Piezoelectric Relations and the Radial Deformation of a Polarized Spherical Shell

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The piezoelectric relations and equations of equilibrium for an elastic spherical shell permanently polarized in the radial direction are derived. The solution for the radial displacement corresponding to uniform pressures on the inner and outer surfaces of the shell and a given voltage difference between the surfaces is obtained.

1. INTRODUCTION

IN this paper we determine the radial displacement of a polarized spherical shell as a function of its piezoelectric and elastic moduli, the pressures exerted on its inner and outer surfaces, and the voltage differences between these surfaces. We treat the problem according to the linear theory of the piezoelectric effect. The purely elastic problem of a homogeneous, isotropic elastic spherical shell subjected to an internal pressure p_1 and an external pressure p_2 is treated in Love,[1] who attributes the results to Lamé. The essential and interesting differences between the problem treated here and in the foregoing is that the polarized shell is neither homogeneous nor isotropic, and there is, in addition to the mechanical boundary conditions, the electrical boundary condition for the voltage difference. Before polarization, the ceramic shell is homogeneous and isotropic. However, after the shell is permanently polarized in the radial direction by applying a large static voltage between its inner and outer surfaces, the point symmetry of the material is transversely isotropic with an axis of symmetry in the direction of the radius vector to the center of the spherical shell. Thus it is clear that the polarized material is neither isotropic nor homogeneous. Huntington and Southwick[2] have shown by measurements of the speeds of ultrasonic waves in polarized ceramics that the elastic coefficients indeed acquire an anisotropic component in the polarized material. The degree of anisotropy in the elastic coefficients introduced by the polarizing process is small for the specific materials studied by Huntington and Southwick, but we shall not neglect any influence the elastic anisotropy may have on our solution. In applications of the final formulas to particular materials with this general type of symmetry, it may be warranted and convenient to simplify the equations by the assumption of elastic isotropy.

2. BASIC EQUATIONS AND BOUNDARY CONDITIONS

We shall adhere to the notations for stress, strain, elastic moduli, etc., adopted as standards for piezoelectric theory.[3] The equations satisfied by the stress T_{ij} and the electric field E_i are

$$T_{ij,\,j}=\frac{\partial T_{ij}}{\partial x_j}=0, \tag{2.1}$$

$$(\text{curl } E)_i=\epsilon_{ijk}E_{k,\,j}=0, \tag{2.2}$$

where we shall use rectangular Cartesian coordinates x_i exclusively and the summation convention for repeated tensor indices. The electric displacement D_i satisfies the equation

$$\text{div } D=D_{i,\,i}=0. \tag{2.3}$$

Let $S_{ij}=\frac{1}{2}(u_{i,\,j}+u_{j,\,i})$ denote the strain tensor, where u_i is the displacement vector. We shall write the linear piezoelectric constitutive relations between **T**, **S**, **E**, and **D** in the form

$$T_{ij}=c^D{}_{ijkl}S_{kl}-h_{kij}D_k, \tag{2.4}$$

$$E_i=-h_{ikl}S_{kl}+\beta^S{}_{ik}D_k, \tag{2.5}$$

where the coefficients in these relations have the names and dimensions given in Table II of reference 3. For the problem considered here of a spherical shell with inner radius r_1 and outer radius r_2 subjected to an internal pressure p_1 and an external pressure p_2, the mechanical boundary conditions to be satisfied are

$$T_{ij}n_j=-p_1n_i \tag{2.6}$$

on the inner surface and

$$T_{ij}n_j=-p_2n_i \tag{2.7}$$

on the outer surface, where n_i is a unit vector in the radial direction.

From (2.2) it follows that E_i is expressible in terms of the electric potential φ as

$$E_i=-\varphi_{,\,i}. \tag{2.8}$$

The inner and outer surfaces of the shell are coated with a conducting material so that they are level surfaces of the potential field φ. The voltage condition takes the form

$$\varphi_2-\varphi_1=-\int E_i dx_i=V, \tag{2.9}$$

[1] A. E. H. Love, *The Mathematical Theory of Elasticity* (Dover Publications, New York, 1944), fourth edition.
[2] H. B. Huntington and R. D. Southwick, J. Acoust. Soc. Am. 27, 677–679 (1955).
[3] IRE Standard on Piezoelectric Crystals, 1949, Proc. Inst. Radio Engrs. 37 1378–1395 (1949).

where the integral is along any convenient path leading from the internal surface of the shell to its outer surface. We seek a solution of the equations of this section, when the coefficients in the constitutive equations (2.4) and (2.5) have the general form to be derived in the next section.

3. ELASTIC AND PIEZOELECTRIC COEFFICIENTS FOR THE POLARIZED SPHERICAL SHELL

Let the origin of the rectangular Cartesian coordinate system x_i be fixed at the center of the shell. Let $r=(x_ix_i)^{\frac{1}{2}}$ denote the distance of a point from the origin, and let $n_i\equiv x_i/r$ be a unit vector field in the radial direction. We shall assume that the elastic and piezoelectric coefficients of the polarized shell are independent of r. The polarizing field is undoubtedly not independent of r, but if the shell is thin or if the polarization saturates, then there is a reasonable physical basis for this assumption. Now since the polarized shell is invariant to arbitrary rotations about any axis passing through the center of the shell, we see that the elastic and piezoelectric coefficients at each point must reduce to the form they assume in a transversely isotropic material where n_i is the axis of symmetry. Thus we shall have

$$\begin{aligned} c_{ijkl}{}^D=&\lambda^D\delta_{ij}\delta_{kl}+\mu^D(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il})\\ &+\xi_1{}^D(\delta_{ij}n_kn_l+\delta_{kl}n_in_j)\\ &+\xi_2{}^D(\delta_{ik}n_jn_l+\delta_{jk}n_in_l+\delta_{il}n_jn_k+\delta_{jl}n_in_k)\\ &+\xi_3{}^Dn_in_jn_kn_l, \end{aligned} \tag{3.1}$$

$$h_{kij}=\zeta_1\delta_{ij}n_k+\zeta_2(\delta_{ik}n_j+\delta_{jk}n_i)+\zeta_3n_in_jn_k, \tag{3.2}$$

$$\beta_{ij}{}^S=\beta\delta_{ij}+\gamma n_in_j, \tag{3.3}$$

where the quantities λ, μ, ξ_1, ξ_2, ξ_3, ζ_1, ζ_2, ζ_3, β, and γ are constants independent of x_i.

At a point in the material lying on the x_3 axis, the unit vector field has the special form $n_i=(0,0,1)$, and the coefficients (3.1), (3.2), and (3.3) assume the values

$$\begin{aligned} &c_{11}{}^D=c_{1111}{}^D=\gamma^D+2\mu^D,\\ &c_{12}{}^D=c_{1122}{}^D=\lambda^D,\\ &c_{33}{}^D=c_{3333}{}^D=\lambda^D+2\mu^D+2\zeta_1{}^D+4\xi_2{}^D+\xi_3{}^D,\\ &c_{13}{}^D=c_{1133}{}^D=\lambda^D+\xi_1{}^D,\\ &c_{44}{}^D=c_{2323}{}^D=\mu^D+\xi_2{}^D,\\ &h_{13}=h_{133}=\zeta_1,\\ &h_{33}=h_{333}=\zeta_1+2\zeta_2+\zeta_3,\\ &h_{15}=2h_{113}=2\zeta_2,\\ &\beta_1{}^S=\beta_{11}{}^S=\beta,\\ &\beta_3{}^S=\beta_{33}{}^S=\beta+\gamma. \end{aligned} \tag{3.4}$$

The notation on the extreme left of the formulas (3.4) is the Voigt matrix notation for the elastic and piezeoelectric moduli. All of the elements not listed in (3.4) are related to those listed by symmetry or vanish identically. The relations (3.4) can be solved for the coefficients λ^D, μ^D, ζ_1, etc.

4. DIFFERENTIAL EQUATION SATISFIED BY RADIAL DISPLACEMENT

If a spherical shell having the material symmetry assumed here is subjected to an inner and outer hydrostatic pressure, and if its surfaces are level surfaces of the electric potential, then it follows simply from symmetry considerations that each of the vector fields u_i, E_i, and D_i will have a direction coincident with n_i. Thus we set

$$u_i=u(r)n_i,\quad E_i=E(r)n_i,\quad D_i=D(r)n_i, \tag{4.1}$$

where the functions $u(r)$, $E(r)$, and $D(r)$ will be determined by the conditions of equilibrium and boundary conditions of Sec. 2. Since the material is not homogeneous, in computing the divergence of the stress tensor in Eq. (2.1) we shall have

$$T_{ij,\,j}=c^D{}_{ijkl,\,j}S_{kl}+c^D{}_{ijkl}S_{kl,\,j}-h_{ijk,\,j}D_k-h_{ijk}D_{k,\,j}, \tag{4.2}$$

and the terms arising from the derivatives of the elastic and piezoelectric moduli do not vanish as in homogeneous materials. The somewhat lengthy calculations indicated in (4.2) are facilitated by use of the following algorisms:

$$r_{,\,i}=n_i,\quad n_{i,\,j}=\frac{1}{r}(\delta_{ij}-n_in_j)\equiv\frac{1}{r}\Delta_{ij},$$

$$n_in_i=1,\quad \Delta_{ij}n_j=0. \tag{4.3}$$

Thus, for example, we have

$$\begin{aligned} S_{kl}&=\tfrac{1}{2}(u'r_{,\,k}n_l+u'r_{,\,l}n_k+un_{k,\,l}+un_{l,\,k})\\ &=u'n_kn_l+\frac{u}{r}\Delta_{kl}, \end{aligned} \tag{4.4}$$

$$D_{k,\,l}=D'n_ln_k+\frac{D}{r}\Delta_{kl}, \tag{4.5}$$

where a prime denotes differentiation with respect to the variable r. After some work we find that Eq. (2.1) reduces to

$$\begin{aligned} T_{ij,\,j}=[&c_{33}{}^Du''+2c_{33}{}^Du'r^{-1}\\ &-2\{c_{33}{}^D+(c_{11}{}^D-c_{33}{}^D)+(c_{12}{}^D-c_{13}{}^D)\}ur^{-2}\\ &-h_{33}D'-2(h_{33}-h_{13})Dr^{-1}]n_i=0. \end{aligned} \tag{4.6}$$

If we multiply (4.6) by n_i and sum and divide the result by $c_{33}{}^D$, we get

$$\begin{aligned} u''+2u'r^{-1}-2\left[1+\frac{(c_{11}{}^D-c_{33}{}^D)+(c_{12}{}^D-c_{13}{}^D)}{c_{33}{}^D}\right]ur^{-2}\\ =\frac{h_{33}}{c_{33}{}^D}D'+2\frac{(h_{33}-h_{13})}{c_{33}{}^D}Dr^{-1}. \end{aligned} \tag{4.7}$$

If the elastic coefficients c^D_{ijk} have the isotropic form, $c_{11}{}^D = c_{33}{}^D$, $c_{12}{}^D = c_{13}{}^D$, and Eq. (4.7) with $D=0$ reduces to Lamé's equation for the radial displacement.[4]

The general solution of Eq. (2.3) for D_i which has the form $(4.1)_3$ is

$$D_i = (D_0/r^2)n_i, \tag{4.8}$$

where D_0 is an arbitrary constant. If we substitute this result into Eq. (4.7), we get a differential equation in u only.

$$u''+2u'r^{-1}-2(1+\nu)ur^{-2} = -2(h_{13}/c_{33}{}^D)D_0r^{-3}, \tag{4.9}$$

where we have defined the quantity ν by

$$\nu \equiv [(c_{11}{}^D - c_{33}{}^D)+(c_{12}{}^D - c_{13}{}^D)][c_{33}{}^D]^{-1}. \tag{4.10}$$

The constant ν vanishes when the elastic coefficients c^D_{ijkl} have the isotropic form.

The general solution of the differential equation (4.9) is easy to obtain by standard methods and has the form

$$u = C_1r^{\alpha_1}+C_2r^{\alpha_2}+\frac{h_{13}D_0}{(1+\nu)c_{33}{}^D}r^{-1}, \tag{4.11}$$

where C_1 and C_2 are arbitrary constants and the exponents α_1 and α_2 are the roots of the quadratic equation

$$\alpha^2+\alpha-2(1+\nu)=0.$$

Thus

$$\alpha_1 = \frac{-1+[1+8(1+\nu)]^{\frac{1}{2}}}{2}, \qquad \alpha_2 = \frac{-1-[1+8(1+\nu)]^{\frac{1}{2}}}{2}. \tag{4.12}$$

For the isotropic case, $\nu=0$ and $\alpha_1=+1$, $\alpha_2=-2$.

5. DETERMINATION OF THE CONSTANTS C_1, C_2, AND D_0 IN TERMS OF THE BOUNDARY DATA

If we substitute the general solutions (4.11) and (4.8) for $u(r)$ and $D(r)$ into the mechanical boundary conditions (2.6) and (2.7), we obtain two linear equations in the constants C_1, C_2, and D_0. These are

$$(\alpha_1c_{33}{}^D+2c_{13}{}^D)r_2{}^{\alpha_1-1}C_1+(\alpha_2c_{33}{}^D+2c_{13}{}^D)r_2{}^{\alpha_2-1}C_2 - \frac{h_{13}}{(1+\nu)}\left[1-\frac{2c_{13}{}^D}{c_{33}{}^D}+\frac{(1+\nu)h_{33}}{h_{13}}\right]r_2{}^{-2}D_0 = -p_2, \tag{5.1}$$

$$(\alpha_1c_{33}{}^D+2c_{13}{}^D)r_1{}^{\alpha_1-1}C_1+(\alpha_2c_{33}{}^D+2c_{13}{}^D)r_1{}^{\alpha_2-1}C_2 - \frac{h_{13}}{(1+\nu)}\left[1-\frac{2c_{13}{}^D}{c_{33}{}^D}+\frac{(1+\nu)h_{33}}{h_{13}}\right]r_1{}^{-2}D_0 = -p_1. \tag{5.2}$$

[4] Reference 1, p. 142.

From the electrical boundary condition (2.9), we obtain a third linear equation in the three constants C_1, C_2, and D_0 as follows. First note that from (2.5), (4.4), and (4.5) we have

$$E(r) = -h_{33}u'-2h_{13}ur^{-1}+\beta_3{}^SD_0r^{-2}. \tag{5.3}$$

Substituting this result into the boundary condition (2.9), we find that

$$-\int_{r_1}^{r_2}E(r)dr = h_{33}u\Big]_{r1}^{r2} +2h_{13}\int_{r_1}^{r_2}ur^{-1}dr+\beta_3{}^SD_0r^{-1}\Big]_{r1}^{r2} = V. \tag{5.4}$$

Now substitute for u in this last expression its general solution (4.11) and evaluate the integral in the central term on the left. Collecting the coefficients of C_1, C_2, and D_0 in the result of this calculation we obtain the equation

$$\left(\frac{2}{\alpha_1}+\frac{h_{33}}{h_{31}}\right)(r_2{}^{\alpha_1}-r_1{}^{\alpha_1})C_1+\left(\frac{2}{\alpha_2}+\frac{h_{33}}{h_{31}}\right)(r_2{}^{\alpha_2}-r_1{}^{\alpha_2})C_2 +\left[\frac{h_{33}-2h_{31}}{(1+\nu)c_{33}{}^D}+\frac{\beta_3{}^S}{h_{31}}\right](r_2{}^{-1}-r_1{}^{-1})D_0 = \frac{V}{h_{31}}. \tag{5.5}$$

Equations (5.1), (5.2), and (5.5) constitute a system of three linear equations in the three constants C_1, C_2, and D_0. All of the coefficients in these equations are known in terms of the material parameters and the inner and outer radii of the shell. Each of the constants C_1, C_2, and D_0 will be some linear function of the inner and outer pressures p_1 and p_2, and the voltage difference V. If we substitute these linear expressions for C_1, C_2, and D_0 into the general solution (4.11) for $u(r)$ we obtain the displacement at any radius r as a linear function of p_1, p_2, and V.

In an actual calculation of the displacement $u(r_2)$ of the outer surface of the shell, it is convenient to introduce the dimensionless constants

$$B_1 = r_2{}^{\alpha_1-1}C_1, \quad B_2 = r_2{}^{\alpha_2-1}C_2, \quad B_3 = \frac{h_{31}D_0}{(1+\nu)c_{33}{}^D}r_2{}^{-2};$$
$$P_1 = \frac{p_1}{(\alpha_1c_{33}+2c_{13})}, \quad P_2 = \frac{p_2}{(\alpha_1c_{33}+2c_{13})}, \tag{5.6}$$
$$P_3 = \frac{h_{31}V}{\beta_3{}^Sc_{33}{}^D(1+\nu)(r_2-r_1)}; \quad \rho = r_1/r_2.$$

We shall then have

$$u(r_2) = r_2[B_1+B_2+B_3], \tag{5.7}$$

and the equations for the determination of B_1, B_2, and

B_3 can be written in the form

$$B_1+a_1B_2+a_2B_3=-P_2, \quad (5.8)$$

$$\rho^{\alpha_1-1}B_1+\rho^{\alpha_2-1}a_1B_2+\rho^{-2}a_2B_3=-P_1, \quad (5.9)$$

$$\left(\frac{1-\rho^{\alpha_1}}{1-\rho}\right)a_3B_1+\left(\frac{1-\rho^{\alpha_2}}{1-\rho}\right)a_4B_2-\rho^{-1}a_5B_3=-P_3, \quad (5.10)$$

where the dimensionless material constants a_1, a_2, a_3, a_4, and a_5 are defined by

$$\begin{aligned}
a_1&=\frac{\alpha_2c_{33}{}^D+2c_{13}{}^D}{\alpha_1c_{33}{}^D+2c_{13}{}^D},\\
a_2&=-\frac{c_{33}{}^D}{\alpha_1c_{33}{}^D+2c_{13}{}^D}\left[1-\frac{2c_{13}{}^D}{c_{33}{}^D}+\frac{(1+\nu)h_{33}}{h_{31}}\right],\\
a_3&=\frac{h_{31}{}^2}{\beta_3{}^Sc_{33}{}^D(1+\nu)}\left(\frac{2}{\alpha_1}+\frac{h_{33}}{h_{31}}\right), \qquad (5.11)\\
a_4&=\frac{h_{31}{}^2}{\beta_3{}^Sc_{33}{}^D(1+\nu)}\left(\frac{2}{\alpha_2}+\frac{h_{33}}{h_{31}}\right),\\
a_5&=+\left[1+\frac{(h_{33}-2h_{31})h_{31}}{\beta_3{}^Sc_{33}{}^D(1+\nu)}\right].
\end{aligned}$$

6. CONCLUSIONS

The displacement of the exterior surface of a radially polarized spherical shell can be conveniently calculated using Eqs. (5.7)–(5.11) in terms of the inner and outer pressures, the applied voltage, and the six dimensionless material parameters ν, a_1, $\cdots$, a_5. Experimental data on the value of these constants are not available. However, the values of the corresponding constants for *homogeneously* polarized rectangular blocks of ceramic barium titanate are available and can be used for an approximate determination of the sphere parameters ν, a_1, $\cdots$, a_5. Static measurements of the decrement in volume of a polarized barium titanate spherical shell as a function of a variable external pressure and applied voltage made at this laboratory indicate that such a procedure is legitimate and that the theory of static deformation presented in this note is correct within the range of pressures and voltages for which these materials behave linearly and no depolarization effects are encountered.

Unfortunately, the explicit solution of the sphere problem given here has little immediate application in acoustics where dynamical solutions are of paramount interest; however, the piezoelectric relations (3.1)–(3.3) remain valid in the dynamical case and Eq. (4.2) equated to the force of acceleration $\rho\partial^2u/\partial t^2$ yields the general dynamical equations of motion for the spherically polarized shell. The mathematical techniques used in reducing Eqs. (4.2)–(4.6) for the case of the purely radial mode should be of value also in the reduction of these complicated equations in the dynamical case and for arbitrary modes of vibration. Our technique avoids the equivalent but more sophisticated mathematical apparatus of curvilinear coordinates and covariant differentiation.

Finally, we suggest that, aside from obvious practical applications, the piezoelectric *spherical* resonator is an ideal source of acoustic radiation for laboratory experiments which can be compared with theoretical results since the boundary conditions are simple and solutions of the radiation equations are available.