Phi_1$  should have exactly 2 pure outputs.  $\square$

Combining Lemma V.14 and Proposition V.17, we obtain an  $\varepsilon$ -small universal set for 2PO extremal channels.

**Corollary V.18.** *For any  $\varepsilon > 0$ , the following set is universal for extremal 2 PO channels:*

$$\mathcal{X}_{2\text{PO}}(\varepsilon) = \{\Phi_{2\text{PO}}(\omega, 0), \omega \in (0, \varepsilon)\} \cup \{\Phi_{2\text{PO}}(\omega, \theta), \omega - \theta \in (0, \varepsilon)\}, \quad (64)$$

where  $\Phi_{2\text{PO}}(\omega, \theta)$  is an extremal channel mapping pure input states  $(\pm \cos \theta, 0, \sin \theta)$  to pure output states  $(\pm \cos \omega, 0, \sin \omega)$ .

### C. A universal set for extremal qubit channels

Finally, we state our main theorem, which is a compilation of Corollaries V.13 for 1PO extremal channels and V.18 for 2PO extremal channels. Note that, in virtue of Propositions V.9, V.12, V.16 and V.17, our results go beyond extremal channels, showing that any universal set of channels *must* contain extremal channels belonging to the each class studied in this section.

**Theorem V.19.** *For any  $\varepsilon > 0$ , the set  $\mathcal{X}(\varepsilon) = \mathcal{U}_2 \cup \mathcal{X}_{1\text{PO}}(\varepsilon) \cup \mathcal{X}_{2\text{PO}}(\varepsilon)$ , where  $\mathcal{X}_{1\text{PO}}(\varepsilon)$  is defined in (62) and  $\mathcal{X}_{2\text{PO}}(\varepsilon)$  is defined in (64), is a universal set for extremal qubit channels.*

*Moreover, any universal set of (general) qubit channels must contain the following extremal channels:*

1. a 1PO degenerate (i.e. constant) channel  $\mathcal{Q}_{|\psi\rangle\langle\psi|}$ ;
2. infinitely many 1PO non-degenerate extremal channels  $\Phi_{1\text{PO}}(1 - \varepsilon)$ ;

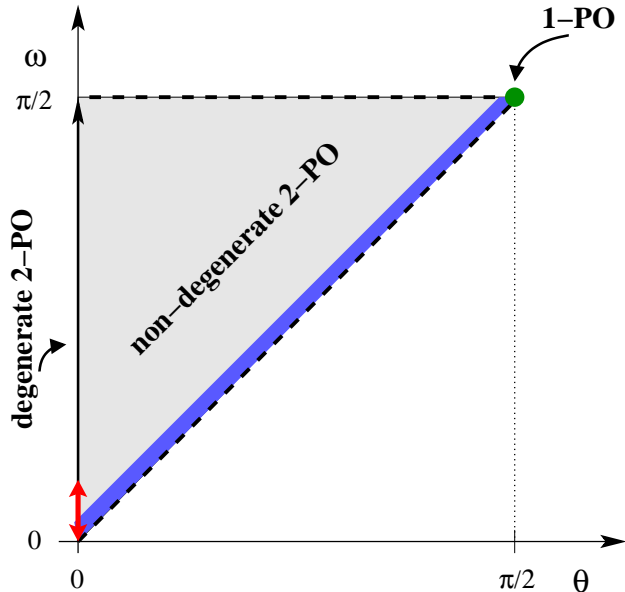


FIG. 3. Illustration of extremal qubit channels in the  $(\theta, \omega)$  plane of latitude angles of the input/output pure states on the Bloch sphere. The inside of the shaded triangle are non-degenerate extremal 2PO channels. The universal set for 2PO extremal channels is a union of an  $\varepsilon$ -strip above the diagonal (blue/dark color) and an  $\varepsilon$ -interval on the  $\omega$ -axis (red/bright color). The 1PO extremal channels are in this parametrization represented by a single point at  $\theta = \omega = \pi/2$ . For details see classification in Lemma V.5.

3. *infinitely many 2PO degenerate extremal channels*  $\Phi_{2\text{PO}}(\varepsilon, 0)$ ;

4. *infinitely many 2PO non-degenerate extremal channels*  $\Phi_{2\text{PO}}(\omega, \theta)$ , with  $0 < \omega - \theta < \varepsilon$ .

## VI. CONCLUDING REMARKS

We have investigated the set of quantum channels acting on a single qubit, i.e. linear, trace preserving, and completely positive maps of the density matrix. We found a compact generalization of the Fujiwara-Algoet conditions, i.e. conditions for the complete positivity of the map, to arbitrary (not necessarily unital) qubit channels. We used these conditions together with purely geometrical considerations to examine the pure output of the quantum channel. We established that no qubit quantum channel exists whose pure output is a circle of non-zero radius on the Bloch sphere, generalizing the “no-pancake theorem”. We derived a universal set of quantum channels for extremal qubit channels, i.e. a set of quantum channels from which all extremal qubit channels can be constructed by concatenation. All other qubit channels can be constructed from these extremal channels by simple classical random sampling. For unital qubit channels we found a universal set

of quantum channels regardless of whether the qubit channel to be decomposed is extremal or not. We showed that our universal sets are essentially minimal, and must be contained in any universal set for arbitrary (not necessarily extremal) qubit channels.

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