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THE NECESSITY OF WHEELS IN UNIVERSAL QUANTIZATION FORMULAS

GIUSEPPE DITO

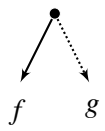
ABSTRACT. In the context of deformation quantization, we provide an elementary argument showing that any universal quantization formula necessarily involves graphs with wheels.

1. NOTATIONS

Throughout this paper, we set $X = \mathbb{R}^d$ ($d \geq 2$) and $(x^1, \ldots, x^d)$ designates a coordinate system on X. Let $\mathfrak{p}$ be a smooth Poisson bivector on X. The Poisson bracket of two smooth functions f, g is given by

$$\{f,g\} = \langle \mathfrak{p}, df \wedge dg \rangle = \sum_{1 \leq i,j \leq d} \mathfrak{p}^{ij} \partial_i f \partial_j g,$$

where ∂_k denotes the partial derivative with respect to x^k. It is graphically represented by

$\{f,g\} =$ 

Let $A = C^\infty(X)$ be the algebra of smooth functions on X with product $m_0 \colon A \otimes A \to A$, i.e., the usual product of smooth functions on $\mathbb{R}^d$. The Hochschild complex $\mathscr{C}^\bullet(A,A) = \oplus_{m \geq 0} \mathscr{C}^m(A,A)$ consists of polydifferential operators on X. One requires that for $m \geq 1$, these polydifferential operators are vanishing on constants. The Hochschild differential is $\delta = [m_0, \cdot\,]_G$, where $[\cdot,\cdot]_G$ is the Gerstenhaber bracket.

1.1. Deformation quantization.

Let $\mathfrak{p}$ be a Poisson bivector on X. Let $c_\hbar = m_0 + \sum_{k \geq 1} \hbar^k c_k$ be an associative deformation of (A, m_0) with antisymmetric part of c_1 being equal to $\mathfrak{p}$. The associativity of $c_\hbar$ is equivalent to the Maurer-Cartan equations:

$$\delta c_k + \frac{1}{2} \sum_{\substack{a+b=k \\ a,b \geq 1}} [c_a, c_b]_G = 0, \text{ for } k = 1, 2 \ldots. \tag{1}$$

A deformation $c_\hbar$ is said to define an associative deformation up to order r, if (1) is satisfied for $k = 1, 2, \ldots, r$.

Following [1], we consider a family of graphs (with a slight departure from the conventions of [1]).

Definition 1. *The set $\mathscr{K}$ consists of simple directed graphs Γ satisfying:*

(1) *The set of vertices V_Γ is finite and is a disjoint union of nonempty sets: $V_\Gamma = V_\Gamma^1 \sqcup V_\Gamma^2$. Vertices belonging to V_Γ^1 (resp. V_Γ^2) are said of type 1 (resp. 2);*
(2) *Each vertex of type 1 is of outdegree 2;*

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(3) *Each vertex of type 2 is of indegree at least 1.*

We denote by $\mathscr{K}_{n,m}$ for $m,n \geq 1$ the subset of $\mathscr{K}$ consisting of graphs having n vertices of type 1 and m vertices of type 2.

Following [1], to a Poisson bivector $\mathfrak{p}$ and a graph $\Gamma \in \mathscr{K}_{n,m}$ is associated a cochain in $\mathscr{C}^m(A,A)$, denoted $B^{\mathfrak{p}}_{\Gamma}$. Kontsevich associates to $\mathfrak{p}$ an associative deformation of (A, m_0)

$$c_\hbar = m_0 + \sum_{n\geq 1} \hbar^n \sum_{\Gamma\in\mathscr{K}_{n,2}} w(\Gamma)\, B^{\mathfrak{p}}_{\Gamma},$$

$w(\Gamma)$ being Kontsevich's weight of the graph Γ.

2. Universal 2-cocycles

We say that a simple directed graph is with wheel (or contain a wheel) if it has a subgraph that is a directed cycle.

Definition 2. *A cochain $C \in \mathscr{C}^m(A,A)$, $m \geq 1$, is called $\mathfrak{p}$-universal if it is a finite $\mathbb{R}$-linear combination of polydifferential operators associated to graphs in $\mathscr{K}_{n,m}$, i.e., C is a finite sum of the form*

$$C = \sum_{n\geq 1} \sum_{\Gamma\in\mathscr{K}_{n,m}} a_\Gamma\, B^{\mathfrak{p}}_{\Gamma}, \quad a_\Gamma \in \mathbb{R}. \tag{2}$$

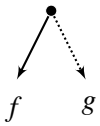
Definition 3. *A cochain $C \in \mathscr{C}^m(A,A)$, $m \geq 1$, is called $\mathfrak{p}$-universal without wheels if all the graphs Γ appearing in (2) have no wheels.*

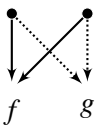
Notice that the Hochschild coboundary of a $\mathfrak{p}$-universal cochain is $\mathfrak{p}$-universal and similarly for $\mathfrak{p}$-universal cochains without wheels.

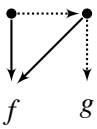
An associative deformation $c_\hbar = m_0 + \sum_{k\geq 1} \hbar^k c_k$ is a universal quantization formula if the c_k are $\mathfrak{p}$-universal 2-cochains.

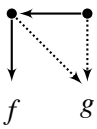
Kontsevich formula [1] for a star-product is an example of universal quantization formula *with wheels* (wheels appear already at order 2 in $\hbar$.)

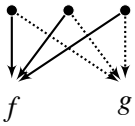
Example 1. *In* [2]*, Penkava and Vanhaecke showed that there is a universal quantization formula without wheels* <u>*up to order 3*</u> *denoted here by* $m_\hbar = m_0 + \hbar m_1 + \hbar^2 m_2 + \hbar^3 m_3$*, where*

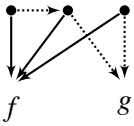
$m_1 f, g) =$ 

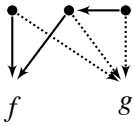
$m_2(f,g) = \frac{1}{2}$ 

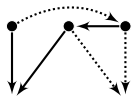
$+\frac{1}{3}$ 

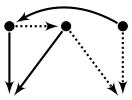
$+\frac{1}{3}$ 

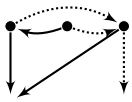
$m_3(f,g) = \frac{1}{6}$ 

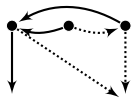
$+\frac{1}{3}$ 

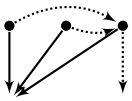
$+\frac{1}{3}$ 

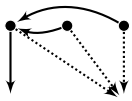
$+\frac{1}{6}$ 

$+\frac{1}{6}$ 

$+\frac{1}{3}$ 

$+\frac{1}{3}$ 

$+\frac{1}{6}$ 

$+\frac{1}{6}$ 

Moreover, it was shown in [2] *that* $m_\hbar$ *cannot be extended to a 4th order associative deformation. These terms can be also obtained by using a kind pseudo-BCH formula.*

Lemma 1. *Let* Γ *be a graph in* $\mathscr{K}_{n,1}$ *with* $n \geq 2$*, then* Γ *has at least a wheel.*

Proof. Let us call f the vertex of type 2 in Γ. Since Γ is a simple graph, edges ending at f must originate from different vertices of type 1. By removing the vertex f and all the edges ending at f we get a simple directed graph Γ'. Each vertex in Γ' is of outdegree at least 1, hence it contains a wheel. □

We have excluded the case $n = 1$ in this Lemma since $\mathscr{K}_{1,1}$ is empty.

Lemma 2. *Let* $\Lambda \in \Gamma(\wedge^2 TX) \subset \mathscr{C}^2(A,A)$ *be a bivector. If* Λ *is* $\mathfrak{p}$*-universal without wheels, then* $\Lambda = a\,\mathfrak{p}$ *for some* $a \in \mathbb{R}$*.*

Proof. We have $\Lambda = \sum_{n\geq 1} \sum_{\Gamma \in \mathscr{K}_{n,2}} a_\Gamma\, B^{\mathfrak{p}}_\Gamma$, where each graph appearing in this sum is a graph without wheels. Let Γ be such a graph. If $\Gamma \in \mathscr{K}_{1,2}$, then $B^{\mathfrak{p}}_\Gamma = \pm\mathfrak{p}$. A direct application of Lemma 1 shows that for $n \geq 2$ there is no Γ without wheels in $\mathscr{K}_{n,2}$ giving rise to a bivector. □

Lemma 3. *Let* $T \in \mathscr{C}^1(A,A)$ *be such that its coboundary* δT *is* $\mathfrak{p}$*-universal without wheels. Then* $\delta T = 0$*.*

Proof. By getting rid of an irrelevant derivation, one can consider that T is a $\mathfrak{p}$-universal 1-cochain. Then the result easily follows from Lemma 1. □

By the HKR theorem, any cocycle $C \in \mathscr{C}^m(A,A)$ is a sum $C = \Lambda + \delta C'$, where $\Lambda \in \Gamma(\wedge^m TX)$ and $C' \in \mathscr{C}^{m-1}(A,A)$. As an immediate consequence we have:

Corollary 1. *Let $C \in \mathscr{C}^2(A,A)$ be a Hochschild cocycle which is $\mathfrak{p}$-universal without wheels, then $C = a\,\mathfrak{p}$ for some $a \in \mathbb{R}$.*

3. No-go theorem

Here we show that the terms up to order 3 in $\hbar$ of any universal quantization formula without wheels are, up to a change of the deformation parameter, given by the m_i's in Example 1 and hence get a contradiction.

Theorem 1. *Any universal quantization formula involves graphs with wheels.*

Proof. We proceed by contradiction assuming that $s_\hbar = m_0 + \sum_{k\geq 1} \hbar^k s_k$ is a universal quantization formula without wheels.

Associativity implies $\delta s_1 = 0$ and Corollary 1 gives that $s_1 = a_1\,\mathfrak{p}$ for some $a_1 \in \mathbb{R}$. The normalization of the star-product forces $a_1 = 1$, hence $s_1 = m_1 = \mathfrak{p}$.

At second order in $\hbar$, the associativity of $s_\hbar$ implies

$$\delta s_2 + \frac{1}{2}[s_1, s_1]_G = 0, \text{ i.e., } \delta s_2 + \frac{1}{2}[m_1, m_1]_G = 0.$$

Since $\frac{1}{2}[m_1, m_1]_G = -\delta m_2$, we find that $s_2 - m_2$ is a 2-cocycle and it is $\mathfrak{p}$-universal without wheels. By Corollary 1 we have $s_2 = m_2 + a_2 m_1$ for some real number a_2.

At third order in $\hbar$, we have

$$\delta s_3 + [s_2, s_1]_G = 0, \text{ i.e., } \delta s_3 + [m_2 + a_2 m_1, m_1]_G = 0.$$

From $[m_2, m_1]_G = -\delta m_3$ and $[m_1, m_1]_G = -2\delta m_2$ we deduce that $s_3 - m_3 - 2a_2 m_2$ is a $\mathfrak{p}$-universal 2-cocycle without wheels. Using Corollary 1 again, we find that $s_3 = m_3 + 2a_2 m_2 + a_3 m_1$ for some real number a_3.

Therefore the first terms in $s_\hbar$ are necessarily of the form:

$$\begin{aligned} s_1 &= m_1, \\ s_2 &= m_2 + a_2 m_1, \\ s_3 &= m_3 + 2a_2 m_2 + a_3 m_1. \end{aligned}$$

The change of parameter $\hbar \to \hbar - a_2\hbar^2 - (a_3 - 2a_2^2)\hbar^3$ in $s_\hbar$ gives us a new universal quantization formula $s'_\hbar$ such that $s'_i = m_i$ for $i = 1,2,3$. But $m_0 + \hbar m_1 + \hbar^2 m_2 + \hbar^3 m_3$ is an associative deformation up to order 3 that cannot be extended to order 4 (cf. Example 1). Hence we have reached a contradiction. □

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