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# Some remarks about Descartes' rule of signs

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**Abstract.** What can we deduce about the roots of a real polynomial in one variable by simply considering the signs of its coefficients? On one hand, we give a complete answer concerning the positive roots, by proposing a statement of Descartes' rule of signs which strengthens the available ones while remaining as elementary and concise as the original. On the other hand, we provide new kinds of restrictions on the combined numbers of positive and negative roots.

## 1. Coefficients and positive roots.

It is convenient to consider that an analytic function in one variable has a *multiset* of roots. A multiset is defined just as a set, except that identical elements are allowed. The *cardinality* of a multiset is the number of its elements. The *multiplicity* of an element is the cardinality of the sub-multiset of the elements identical to it.

Consider expressions of the type

$$Y = a_0x^{\alpha_0} + a_1x^{\alpha_1} + \cdots + a_nx^{\alpha_n}, \quad (1)$$

with real exponents  $\alpha_0 < \alpha_1 < \dots < \alpha_n$  and non-zero real coefficients  $a_0, \dots, a_n$ . The *sequence of signs* of an expression (1) is the ordered list  $\sigma_0, \dots, \sigma_n$  of the signs  $\sigma_i = |a_i|^{-1}a_i$ . A *variation* occurs in such a sequence when  $\sigma_i = -\sigma_{i+1}$ . For example the sequence  $1, -1, 1, 1$  has two variations.

We say that two expressions of type (1) belong to the same *class* if they have same number of terms, same list of exponents and same sequence of signs.

**Proposition 1.** Let us fix arbitrarily a class and a multiset of positive numbers. We call  $k$  the number of variations in the sequence of signs associated to the class, and  $p$  the cardinality of the multiset. There exists in the given class a multiterm expression (1) with the given multiset of positive roots if and only if  $k - p$  is even and non-negative.

This statement gives an “if and only if” form to the classical Descartes’ rule of signs. It continues previous works. In 1998, Anderson, Jackson and Sitharam [1] proposed examples of polynomials with any sequence of signs, and with any number  $p$  of positive roots, provided that  $k - p$  is even and non-negative. In 1999, Grabiner [5] extended this result by giving also examples of polynomials with missing terms (i.e. zero coefficients for some of the intermediate integer powers of  $x$ ). Our Proposition 1 strengthens this statement, by extending these previous results to real exponents, and by showing that when a given sequence of signs allows  $p$  positive roots, it indeed allows any multiset of positive roots of cardinality  $p$ . Surprisingly the proof, as presented below, remains very elementary.

## 2. Proof of the “only if” part.

This proof is well-known and we give it only for completeness. We assume that the given multiset is the multiset of positive roots of  $Y$ .

(i) The parity of  $k$  decides if the sign of  $a_0$  is or not the sign of  $a_n$ . The parity of  $p$  decides if the sign of  $Y$  is or not the same at the two ends, i.e. at  $0^+$  and at  $+\infty$ . Obviously these boundary signs are respectively the signs of  $a_0$  and of  $a_n$ . Thus  $k - p$  is even.

(ii) Let  $m$  and  $m'$  be the cardinalities of the respective multisets of positive roots of a function  $f$  and of its derivative  $f'$ . Then  $m \leq m' + 1$ .

(iii) One multiplies  $Y$  by  $x^{-\alpha}$ , computes the derivative in  $x$  and observes the resulting sequence of signs. The signs corresponding to the exponents  $\alpha_i < \alpha$  are changed, while those corresponding to  $\alpha_i > \alpha$  are not changed. We can delete any given variation of sign without touching the others by

choosing  $\alpha$  in some interval of the form  $]\alpha_i, \alpha_{i+1}[$ .

To conclude, we prove that  $p \leq k$  by an induction on  $k$ . If  $k = 0$  then  $p = 0$ . If  $Y$  has  $k$  variations, argument (iii) shows that  $(x^{-\alpha}Y)'$  has  $k - 1$  variations for some choice of  $\alpha$ . Argument (ii) gives the required estimate on the respective numbers of roots. QED

These arguments are actually older than usually thought. Argument (ii) was given in 1741 by de Gua and again in 1798 by Lagrange. Argument (iii) and the statement of the “only if” part are given by de Gua in the case of integer exponents, by Laguerre in the case of real exponents<sup>1</sup>. Laguerre is explicit about the parity conclusion (i), which is obvious. If the statement of the “only if” part omits this conclusion, then it remains correct if the roots are counted without multiplicity<sup>2</sup>.

### 3. Proof of the “if” part.

We call  $(\sigma_0, \dots, \sigma_n)$  the prescribed sequence of signs (with  $k$  variations). We assume that the cardinality  $p$  of the given multiset of positive numbers is such that  $k - p$  is even and positive. We look for an expression (1) with  $\sigma_i a_i > 0$  for all  $i$ , such that the given multiset is the multiset of its positive roots. We will be able to find such an expression with the additional constraint that  $a_i = a_{i+1}$  when  $\sigma_i = \sigma_{i+1}$ . We collect the terms in (1) according to this constraint. We set  $i_0 = 0$  and call  $i_1, i_2, \dots, i_k$  the integers  $i_j$  such that  $i_{j-1} < i_j$ ,  $\sigma_{i_{j-1}} \neq \sigma_{i_j}$ ,  $\sigma_{i_k} = \sigma_n$ . We call  $b_0, b_1, \dots, b_k$  the common values of the coefficients:

$$b_j = a_{i_j} = a_{i_{j+1}} = \dots = a_{i_{j+1}-1}.$$

We set

$$\varphi_j(x) = x^{\alpha_{i_j}} + x^{\alpha_{i_j+1}} + \dots + x^{\alpha_{i_{j+1}-1}}.$$

Now we look for a  $(b_0, b_1, \dots, b_k)$  such that  $\sigma_0 b_0 > 0$ ,  $b_i b_{i+1} < 0$  and the expression

$$Y = b_0 \varphi_0(x) + b_1 \varphi_1(x) + \dots + b_k \varphi_k(x)$$

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<sup>1</sup>See [6] pp. 89–92, [10] p. 195, [11]. Jensen [8] noticed the similarity of de Gua’s and Laguerre’s arguments. Instead of derivation, de Gua used Johan Hudde’s operation, which is described as a term by term multiplication of the polynomial and an arithmetic progression. The modern reader will simply compare this with the operation  $Y \mapsto x^{-m+1}(x^m Y)'$ .

<sup>2</sup>Authors of the 18th century used to count the roots with their multiplicity and to omit the parity conclusion. Incidentally, many authors following Cajori [2] attribute the parity conclusion to Gauss [4] in 1828. This is very strange as firstly, this conclusion is obvious, secondly, Fourier published it in 1820 (see [3] p. 294) and thirdly, it is absent from Gauss’ paper.

has the required multiset of positive roots.

We begin with the case where all the roots  $x_1, \dots, x_p$  are simple. The column vector  $(b_0, \dots, b_k)$  should be in the kernel of the matrix

$$\Phi = \begin{pmatrix} \varphi_0(x_1) & \varphi_1(x_1) & \cdots & \varphi_k(x_1) \\ \varphi_0(x_2) & \varphi_1(x_2) & \cdots & \varphi_k(x_2) \\ & & \cdots & \\ \varphi_0(x_p) & \varphi_1(x_p) & \cdots & \varphi_k(x_p) \end{pmatrix}.$$

This  $p \times (k+1)$  matrix has a non-trivial  $(b_0, \dots, b_k)$  in its kernel since  $p \leq k$ .

If  $p = k$ , the “only if” part shows that the coefficients  $b_i$  are non-zero, that their signs alternate and that there is no other root. This makes an  $(a_0, \dots, a_n)$  such that  $Y$  has the prescribed set of positive roots.

If  $1 \leq p < k$ , observe that the  $p \times p$  submatrix formed by the first  $p$  columns of  $\Phi$  is invertible. If its determinant was zero we could find a non-trivial column vector  $(c_0, \dots, c_{p-1})$  in its kernel. But  $\sum_0^{p-1} c_i \varphi_i(x)$  would have a root at each  $x_j$ . It would have  $p$  roots and at most  $p-1$  variations, which contradicts what we proved in the “only if” part.

So after setting  $b_k(\epsilon) = \sigma_n$  and  $b_i(\epsilon) = (-1)^{k-i} \sigma_n \epsilon$ ,  $i = p, \dots, k-1$ , there is for any  $\epsilon$  a unique  $(b_0(\epsilon), \dots, b_{p-1}(\epsilon))$  such that the column vector  $(b_0(\epsilon), \dots, b_{p-1}(\epsilon), b_p(\epsilon), \dots, b_k(\epsilon))$ , regular in  $\epsilon$ , is in the kernel of  $\Phi$ .

Call  $Y_\epsilon$  the expression  $Y$  with coefficients  $b_i(\epsilon)$ . It vanishes at the  $x_i$ 's. As  $k-p$  of the coefficients of  $Y_0$  vanish, the  $p+1$  remaining ones should be non-zero with alternate signs according to the “only if” part, which at the same time establishes that  $Y_0$  cannot have any other positive root. As  $k-p$  is even, the signs agree with the prescribed sequence of signs when  $\epsilon > 0$ .

Consider  $Y_\epsilon/Y_0$ . This quotient tends to a finite limit at the  $x_i$ 's, which are zeros of the numerator and the denominator. As  $\epsilon \rightarrow 0$ , it tends uniformly to 1, as shown by usual techniques. Thus  $Y_\epsilon$  has no other positive root for a sufficiently small  $\epsilon$ . The problem is solved in the case of simple roots.

In the cases with multiple roots the construction and the proofs are exactly the same, except that  $\Phi$  has e.g. the row  $(\varphi'_0(x_1), \dots, \varphi'_k(x_1))$  if  $x_1$  is a multiple root, the row  $(\varphi''_0(x_1), \dots, \varphi''_k(x_1))$  if it is at least a triple root, etc.

If finally  $p = 0$ ,  $k$  is even and we can build a  $Y$  without root. We choose  $b_0(\epsilon) = \sigma_0$ ,  $b_k(\epsilon) = \sigma_0$  and  $b_i(\epsilon) = (-1)^i \epsilon \sigma_0$ . We choose a sufficiently small  $\epsilon$  such that there is no positive root. QED

#### 4. Further information about trinomials.

Etymologically a trinomial is simply the sum of three monomials, and

there is no restriction on the degree. Trinomials with unprescribed degree were studied early in the history of algebraic equations (see [14], pp. 11 and 24). They are natural objects in the context of Laguerre's extension of Descartes' rule to real exponents, as well as in the context of Khovansky's theory of fewnomials [9], and are consequently the object of recent studies. In 2002, Haas [7], Li, Rojas & Wang [12] proved that the optimal upper bound on the number of roots in the positive quadrant of a system of two trinomials in two variables is five.

We were not able to find the following elegant and elementary formula in any of these old or recent studies.

**Proposition 2.** A trinomial  $ax^\alpha + bx^\beta + cx^\gamma$ ,  $\alpha < \beta < \gamma$ ,  $a > 0$ ,  $c > 0$ ,  $b < 0$ , is positive on  $]0, \infty[$  if and only if

$$\left(\frac{a}{\gamma - \beta}\right)^{\gamma - \beta} \left(\frac{b}{\alpha - \gamma}\right)^{\alpha - \gamma} \left(\frac{c}{\beta - \alpha}\right)^{\beta - \alpha} > 1.$$

**Remark.** The discriminant of  $ax^\alpha + bx^\beta + cx^\gamma$  is  $b^2 - 4ac$  if  $(\alpha, \beta, \gamma) = (0, 1, 2)$ . It is  $-c(4b^3 + 27ca^2)$  if  $(\alpha, \beta, \gamma) = (0, 1, 3)$ . It is  $c^2(-27b^4 + 256ca^3)$  if  $(\alpha, \beta, \gamma) = (0, 1, 4)$  and  $16ac(4ca - b^2)^2$  if  $(\alpha, \beta, \gamma) = (0, 2, 4)$ . These formulas are quite familiar. The general formula in the proposition gives in each case the main factor.

The formula is obtained by a straightforward computation, but there occur unexpected simplifications. It is interesting to try to find a shortcut. Consider the trinomial  $Y = ax^\alpha + bx^\beta + cx^\gamma$ . The explicit expression in the existence proof of the previous section shows that  $Y$  has a double root at  $x_1$  if and only if

$$\frac{ax_1^\alpha}{\gamma - \beta} = \frac{bx_1^\beta}{\alpha - \gamma} = \frac{cx_1^\gamma}{\beta - \alpha}.$$

We set

$$A = \frac{a}{\gamma - \beta}, \quad B = \frac{b}{\alpha - \gamma}, \quad C = \frac{c}{\beta - \alpha},$$

which are positive numbers according to the hypotheses of the proposition, and continue the computation by eliminating  $x_1$ , which gives the expected condition  $A^{\gamma - \beta} B^{\alpha - \gamma} C^{\beta - \alpha} = 1$  for a double root, which is located at

$$x_1 = \left(\frac{C}{B}\right)^{1/(\beta - \gamma)} = \left(\frac{A}{C}\right)^{1/(\gamma - \alpha)} = \left(\frac{B}{A}\right)^{1/(\alpha - \beta)}.$$

## 5. Combining positive and negative roots.

If we just focus on the positive roots of a given polynomial  $Y$ , Proposition 1 tells us that there is nothing we can add to Descartes' rule of signs. All that can be deduced from the sign sequence of  $Y$  is the upper bound and the parity of  $P$ , the cardinality of the multiset of positive roots. By changing  $x$  to  $-x$ , the same can be said on  $N$ , the cardinality of the multiset of negative roots of  $Y$ . But let us consider the constraints on  $(P, N)$ .

Consider a polynomial  $Y(x)$  of degree 4 with sign sequence  $+, -, -, -, +$ . By Descartes' rule  $P = 0$  or  $2$ . If we change  $x$  to  $-x$  the sign sequence becomes  $+, +, -, +, +$ . Thus  $N = 0$  or  $2$ . Grabiner [5] points out that  $(P, N) = (0, 2)$  is *impossible* for such a  $Y(x)$ .

The proof does not require any computation. Because the constant term  $Y(0)$  is positive,  $P = 0$  implies  $Y > 0$  for  $x > 0$ . But the odd part of  $Y$  is negative when  $x > 0$ . So the even part is positive. For  $x < 0$  the odd and the even parts are then positive. We must have  $N = 0$ . QED.

Consider an expression (1) as in Proposition 1, where the  $\alpha_i$ 's are non-negative integers, i.e.  $Y(x)$  is a polynomial. Given the sequence of signs  $\sigma_0, \dots, \sigma_n$ , Grabiner proves<sup>3</sup> that all the  $(P, N)$ 's produced as follows are *possible*. One chooses any subset of  $\{1, \dots, n-1\}$  and "erases" the corresponding  $\sigma_i$ 's from the sequence of signs. The number of variations in the resulting sequence gives  $P$ . One then considers the modified sequence of signs  $\sigma_0(-1)^{\alpha_0}, \sigma_1(-1)^{\alpha_1}, \dots, \sigma_n(-1)^{\alpha_n}$  and erases from it the signs with index in the same subset. The new number of variations gives  $N$ .

Let us call this construction of possible  $(P, N)$ 's as Grabiner's erasing term rule. Grabiner observes that it gives all the possible  $(P, N)$ 's for polynomials up to degree 4, and conjectures that the same is true for higher degrees.

Here is a counterexample. Consider a polynomial of 5th degree with sign sequence  $+, +, -, +, +, -$ . Here  $(P, N) = (3, 0)$  is possible, as shown by

$$(20+37x+18x^2)(1-x)(2-x)(3-x)=120+2x-179x^2+4x^3+71x^4-18x^5.$$

This combination is not obtained by Grabiner's erasing term rule. Indeed, after changing  $x$  in  $-x$  the sequence becomes  $+, -, -, -, +, +$ . As  $N = 0$  we should erase terms and obtain a sequence without variation. We should

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<sup>3</sup>The idea is that small terms can be neglected before applying Descartes' rule. Estimates on how small these terms should be are discussed in [13]. It is interesting to compare these estimates to the first Lemma in [6].

erase either the three  $-$ , or the three  $-$  and the internal  $+$ . But erasing the corresponding signs in  $+, +, -, +, +, -$  gives in both cases only one variation, while  $P = 3$ .

In contrast  $(P, N) = (3, 0)$  is impossible for the sequence  $+, +, -, +, -, -$ . Descartes' rule predicts all the impossibilities for degree 5 polynomials without gaps, except this one and its trivial analogous.

To prove this impossibility, let us specify the ordering of sign sequences by associating the first sign to the constant term. We notice that the odd part  $\mathcal{O}$  has sequence of signs  $+, +, -$ . Its derivative of course has the same sequence and consequently a unique positive root  $r$ . On  $]0, r[$ ,  $\mathcal{O}$  is positive, and for  $x > r$ ,  $\mathcal{O}$  is decreasing. The even part  $\mathcal{E}$  has sequence of signs  $+, -, -$ , thus it decreases for  $x > 0$ . Our polynomial  $\mathcal{E} + \mathcal{O}$  decreases and has at most 1 root on  $]r, +\infty[$ . As  $N = 0$ ,  $\mathcal{E} - \mathcal{O}$  is positive on  $]0, +\infty[$ . On  $]0, r[$ ,  $\mathcal{O}$  and thus  $\mathcal{E} + \mathcal{O}$  are positive. There is one positive root on  $]r, +\infty[$ , thus  $P = 3$  is impossible. QED

Let us pass to degree 6 polynomials. The exhaustive list of non-Descartes impossibilities is, up to trivial transformations:

$+, +, -, +, -, +, +$  is incompatible with  $(P, N) = (2, 0)$  or  $(4, 0)$

$+, +, +, +, -, +, +$  is incompatible with  $(P, N) = (2, 0)$

$+, +, -, -, -, -, +$  is incompatible with  $(P, N) = (0, 4)$

The argument we gave in Grabiner's example proves the first and second statements.

To prove the third statement, we write a polynomial with  $(P, N) = (0, 4)$  as

$$p = (c - bx + x^2)(x + x_1)(x + x_2)(x + x_3)(x + x_4),$$

where  $x_i > 0$ ,  $i = 1, \dots, 4$ ,

$$4c > b^2 > 0. \tag{2}$$

Expanding,  $p = a_0 + a_1x + \dots + a_5x^5 + x^6$ , with in particular

$$a_2 = c\beta - b\gamma + \delta < 0, \tag{3}$$

$$a_5 = -b + \alpha < 0, \tag{4}$$

the positive numbers  $\alpha, \beta, \gamma, \delta$  being defined as

$$\begin{aligned} \alpha &= x_1 + x_2 + x_3 + x_4 \\ \beta &= x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4 + x_3x_4 \\ \gamma &= x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 \\ \delta &= x_1x_2x_3x_4 \end{aligned} \tag{5}$$



By (2) and (4),

$$\sqrt{c} > b/2 > \alpha/2 > 0, \quad (6)$$

and, by (2) and (3),

$$2\sqrt{c} > b > \frac{c\beta + \delta}{\gamma},$$

which gives

$$\beta\sqrt{c}^2 - 2\gamma\sqrt{c} + \delta < 0. \quad (7)$$

We expand the discriminant  $4\gamma^2 - 4\beta\delta$  of this expression using (5). All the terms are positive. Now (7) implies in particular that

$$\sqrt{c} < \frac{\gamma + \sqrt{\gamma^2 - \beta\delta}}{\beta}.$$

Combining with (6) gives  $\beta\alpha/2 < \beta\sqrt{c} < \gamma + \sqrt{\gamma^2 - \beta\delta}$  or

$$\alpha\beta/2 - \gamma < \sqrt{\gamma^2 - \beta\delta}. \quad (8)$$

After expanding and canceling, we see that all the terms of  $\alpha\beta/2 - \gamma$  are positive. We square both sides of (8). This gives us  $\alpha^2\beta - 4\alpha\gamma + 4\delta < 0$ . But the left hand side, after expanding and canceling, has only positive terms. This is a contradiction. QED

To check that our list of impossibilities is complete at degree 6, it is enough to consider the possibilities established by Grabiner's erasing term rule and the two following polynomials, which after changing  $x$  into  $-x$ ,  $x$  into  $1/x$  or  $Y$  into  $-Y$  provide examples for all the other possibilities:

$$(31+11x+x^2)(1-x)(2-x)(3-x)(4-x)=744-1286x+559x^2+25x^3-44x^4+x^5+x^6,$$

$$(9+8x+2x^2)(1-x)(2-x)(3-x)(4-x)=216-258x-37x^2+90x^3-x^4-12x^5+2x^6.$$

These possibilities and impossibilities do not seem to organize themselves in simple classes.

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