Saint-Venant's Principle

R. A. Toupin

Saint-Venant's Principle

R. A. TOUPIN

Abstract

The principle of the elastic equivalence of statically equivalent systems of load, or SAINT-VENANT'S Principle, is given a precise mathematical formulation and proof. Counterexamples to traditional verbal statements of the principle are given, and the results are compared with previous mathematical work on the Saint-Venant principle.

1. Introduction

SAINT-VENANT'S principle, or the "principle of elastic equivalence of statically equivalent systems of load", is described by LOVE [*2*, § 89] as follows:

"According to this principle, the strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part."

LOVE goes on to state in this same section:

"In the problem in hand [bending of a bar by terminal couples], we infer that, when the length of the bar is large compared with any diameter of its cross section, the state of stress and strain set up in its interior by the terminal couple is practically independent of the distribution of the tractions, of which the couple is the resultant, in all portions of the bar except comparatively small portions near its ends."

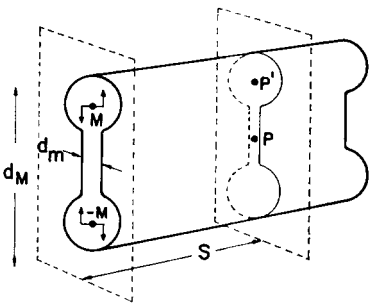

Fig. 1

Because the mathematical theory of elasticity under consideration is linear, what is claimed is that any system of forces applied to one end only of a long bar, which has *zero* resultant force and *zero* resultant couple, gives rise to a strain field which is "practically" zero everywhere in the bar except at points within a few diameters of the loaded end. The following example indicates that this conjecture may be false. Consider a long bar with a cross section as depicted in Fig. 1. Let the loaded end be subject to tractions equipollent to a couple M on the upper circular lobe of the cross section and a couple $-M$ on the lower so that all the conditions of LOVE'S statement are met. Consider a point P which is any finite number of diameters d_M away from the loaded end and lying in the thin member joining the two lobes of a cross section. It seems intuitively reasonable that the strain at such a point P will be large for arbitrarily small values of M and for sufficiently small values of the thickness d_m. It would appear from this example that the rate of decay of the strain depends strongly, not on any single linear dimension of the cross section, but rather

on its *shape*. Moreover, if one compares the strain at a point like P' deeply embedded in the interior of the lobe of the cross section in Fig. 1 with the strain at a point in the thin center section at the same distance from the loaded end, it is reasonable to expect that, for this type of loading, the strain at P' will be much smaller than the strain at P. It seems reasonable to require that any quantitative treatment of SAINT-VENANT'S principle embrace these qualitative, intuitive observations.

SAINT-VENANT introduced the principle of the elastic equivalence of statically equivalent systems of loads only for the special case of perfect cylinders. The broader statement of the principle given by LOVE for bodies of arbitrary shape cannot possibly be true, and SAINT-VENANT, himself, [*1*, p. 67] alluded to the following caution. Suppose, as in Fig. 2, tractions of arbitrarily small magnitude, but not zero, are applied to one end of a long rectangular region in plane elasticity theory. Let there be a slit in the rectangular region at an arbitrarily large but finite distance s from the loaded end. Then the strain at points P near enough to the tip of the slit are arbitrarily large. One learns from this example not to expect an unqualified pointwise estimate for the strains at points near a jagged part of the boundary.

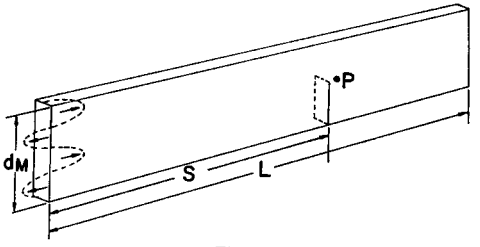

Fig. 2

Finally, to set the stage properly for the treatment of SAINT-VENANT'S principle given here, we repeat an objection raised by VON MISES [*1*] to the traditional statement of the principle given by LOVE; *viz*, since the mathematical theory under consideration is linear, the strain at any point in any body, under any system of loading, if different from zero by any amount, however small, will be arbitrarily large for the same point in the same body under a suitable system of proportional loads. In other words, in the linear theory, a correct formulation of the principle must express in some sense the *relative* smallness of the strains at different points or entail in some other way a measure of the magnitude of the applied loads.

With these preparations and ideas in mind, we announce our main result:

Let a cylinder of arbitrary length and cross section be loaded on one end only with an arbitrary system of self-equilibrated forces. Then the stored elastic energy $U(s)$ *in the cylinder beyond a distance* s *from the loaded end bears a ratio to the total stored energy* $U(0)$ *which always satisfies the inequality*

$$\frac{U(s)}{U(0)} \leqq e^{-(s-l)/s_c(l)} \tag{1.1}$$

where

(i) *the characteristic decay length* $s_c(l)$ *is given by*

$$s_c(l) = \sqrt{\frac{\mu^*}{\varrho\, \omega_0^2(l)}}.$$

(ii) $\mu^* = \mu_M^2/\mu_m$, *where* μ_M *is the maximum elastic modulus, and* μ_m *is the minimum elastic modulus.*

(iii) ϱ *is the mass density.*

(iv) $\omega_0(l)$ *is the smallest characteristic frequency of free vibration of a section of the cylinder of length* l.

The parameter $l>0$ is at one's disposal to choose in a manner which will provide a small value for $s_c(l)$.

Once the energy has been estimated by (1.1), we apply the following auxiliary result to obtain a pointwise estimate of the strain at interior points of the cylinder:

Let a solid sphere be deformed in any manner whatsoever, and let U_0 *be the elastic energy of deformation. Then the strain at the center of the sphere always satisfies an inequality of the form*

$$|e|^2 \leqq K \frac{U_0}{V}, \tag{1.2}$$

where V *is the volume of the sphere and* K *is a material constant.*

It is clear how the estimates (1.1) and (1.2) together provide a pointwise *a priori* estimate of the strain at all interior points of the cylinder beyond a distance $l>0$ from the loaded end of the cylinder. The smaller is the volume of the largest sphere lying within the cylinder and having a given point P of the cylinder for its center, the poorer is the estimate (1.2). This reflects the qualitative limitation of what can be expected from a general *a priori* estimate of the strain at a point, noted earlier.

The remainder of this paper is devoted to a proof of (1.1) and (1.2), a discussion of a generalization of (1.1) to non-cylindrical bodies, and a comparison of these results with previous work on SAINT-VENANT'S principle.

2. Notation and Definitions

We use Cartesian tensor notation and the summation convention for repeated indices. Coordinates of points $\boldsymbol{x}$ are denoted by x_i. Let $u_i(\boldsymbol{x})$ denote the components of the *displacement vector field*. Then the components of the *strain tensor* are given by

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) = u_{(i,j)}, \tag{2.1}$$

where a comma denotes partial differentiation;

$$u_{i,j} = \frac{\partial u_i}{\partial x_j}. \tag{2.2}$$

The *rotation tensor* is defined by

$$\omega_{ij} = \tfrac{1}{2}(u_{j,i} - u_{i,j}) = u_{[i,j]}. \tag{2.3}$$

Round and square brackets around indices are used to denote the symmetric and antisymmetric parts of a tensor as in (2.1) and (2.3).

The *strain energy density* for a given material is a positive definite quadratic form

$$W = \tfrac{1}{2} c_{ijkl}\, e_{ij}\, e_{kl} \tag{2.4}$$

in the strain components. We consider only homogeneous materials, for which the *elasticity tensor* c_{ijkl} is constant independent of $\boldsymbol{x}$. The components satisfy the symmetry relations

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}. \tag{2.5}$$

The norm of a vector or tensor is defined as follows:

$$|u| \equiv \sqrt{u_i u_i}, \qquad |e| \equiv \sqrt{e_{ij} e_{ij}}. \tag{2.6}$$

Since the elasticity tensor is positive definite, there exist maximum and minimum elastic moduli, μ_M and μ_m, such that

$$\mu_m |e|^2 \leqq c_{ijkl} e_{ij} e_{kl} \leqq \mu_M |e|^2, \tag{2.7}$$

where the equality signs hold for suitable values of the strain.

The components of the *stress tensor* $\boldsymbol{t}$ are defined by

$$t_{ij} = c_{ijkl} e_{kl} = t_{ji}. \tag{2.8}$$

The *traction* or *stress vector* ${}_{(n)}\boldsymbol{t}$, which acts at a point on a surface in the body with unit normal $\boldsymbol{n}$, is given by

$${}_{(n)}t_i = t_{ij} n_j, \qquad n_i n_i = 1. \tag{2.9}$$

Let ϱ denote the constant mass per unit volume of the material. Then the *equations of motion* are

$$\varrho \ddot{u}_i = t_{ij,j} = c_{ijkl} u_{k,lj}, \tag{2.10}$$

where $\ddot{u}_i = \partial^2 u_i / \partial t^2$ is the acceleration. We omit any consideration of body forces. The *equilibrium equations* are

$$t_{ij,j} = c_{ijkl} u_{k,lj} = 0. \tag{2.11}$$

The traction boundary-value problem of elastostatics consists in finding a solution $u_i(\boldsymbol{x})$ of the equilibrium equations (2.11) for which the tractions ${}_{(n)}t_i$ have prescribed values on the surface of the body. SAINT-VENANT'S principle concerns properties of certain solutions of this traction boundary-value problem of linear elastostatics.

3. Proof of the Energy Estimate for Cylinders

Consider a cylinder (Fig. 3) with an arbitrary regular cross section deformed by an arbitrary system of self-equilibrated forces applied to one end of the cylinder. Let C_s denote the intersection with the body of a plane perpendicular to the axis of the cylinder and distance s from the loaded end. Let B_s denote the part of the cylinder beyond distance s from the loaded end.

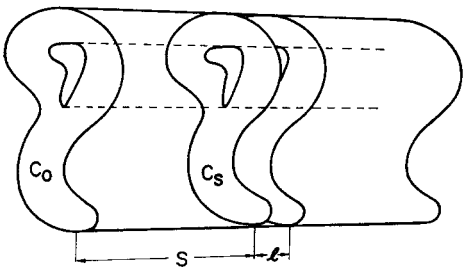

Fig. 3

We assume that the displacements $u_i(\boldsymbol{x})$ satisfy the equilibrium equations

$$t_{ij,j} = 0 = c_{ijkl} u_{k,lj}, \qquad \boldsymbol{x} \in B \tag{3.1}$$

at every point of the cylinder B, and that the tractions ${}_{(n)}t_i$ vanish at every point of the surface ∂B of the cylinder except on the end C_0:

$${}_{(n)}t_i = t_{ij} n_j = 0, \qquad \boldsymbol{x} \in \partial B - C_0. \tag{3.2}$$

It follows from the equilibrium equations (3.1) and the symmetry of the stress tensor that

$$\int_{\partial B_s} {}_{(n)}t_i \, da = 0, \qquad \int_{\partial B_s} x_{[i\,(n)}t_{j]} \, da = 0, \tag{3.3}$$

for every value of s. These properties of a solution express the simple fact that the resultant force and resultant moment on every part B_s of the cylinder must vanish in equilibrium. But the tractions ${}_{(n)}t_i$, according to our hypothesis (3.2), vanish at every point of the boundary of B_s except at points in the cross section C_s. Therefore, it follows from (3.3) that

$$\int_{C_s} {}_{(n)}t_i \, da = 0, \qquad \int_{C_s} x_{[i\,(n)}t_{j]} \, da = 0, \tag{3.4}$$

for every value of s. These conditions for $s=0$ require that the data correspond to a system of self-equilibrated forces on C_0 else no solution can possibly exist. More generally, however, we see that the tractions on *every* cross section C_s are a self-equilibrating system.

Next we observe that the elastic energy of deformation stored in the part B_s of the cylinder is also expressible as a surface integral over the cross section C_s:

$$\begin{aligned} U(s) &= \tfrac{1}{2}\int_{B_s} c_{ijkl}\, e_{ij}\, e_{kl}\, dv = \tfrac{1}{2}\int_{B_s} t_{ij}\, u_{i,j}\, dv \\ &= \tfrac{1}{2}\int_{B_s} [(t_{ij}\, u_i)_{,j} - t_{ij,j}\, u_i]\, dv \\ &= \tfrac{1}{2}\int_{\partial B_s} t_{ij}\, u_i\, n_j\, da, \\ U(s) &= \tfrac{1}{2}\int_{C_s} {}_{(n)}t_i\, u_i\, da. \end{aligned} \tag{3.5}$$

We shall later need the fact that $U(s)$ is non-negative and is a non-increasing function of s. These properties follow from the positive-definiteness of the energy density.

Next we observe that, because of the properties (3.4) of the tractions on C_s, the actual displacements $u_i(x)$ of points of C_s which appear in the integrand of (3.5) may be replaced by modified displacements $\bar{u}_i$ given by

$$\bar{u}_i = u_i + a_i + b_{ij}\, x_j, \tag{3.6}$$

where the a_i and $b_{ij} = -b_{ji}$ are arbitrary constants. Thus,

$$U(s) = \tfrac{1}{2}\int_{C_s} {}_{(n)}t_i\, \bar{u}_i\, da. \tag{3.7}$$

This step in the argument is crucial. The displacement field $\bar{u}_i$ differs from u_i by a rigid motion, and the equivalence of (3.5) and (3.7) follows essentially from the general proposition that any system of self-equilibrated forces does no work in a rigid motion of the points of action.

Next we apply the Schwarz and arithmetic-geometric mean inequality (*cf.* Appendix A) to assert that

$$U(s) \leqq \tfrac{1}{4}\{\alpha \int_{C_s} |{}_{(n)}t|^2\, da + (1/\alpha)\int_{C_s} |\bar{u}|^2\, da\} \tag{3.8}$$

for arbitrary positive values of the parameter α.

But,

$$|_{(n)}t|^2 = t_{ij}\, n_j\, t_{ik}\, n_k \leqq |t|^2, \tag{3.9}$$

and, as shown* in Appendix B,

$$|t|^2 \leqq \mu^* c_{ijkl}\, e_{ij}\, e_{kl} = 2\mu^* W \tag{3.10}$$

where,

$$\mu^* = \mu_M^2/\mu_m. \tag{3.11}$$

The maximum and minimum elastic moduli μ_M and μ_m were defined in (2.7). It follows from (3.8), (3.9), and (3.10) that

$$U(s) \leqq \tfrac{1}{4}\left\{2\alpha\mu^* \int_{C_s} W\, da + (1/\alpha)\int_{C_s} \bar{u}^2\, da\right\}, \tag{3.12}$$

Now integrate the inequality (3.12) between the limits s and $s+l$ where $l>0$ is some positive distance to the right of s. This yields the inequality

$$l\, Q(s,l) \leqq \tfrac{1}{4}\left\{2\alpha\mu^* \int_{C_{s,l}} W\, dv + (1/\alpha)\int_{C_{s,l}} \bar{u}^2\, dv\right\}, \tag{3.13}$$

where

$$Q(s,l) \equiv \frac{1}{l}\int_s^{s+l} U(s')\, ds' \tag{3.14}$$

is the mean value of $U(s)$ in the segment of the cylinder between s and $s+l$, and $C_{s,l}$ denotes the part of the cylinder lying between the cross sections C_s and C_{s+l}.

It is shown in Appendix D that one can always choose the constants a_i and b_{ij} in the definition of $\bar{u}_i$ in such a way that

$$\int_{C_{s,l}} \bar{u}_i\, dv = 0, \qquad \int_{C_{s,l}} \bar{u}_{[i}\, x_{j]}\, dv = 0. \tag{3.15}$$

It follows from RAYLEIGH's principle (*cf.* the discussion of Rayleigh's principle in Appendix C) that, for this choice of the constants a_i and b_{ij},

$$\frac{1}{2}\int_{C_{s,l}} |\bar{u}|^2\, dv \leqq \frac{1}{\varrho\, \omega_0^2(l)} \int_{C_{s,l}} W(\bar{e})\, dv. \tag{3.16}$$

But $W(\bar{e}) = W(e)$ since $\bar{u}_i$ and u_i differ by a rigid motion. It follows then from (3.13) and (3.16) that

$$Q(s,l) \leqq \frac{1}{2l}\left[\alpha\mu^* + \frac{1}{\alpha\, \varrho\, \omega_0^2(l)}\right] \int_{C_{s,l}} W\, dv. \tag{3.17}$$

But one easily verifies that

$$\frac{1}{l}\int_{C_{s,l}} W\, dv = \frac{1}{l}\,[U(s) - U(s+l)] = -\frac{dQ(s,l)}{ds}. \tag{3.18}$$

Thus, (3.17) implies the differential inequality

$$s_c(l,\alpha)\, \frac{dQ(s,l)}{ds} + Q(s,l) \leqq 0, \tag{3.19}$$

---

* *Note added in proof:* Professor J. L. ERICKSEN has supplied me a simpler proof of the sharper estimate $|t|^2 \leqq 2\mu_M W$.

where

$$s_c(l,\alpha) \equiv \frac{1}{2}\left[\alpha\mu^* + \frac{1}{\alpha\,\varrho\,\omega_0^2(l)}\right]. \tag{3.20}$$

The *characteristic decay length* $s_c(l,\alpha)$ has a minimum with respect to the parameter $\alpha>0$ when $\alpha=1/\sqrt{\mu^*\varrho\,\omega_0^2(l)}$. This minimum value is given by

$$s_c(l) = \sqrt{\frac{\mu^*}{\varrho\,\omega_0^2(l)}}. \tag{3.21}$$

Choosing α in this way and integrating the differential inequality (3.19) between limits s_1 and $s_2 \geqq s_1$, we get

$$\frac{Q(s_2,l)}{Q(s_1,l)} \leqq e^{-(s_2-s_1)/s_c(l)}. \tag{3.22}$$

Now, as noted earlier, $U(s)$ is a non-increasing function of s. Therefore, since $Q(s,l)$ is the mean value of $U(s)$ in the interval $[s, s+l]$, one has

$$U(s+l) \leqq Q(s,l) \leqq U(s), \tag{3.23}$$

and (3.22) implies that

$$\frac{U(s_2+l)}{U(s_1)} \leqq e^{-(s_2-s_1)/s_c(l)}. \tag{3.24}$$

If one now sets $s_1=0$ and $s_2=s-l$ in (3.24), one obtains the energy estimate (1.1), which was to be proved.

4. Proof of the Strain Inequality

Here we shall prove the inequality (1.2) only for isotropic elastic materials. A similar result holds for anisotropic media, but the calculations are more complicated. The following proof of (1.2) begins with the expression

$$\frac{R^5}{5}\,e_{ik}(0) = \frac{3}{8\pi}\int_B (5\,u_{(i}\,x_{k)} - e_{ik}\,r^2)\,dv \tag{4.1}$$

given by DIAZ & PAYNE [3] for the strain components at the center of an isotropic spherical ball B of radius R in terms of a certain weighted average of the strains and displacements throughout the ball. If we square this expression for $e_{ik}(0)$ and sum over all values of i and k and use Schwarz's inequality, we get

$$\frac{R^{10}}{25}\,|e(0)|^2 \leqq \frac{9V}{(8\pi)^2}\int_B |5\,u_{(i}\,x_{k)} - e_{ik}\,r^2|^2\,dv, \tag{4.2}$$

where $V=4\pi R^3/3$ is the volume of the ball. Next we use the inequality (A.6) to reduce the right member of (4.2) to the form

$$|e(0)|^2 \leqq \frac{75}{16\pi R^7}\int_B \left[25\,(1+\alpha)\,|u_{(i}\,x_{k)}|^2 + \left(1+\frac{1}{\alpha}\right)e^2\,r^4\right]dv. \tag{4.3}$$

But

$$|u_{(i}\,x_{k)}|^2 \leqq |u|^2 r^2, \tag{4.4}$$

and, by Rayleigh's principle,

$$\int_B u^2\,dv \leqq \frac{2}{\varrho\,\Omega_0^2(R)}\int_B W\,dv \tag{4.5}$$

provided

$$\int_B u_i \, dv = 0, \qquad \int_B x_{[i} u_{k]} \, dv = 0 \tag{4.6}$$

where $\Omega_0(R)$ is the smallest characteristic frequency of free vibration of a sphere of radius R. The side conditions (4.6) may be assumed without loss in generality because the right-hand member of (4.1) is invariant under addition of a rigid motion to u_i. We also have

$$|e|^2 \leqq \frac{2}{\mu_m} W. \tag{4.7}$$

It follows from (4.3), (4.4), (4.5), and (4.7) that

$$\mu_m |e(0)|^2 \leqq \frac{25}{2}\left[\left(1+\frac{1}{\alpha}\right) + 25\,(1+\alpha)\,\frac{\mu_m}{\varrho\,\Omega_0^2(R)/R^2}\right]\frac{1}{V}\int_B W\,dv. \tag{4.8}$$

The parameter $\alpha > 0$ may be chosen arbitrarily. By choosing for α the value for which the coefficient of the mean value of the energy density in (4.8) is a minimum, one obtains

$$\mu_m |e(0)|^2 \leqq \frac{25}{2}\left(1+5\,\frac{\mu_m}{\varrho\,\Omega_0^2(R)/R^2}\right)\frac{\int_B W\,dv}{V(B)} \tag{4.9}$$

which establishes the inequality (1.2) and a specific admissible value for the constant.

5. Remarks on the Energy Estimate for Non-Cylindrical Bodies

A review of the essential steps and ideas of the proof of the energy estimate (1.2) for cylindrical bodies reveals that a similar result can be obtained for certain bodies of more general shape. We shall not derive such estimates here in detail but only remark on how one might proceed.

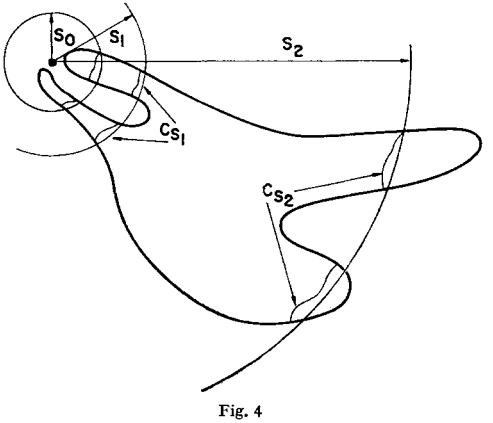

Fig. 4

Consider a body of the general shape depicted in Fig. 4. Let the tractions on the surface of the body vanish everywhere except on the part of the surface contained within a sphere of radius s_0. Let C_s denote the intersection of a concentric sphere of radius s with the body, and let $C_{s,l}$ denote the part of the body lying between the concentric spheres of radius s and $s+l$. In general, the intersection C_s will consist of several disconnected two-dimensional regions on the sphere of radius s. The tractions on each connected part of the intersection C_s may or may not always be self-equilibrated. For example, in Fig. 4, C_{s_1} consists of two disconnected regions, and it is clear that, for some ways of loading the surface inside the sphere of radius s_0, the tractions on each piece of C_{s_1} will not be separately self-equilibrated. On the other hand, the intersection C_{s_2} is also not a connected region on the sphere of radius s_2, but in this

case, the tractions on each piece of C_{s_2} are separately self-equilibrated. For cylindrical bodies we made a fixed choice of the parameter l independent of the value of s. This restriction is not essential to the method, and it is permissible to let the value of l depend on s. When the tractions on each connected part of the intersection C_s are not separately self-equilibrated (the tractions on the whole of every intersection C_s are always self-equilibrated), it is necessary to choose $l(s)$ large enough so that the region $C_{s\,l(s)}$ is connected. This will insure that the lowest characteristic frequency $\omega_0(C_{s,\,l})$ of $C_{s,\,l}$ is greater than zero, and it will then be possible to estimate the integral of $\bar{u}^2$ over $C_{s,\,l}$ in terms of the strain energy in $C_{s,\,l}$. On the other hand, if the tractions on each connected piece of C_s are separately self-equilibrated, any value of $l(s)>0$ is admissible. One may jump over cavities in the body using these same ideas. For certain kinds of non-cylindrical bodies, then, it should be fairly clear how one can proceed to obtain an inequality of the form

$$\frac{U(s+l(s))}{U(s_0)} \leqq \exp\left[-\int_{s_0}^{s} \frac{ds'}{s_c(s')}\right]. \tag{5.1}$$

Results of this kind will be given in greater detail in future papers. Also, it appears that the present method allows an easy and efficient formulation of a Saint-Venant principle as it applies to the theory of elastic plates and shells.

It appears also that other special assumptions such as the homogeneity of the material, which we have made here in the interest of simplicity, are not essential to the method.

Finally, it will undoubtedly be obvious to the mathematically inclined that the "decay theorem" proved here for the elasticity equations has its counterpart for every even-order system of differential equations derived from a positive definite energy integral.

6. Remarks on Previous Results

As all who have attempted to comprehend the traditional verbal statements of SAINT-VENANT'S principle and to transform them into a precise mathematical statement would surely agree, it has been the lack of such a definite formulation of the principle rather than its "proof" that has been responsible for its elusive character and the long history of the problem. It is, of course, my opinion that the energy estimate (1.1), or its counterpart for more general bodies, and auxiliary strain estimates like (1.2) represent a true mathematical transcription of the verbal Saint-Venant principle of elasticity theory. These inequalities are precise statements about *properties held in common by every solution of the boundary-value problem for a given body loaded in every possible way on a given portion of its boundary*. The results are independent of the construction or even the existence of a single solution.

FILON [4] (*cf.* also LOVE [2, § 266]) constructed, in essence, a large but not exhaustive class of solutions for circular cylinders. Simply by examination of solutions in this class he perceived a rapid decay in the strain induced in a circular rod by self-equilibrated forces applied to one end, but no common feature of all the solutions like (1.1) can be easily deduced from his analysis. The remarks of LOVE [*op. cit*] do not constitute a proof of the "exponential decay" of the

energy even for this restricted class of loadings of circular cylinders. Moreover, this attack on the problem seems not to be consistent with the spirit of the principle and the way it is used. After all, if one can construct, or is willing to construct solutions, there is no need for the principle. What the principle calls for is an estimate or bound for the strain which can be obtained without having to construct explicit solutions. The present method seems to fulfill these requirements. Methods exist [5] for estimating the characteristic frequencies which occur in the construction of the energy inequality given here, or these frequencies might even be obtained experimentally.

Two other classes of general theorems have been proved in connection with the Saint-Venant principle and put forward as having some bearing on the original question posed by SAINT-VENANT'S remarks. The first of these are due to BOUSSINESQ [6], VON MISES [7], and STERNBERG [8]. These theorems concern a representation of the strain at an interior point of a given elastic body which is caused by a *sequence of loads* on a *sequence of regions* of its boundary. The second class of theorems is due to ZANABONI [9] and concerns estimates for the total energy of a *sequence of bodies* under the action of a *fixed system of loads* on a given common portion of their boundaries. While the theorems of BOUSSINESQ, VON MISES, STERNBERG, and ZANABONI have independent interest, I have been unable to perceive an easy relationship between these theorems and the Saint-Venant principle. It seems to me that the latter calls for a consideration of a fixed body, a given region of its boundary, and all possible ways of loading that given part of its boundary. These qualities of the Saint-Venant problem and the failure of the Boussinesq, von Mises, and Sternberg theorems to share them were stated by TRUESDELL [13]. TRUESDELL'S interesting conjecture concerning an appropriate definition of the torsional rigidity consistent with the Saint-Venant principle has not been touched upon here.

It should not be inferred from the above and the remarks of VON MISES in [7] that BOUSSINESQ'S throughts on the Saint-Venant principle were restricted to the idea which has been given its most precise and general statement by STERNBERG. In § 70 of his memoir [6], BOUSSINESQ conjectured an exponential rate of decay of the stress in a rod loaded on one end only. He perceived the possible truth of such a result from his own explicit solution for the displacements in an elastic half-space subjected to a doubly periodic system of tractions.

During the last two summers I have had the benefit and pleasure of discussions of the results of STERNBERG and ZANABONI with Professor A. ROBINSON. ROBINSON has claimed a proof of the following theorem: In a sufficiently long cylinder loaded on one end only, the strain at every interior point which lies a distance greater than δ from the boundary and greater than a sufficiently large distance $s(\delta, \varepsilon)$ from the loaded end will be smaller than any preassigned value ε. A proof of this result will be published soon.

Finally, a result of DOU'S [10] should be mentioned. DOU has shown that for an isotropic cylinder of square cross section, the total energy $U(0)$ satisfies an inequality of the form

$$U(0) \leqq \frac{K}{\mu} \int_{C_0} [|_{(n)}t|^2 + |\overset{s}{V}_{(n)}t|^2]\, da, \tag{6.1}$$

where μ is the shear modulus, and K is a constant independent of the length of the cylinder. Results like (6.1) can be used in conjunction with the energy estimate (1.1) and strain inequalities like (1.2) to obtain an estimate for the strain at interior points in terms of the applied loads and the surface gradient $\overset{s}{V}_{(n)}t$ of the applied loads.

Acknowledgement. I wish especially to thank Professor F. JOHN for invaluable advice and discussion which allowed me to overcome a major obstacle in the proof of the energy estimate. I wish to thank also my colleague, Dr. W. MIRANKER, for many discussions and helpful advice on the whole of this question.

7. Appendix

A. The Schwarz and Geometric-Arithmetic Mean Inequality

Let $u_i(x)$ and $w_i(x)$ be any two vector fields in a region R of Euclidean space. Then Schwarz's inequality is the assertion that

$$\int_R u_i w_i \, dv \leqq \sqrt{\int_R u^2 dv \int_R w^2 dv}. \tag{A.1}$$

From the geometric-arithmetic mean inequality, we have

$$\sqrt{a\,b} \leqq \frac{a+b}{2} \tag{A.2}$$

for all positive numbers a and b. More generally,

$$\sqrt{a\,b} \leqq \frac{\alpha a + b/\alpha}{2} \tag{A.3}$$

for all positive numbers α, a, and b. Combining (A.1) and (A.3), one has

$$\int_R u_i w_i \, dv \leqq \tfrac{1}{2}\left\{\alpha \int_R u^2 dv + (1/\alpha)\int_R w^2 dv\right\} \tag{A.4}$$

for all $\alpha > 0$ and vector fields u_i and w_i.

It follows also from (A.3) that

$$2|a|\,|b| \leqq \alpha a^2 + b^2/\alpha \tag{A.5}$$

for all $\alpha > 0$. Hence,

$$\begin{aligned}(a+b)^2 = a^2 + b^2 + 2ab &\leqq a^2 + b^2 + 2|a|\,|b| \\ &\leqq (1+\alpha)a^2 + (1+1/\alpha)b^2.\end{aligned} \tag{A.6}$$

This last inequality holds also when a and b are pairs of vectors or pairs of tensors.

B. A Bound for the Stress in Terms of the Energy Density

The stress t_{ij} is a linear function $c_{ijkl}e_{kl}$ of the strain t_{kl}, and the elastic constants c_{ijkl} may be viewed as a positive definite, symmetric linear transformation in the six-dimensional linear vector space of all symmetric second order tensors $f_{ij} = f_{ji}$. Viewed in this manner, we can assert that the linear transformation defined by c_{ijkl} has six orthogonal proper vectors and the corresponding six eigenvalues. In other words, there exist six symmetric tensors $\underset{\alpha}{e}_{ij}$ and real numbers $\underset{\alpha}{\mu}$ such that

$$c_{ijkl}\,\underset{\alpha}{e}_{kl} = \underset{\alpha}{\mu}\,\underset{\alpha}{e}_{ij}, \qquad \underset{\alpha}{e}_{ij}\,\underset{\beta}{e}_{ij} = \delta_{\alpha\beta}, \qquad \alpha, \beta = 1, 2, \ldots, 6. \tag{B.1}$$

Let μ_m and μ_M denote the minimum and maximum of the numbers $\underset{\alpha}{\mu}$. It is then a familiar proposition in the theory of quadratic forms that

$$\mu_m |e|^2 \leqq c_{ijkl}\, e_{ij}\, e_{kl} = 2W \leqq \mu_M |e|^2 \tag{B.2}$$

for arbitrary tensors e_{ij}.

Next we observe that

$$|\boldsymbol{t}|^2 = \boldsymbol{e}\cdot\boldsymbol{c}^2\cdot\boldsymbol{e} \tag{B.3}$$

where $\boldsymbol{c}^2$ is a short notation for the square of the linear transformation defined by c_{ijkl}. The eigenvalues of $\boldsymbol{c}^2$ are the squares of the eigenvalues of $\boldsymbol{c}$. Hence,

$$|t|^2 \leqq \mu_M^2 |e|^2 \leqq 2(\mu_M^2/\mu_m) W, \tag{B.4}$$

where we have used the first inequality of (B.2).

C. Rayleigh's Principle

According to Rayleigh's principle (*cf., e.g.,* GOULD [*11*]), the stationary values of the ratio

$$\varrho\,\omega^2 = \frac{\int_B W(\boldsymbol{u})\, dv}{\frac{1}{2}\int_B |u|^2 dv} \tag{C.1}$$

are the characteristic frequencies of free vibration of an elastic body B. The smallest characteristic frequency (eigenvalue) is given by the minimum of Rayleigh's quotient over the set of all continuously differentiable vector fields $u_i(\boldsymbol{x}) \not\equiv 0$, $\boldsymbol{x} \in B$, B a regular region. The next largest eigenvalue is given by the minimum of Rayleigh's quotient over the set of vector fields $u_i(\boldsymbol{x})$ which are normal to the set of eigenfunctions belonging to the smallest eigenvalue; normal in the sense that

$$\int_B u_i \underset{\alpha}{v_i}\, dv = 0, \qquad \alpha = 1, 2, \ldots, N, \tag{C.2}$$

where there are N linearly independent eigenfunctions $\underset{\alpha}{v_i}$, $\alpha = 1, 2, \ldots, N$, for which Rayleigh's quotient is a minimum.

But the minimum value of Rayleigh's quotient in elasticity theory is zero so that the smallest eigenvalue is zero. The set of vector fields belonging to this eigenvalue comprises the rigid motions. There are six linearly independent rigid motions corresponding to the six degrees of freedom of a rigid body. A vector field u_i is normal to every rigid motion if and only if

$$\int_B u_i\, dv = 0, \qquad \int_B x_{[i}\, u_{j]}\, dv = 0. \tag{C.3}$$

Let C denote the class of all continuously differentiable vector fields in B which satisfy the side conditions (C.3). Then, according to Rayleigh's principle, the lowest characteristic frequency of free vibration (excluding the degenerate case of no motion at all) is given by

$$\varrho\,\omega_0^2(B) = \min_{\boldsymbol{u}\in C} \frac{\int_B W(\boldsymbol{u})\, dv}{\frac{1}{2}\int_B |u|^2 dv}. \tag{C.4}$$

But this means that, for every vector field in B satisfying the side conditions (C.3), one has the inequality,

$$\frac{1}{2}\int_B |u|^2 \leqq \frac{1}{\varrho\,\omega_0^2(B)}\int_B W(\boldsymbol{u})\,dv. \tag{C.5}$$

D. A Lemma Concerning the Side Conditions

Required in the course of the proof of the energy estimate is the following proposition. If

$$\bar{u}_i = a_i + b_{ij}\,x_j + u_i, \tag{D.1}$$

where a_i and $b_{ij} = -b_{ji}$ are constants, and $u_i(x)$ is an arbitrary vector field in a region R of Euclidean space, then it is always possible to choose the constants a_i and b_{ij} in such a way that

$$\int_R \bar{u}_i\,dv = 0, \qquad \int_R x_{[i}\bar{u}_{j]}\,dv = 0. \tag{D.2}$$

This proposition may be proved as follows.

First of all, let the origin of the coordinate system be chosen as coincident with the center of mass of the region R. This incurs no loss in generality since it entails only a transformation of the constants a_i and b_{ij}. One then has

$$\int_R \bar{u}_i\,dv = \int_R (a_i + b_{ij}\,x_j + u_i)\,dv = V a_i + U_i \tag{D.3}$$

where $U_i = \int u_i\,dv$. Thus, the first condition (D.2) is met if one chooses

$$a_i = -\frac{1}{V}\int u_i\,dv. \tag{D.4}$$

The second condition (D.2) reads as follows:

$$\int_R x_{[i}(a_{j]} + b_{j]k}\,x_k + u_{j]})\,dv = 0. \tag{D.5}$$

The first term in this expression vanishes because of our choice of origin, and the condition can be placed in the form

$$E_{k[i}\,b_{j]k} = -\int_R x_{[i}\,u_{j]}\,dv, \tag{D.6}$$

where

$$E_{ki} \equiv \int_R x_k\,x_i\,dv \tag{D.7}$$

is Euler's tensor (*cf.* [*12*, § 168]) for the region R. If we set $b_i = e_{ijk}\,b_{jk}$ and $V_i = \int e_{ijk}\,x_j\,u_k\,dv$, where e_{ijk} is the alternating tensor, we find that (D.6) is equivalent to the equation

$$I_{jk}\,b_k = V_j \tag{D.8}$$

where

$$I_{jk} \equiv E_{ll}\,\delta_{jk} - E_{jk} \tag{D.9}$$

is the tensor of inertia for the region R. It is proved in works on kinematics that the tensor of inertia, I_{jk}, for every region R is positive definite. Therefore, in particular, I_{jk} is non-singular so that the equation (D.8) always has a solution for arbitrary V_j.

References

[1] SAINT-VENANT, A.-J.-C. BARRÉ DE, Mémoire sur la torsion des prismes, avec des considérations sur leur flexion ... (Read June 13, 1853). Mém. Divers Savants **14**, 233—560 (1855). Also issued separately: De la Torsion des Prismes ... Paris: Imprimérie Impériale (1855).

[2] LOVE, A. E. H., A Treatise on the Mathematical Theory of Elasticity, Fourth Edition. Cambridge: The University Press 1927.

[3] DIAZ, J. B., & L. E. PAYNE, Mean Value Theorems in the Theory of Elasticity. Proceedings of the Third U.S. National Congress of Applied Mechanics, 293—303 (1958).

[4] FILON, L. N. G., On the Elastic Equilibrium of Circular Cylinders under Certain Practical Systems of Load. Phil. Trans. Roy. Soc. (Ser. A) **198**, 147 (1902).

[5] BRAMBLE, J. H., & L. E. PAYNE, Some Inequalities for Vector Functions with Applications in Elasticity. Arch. Rational Mech. Anal. **11**, 16—26 (1962).

[6] BOUSSINESQ, M. J., Applications des potentiels à l'étude de l'équilibre et du mouvement des solides élastiques. Paris: Gauthier-Villars 1885.

[7] VON MISES, R., On Saint-Venant's Principle. Bull. Amer. Math. Soc. **51**, 555—562 (1945).

[8] STERNBERG, E., On Saint-Venant's Principle. Quart. of Appl. Math. **11**, 393—402 (1954).

[9] ZANABONI, O., Dimostrazione generale del Principio del de Saint-Venant. Atti. Acc. Naz. Lincei **25**, 117—120 (1937).

[10] DOU, A., On the Principle of Saint-Venant. Mathematics Research Center, U.S. Army, The University of Wisconsin, Tech. Report No. 472 (1964).

[11] GOULD, S., Variational Methods for Eigenvalue Problems. Math. Expositions **10**, Toronto (1957).

[12] TRUESDELL, C., & R. TOUPIN, The Classical Field Theories. Handbuch der Physik, Vol. III/1. Berlin-Göttingen-Heidelberg: Springer 1960.

[13] TRUESDELL, C., The Rational Mechanics of Materials—Past, Present, Future. Appl. Mech. Rev. **12**, 75—80 (1959).

IBM Watson Research Center
Yorktown Heights, New York