Discrete mode observability of structured switching descriptor linear systems: A graph-theoretic approach

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Abstract

The main result of the paper consists in graphical necessary and sufficient conditions which ensure the generic discrete mode observability of structured switching descriptor systems. The methods used in the previous studies on the observability of switching linear systems on standard form are not applicable to switching descriptor systems. So, we develop a new approach starting from bipartite representations of these systems and then building a new kind of digraph dedicated to the discrete mode observability study. The proposed method assumes only the knowledge of the system's structure and is applicable to a large class of descriptor systems including regular and non regular systems even if they are square or under-determined. The provided conditions can be implemented by classical graph-theory algorithms.
Key words: Switching systems, descriptor systems, mode observability, graph theory.

1 Introduction

Hybrid systems, combining event-driven and time-driven dynamics, have received growing attention in the control community as they describe a wide range of systems (Johansson and Rantzer (Eds), 2007). On the other hand, descriptor systems, which handle systems with both differential and non-differential relations, result from a convenient and natural modelling process (Müller, 2000). Their applications can be found in robotics, electrical networks, biologic and economic systems (Müller, 2000). When the model representing the whole or more generally a part of a system is a singular model (for modelling convenience), the functioning system is then represented by a switching descriptor system and in order to check the functioning mode, we have to observe the discrete mode variable of the switching system. Switching descriptor systems are also particularly suited to handle systems (even in standard form) where the dynamics of the continuous part is not entirely known in each discrete mode. Some practical examples where the switching descriptor models are useful and pertinent are provided in (Boukas, 2008; Clotet *et al.*, 2009; De Koning, 2003). The paper focuses on the discrete mode observability of switching descriptor systems. The discrete mode observability is relevant to detect some abrupt changes due to faults and which make the system switching to non nominal dynamics or for supervision when the switching between different modes implies control structure modifications. Few works deal with the observability of switching descriptor system, whereas the developed approaches used to study systems in standard form are not directly applicable. Moreover, for the most part, observability studies use algebraic or geometric approaches and so require the exact knowledge of the state space matrices characterizing the systems' model. In many modelling problems or during the conception stage, these matrices have a number of fixed zero entries determined by the physical laws while the remaining entries are not precisely known. In these cases, to study the structural properties, like observability, the idea is that we only keep the zero/non-zero entries in the state space matrices. Many interesting works on these models, called structured models, aim to analyse their properties (Dion *et al.*, 2003; Murota, 1987; Reinschke, 1988).

The paper is organised as follows: after Section 2, which is devoted to the problem formulation, some definitions related to the graph-theoretic approach are given in Section 3. The main result is provided in Section 4 before a brief conclusion.

2 Problem statement

Consider the following switching descriptor system (SDS)

$$\Sigma : \begin{cases} E_{r(t)}\dot{x}(t) = A_{r(t)}x(t) \\ \quad\quad y(t) = C_{r(t)}x(t) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$ are respectively the state vector and the output (measurement) vector and where $E_{r(t)} \in \mathbb{R}^{m\times n}$, $A_{r(t)} \in \mathbb{R}^{m\times n}$ and $C_{r(t)} \in \mathbb{R}^{p\times n}$. In order to guarantee that there exists at least one trajectory $x(t)$ satisfying the relations defining system Σ, $E_{r(t)}x(0_-)$ is assumed to be admissible *i.e.* it does not result in contrary equations in Σ and is such that system Σ is solvable. The exogenous and unobserved discrete mode variable (or switching signal) $r : [0, \infty) \rightarrow Q = \{1, \ldots, N\}$, is assumed, as in (Babaali and Pappas, 2005), to be right-continuous and only a finite number of jumps can occur in any finite interval.

The discrete mode observability is the capacity to deduce the discrete mode knowing the measurements. It is based on the mode distinguishability:

Definition 1 *(Mode distinguishability) Two distinct modes $q \in Q$ and $q' \in Q$ are distinguishable if, for almost all initial conditions x_0, either there exist an integer $s \geq 0$ and an expression $f_q(y, \dot{y}, \ldots, y^{(s)}) = 0$ which is satisfied for mode q but is not satisfied for mode q', or there exist an integer $s' \geq 0$ and an expression $f_{q'}(y, \dot{y}, \ldots, y^{(s')}) = 0$ which is satisfied for mode q' but is not satisfied for mode q.*

Here, "for almost all initial conditions x_0" is to be understood as "for all $x_0 \in \mathbb{R}^n$ except for the zero set of some polynomials with real coefficients in the n initial state components" ($x_0 = 0$ for example).

Definition 2 *(Discrete mode observability) Σ is discrete mode observable if its modes are distinguishable 2-by-2.*

Discrete mode observability analysis can then be reduced to the study of the distinguishability of each pair of modes.

Thus, in this paper, we consider that we have only 2 modes. Moreover, since we study a structural property, it is pertinent to deal with structured systems, for which we assume that only the sparsity pattern of matrices E_q, A_q and C_q is known for $q \in \{1, 2\}$. So, to each entry of these matrices, we only know whether its value is fixed to zero, or that it has an non-fixed real value represented by a parameter λ_i. The vector of these parameters is $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)^T$ and it is assumed that Λ can take any value in $\mathbb{R}^h$. We denote by A_q^λ, C_q^λ and E_q^λ respectively the matrices obtained by replacing the non zeros in A_q, C_q and E_q, for $q \in \{1, 2\}$ by the corresponding parameters λ_i and we denote

$$\Sigma_\Lambda : \begin{cases} E^\lambda_{r(t)}\dot{x}(t) = A^\lambda_{r(t)}x(t) \\ \qquad y(t) = C^\lambda_{r(t)}x(t) \end{cases} \tag{2}$$

If all parameters λ_i are numerically fixed, we obtain a so-called admissible realization of Σ_Λ. We say that a property is true generically for Σ_Λ if it is true for almost its realizations or equivalently for almost all parameters λ_i.

For the discrete mode observability analysis, it is pertinent and necessary to highlight the similarities and the differences between the models associated to these modes. Indeed, for $q \neq q'$, it is not realistic to assume that all the parameters of A_q^λ, C_q^λ or E_q^λ are free from the ones of $A_{q'}^\lambda$, $C_{q'}^\lambda$ or $E_{q'}^\lambda$. Thus, we decompose each structured matrix into 2 parts: the first one is common to the 2 modes and the second one is specific to each mode *i.e.* for $q \in \{1, 2\}$, $A_q^\lambda = A_0 + A_q^s$, $C_q^\lambda = C_0 + C_q^s$ and $E_q^\lambda = E_0 + E_q^s$. We assume that the entries of these matrices are free and that a coefficient of A_q^λ (resp. C_q^λ and E_q^λ) is exclusively in A_0 or in A_q^s (resp. in C_0 or in C_q^s, and in E_0 or in E_q^s). These notations can be extended to the multi-mode case (Boukhobza and Hamelin, 2011).

3 Graphical representation and definitions

For each mode $q = 1, 2$, we associate to structured system Σ_Λ a bipartite graph noted $B(\Sigma_\Lambda, q) = (\mathbf{V}^+, \mathbf{V}^-, W_q)$, where $\mathbf{V}^+$ and $\mathbf{V}^-$ are 2 disjoint vertex subsets and W_q is the edge set related to mode q. The vertices are associated to the whole internal state, dynamical variables and outputs of Σ_Λ and the edges represent links between these variables for each mode. More precisely, $\mathbf{V}^+ = \mathbf{X}$ and $\mathbf{V}^- = \mathbf{Y} \cup \mathbf{Z}$, with $\mathbf{X} = \{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}\}$, $\mathbf{Z} = \{\mathbf{z_1}, \mathbf{z_2}, \ldots, \mathbf{z_m}\}$ representing relation $z = E^\lambda_{r(t)}x$ and $\mathbf{Y} = \{\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_p}\}$. Edge set is related to each mode q and is defined by $W_q = A_q\text{-}edges \cup C_q\text{-}edges \cup E_q\text{-}edges$, where $A_q\text{-}edges = \{(\mathbf{x_j}, \mathbf{x_i}) \mid A_q^\lambda(i,j) \neq 0\}$, $C_q\text{-}edges = \{(\mathbf{x_j}, \mathbf{y_i}) \mid C_q^\lambda(i,j) \neq 0\}$ and $E_q\text{-}edges = \{(\mathbf{x_j}, \mathbf{z_i}) \mid E_q^\lambda(i,j) \neq 0\}$. Each edge is associated to a free non-zero parameter of the system's model called the weight of the edge. Number q is written under each edge associated to an element of specific matrices A_q^s, C_q^s and E_q^s and represents its index. The edges which are common to the two modes *i.e.* associated to matrices A_0, C_0 and E_0 have index 0. The edges which are specific to mode q have index q.

Example 1 *To the system defined by the following matrices, we associate bipartite graphs in Figure 1.*

$$A_0 = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_4 & 0 \end{pmatrix}, C_0 = \begin{pmatrix} \lambda_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_8 & 0 & 0 \end{pmatrix}, E_0 = \begin{pmatrix} \lambda_9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_{10} \\ 0 & \lambda_{11} & \lambda_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{13} & 0 & 0 \end{pmatrix}.$$

The specific matrices for mode 1 are such that the entries of A_1^s are zero except $A_1^s(3,1) = \lambda_{14}$, $C_1^s = 0$ and the entries of E_1^s are zero except $E_1^s(2,3) = \lambda_{15}$. The specific matrices for mode 2 are such that $A_2^s = 0$, $C_2^s = 0$ and the entries of E_2^s are zero except $E_2^s(3,1) = \lambda_{16}$.

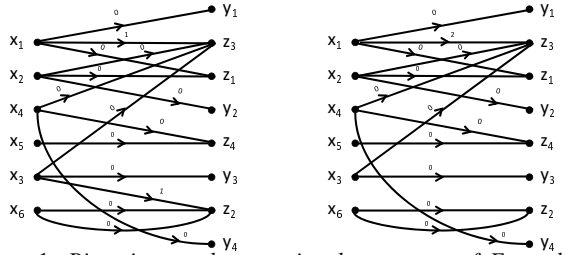

Figure 1. *Bipartite graphs associated to system of Example 1. Mode* 1 *on the left, mode 2 on the right.*

• Two edges are disjoint if they have no common vertex. A matching is a set M of disjoint edges.

• A path P is denoted $P = \mathbf{v_{s_0}} \to \mathbf{v_{s_1}} \to \ldots \to \mathbf{v_{s_i}}$, where, for a given $q \in \{1, 2\}$, $(\mathbf{v_{s_j}}, \mathbf{v_{s_{j+1}}}) \in W_q$ for $j = 0, 1, \ldots, i-1$. We say in this case that P covers $\mathbf{v_{s_0}}, \mathbf{v_{s_1}}, \ldots, \mathbf{v_{s_i}}$. A path is simple when every vertex occurs only once. The weight of P is the product of the weights of all its edges. A *cycle* is a path of the form $\mathbf{v_{s_0}} \to \ldots \mathbf{v_{s_i}} \to \mathbf{v_{s_0}}$, where $\mathbf{v_{s_0}}, \ldots, \mathbf{v_{s_i}}$ are distinct.

• Let $\mathcal{V}_1$ and $\mathcal{V}_2$ represent two subsets, P is a $\mathcal{V}_1$-*topped* path if its end belongs to $\mathcal{V}_1$.

Consider now any bipartite graph noted B defined by the triplet $(\mathbf{V}^+, \mathbf{V}^-, W)$, and let us recall the subdivision of such graph into $\nu + 2$ partially ordered irreducible components denoted $\mathcal{C}_i(B) = (\mathbf{V_i^+}(B), \mathbf{V_i^-}(B), W_i(B))$ using the Dulmage-Mendelsohn (DM) decomposition (Dulmage and Mendelsohn, 1958; Murota, 1987):

↬ Find a maximal matching M in B. We associate to this maximal matching a non bipartite digraph noted $B_M = (\mathbf{V}^+, \mathbf{V}^-, W_M)$ where $(\mathbf{v_1}, \mathbf{v_2}) \in W_M \Leftrightarrow (\mathbf{v_1}, \mathbf{v_2}) \in W$ or $(\mathbf{v_2}, \mathbf{v_1}) \in M$. We denote by $\partial^+\mathbf{M}$ (resp. $\partial^-\mathbf{M}$) the set of vertices in $\mathbf{V}^+$ (resp. in $\mathbf{V}^-$) covered by the edges of M. We note $\mathbf{S_0^+} = \mathbf{V}^+ \setminus \partial^+\mathbf{M}$ and $\mathbf{S_0^-} = \mathbf{V}^- \setminus \partial^-\mathbf{M}$.

↬ $\mathbf{V_0^+}(B) = \mathbf{S_0^+} \cup \{\mathbf{v} \in \mathbf{V}^+, \exists$ a path in B_M from $\mathbf{S_0^+}$ to $\mathbf{v}\}$.

↬ $\mathbf{V_0^-}(B) = \{\mathbf{v} \in \mathbf{V}^-, \exists$ a path in B_M from $\mathbf{S_0^+}$ to $\mathbf{v}\}$.

↬ $W_0(B) = \{$edges linking $\mathbf{V_0^+}(B)$ to $\mathbf{V_0^-}(B)\}$.

↬ $\mathbf{V_\infty^+}(B) = \{\mathbf{v} \in \mathbf{V}^+, \exists$ a path in B_M from $\mathbf{v}$ to $\mathbf{S_0^-}\}$

↬ $\mathbf{V_\infty^-}(B) = \mathbf{S_0^-} \cup \{\mathbf{v} \in \mathbf{V}^-, \exists$ a path in B_M from $\mathbf{v}$ to $\mathbf{S_0^-}\}$.

↬ $W_\infty(B) = \{$edges linking $\mathbf{V_\infty^+}(B)$ to $\mathbf{V_\infty^-}(B)\}$.

↬ For $i = 1, \ldots, \nu$, let $\mathcal{C}_\mathbf{i}(B)$ be the strongly connected components (2 vertices $\mathbf{v_i}$ and $\mathbf{v_j}$ are said to be strongly connected if it exists path from $\mathbf{v_i}$ to $\mathbf{v_j}$ and a path from $\mathbf{v_j}$ to $\mathbf{v_i}$, the relation "*is strongly connected to*" is an equivalence relation and we call each equivalent class a strongly connected component) of the graph obtained from B_M after deleting the vertices and the edges of $\mathcal{C}_\mathbf{0}(B)$ and $\mathcal{C}_\infty(B)$.

↬ Define a partial order "$\preccurlyeq$" on the strongly connected compo-nents $\mathcal{C}_\mathbf{i}(B)$ for $i = 0, 1, \ldots, \nu, \infty$ as follows: $\mathcal{C}_\mathbf{i}(B) \preccurlyeq \mathcal{C}_\mathbf{j}(B) \Leftrightarrow$ there exists a path starting from vertices of $\mathcal{C}_\mathbf{j}(B)$ to the ones of $\mathcal{C}_\mathbf{i}(B)$ in B_M, $\mathcal{C}_\mathbf{0}(B)$ is called the **minimal inconsistent part** of B and $\mathcal{C}_\infty(B)$ is the **maximal inconsistent** part of B.

4 Main result

4.1 Preliminaries

To establish the mode observability of each mode, we apply some results deduced mainly from (Murota, 1987; Boukhobza and Hamelin, 2011). These results allow in a first stage to reduce the graph to its useful part *i.e.* to the part where it is possible to write a non-trivial expression linking only the output and their derivatives. The DM-decomposition properties allow to prove quite easily the following result:

Lemma 1 *For a mode q, there can exist a redundancy equation linking the output components and their derivatives only in the maximal inconsistent part $\mathcal{C}_\infty(B(\Sigma_\Lambda, q))$ of $B(\Sigma_\Lambda, q)$.*

According to the previous lemma, we consider in the sequel of the paper only the maximal inconsistent part of $B(\Sigma_\Lambda, q)$. We can then work on bipartite graphs $\bar{B}(\Sigma_\Lambda, q)$ instead of $B(\Sigma_\Lambda, q)$, where $\bar{B}(\Sigma_\Lambda, q) = (\bar{\mathbf{V}}^+_{\mathbf{q}}, \bar{\mathbf{V}}^-_{\mathbf{q}}, \bar{W}_q)$ with $\bar{\mathbf{V}}^+_{\mathbf{q}} = \mathbf{V}^+_\infty(B(\Sigma_\Lambda, q))$, $\bar{\mathbf{V}}^-_{\mathbf{q}} = \mathbf{V}^-_\infty(B(\Sigma_\Lambda, q))$ and $\bar{W}_q$ is the subset of edges included in W_q which link $\mathbf{V}^+_\infty(B(\Sigma_\Lambda, q))$ to $\mathbf{V}^-_\infty(B(\Sigma_\Lambda, q))$. In this framework, $\bar{A}_q$-*edges*, $\bar{C}_q$-*edges* and $\bar{E}_q$-*edges* are respectively the subsets of A_q-*edges*, C_q-*edges* and E_q-*edges* linking $\mathbf{V}^+_\infty(B(\Sigma_\Lambda, q))$ to $\mathbf{V}^-_\infty(B(\Sigma_\Lambda, q))$. For $q = 1, 2$, we denote by $\bar{\mathbf{X}}_\mathbf{q}$, $\bar{\mathbf{Y}}_\mathbf{q}$ and $\bar{\mathbf{Z}}_\mathbf{q}$ the restrictions of respectively $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ to the vertices of $\bar{\mathbf{V}}^+_\mathbf{q}$ and $\bar{\mathbf{V}}^-_\mathbf{q}$ *i.e.* $\bar{\mathbf{X}}_q = \bar{\mathbf{V}}^+_\mathbf{q}$, $\bar{\mathbf{Y}}_\mathbf{q} = \bar{\mathbf{V}}^-_\mathbf{q} \cap \mathbf{Y}$ and $\bar{\mathbf{Z}}_\mathbf{q} = \bar{\mathbf{V}}^-_\mathbf{q} \cap \mathbf{Z}$.

Example 1 (Continued): *For both modes 1 and 2, we can choose as maximal matching $M = \{(\mathbf{x_1}, \mathbf{z_3}), (\mathbf{x_2}, \mathbf{y_2}), (\mathbf{x_3}, \mathbf{y_3}), (\mathbf{x_4}, \mathbf{y_4}), (\mathbf{x_5}, \mathbf{z_4}), (\mathbf{x_6}, \mathbf{z_2})\}$. We can deduce that the maximal inconsistent part is defined by the following set of vertices: $\bar{\mathbf{V}}^+_\mathbf{1} = \bar{\mathbf{V}}^+_\mathbf{2} = \{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}\}$ and $\bar{\mathbf{V}}^-_\mathbf{1} = \bar{\mathbf{V}}^-_\mathbf{2} = \{\mathbf{y_1}, \mathbf{y_2}, \mathbf{y_3}, \mathbf{y_4}, \mathbf{z_1}, \mathbf{z_3}\}$. The reduced bipartite graphs are depicted in Figure 2.*

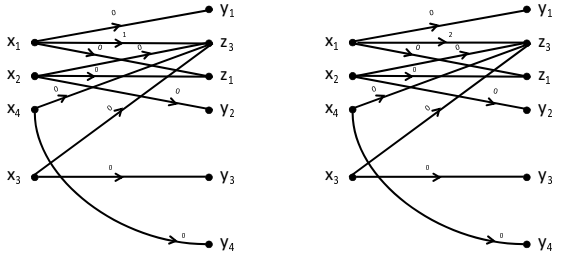

Figure 2. *Reduced bipartite graphs of the system of Example 1. Mode* 1 *on the left, mode 2 on the right.*

We define, for each $q = 1, 2$, and each output vertex subset $\mathbf{V_Y} \subseteq \bar{\mathbf{Y}}_\mathbf{q}$, sub-graph $\bar{B}(\Sigma_\Lambda, q, \mathbf{V_Y})$ by removing from $\bar{B}(\Sigma_\Lambda, q)$ all the output vertices which are not in $\mathbf{V_Y}$ and the C_q-*edges* which do not arrive to $\mathbf{V_Y}$. Then, we exhibit two classes of output vertex subsets for which we can compute the observability subspaces dimensions:

- $\mathbf{V_Y} \subseteq \bar{\mathbf{Y}}_\mathbf{q}$ is said to be q-complete, if the minimal inconsistent part of $\bar{B}(\Sigma_\Lambda, q, \mathbf{V_Y})$ is empty.
- $\mathbf{V_Y} \subseteq \bar{\mathbf{Y}}_\mathbf{q}$ is said to be q-eligible, if only the maximal inconsistent part of $\bar{B}(\Sigma_\Lambda, q, \mathbf{V_Y})$ is not empty.

We can deduce the following results from (Boukhobza and Hamelin, 2011):

Lemma 2 *For each q-complete subset $\mathbf{Y_u} = \{\mathbf{y_{i_{1,q}}}, \mathbf{y_{i_2}}, \ldots, \mathbf{y_{i_k}}\} \subset \bar{\mathbf{Y}}_\mathbf{q}$ there exist generically state space subdivision $\bar{X}_q = (\hat{X}_q^T, \check{X}_q^T)^T$, functions $\varphi_{x,q}$, $\varphi_{y,q}$ and an integer $\nu \leq n$ such that, in mode q, dynamics of system (2) restricted to its maximal inconsistent part can be written as:*

$$\begin{aligned} \tilde{E}_q \dot{\tilde{X}}_q &= \tilde{A}_q \tilde{X}_q + \varphi_{x,q}(Y_u, \dot{Y}_u, \ldots, Y_u^{(\nu)}) \\ Y_s &= \tilde{C}_q \tilde{X}_q + \varphi_{y,q}(Y_u, \dot{Y}_u, \ldots, Y_u^{(\nu)}) \end{aligned} \quad (3)$$

where $\tilde{E}_q$ is a square full rank matrix and $\mathbf{Y_s} = \bar{\mathbf{Y}}_\mathbf{q} \setminus \mathbf{Y_u}$.

Corollary 3 *For each $\mathbf{y_i} \in \bar{\mathbf{Y}}_\mathbf{q}$, there exist integers k and s, and a function f_q, which satisfy expression on form $y_i^{(k)} = f_q(y_i, \dot{y}_i, \ldots, y_i^{(k-1)}, Y_u, \dot{Y}_u, \ldots, Y_u^{(s)}) = 0$, where $\mathbf{Y_u} \subseteq \bar{\mathbf{Y}}_\mathbf{q} \setminus \{\mathbf{y_i}\}$.*

Finally, we characterize the generic dimension of the observability subspace associated to an output vertex subset:

Definition 3 *For each mode $q \in \{1, 2\}$ and to each q-**eligible** vertex subset $\mathbf{V_Y} \subseteq \bar{\mathbf{Y}}_\mathbf{q}$, integer $\beta_q(\mathbf{V_Y})$ is defined as $card(\bar{\mathbf{X}}_\mathbf{q})$ minus the sum of minimal number of $\bar{E}_q$-edges included in a maximal matching of the maximal inconsistent part of $\bar{B}(\Sigma_\Lambda, q, \mathbf{V_Y})$. For each q-**complete** vertex subset $\mathbf{V_Y} \subseteq \bar{\mathbf{Y}}_\mathbf{q}$, we define $\beta_q(\mathbf{V_Y})$ as $card(\bar{\mathbf{X}}_\mathbf{q})$ minus the sum of: - the minimal number of $\bar{E}_q$-edges included in a maximal matching of the maximal inconsistent part of $\bar{B}(\Sigma_\Lambda, q, \mathbf{V_Y})$ - the maximal number of $\bar{E}_q$-edges included in a maximal matching of the consistent part of $\bar{B}(\Sigma_\Lambda, q, \mathbf{V_Y})$.*

We can deduce from (Boukhobza and Hamelin, 2011):

Lemma 4 *Consider SSDS Σ_Λ, for each q-complete vertex subset $\mathbf{V_Y} \subseteq \bar{\mathbf{Y}}_\mathbf{q}$, $\beta_q(\mathbf{V_Y})$ is equal to the generic dimension of the observable subspace for mode q when restricting the measurements to the output components associated to $\mathbf{V_Y}$.*

Definition 4 *For each $\mathbf{y_i} \in \bar{\mathbf{Y}}_\mathbf{q}$ and mode q, we define integer $d_q(\mathbf{y_i}) = \beta_q(\bar{\mathbf{Y}}_\mathbf{q}) - \beta_q(\bar{\mathbf{Y}}_\mathbf{q} \setminus \{\mathbf{y_i}\})$.*

4.2 Discrete mode observability digraph

The discrete mode observability digraph contains the same vertices than $\bar{B}(\Sigma_\Lambda, q)$ but not the same edges. In fact, we remove all the $\bar{E}_q$-*edges* and substitute them by new edges denoted Γ_q-*edges* and Ψ_q-*edges* defined for each mode q.

Definition of Γ_q-*edges* and their indices:

↬ Consider $\bar{B}_E(\Sigma_\Lambda, q)$ the restriction of $\bar{B}(\Sigma_\Lambda, q)$ to only the $\bar{E}_q$-*edges*,

↬ Let M be a maximal matching in $\bar{B}_E(\Sigma_\Lambda, q)$ to which we associate a non bipartite digraph noted $\bar{B}_{q,M} = (\bar{\mathbf{X}}_\mathbf{q}, \bar{\mathbf{Y}}_\mathbf{q} \cup \bar{\mathbf{Z}}_\mathbf{q}, W_M)$ where $(\mathbf{v_1}, \mathbf{v_2}) \in W_M \Leftrightarrow (\mathbf{v_1}, \mathbf{v_2}) \in \bar{E}_q$-*edges* or $(\mathbf{v_2}, \mathbf{v_1}) \in M$.

↬ In W_M, if $(\mathbf{v_1}, \mathbf{v_2}) \in \bar{E}_q$-*edges*, then it conserves its index and if $(\mathbf{v_1}, \mathbf{v_2}) \in W_M \setminus \bar{E}_q$-*edges*, then it has the same index than $(\mathbf{v_1}, \mathbf{v_2})$ which belongs to $\bar{E}_q$-*edges*.

↬ Let $\mathfrak{E}^+(q, 0)$ be the set of vertices in $\bar{\mathbf{X}}_q$ which are not covered by the edges of M,

↬ For each mode q, we note by $\mathcal{C}_i(\bar{E}_q) = (\mathfrak{E}^+(q, i), \mathfrak{E}^-(q, i), \mathcal{E}(q, i))$, $i = 1, \ldots, \mu_q$ the strongly connected components related to $\bar{B}_{q,M}$ and $\mathcal{C}_0(\bar{E}_q) = (\mathfrak{E}^+(q, 0), \emptyset, \emptyset)$.

↬ Use the partial order "$\preccurlyeq$" (cf. Section 3) on the strongly connected components $\mathcal{C}_i(\bar{E}_q)$ for $i = 0, 1, \ldots, \mu_q$.

↬ if $\mathcal{C}_i(\bar{E}_q)$ contains an edge with index q then $\mathcal{C}_i(\bar{E}_q)$ is said to be a q-component.

↬ Γ_q-*edges* $= \{(\mathbf{z_j}, \mathbf{x_i})$ if $\mathbf{x_i}$ and $\mathbf{z_j}$ belong to the same strongly component $\mathcal{C}_\ell(\bar{E}_q)\} \cup \{(\mathbf{z_j}, \mathbf{x_i})$ if $\mathbf{x_i} \in \mathfrak{E}^+(q, \ell)$,

$\mathbf{z_j} \in \mathfrak{E}^-(q,k)$ and $\mathcal{C}_\ell(\bar{E}_q) \preccurlyeq \mathcal{C}_k(\bar{E}_q),\ \ell \neq 0\}$.
The indices of the Γ_q-*edges* are computed as follows:
- all the edges linking elements of a q-component have index q.
- if $\mathcal{C}_\ell(\bar{E}_q) \preccurlyeq \mathcal{C}_k(\bar{E}_q)$ and if there exists a path in $\bar{B}_{q,M}$ starting from $\mathcal{C}_\ell(\bar{E}_q)$ to $\mathcal{C}_k(\bar{E}_q)$ containing at least an edge with index q, then the Γ_q-*edges* linking $\mathbf{x_i} \in \mathfrak{E}^+(q,\ell)$ to $\mathbf{z_j} \in \mathfrak{E}^-(q,k)$ have all index q.
- if $\mathcal{C}_\ell(\bar{E}_q) \preccurlyeq \mathcal{C}_k(\bar{E}_q)$ and if there exists a path starting from $\mathcal{C}_\ell(\bar{E}_q)$ to $\mathcal{C}_k(\bar{E}_q)$ containing only edges with index 0, then the Γ_q-*edges* linking $\mathbf{x_i} \in \mathfrak{E}^+(q,\ell)$ to $\mathbf{z_j} \in \mathfrak{E}^-(q,k)$ have index 0.

Definition of Ψ_q-*edges* and their indices:

$\nrightarrow \Psi_q$-*edges* $= \{(\mathbf{x_i}, \mathbf{z_j})$ if $\mathbf{x_i} \in \mathfrak{E}^+(q,0), \mathbf{z_j} \in \mathfrak{E}^-(q,k)$ and $\mathcal{C}_0(\bar{E}_q) \preccurlyeq \mathcal{C}_k(\bar{E}_q)\}$.
The indices of the Ψ_q-*edges* are computed as follows:
- if there exists a path in $\bar{B}_{q,M}$ starting from $\mathcal{C}_0(\bar{E}_q)$ to $\mathcal{C}_k(\bar{E}_q)$ containing at least an edge with index q then the Ψ_q-*edges* linking $\mathbf{x_i} \in \mathfrak{E}^+(q,0)$ to $\mathbf{z_j} \in \mathfrak{E}^-(q,k)$ have index q.
- if there exists a path in $\bar{B}_{q,M}$ starting from $\mathcal{C}_0(\bar{E}_q)$ to $\mathcal{C}_k(\bar{E}_q)$ containing only edges with index 0 then the Ψ_q-*edges* linking $\mathbf{x_i} \in \mathfrak{E}^+(q,0)$ to $\mathbf{z_j} \in \mathfrak{E}^-(q,k)$ have index 0.

The interpretation of Γ_q-*edges* and Ψ_q-*edges* is related to the decomposition of matrix $\bar{E}_q$ into $(\tilde{E}_q \ \ \hat{E}_q)$, where $\tilde{E}_q$ is a square generically full column matrix. Edge subset Γ_q-*edges* represents matrix $\tilde{E}_q^{-1}$ and Ψ_q-*edges* represents matrix $\tilde{E}_q^{-1}\hat{E}_q$ as it is stated in the following lemma proved in Appendix A:

Lemma 5 *For SSDS Σ_Λ in mode q, matrix $\bar{E}_q$ can be written as $(\tilde{E}_q \ \ \hat{E}_q)$, after possibly column permutations, where $\tilde{E}_q$ is a square generically full column matrix which columns correspond to state vertices included in $\mathfrak{E}^+(q,k)$, $k \neq 0$ and $\bar{E}_q$ corresponds to state vertices included in $\mathfrak{E}^+(q,0)$. We can state:*
i. *The Γ_q-edges correspond generically to the edges of $\tilde{E}_q^{-1}$ and when the index of an edge of Γ_q-edges is equal to q, then the corresponding element of $\tilde{E}_q^{-1}$ is specific to mode q.*
ii. *The Ψ_q-edges correspond generically to the edges of $\tilde{E}_q^{-1}\hat{E}_q$ and when the index of an edge of Ψ_q-edges is equal to q, then the corresponding element of $\tilde{E}_q^{-1}\hat{E}_q$ is specific to mode q.*

Using the previous Γ_q and Ψ_q edges definitions, we construct, for each mode $q = 1, 2$, a new digraph noted $\mathcal{G}_{dmo}(\Sigma_\Lambda, q)$ defined by $\big(\bar{\mathbf{V}}^+_{\mathbf{q}}, \bar{\mathbf{V}}^-_{\mathbf{q}}, \bar{W}_q \cup \Gamma_q\text{-}edges \cup \Psi_q\text{-}edges \backslash \bar{E}_q\text{-}edges\big)$, where each edge belonging to $\bar{A}_q \cup \bar{C}_q$ keeps its initial index.

Example 1 (Continued): *For mode 1, to cover $\bar{\mathbf{Z}}_{\mathbf{1}}$ with only $\bar{E}_1$-edges, we choose the maximal matching $M = \{(\mathbf{x_1}, \mathbf{z_1}), (\mathbf{x_2}, \mathbf{z_3})\}$. Therefore, $\mathfrak{E}^+(1,0) = \{\mathbf{x_3}, \mathbf{x_4}\}$ which vertices are not covered by M. When we carry out the decomposition of $\bar{B}_M(\Sigma_\Lambda, 1)$ into strongly connected components, we obtain two strongly connected components: $\mathcal{C}_1(\bar{E}_1)$ defined by $\mathfrak{E}^+(1,1) = \{\mathbf{x_2}\}$ and $\mathfrak{E}^-(1,1) = \{\mathbf{z_3}\}$ and $\mathcal{C}_2(\bar{E}_1)$ defined by $\mathfrak{E}^+(1,2) = \{\mathbf{x_1}\}$ and $\mathfrak{E}^-(1,2) = \{\mathbf{z_1}\}$, with $\mathcal{C}_1(\bar{E}_1) \not\preccurlyeq \mathcal{C}_2(\bar{E}_1)$ and $\mathcal{C}_2(\bar{E}_1) \not\preccurlyeq \mathcal{C}_1(\bar{E}_1)$. These two strongly connected components are not 1-components. We can then deduce that there exist two Γ_1-edges with indices 0: $\{(\mathbf{z_1}, \mathbf{x_1}), (\mathbf{z_3}, \mathbf{x_2})\}$. Finally, as there exists a path between $\mathfrak{E}^+(1,0)$ and $\mathcal{C}_1(\bar{E}_1)$ $(\mathbf{x_3} \to \mathbf{z_3})$, which does not contain any edge with index 1, we have one Ψ_1-edge having index 0: $(\mathbf{x_3}, \mathbf{z_3})$. The discrete mode observability digraph associated to mode 1 is depicted in the left side of Figure 3. Similarly, the*

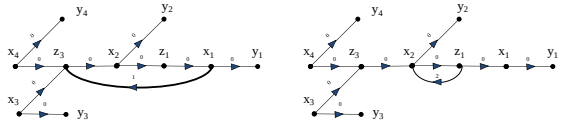

Figure 3. *Discrete mode observability digraphs for the system of Example 1. Mode 1 (left), Mode 2 (right).*
discrete mode observability digraph for mode 2 is depicted in the right side of Figure 3.

In the new digraph, we define two particular vertex subsets:
• For $q \in \{1, 2\}$, $V_{ess,q}\big[\mathcal{V}_1, \mathcal{V}_2\big]$ is the vertex subset including the vertices present in all the maximum $\mathcal{V}_1$–$\mathcal{V}_2$ linkings included in $\mathcal{G}_{dmo}(\Sigma_\Lambda, q)$.
• For $q \in \{1, 2\}$, in $\mathcal{G}_{dmo}(\Sigma_\Lambda, q)$, there exists a unique vertex subset noted $S^o_q\big[\mathcal{V}_1, \mathcal{V}_2\big]$, called minimum output separator, which is the set of begin vertices of all direct $V_{ess,q}\big[\mathcal{V}_1, \mathcal{V}_2\big]$–$\mathcal{V}_2$ paths. Note that, an output vertex subset $\mathbf{V_Y}$ is q-eligible if in $\mathcal{G}_{dmo}(\Sigma_\Lambda, q)$, $S^o_q\big[\mathfrak{E}^+(q,0), \mathbf{V_Y}\big] = \mathfrak{E}^+(q,0)$.

4.3 Discrete mode observability analysis

Proposition 6 *SSDS Σ_Λ, with two possible modes $q \in \{1, 2\}$ is generically discrete mode observable iff one of the following conditions is satisfied in its associated discrete mode observability digraphs $\mathcal{G}_{dmo}(\Sigma_\Lambda, q)$, $q = 1, 2$:*
1. *for some $q \in \{1, 2\}$, there exists a cycle $\mathcal{C}$ containing at least one edge of index q;*
2. *for some $q \in \{1, 2\}$, there exists a $\mathfrak{E}^+(q,0)$–$\bar{\mathbf{Y}}_{\mathbf{q}}$ path containing at least one edge of index q;*
3. *for $q \neq q'$, $\bar{\mathbf{Y}}_{\mathbf{q}} \neq \bar{\mathbf{Y}}_{\mathbf{q'}}$;*
4. *for some $q \in \{1, 2\}$, there exists a q-eligible vertex subset $\mathbf{V_Y} \subseteq \bar{\mathbf{Y}}_{\mathbf{q}}$ such that, either $\mathbf{V_Y}$ is not q-eligible for mode q', or $\beta_q(\mathbf{V_Y}) \neq \beta_{q'}(\mathbf{V_Y})$;*
5. *for some $q \in \{1, 2\}$, there exist an edge e_κ of index q and an $\mathbf{y_i}$-topped path P of length strictly greater or equal to $2\ d_q(\mathbf{y_i}) + 1$ which covers e_κ and such that $\mathbf{v}_\ell$ belongs to a direct $S^o_q[\{\mathbf{v_P}\}, \bar{\mathbf{Y}}_{\mathbf{q}}]$-$\mathbf{y_i}$ path, where $\mathbf{v_P}$ and $\mathbf{v}_\ell$ are respectively the begin vertex of P and the end vertex of e_κ.*

Proof: *Sufficiency: Condition **1**: Let us denote by $\mathbf{x_i}$ one of the vertices of cycle $\mathcal{C}$ verifying Condition **1**. There exist a $\mathbf{x_i}$–$\mathbf{y_j}$ path in $\mathcal{G}_{dmo}(\Sigma_\Lambda, q)$, where $\mathbf{y_j} \in \bar{\mathbf{Y}}_{\mathbf{q}}$ and a q-complete subset $\mathbf{Y_u} \subset \bar{\mathbf{Y}}_{\mathbf{q}} \setminus \{\mathbf{y_j}\}$. From Lemma 2, there exist functions $\varphi_{x,q}$, $\varphi_{y,q}$ and an integer $\nu \geq 0$ such that for mode q, Σ_Λ restricted to its maximal inconsistent part can be written as in form (3). The digraph representation of (3) contains also $\mathcal{C}$. Note that $\mathcal{C}$ is necessarily a succession of $\tilde{E}_q^{-1}$ and $\tilde{A}_q$ edges. From Theorem 21.1 of (Reinschke, 1988), the characteristic equation of matrix $\tilde{E}_q^{-1}\tilde{A}_q$, which has the form $(\tilde{E}_q^{-1}\tilde{A}_q)^{\tilde{n}} + \ldots + a_k(\tilde{E}_q^{-1}\tilde{A}_q)^k + \ldots + a_0 I = 0$, contains a term $a_{\tilde{n}-\bar{k}}(\tilde{E}_q^{-1}\tilde{A}_q)^{\tilde{n}-\bar{k}}$, where $2\bar{k}$ is the length of $\mathcal{C}$ and $\tilde{n}$ is the dimension of $\tilde{X}_q$. $a_{\tilde{n}-\bar{k}}$ depends on the weight of $\mathcal{C}$ and so, on a specific parameter of E^s_q or A^s_q. Thus,*
$$\big(\widetilde{C}_{q,j}(\tilde{E}_q^{-1}\tilde{A}_q)^{\tilde{n}} + \ldots + a_k\widetilde{C}_{q,j}(\tilde{E}_q^{-1}\tilde{A}_q)^k + \ldots + a_0\widetilde{C}_{q,j}\big)\tilde{X}_q = 0 \quad (4)$$
where $\widetilde{C}_{q,j}$ is the line of matrix $\widetilde{C}$ related to output y_j in mode q in (3) i.e. $y_j = \tilde{C}_{q,j}\tilde{X}_q + \varphi_{y,q,j}(Y_u, \dot{Y}_u, \ldots, Y_u^{(\nu)})$. *Equation*

(4) leads to an algebraic equation parametrized by at least an element specific to mode q.

*Condition **2**: Let denote by $\mathbf{v_i}$ the vertex of $\mathfrak{E}^+(q,0)$ from which there is a $\mathfrak{E}^+(q,0)$–$\bar{\mathbf{Y}}_\mathbf{q}$ path P containing at least an edge e_κ of index q. Let λ_κ be the non-zero parameter (or weight) associated to e_κ, $\mathbf{y_j}$ be the end of P and $2\ell+1$ its length. It exists a q-complete subset $\mathbf{Y_u} = \{\mathbf{y_{i_1}}, \mathbf{y_{i_2}}, \ldots, \mathbf{y_{i_k}}\} \subset \bar{\mathbf{Y}}_\mathbf{q} \setminus \{\mathbf{y_j}\}$ such that elements of $\mathfrak{E}^+(q,0)$ are covered by a complete matching in $\bar{B}(\Sigma_\Lambda, q)$ without using edges and vertices of P neither the ones belonging to $\bar{E}_q$-edges. From Lemma 2, there exist functions $\varphi_{x,q}$, $\varphi_{y,q}$ and an integer $\nu \leq \tilde{n}$ such that the dynamics equation of state part $\tilde{X}_q$ is in form (3). From the characteristic equation of matrix $\tilde{A}_q$, we can write an equation as (4), where also $\tilde{C}_{q,j}$ is the line of matrix $\widetilde{C}$ related to output y_j in mode q in (3)* i.e. *$y_j = \tilde{C}_{q,j}X_{1,q} + \varphi_{y,q,j}(Y_u, \dot{Y}_u, \ldots, Y_u^{(\nu)})$. Thus, $\tilde{C}_{q,j}(\tilde{E}_q^{-1}\tilde{A}_q)^k\tilde{X}_q = y_j^{(k)} - \tilde{C}_{q,j}\big(\varphi_{x,q}^{(k-1)}(Y_u, \dot{Y}_u, \ldots, Y_u{}^{(\nu)}) + \tilde{E}_q^{-1}\tilde{A}_q\varphi_{x,q}^{(k-2)}(Y_u, \dot{Y}_u, \ldots, Y_u{}^{(\nu)}) + \ldots + (\tilde{E}_q^{-1}\tilde{A}_q)^{k-2}\dot{\varphi}_{x,q}(Y_u, \dot{Y}_u, \ldots, Y_u{}^{(\nu)})\big) - \varphi_{y,q,j}^{(k)}(Y_u, \dot{Y}_u, \ldots, Y_u^{(\nu)})$. Since there exists a path of length $2\ell+1$ from $\mathbf{v_i}$ to $\mathbf{y_j}$ containing an edge e_κ, term $C_j(\tilde{E}_q^{-1}\tilde{A}_q)^\ell$ is not zero and depends on the element denoted previously λ_κ. Therefore, we obtain an algebraic equation specific to mode q.*

*Condition **3**: This condition implies that there exists $\mathbf{y_i} \in \bar{\mathbf{Y}}_\mathbf{q} \setminus \bar{\mathbf{Y}}_{\mathbf{q}'}$. Using Corollary 3, there is a redundancy equation including y_i and its derivatives in mode q but not in mode q'.*

*Condition **4**: If $\mathbf{V_Y}$ is q-eligible for mode q and not for mode q', then Corollary 3 ensures the existence of an equation linking the components of $\mathbf{V_Y}$ and their derivatives in mode q but not in mode q'. Otherwise, if $\beta_q(\mathbf{V_Y}) \neq \beta_{q'}(\mathbf{V_Y})$, then at least one output derivative can be expressed in mode q' using other outputs but not in mode q .*

*Condition **5**: If, for some q, $\mathbf{y_i} \in \bar{\mathbf{Y}}_\mathbf{q}$ then there exists a q-complete vertex subset $\mathbf{Y_u} \subseteq \bar{\mathbf{Y}}_\mathbf{q} \setminus \{\mathbf{y_i}\}$. From Lemma 2, there exist functions $\varphi_{x,q}$, $\varphi_{y,q}$ and an integer $\nu \leq \tilde{n}$ such that the dynamics equation of state part $\tilde{X}_q$ is in form (3). Moreover, there exists a minimal subset $\mathbf{Y_s} \subseteq \bar{\mathbf{Y}}_\mathbf{q} \setminus (\mathbf{Y_u} \cup \{\mathbf{y_i}\})$ such that $\forall k \geq d_q(\mathbf{y_i})$, we can write an equation of the form*

$$y_i^{(k)} = \sum_{s<\tilde{k_i}} \alpha_{i,s} y_i^{(s)} + \sum_{l \,|\, \mathbf{y}_l \in \mathbf{Y_s}} \sum_{s=0}^{\tilde{n}} \alpha_{l,s} y_l^{(s)} + \upsilon(Y_u, \ldots, Y_u^{(n)}) \quad (5)$$

*Let us denote by $\mathbf{v_P} = \mathbf{x_j}$ the begin vertex of the so-called path P satisfying Condition **5*** (i.e. *P is a $\mathbf{y_i}$-topped path of length $2k+1$ strictly greater than $d_q(\mathbf{y_i})$ and covers e_κ) and e_j the j^{th} Euclidean vector. Relation (5) can be written as:*

$$\tilde{C}_{q,i}(\tilde{E}_q^{-1}\widetilde{A}_q)^k\mathrm{e}_j = \Big(\sum_{s<d_q(\mathbf{y_i})} \alpha_{i,s}\tilde{C}_{q,i}(\tilde{E}_q^{-1}\widetilde{A}_q)^s + \sum_{\mathbf{y}_l \in \mathbf{Y_s}} \sum_{s=0}^{n} \alpha_{l,s}\tilde{C}_{q,l}(\tilde{E}_q^{-1}\widetilde{A}_q)^s\Big)\mathrm{e}_j \quad (6)$$

where each non-zero component of $\tilde{C}_{q,l}(\tilde{E}_q^{-1}\widetilde{A}_q)^s$ is associated to the paths arriving to $\mathbf{y}_l \in \mathbf{Y_s}$ of length $2s+1$. If $S_q^o\big[\{\mathbf{x_j}\}, \mathbf{Y_s} \cup \{\mathbf{y_i}\}\big]$ is a state vertex, let us denote it $\mathbf{x_r}$. There exist k_r and k' such that $k_r + k' = k$ and $\tilde{C}_{q,i}(\tilde{E}_q^{-1}\widetilde{A}_q)^k\mathrm{e}_j = \tilde{C}_{q,i}(\tilde{E}_q^{-1}\widetilde{A}_q)^{k_r}\Delta_r(\tilde{E}_q^{-1}\widetilde{A}_q)^{k'}\mathrm{e}_j$ where Δ_r is a diagonal matrix which has only one non-zero element $\Delta_r(r,r) = 1$. We can do the same reasoning for each term $\tilde{C}_{q,l}(\tilde{E}_q^{-1}\widetilde{A}_q)^s\mathrm{e}_j$ and so there exist s_r and s' such that $s_r + s' = s$ and $\tilde{C}_{q,l}(\tilde{E}_q^{-1}\widetilde{A}_q)^s\mathrm{e}_j = \tilde{C}_{q,l}(\tilde{E}_q^{-1}\widetilde{A}_q)^{s_r}\Delta_r(\tilde{E}_q^{-1}\widetilde{A}_q)^{s'}\mathrm{e}_j$. The fact that end vertex $\mathbf{x}_\ell$ of e_κ belongs to a direct $S_q^o[\{\mathbf{x_j}\}, \bar{\mathbf{Y}}_\mathbf{q}]$-$\mathbf{y_i}$ path implies that specific edge e_κ belongs to a $S_q^o\big[\mathbf{v_P}, \mathbf{Y_s} \cup \{\mathbf{y_i}\}\big]$–$\mathbf{Y_s} \cup \{\mathbf{y_i}\}$ path. This means that edge e_κ appears in only some $S_q^o\big[\mathbf{v_P}, \mathbf{Y_s} \cup \{\mathbf{y_i}\}\big]$–$\mathbf{Y_s} \cup \{\mathbf{y_i}\}$ paths. Thus, some terms of $\tilde{C}_{q,i}(\tilde{E}_q^{-1}\widetilde{A}_q)^{k_r}$ and $\tilde{C}_{q,l}(\tilde{E}_q^{-1}\widetilde{A}_q)^{s_r}$, but not all, contain the non-zero parameter corresponding specific to mode q. Denoting by $C_r = \mathrm{e}_r^T$, where e_r is the r^{th} Euclidean vector, we have that $\tilde{C}_{q,i}(\tilde{E}_q^{-1}\widetilde{A}_q)^k\mathrm{e}_j = \tilde{C}_{q,i}(\tilde{E}_q^{-1}\widetilde{A}_q)^{k_r}\Delta_r(\tilde{E}_q^{-1}\widetilde{A}_q)^{k'}\mathrm{e}_j = \alpha' C_r(\tilde{E}_q^{-1}\widetilde{A}_q)^{k'}\mathrm{e}_j$ and $\tilde{C}_{q,l}(\tilde{E}_q^{-1}\widetilde{A}_q)^s\mathrm{e}_j = \tilde{C}_{q,l}(\tilde{E}_q^{-1}\widetilde{A}_q)^{s_r}\Delta_r(\tilde{E}_q^{-1}\widetilde{A}_q)^{s'}\mathrm{e}_j = \alpha'_{l,s}C_r(\tilde{E}_q^{-1}\widetilde{A}_q)^{s'}\mathrm{e}_j$. Thus, after substitution of the previous terms in relation (6),

$$\alpha' C_r(\tilde{E}_q^{-1}\widetilde{A}_q)^{k'}\mathrm{e}_j = \Big(\sum_{s_r \leq s < d_q(\mathbf{y_i})} \alpha'_{i,s}\alpha_{i,s}C_r^{s-s_r} + \sum_{l \,|\, \mathbf{y}_l \in \mathbf{Y_s}} \sum_{s=s_r}^{\tilde{n}} \alpha'_{l,s}\alpha_{l,s}C_r(\tilde{E}_q^{-1}\widetilde{A}_q)^{s-s_r}\Big)\mathrm{e}_j$$

where some, but not all, coefficients α' and $\alpha'_{l,s}$ depend on the weight of e_κ. This equality leads to an algebraic relation satisfied only in mode q. If $S_q^o\big[\{\mathbf{x_j}\}, \mathbf{Y_s} \cup \{\mathbf{y_i}\}\big]$ is a dynamical vertex, we have the same result.

Necessity: From Lemma 1, only the edges in the maximal inconsistent part in each mode of Σ_Λ can lead to an algebraic relation allowing the discrete mode observability. It is necessary to consider a q-complete subset $\mathbf{Y_u} \subseteq \bar{\mathbf{Y}}_\mathbf{q}$ to obtain for mode q a system on form (3).

*When Conditions **1** and **2** are not satisfied, it is not possible, for any choice of $\mathbf{Y_u}$, to obtain an algebraic relation achieving mode distinguishability from the characteristic equation of any matrix $(\tilde{E}_q^{-1}\widetilde{A}_q)$ of (3). In this case also, $\varphi_{x,q}(Y_u, \dot{Y}_u, \ldots, Y_u^{(\nu)})$ does not depend on any specific element of $\bar{A}_q$, $\bar{C}_q$ or $\bar{E}_q$, $q \in \{1, 2\}$.*

*When Conditions **3** and **4** are not satisfied, the output elements of some vertex subset $\mathbf{Y_s}$ implicated in any of the existing algebraic output equations have similar derivative degrees for modes 1 and 2. Thus, the only way to have different algebraic equations is that there exists at least one coefficient $\alpha_{i,s}$ or $\alpha_{l,s}$ of (5), which characterizes a parameter which intervenes in only one mode. Without loss of generality, we can assume that $\mathbf{Y_s}$ is minimal. Defining $\widetilde{k}_i = \beta_q(\mathbf{Y_s} \cup \{\mathbf{y_i}\} \cup \mathbf{Y_u}) - \beta_q(\mathbf{Y_s} \cup \mathbf{Y_u}) \geq d_q(\mathbf{y_i})$, in redundancy relation (5), we have necessarily $k \geq \widetilde{k}_i$. To guarantee discrete mode observability, there must exist at least a $\mathbf{y_i}$-topped path P, of length greater or equal to $2\widetilde{k}_i + 1$ and so of length strictly greater than $2d_q(\mathbf{y_i})$, associated to this relation. Let us denote by $\mathbf{x_j}$ the begin vertex of P and e_j the j^{th} Euclidean vector. Equation (5) can be written as relation (6), where $\tilde{C}_{q,l}(\tilde{E}_q^{-1}\widetilde{A}_q)^s$ is associated with $\mathbf{y}_l$-topped paths, $\mathbf{y}_l \in \mathbf{Y_s}$ of length $2s+1$. If for each $\mathbf{x}_\ell$, end vertex of a specific edge belonging to P, $\mathbf{x}_\ell$ does not belong to any direct $S_q^o[\{\mathbf{v_P}\}, \mathbf{Y_{1,q}}]$-$\mathbf{y_i}$ path, then all the paths starting from $\mathbf{x_r}$ to $\mathbf{Y}$ contain only edges common to the two modes. Then all the existing relations of the form (5) do not contain terms specific to a mode q. We can do the same reasoning for all q-complete subsets $\mathbf{Y_u} \subseteq \bar{\mathbf{Y}}_\mathbf{q}$. Thus, it is not possible to achieve the mode distinguishability.* △

Comments and interpretation: If there is a specific edge belonging to any cycle in the discrete mode observability digraphs $\mathcal{G}_{dmo}(\Sigma_\Lambda, q)$ then the distinguishability is possible (first condition). If a specific edge belongs to any path linking the minimal inconsistent part of E_q to the output in $\mathcal{G}_{dmo}(\Sigma_\Lambda, q)$ then the distinguishability is possible (second condition). If a specific edge allows to modify the output subdivision due to the previous DM-decomposition (third condition) or to modify the observability subspace of any output measurements set (fourth condition), then the distinguishability is possible. Finally, if a specific edge belongs to an output rooted path including a state vertex, which can be linked independently to other outputs, then the distinguishability is possible (fifth condition). If all the conditions are not satisfied, then the two modes are too similar or their differences are not observable from the

measurements and so they are not distinguishable.

Example 1 (Continued): For mode 1, we have only one specific edge $(\mathbf{x_1}, \mathbf{z_3})$ belonging to a cycle: $\mathbf{x_1} \rightarrow \mathbf{z_3} \rightarrow \mathbf{x_2} \rightarrow \mathbf{z_1} \rightarrow \mathbf{x_1}$ and Condition **1** is satisfied. This characterizes an algebraic relation specific to mode 1: $\lambda_9\lambda_{11}\ddot{y}_1 = \lambda_1\lambda_5\left(\frac{\lambda_3}{\lambda_8}y_4 + \frac{\lambda_{14}}{\lambda_5}y_1 - \frac{\lambda_{12}}{\lambda_7}\dot{y}_3\right)$. For mode 2, we have only one specific edge $(\mathbf{z_1}, \mathbf{x_2})$ belonging to a cycle: $\mathbf{z_1} \rightarrow \mathbf{x_2} \rightarrow \mathbf{z_1}$. This characterizes an algebraic relation specific to mode 2: $\lambda_9\lambda_{11}\ddot{y}_1 = \lambda_1\lambda_5\left(\frac{\lambda_3}{\lambda_8}y_4 - \frac{\lambda_{16}}{\lambda_5}\dot{y}_1 - \frac{\lambda_{12}}{\lambda_7}\dot{y}_3\right)$. Condition **2** is not satisfied for mode 1 neither in mode 2 because the direct $\mathfrak{E}^+(1,0)$–$\bar{\mathbf{Y}}_\mathbf{1}$ paths do not contain specific edge. Condition **3** is not satisfied also as $\bar{\mathbf{Y}}_\mathbf{1} = \bar{\mathbf{Y}}_\mathbf{2}$. Condition **4** is satisfied for $\mathbf{V_y} = \{\mathbf{y_2}, \mathbf{y_3}, \mathbf{y_4}\}$ which is 1-eligible and 2-eligible and $\beta_1(\mathbf{V_y}) = 4$ while $\beta_2(\mathbf{V_y}) = 3$. This characterizes an algebraic relation specific to mode 2 and not satisfied for mode 1: $\frac{\lambda_{11}}{\lambda_6}\dot{y}_2 + \frac{\lambda_{12}}{\lambda_7}\dot{y}_3 + \frac{\lambda_{16}\lambda_1}{\lambda_9\lambda_6}y_2 = \frac{\lambda_3}{\lambda_8}y_4$. For Condition **5**, only $\mathbf{y_1}$ or $\mathbf{y_2}$ can play the role of $\mathbf{y_i}$ in Proposition 6 because they are successor of $\mathbf{x_1}$. $d_1(\mathbf{y_1}) = 0 = d_1(\mathbf{y_2})$ and $S_1^o[\{\mathbf{x_1}\}, \bar{\mathbf{Y}}_\mathbf{1}] = \{\mathbf{x_1}\}$. The existence of direct path $\mathbf{x_1}$-$\mathbf{y_2}$ of length 3 covering specific edge $(\mathbf{x_1}, \mathbf{z_3})$ allows to satisfy Condition **5**. This characterizes another algebraic relation specific to mode 1: $\frac{\lambda_{11}}{\lambda_6}\dot{y}_2 + \frac{\lambda_{12}}{\lambda_7}\dot{y}_3 = \frac{\lambda_3}{\lambda_8}y_4 + \frac{\lambda_{14}}{\lambda_5}y_1$. For mode 2, Condition **5** is satisfied also and we have relation: $\frac{\lambda_{11}}{\lambda_6}\dot{y}_2 + \frac{\lambda_{12}}{\lambda_7}\dot{y}_3 + \frac{\lambda_{16}}{\lambda_5}\dot{y}_1 = \frac{\lambda_3}{\lambda_8}y_4$.

5 Conclusion

In this paper, we propose a graph-theoretic tool to characterize exactly the generic discrete mode observability of structured switching descriptor linear systems. The studied system can be under-determined, overdetermined or square and possibly non-regular. Our approach consists in two steps. First, we simplify the bipartite graphs associated to the system and then we build a new digraph dedicated to the discrete mode observability. Then, we establish the necessary and sufficient graphical conditions for the discrete mode observability. These conditions generalize the ones established in (Boukhobza and Hamelin, 2011) for systems in standard form. To check these conditions, we can use classical programming techniques, which are free from numerical difficulties as their computational complexity is polynomial.

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Appendix A: Proof of Lemma 5

i. We can always reorder the strongly connected components to have $\mathcal{C}_j(\bar{E}_q) \preccurlyeq \mathcal{C}_i(\bar{E}_q)$ only when $i \leq j$. In this case, Dulmage-Mendelsohn decomposition of the bipartite graph related to matrix $\tilde{E}_q$ leads (after a possible permutation of rows and columns of this matrix) to the following bloc decomposition of $\tilde{E}_q$:

$$\tilde{E}_q = \begin{array}{c} \overset{\mathfrak{E}^+(q,1)}{\longleftrightarrow}\ \overset{\mathfrak{E}^+(q,2)}{\longleftrightarrow}\ \overset{\mathfrak{E}^+(q,3)}{\longleftrightarrow}\ \dots\ \overset{\mathfrak{E}^+(q,\mu_q)}{\longleftrightarrow} \\ \begin{pmatrix} E_{q,1,1} & E_{q,1,2} & E_{q,1,3} & \dots & E_{q,1,\mu_q} \\ 0 & E_{q,2,2} & E_{q,2,3} & \dots & E_{q,2,\mu_q} \\ \vdots & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & E_{q,\mu_q,\mu_q} \end{pmatrix} \end{array} \begin{array}{l} \updownarrow \mathfrak{E}^-(q,1) \\ \updownarrow \mathfrak{E}^-(q,2) \\ \vdots \\ \updownarrow \mathfrak{E}^-(q,\mu_q) \end{array}$$

where all matrices $E_{q,i,i}$, $= 1, \dots, \mu_q$ are square matrices and generically full row-rank because there is no inconsistent part in the bipartite graph related to matrix $\tilde{E}_q$ and $E_{q,i,j} \neq 0$ only if there exists an edge from $\mathcal{C}_i(\bar{E}_q)$ to $\mathcal{C}_j(\bar{E}_q)$. Applying the inversion triangular block matrices, to the inversion of $\tilde{E}_q$, we have that $\tilde{E}_q^{-1}$ is also a block triangular matrix. Since, $E_{q,i,i}$ is associated to a strongly connected component, this sub-matrix cannot, by any row or column permutation, be partitioned into a block-triangular matrix. Therefore, the bipartite graph representing this matrix contains edges from all vertices of $\mathfrak{E}^-(q,i)$ to the ones of $\mathfrak{E}^+(q,i)$. Moreover, if there exists in $E_{q,i,i}$ an element which is specific to mode q, then all the elements of $E_{q,i,i}^{-1}$ will be specific to mode q and so must be represented by edges with index q.

Let us look now to the sub-matrices in the upper part of $\tilde{E}_q^{-1}$ and consider first the two last blocks of $\tilde{E}_q$ *i.e.* $\begin{pmatrix} E_{q,\mu_q-1,\mu_q-1} & E_{q,\mu_q-1,\mu_q} \\ 0 & E_{q,\mu_q,\mu_q} \end{pmatrix}$. The inversion of this sub-matrix gives $\begin{pmatrix} E_{q,\mu_q-1,\mu_q-1}^{-1} & -E_{q,\mu_q-1,\mu_q-1}^{-1}E_{q,\mu_q-1,\mu_q}E_{q,\mu_q,\mu_q}^{-1} \\ 0 & E_{q,\mu_q,\mu_q}^{-1} \end{pmatrix}$. Matrix $E_{q,\mu_q-1,\mu_q-1}^{-1}E_{q,\mu_q-1,\mu_q}E_{q,\mu_q,\mu_q}^{-1}$ is generically non-zero iff there exists an edge from $\mathcal{C}_{\mu_q-1}(\bar{E}_q)$ to $\mathcal{C}_{\mu_q}(\bar{E}_q)$. In this case, the bipartite representing this matrix contain edges from all vertices of $\mathfrak{E}^-(q,\mu_q)$ to the ones of $\mathfrak{E}^+(q,\mu_q-1)$. Moreover, if there exists a path from a vertex of $\mathcal{C}_{\mu_q-1}(\bar{E}_q)$ to a vertex of $\mathcal{C}_{\mu_q}(\bar{E}_q)$, then all the elements of $E_{q,\mu_q-1,\mu_q-1}^{-1}E_{q,\mu_q-1,\mu_q}E_{q,\mu_q,\mu_q}^{-1}$ will be specific to mode q and so must be represented by edges with index q. We can generalize this reasoning to the other blocs of $\tilde{E}_q^{-1}$ and we have that the Γ_q*-edges* as defined in Subsection 4.2 corresponds generically to the edges of matrix $\tilde{E}_q^{-1}$, where $\tilde{E}_q$ is the square generically full column matrix described by the bipartite graph restricted to $\bar{E}_q$*-edges* between vertices included in $\mathfrak{E}^+(q,k)$, $k \neq 0$ and the ones included in $\mathfrak{E}^-(q,k)$, $k \neq 0$. Moreover, when the index of an edge of Γ_q*-edges* is equal to q, then the corresponding element of $\tilde{E}_q^{-1}$ is specific to mode q.

ii. According to the previous characterization of $\tilde{E}_q$ and as edges related to $\hat{E}_q$ are the ones starting from $\mathfrak{E}^+(q,0)$, we have that the non zero elements of $\tilde{E}_q^{-1}\hat{E}_q$ correspond in $\bar{B}_{q,M}$ to the paths starting from $\mathfrak{E}^+(q,0)$ by necessarily an $\hat{E}_q$-edge and arriving to some $\mathcal{C}_i(\bar{E}_q)$. This is traduced by the definition of Ψ_q*-edges*. If, besides, this path contains an edge with index q, then all the edges from the corresponding begin vertex and all vertices of $\mathcal{C}_i(\bar{E}_q)$ describe relations which are specific to mode q and must have index q.