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THE NECESSITY OF WHEELS IN UNIVERSAL QUANTIZATION FORMULAS

GIUSEPPE DITO

ABSTRACT. In the context of formal deformation quantization, we provide an elementary argument showing that any universal quantization formula necessarily involves graphs with wheels.

1. INTRODUCTION

Since its inception in the 70's by Flato and his collaborators [1], deformation quantization has undergone several major developments. The most spectacular one being the proof by Kontsevich [7] of the existence of star-products on any smooth Poisson manifold as a consequence of his Formality theorem. This theorem establishes the existence of an L_∞ quasi-isomorphism between two differential graded Lie algebras naturally attached to a manifold, namely, the Hochschild complex endowed with Gerstenhaber bracket on one side, and the space of polyvector fields endowed with the Schouten-Nijenhuis bracket and trivial differential, on the other side.

The proof relies on the discovery of an explicit formula (in the affine case) in terms of Feynman graphs and their weights, and corresponds to the perturbation series of a Poisson sigma model [3]. An important feature of Kontsevich's quantization formula is that it is a universal quantization formula in the sense that it only depends on the Poisson structure and its derivatives of all orders. Moreover, among the family of graphs considered by Kontsevich there are graphs with wheels, i.e. having oriented cycles as subgraphs. The presence of graphs with wheels discards direct generalizations of Kontsevich formula to infinite-dimensional spaces (quantum field theory) as their computation involves traces leading to an ill-defined star-product. Of course for special classes of Poisson brackets it is still possible to find quantization formulas which do not involve wheels. This is the case for the well-known Moyal product or the BCH-quantization [2, 5, 6] on the dual of a Lie algebra, but these examples do not reflect the general situation.

In this paper, we show that a universal quantization formula, i.e. a correspondence that associates to any Poisson bracket defined on the affine space a star-product, does necessarily involves graphs with wheels. The proof consists essentially in reducing the general case to the obstruction found by Penkava and Vanhaecke [9] in their study of the quantization of polynomial Poisson brackets.

The paper is organized as follows. Section 2 is devoted to preliminaries: basic notions on deformation quantization and Kontsevich formula are briefly recalled there. In Section 3, we study universal 2-cocycles in the Hoschchild complex and show that the second cohomology space of the subcomplex consisting of universal cochains without wheels is one-dimensional and is generated by the Poisson bivector. The main theorem is proved in Section 4.

I am indebted to T. Willwacher for letting me know another derivation of the main result of this paper. His proof [11] does not rely on the obstruction of Penkava and Vanhaecke.

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2. Preliminaries

Let X be a smooth d-dimensional manifold. Let $A = C^\infty(X)$ be the algebra of smooth complex-valued functions on X with product $m_0\colon A\times A\to A$, that is, the usual product of smooth functions on X.

2.1. **Deformation quantization.** Recall that a Poisson bivector $\mathfrak{p}$ on X is a section in $\Gamma(\wedge^2 TX)$ such that the Poisson bracket associated to $\mathfrak{p}$

$$\{f,g\} = \langle \mathfrak{p}, df\wedge dg\rangle, \quad f,g\in A,$$

endows A with a structure of Lie algebra satisfying the Leibniz property. In local coordinates $(x^1,\dots,x^d)$ the Poisson bracket reads

$$\{f,g\} = \sum_{1\le i,j\le d} \mathfrak{p}^{ij}\partial_i f\partial_j g,$$

where ∂_k denotes the partial derivative with respect to x^k.

Let $A[\![\hbar]\!]$ be the space of formal series in $\hbar$ with coefficients in A. A star-product [1] is a $\mathbb{C}[\![\hbar]\!]$-bilinear product on $A[\![\hbar]\!]$ which is an associative deformation of the algebra of smooth functions on X:

$$c_\hbar = m_0 + \sum_{k\ge 1}\hbar^k c_k, \tag{1}$$

where the c_k are bidifferential operators vanishing on constants and such that the antisymmetric part of c_1 is equal to the Poisson bivector $\mathfrak{p}$, i.e., a star-product is a noncommutative associative deformation of the pointwise product of functions in the direction of the Poisson bivector.

2.2. **Hochschild cohomology.** The normalized differential Hochschild cochain complex of the associative algebra A with values in itself $\mathscr{C}^\bullet(A,A) = \oplus_{m\ge 0}\mathscr{C}^m(A,A)$ consists of polydifferential operators on X that are vanishing on constants. Locally, an m-cochain $C\in\mathscr{C}^m(A,A)$ has the form

$$C(f_1,\dots,f_m) = \sum C_{\alpha_1\cdots\alpha_m}\partial^{\alpha_1}f_1\cdots\partial^{\alpha_m}f_m, \qquad f_1,\dots,f_m\in A, \tag{2}$$

where the sum is finite and runs over multi-indices $\alpha_i\in\mathbb{N}^d$ such that $|\alpha_i|\ge 1$, and the $C_{\alpha_1\cdots\alpha_m}$ are locally defined smooth functions on X.

As usual, we shall consider the Hochschild complex as a $\mathbb{Z}$-graded vector space with a shift in the degree: $D^\bullet_{\text{poly}}(X) = \mathscr{C}^\bullet(A,A)[1]$. Hence

$$D^k_{\text{poly}}(X) = \begin{cases} \mathscr{C}^{k+1}(A,A) & \text{for } k\ge -1,\\ \{0\} & \text{otherwise.}\end{cases}$$

The Gerstenhaber bracket $[\cdot,\cdot]_G$ on $D^\bullet_{\text{poly}}(X)$ is defined on homogeneous elements $D_i\in D^{k_i}_{\text{poly}}(X)$ by:

$$[D_1,D_2]_G = D_1\circ D_2 - (-1)^{k_1k_2}D_2\circ D_1,$$

where $\circ\colon D^{k_1}_{\text{poly}}(X)\times D^{k_2}_{\text{poly}}(X)\to D^{k_1+k_2}_{\text{poly}}(X)$ is a composition law for polydifferential operators:

$$\begin{aligned}&(D_1\circ D_2)(f_0,\dots,f_{k_1+k_2})\\ &\qquad = \sum_{0\le j\le k_1}(-1)^{jk_2}D_1(f_0,\dots,f_{j-1},D_2(f_j,\dots,f_{j+k_2}),f_{j+k_2+1},\dots f_{k_1+k_2}).\end{aligned}$$

The Hochschild differential is given by $\delta = [m_0,\ \cdot\ ]_G$. It is the standard Hochschild differential d up to a sign: for $D\in D^k_{\text{poly}}(\mathbb{R}^d)$, we have $\delta D = (-1)^k dD$. Recall that $(D^\bullet_{\text{poly}}(X),[\cdot,\cdot]_G,\delta)$ is a differential graded Lie algebra.

Remark 1. *In terms of the Gertenhaber bracket, the associativity of the product (1) is equivalent to the Maurer-Cartan equations:*

$$\delta c_k + \frac{1}{2} \sum_{\substack{a+b=k \\ a,b\geq 1}} [c_a, c_b]_G = 0, \quad \textit{for all } k \geq 1. \tag{3}$$

A deformation $c_\hbar$ is said to define an associative deformation up to order r, if (3) is satisfied for $k = 1, 2, \ldots, r$.

The cohomology $H^\bullet(A,A)$ of the complex $(\mathscr{C}^\bullet(A,A), \delta)$ is the space of polyvectors $\Gamma(\wedge^\bullet TX)$. This is a smooth version of the Hochschild-Kostant-Rosenberg (HKR) theorem due to Vey [10], still called HKR theorem. We denote by $T^\bullet_{\text{poly}}(X)$ the graded vector space $T^\bullet_{\text{poly}}(X) = \oplus_{k\in\mathbb{Z}} T^k_{\text{poly}}(X)$, where

$$T^k_{\text{poly}}(X) = \begin{cases} \Gamma(\wedge^{k+1} TX) & \text{for } k \geq -1, \\ \{0\} & \text{otherwise.} \end{cases}$$

$T^\bullet_{\text{poly}}(X)$ is endowed with the Schouten-Nijenhuis bracket $[\cdot,\cdot]_{SN}$, On decomposable tensors it is given by:

$$\begin{aligned} &[\xi_0 \wedge \cdots \wedge \xi_k, \eta_0 \wedge \cdots \wedge \eta_l]_{SN} \\ &= \sum_{0\leq i\leq k} \sum_{0\leq j\leq l} (-1)^{i+j} [\xi_i, \eta_j] \wedge \xi_0 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_k \wedge \eta_0 \wedge \cdots \wedge \hat{\eta}_j \wedge \cdots \wedge \eta_l. \end{aligned} \tag{4}$$

Recall that the Jacobi identity for a bivector $\pi \in \Gamma(\wedge^2 TX)$ is equivalent to the condition $[\pi,\pi]_{SN} = 0$.

$(T^\bullet_{\text{poly}}(X), [\cdot,\cdot]_{SN})$ is a graded Lie algebra that is considered as a differential graded Lie algebra with trivial differential $\mathbf{0}$.

The Formality theorem [7] establishes the existence of an L_∞ quasi-isomorphism between the differential graded Lie algebras $(T^\bullet_{\text{poly}}(X), [\cdot,\cdot]_{SN}, \mathbf{0})$ and $(D^\bullet_{\text{poly}}(X), [\cdot,\cdot]_G, \delta)$. A remarkable consequence of the Formality theorem is the existence of deformation quantization of any smooth Poisson manifold:

Theorem 1 ([7])**.** *Let $\mathfrak{p}$ be a Poisson bivector on a smooth manifold X. Then there exists a star-product $c_\hbar$ on X such that $\frac{1}{2}\big(c_\hbar(f,g) - c_\hbar(g,f)\big) = \hbar\{f,g\} + O(\hbar^2)$.*

Actually, Kontsevich gives an explicit description of the L_∞ quasi-isomorphism for $X = \mathbb{R}^d$ in terms of graphs and weights. We shall briefly recall a few notions on graphs that we will need in our discussion and refer to [7] for details.

2.3. **Graphs.** A simple directed graph is a graph whose edges are oriented and such that it does not contain loops: or multiple edges: . The indegree (resp. outdegree) of a vertex is the number of edges ending (resp. starting) at that vertex.

Following [7], we consider a family of graphs (with a slight departure from the convention of [7]).

Definition 1. *The set $\mathscr{K}$ consists of all the simple directed graphs Γ satisfying:*

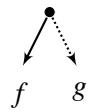
1. *The set of vertices V_Γ is finite and is a disjoint union of nonempty sets: $V_\Gamma = V^1_\Gamma \sqcup V^2_\Gamma$. Vertices belonging to V^1_Γ (resp. V^2_Γ) are said of type 1 (resp. 2);*
2. *Each vertex of type 1 is of outdegree 2;*
3. *Each vertex of type 2 is of indegree at least 1 and of outdegree 0;*
4. *Vertices and edges are labeled.*

We denote by $\mathscr{K}_{n,m}$ for $m,n \geq 1$ the subset of $\mathscr{K}$ consisting of graphs having n vertices of type 1 and m vertices of type 2. Thus a graph in $\mathscr{K}_{n,m}$ has 2n edges.

We shall make a slight abuse of notation by dropping the labels on the vertices of type 1.

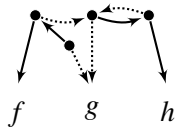
We set $X = \mathbb{R}^d$ $(d \geq 2)$ and $(x^1, \ldots, x^d)$ designates a coordinate system on X. Let $\mathfrak{p}$ be a smooth Poisson bivector on X. The Poisson bracket of two smooth functions f, g is graphically represented by a graph in $\mathscr{K}_{1,2}$:

$$\{f,g\} = \sum_{1 \leq i,j \leq d} \mathfrak{p}^{ij}\, \partial_i f \partial_j g \; =$$



More generally, one associates a cochain $B^{\mathfrak{p}}_{\Gamma}$ in $\mathscr{C}^m(A,A)$ to any graph $\Gamma \in \mathscr{K}_{n,m}$ by letting the edges of Γ act by partial derivatives (see [7] for details.) It is convenient to denote vertices of type 1 by $\bullet$ and vertices of type 2 by letters $f, g, \ldots$ that will eventually be representing functions in A. The labels of the edges will be graphically specified by solid and dotted lines. As an example, to the graph $\Gamma \in \mathscr{K}_{4,3}$ below one associates $B^{\mathfrak{p}}_{\Gamma} \in \mathscr{C}^3(A,A)$ by

(5) $$B^{\mathfrak{p}}_{\Gamma}(f,g,h) =$$



$$= \sum \mathfrak{p}^{i_1 j_1}\, \partial_{i_1} \mathfrak{p}^{i_2 j_2}\, \partial_{j_2 j_3} \mathfrak{p}^{i_3 j_3}\, \partial_{i_3} \mathfrak{p}^{i_4 j_4}\, \partial_{i_2} f\, \partial_{j_1 j_4} g\, \partial_{i_4} h,$$

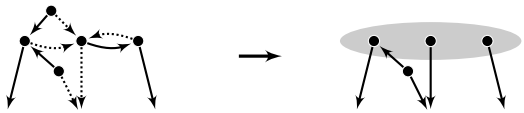
where the sum runs over $1 \leq i_1, j_1, i_2, j_2, i_3, j_3, i_4, j_4 \leq d$. Notice that we have implicitly labeled the vertices of type 1 in the sum, but any other labeling leads to the same cochain.

Kontsevich formula associates to a Poisson bivector $\mathfrak{p}$ on $\mathbb{R}^d$ an associative deformation of (A, m_0)

(6) $$\mathfrak{K}_{\hbar} = m_0 + \sum_{n \geq 1} \hbar^n k^{\mathfrak{p}}_n(f,g), \quad \text{where} \quad k^{\mathfrak{p}}_n(f,g) = \sum_{\Gamma \in \mathscr{K}_{n,2}} w(\Gamma)\, B^{\mathfrak{p}}_{\Gamma},$$

and where $w(\Gamma) \in \mathbb{R}$ is the Kontsevich's weight of the graph Γ, it is independent of the Poisson bivector and the dimension d.

Convention. Sometimes it is immaterial to distinguish between the first and second argument of the Poisson bracket. In such situations we shall simply represent the Poisson bracket by using solid lines for both edges, i.e., . Moreover, when it is not essential to fully draw a graph, but just keep few vertices and edges relevant to our discussion, we shall use a "grey zone" to indicate that not all the vertices or edges in that zone are drawn. This is illustrated by



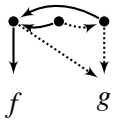
3. Universal 2-cocycles

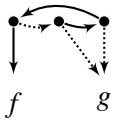
Recall that a directed cycle in a simple directed graph consists of a subgraph for which each vertex is of indegree and outdegree 1. We say that a simple directed graph is with wheels if it has a directed cycle as subgraph. An elementary result from graph theory says that a simple directed graph such that each vertex is of outdegree at least 1 must have a wheel. In the rest of this paper, we set $X = \mathbb{R}^d$ and $A = C^{\infty}(\mathbb{R}^d)$.

Definition 2. *A cochain $C \in \mathscr{C}^m(A,A)$, $m \geq 1$, is called $\mathfrak{p}$-universal if it is a finite $\mathbb{R}$-linear combination of polydifferential operators associated to graphs in $\mathscr{K}_{n,m}$, i.e., C is a finite sum of the form*

$$C = \sum_{n\geq 1} \sum_{\Gamma \in \mathscr{K}_{n,m}} a_\Gamma \, B^{\mathfrak{p}}_\Gamma, \quad a_\Gamma \in \mathbb{R}. \tag{7}$$

A cochain $C \in \mathscr{C}^m(A,A)$, $m \geq 1$, is called $\mathfrak{p}$-universal without wheels if all the graphs Γ appearing in (7) have no wheels.

Example 1. *The 2-cochain $C_1(f,g) =$* 

is $\mathfrak{p}$-universal without wheels, while $C_2(f,g) =$ 

is $\mathfrak{p}$-universal with wheels.

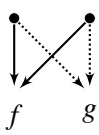
Definition 3. *A universal quantization formula is a correspondence that associates a star-product to a Poisson bivector*

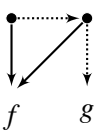
$$\mathfrak{p} \mapsto c^{\mathfrak{p}}_\hbar = m_0 + \sum_{k\geq 1} \hbar^k c^{\mathfrak{p}}_k,$$

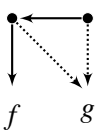
where the $c^{\mathfrak{p}}_k$ are $\mathfrak{p}$-universal 2-cochains.

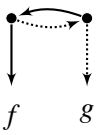
To keep the notation simple, we will omit the reference to $\mathfrak{p}$ in $c^{\mathfrak{p}}_k$ and write c_k, etc.

Example 2. *Kontsevich formula (6) for a star-product is an example of universal quantization formula with wheels. Wheels are already appearing at order 2 in $\hbar$, the 2nd term being explicitly given by* [4]

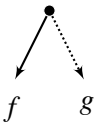
$k_2(f,g) = \frac{1}{2}$ 

$+\frac{1}{3}$ 

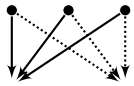
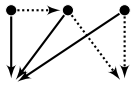
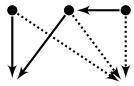
$+\frac{1}{3}$ 

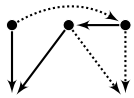
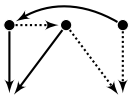
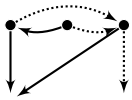
$-\frac{1}{6}$ 

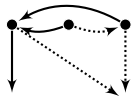
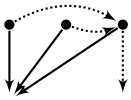
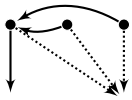
Example 3. *In* [9]*, Penkava and Vanhaecke have constructed a universal quantization formula without wheels associative up to order 3, denoted here by* $m_\hbar = m_0 + \hbar m_1 + \hbar^2 m_2 + \hbar^3 m_3$*, where*

$m_1(f,g) =$ 

$m_2(f,g) = \frac{1}{2}$ $+\frac{1}{3}$ $+\frac{1}{3}$

$m_3(f,g) = \frac{1}{6}$  $+\frac{1}{3}$  $+\frac{1}{3}$ 

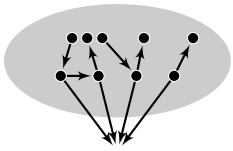
$+\frac{1}{6}$  $+\frac{1}{6}$  $+\frac{1}{3}$ 

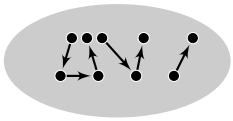
$+\frac{1}{3}$  $+\frac{1}{6}$  $+\frac{1}{6}$ 

For a general Poisson bivector, it was shown in [9] *that* $m_\hbar$ *cannot be extended to a 4th order associative deformation.*

Lemma 1. *Let* Γ *be a graph in* $\mathscr{K}_{n,1}$ *with* $n \geq 2$*, then* Γ *has at least a wheel.*

Proof. Let us call f the vertex of type 2 in Γ. Since Γ is a simple graph, edges ending at f must originate from different vertices of type 1. By removing the vertex f and all the edges ending at f we get a simple directed graph Γ'. This truncation is pictorially represented as:

$\Gamma =$ 

$\longrightarrow$ $\Gamma' =$ 

Each vertex in Γ' is of outdegree at least 1, hence it contains a wheel. □

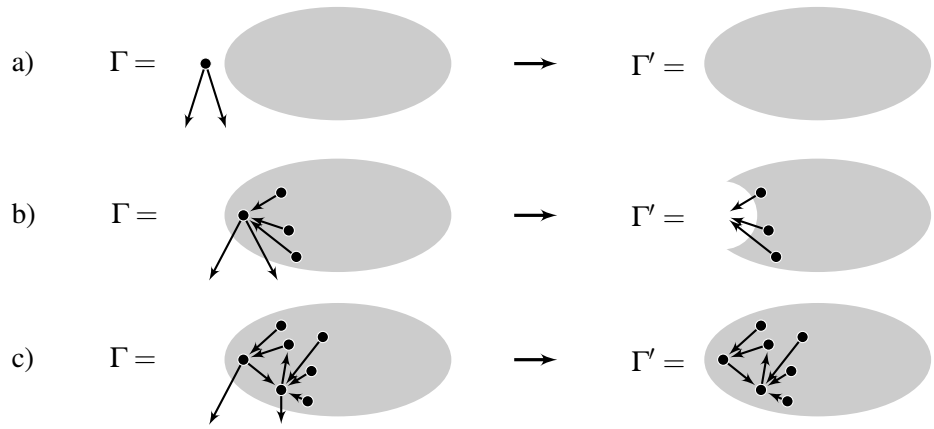
We have excluded the case $n = 1$ in this Lemma since $\mathscr{K}_{1,1}$ is empty.

Lemma 2. *Let* $\Lambda \in \Gamma(\wedge^2 TX) \subset \mathscr{C}^2(A,A)$ *be a bivector. If* Λ *is* $\mathfrak{p}$*-universal without wheels, then* $\Lambda = a\mathfrak{p}$ *for some* $a \in \mathbb{R}$*.*

Proof. Suppose Λ is $\mathfrak{p}$-universal, then it can be represented as a finite sum $\sum_{n\geq 1}\ \sum_{\Gamma\in\mathscr{K}_{n,2}} a_\Gamma\ B^{\mathfrak{p}}_\Gamma$. Each graph Γ appearing in this sum has two vertices of type 2 each of them having indegree 1. We distinguish several cases.

1) If $\Gamma\in\mathscr{K}_{1,2}$, then clearly $B^{\mathfrak{p}}_\Gamma=\pm\mathfrak{p}$.

2) If $\Gamma\in\mathscr{K}_{n,2}$ with $n\geq 2$, then there are three possible subcases to which we apply a similar truncation as the one used in Lemma 1:



For each subcase, when it occurs, the graph Γ' contains at least a wheel. Indeed for a), each vertex of Γ' has outdegree 2; for b), it follows Lemma 1; for c) each vertex of Γ' has outdegree at least 1.

In conclusion, a bivector which is $\mathfrak{p}$-universal without wheels can only be proportional to the Poisson bivector. □

Lemma 3. *Let $T\in\mathscr{C}^1(A,A)$ be such that its coboundary δT is $\mathfrak{p}$-universal without wheels. Then $\delta T=0$.*

Proof. By getting rid of an irrelevant derivation in T, one can consider that T is a $\mathfrak{p}$-universal 1-cochain. Indeed, one can use a homotopy formula (see e.g. [8] for an explicit formula for 2-cocycles) to define a $\mathfrak{p}$-universal 1-cochain T' such that $\delta T'=\delta T$. Since δT is without wheels so will be T', but Lemma 1 tells us that if T' is not vanishing, then it must contain a wheel. Therefore $T'=0$ and $\delta T=0$. □

By the HKR theorem, any cocycle $C\in\mathscr{C}^m(A,A)$ is a sum $C=\Lambda+\delta C'$, where $\Lambda\in\Gamma(\wedge^m TX)$ and $C'\in\mathscr{C}^{m-1}(A,A)$. The m-vector Λ being the antisymmetric part of C then, if C is $\mathfrak{p}$-universal without wheels, so are Λ and $\delta C'$. As an immediate consequence of Lemmas 2 and 3 we have:

Lemma 4. *Let $C\in\mathscr{C}^2(A,A)$ be a Hochschild 2-cocycle which is $\mathfrak{p}$-universal without wheels, then $C=a\,\mathfrak{p}$ for some $a\in\mathbb{R}$.*

Notice that the Hochschild coboundary of a $\mathfrak{p}$-universal cochain is $\mathfrak{p}$-universal and similarly for $\mathfrak{p}$-universal cochains without wheels. Hence one can consider the subcomplex of the Hochschild complex consisting of $\mathfrak{p}$-universal cochains without wheels. In essence, Lemma 4 states that the second cohomology space for this subcomplex is one-dimensional and is generated by $\mathfrak{p}$.

4. The necessity of wheels

Here we establish the main result of this paper by showing that the terms up to order 3 in $\hbar$ of any universal quantization formula without wheels are, up to a change of the deformation parameter, given by the m_i's in Example 3 and hence get a contradiction.

Theorem 2. *Any universal quantization formula involves graphs with wheels.*

Proof. We proceed by contradiction assuming that $s_\hbar = m_0 + \sum_{k\geq 1} \hbar^k s_k$ is a universal quantization formula without wheels.

Associativity implies $\delta s_1 = 0$ and Lemma 4 gives that $s_1 = a_1\,\mathfrak{p}$ for some $a_1 \in \mathbb{R}$. The normalization of the star-product, $\frac{1}{2}\big(s_1(f,g) - s_1(g,f)\big) = \{f,g\}$, forces $a_1 = 1$, hence $s_1 = m_1 = \mathfrak{p}$.

At second order in $\hbar$, the associativity of $s_\hbar$ implies

$$\delta s_2 + \frac{1}{2}[s_1,s_1]_G = 0, \text{ i.e., } \delta s_2 + \frac{1}{2}[m_1,m_1]_G = 0.$$

Since $\frac{1}{2}[m_1,m_1]_G = -\delta m_2$, we find that $s_2 - m_2$ is a 2-cocycle and it is $\mathfrak{p}$-universal without wheels. By Lemma 4 we have $s_2 = m_2 + a_2 m_1$ for some real number a_2.

At third order in $\hbar$, we have

$$\delta s_3 + [s_2,s_1]_G = 0, \text{ i.e., } \delta s_3 + [m_2 + a_2 m_1, m_1]_G = 0.$$

From $[m_2,m_1]_G = -\delta m_3$ and $[m_1,m_1]_G = -2\delta m_2$ we deduce that $s_3 - m_3 - 2a_2 m_2$ is a $\mathfrak{p}$-universal 2-cocycle without wheels. Using Lemma 4 again, we find that $s_3 = m_3 + 2a_2 m_2 + a_3 m_1$ for some real number a_3.

Therefore the first terms in $s_\hbar$ are necessarily of the form:

$$\begin{aligned} s_1 &= m_1, \\ s_2 &= m_2 + a_2 m_1, \\ s_3 &= m_3 + 2a_2 m_2 + a_3 m_1. \end{aligned}$$

The change of parameter $\hbar \to \hbar - a_2\hbar^2 - (a_3 - 2a_2^2)\hbar^3$ in $s_\hbar$ gives us a new universal quantization formula $s'_\hbar$ such that $s'_i = m_i$ for $i = 1,2,3$. But $m_0 + \hbar m_1 + \hbar^2 m_2 + \hbar^3 m_3$ is an associative deformation up to order 3 that cannot be extended to order 4 (cf. Example 3). Hence we have reached a contradiction. □

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